Unstable system dynamics: do we understand it fully?

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Talk overview

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- *Comparison:* relations between the stable and Zeno dynamics in this and other models
- *Regularity of "undisturbed" decay law:* example of the Winter model

Quantum kinematics of decays

Three objects are needed:

- the state space $\mathcal{H}$ of an *isolated system*
- projection P to subspace $P\mathcal{H} \subset \mathcal{H}$ of *unstable system*
- *time evolution* e^{-iHt} on $\mathcal{H}$, not reduced by P for $t > 0$

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Suppose that evolution starts at $t = 0$ from a state $\psi \in P\mathcal{H}$ and we perform a *non-decay measurement* at some $t > 0$

The non-decay probabilities define in this situation the *decay law*, i.e. the function $P: \mathbb{R}_+ \to [0, 1]$ defined by

$$P(t) := \|P\,\mathrm{e}^{-iHt}\psi\|^2\,;$$

we may also denote it as $P_\psi(t)$ to indicate the initial state

Repeated measurements

Suppose we perform non-decay measurements at times $t/n,\, 2t/n\ \dots,\, t$, all with the positive outcome, then the resulting non-decay probability is

$$M_n(t) = P_\psi(t/n)P_{\psi_1}(t/n)\cdots P_{\psi_{n-1}}(t/n)\,,$$

where ψ_{j+1} is the normalized projection of $\mathrm{e}^{-iHt/n}\psi_j$ on $P\mathcal{H}$ and $\psi_0 := \psi$, in particular, for $\dim P = 1$ we have

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Consider the *limit of permanent measurement*, $n\to\infty$. If $\dim P = 1$ and the one-sided derivative $\dot{P}(0+)$ vanishes, we find $M(t) := \lim_{n\to\infty} M_n(t) = 1$ for all $t>0$, or *Zeno effect*. The same is true if $\dim P > 1$ provided the derivative $\dot{P}_\psi(0+)$ has such a property for *any* $\psi\in P\mathcal{H}$.

When does Zeno effect occur?

Recall first a simple (and very old) result:

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Remarks:

- Naturally, $M(t) = P(t)$ if the undisturbed decay law is exponential, i.e. $P(t) = \mathrm{e}^{-\Gamma t}$
- However, $P(t) = \mathrm{e}^{-\Gamma t}$ correspond to a state not belonging to $\mathcal{Q}(H)$. And what is worse, decay law exponentiality requires $\sigma(H) = \mathbb{R}$!

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- New mathematical questions, in particular, about *Zeno dynamics*: what is the time evolution in $P\mathcal{H}$ generated by permanent observation?

Zeno dynamics

Assume that H is *bounded from below* and consider the non-trivial situation, $\dim \mathcal{H} > 1$. We ask: does the limit

$$(Pe^{-iHt/n}P)^n \longrightarrow e^{-iH_P t}$$

hold as $n \to \infty$, in which sense, and what is then Zeno dynamics generator, i.e. the operator H_P?

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Consider quadratic form $u \mapsto \|H^{1/2}Pu\|^2$ with the form domain $D(H^{1/2}P)$ which is closed. By [Chernoff'74] the associated s-a operator, $(H^{1/2}P)^*(H^{1/2}P)$, is a natural candidate for H_P (while, in general, PHP is not!)

Counterexamples in [E.'85] and [Matolcsi-Shvidkoy'03] show, however, that it is necessary to assume that H is *densely defined*

Zeno dynamics, continued

Proposition: Let H be a self-adjoint operator in a *separable* $\mathcal{H}$, *bounded from below*, and let P be a *finite-dimensional* orthogonal projection on $\mathcal{H}$. If $P\mathcal{H} \subset \mathcal{Q}(H)$, then for any $\psi \in \mathcal{H}$ and $t \geq 0$ we have

$$\lim_{n\to\infty} (P\mathrm{e}^{-iHt/n}P)^n \psi = \mathrm{e}^{-iH_P t}\psi\,,$$

uniformly on any compact interval of the variable t

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Proof *(following Graf & Guekos):* (i) We need to check

$$\lim_{t\to 0} t^{-1} \left\| P\mathrm{e}^{-itH}P - P\mathrm{e}^{-itH_P}P \right\| = 0\,,$$

since it implies $\left\| (P\mathrm{e}^{-itH/n}P)^n - \mathrm{e}^{-itH_P} \right\| = n\, o(t/n)$ as $n \to \infty$ by means of a natural telescopic estimate

Zeno dynamics, continued

One may assume $H \geq cI$, $c > 0$. First we first prove that

$$t^{-1}\left[(f, P\mathrm{e}^{-itH}Pg) - (f, g) - it(\sqrt{H}Pf, \sqrt{H}Pg)\right] \longrightarrow 0$$

as $t \to 0$ for all f, g from $D(\sqrt{H}P) = P\mathcal{H}$. The LHS equals $\left(\sqrt{H}Pf, \left[\frac{e^{-itH}-I}{tH} - i\right]\sqrt{H}Pg\right)$ and the square bracket tends to zero strongly.

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$$t^{-1}\left[(f, Pe^{-itH_P}Pg) - (f,g) - it(\sqrt{H_P}f, \sqrt{H_P}g)\right] \longrightarrow 0$$

holds as $t \to 0$ for any $f, g \in P\mathcal{H}$. Next we note that $(\sqrt{H_P}f, \sqrt{H_P}g) = (\sqrt{H}Pf, \sqrt{H}Pg)$, and consequently, $t^{-1}(P\mathrm{e}^{-itH}P - P\mathrm{e}^{-itH_P}P) \to 0$ weakly as $t \to 0$, however, the two topologies are equivalent if $\dim P < \infty$. $\square$

Zeno dynamics, continued

Without the restriction, situation is more complicated:

Theorem [E.-Ichinose '04]: Under same assumptions, except that P *can be arbitrary*, we have for any $T > 0$

$$\lim_{n\to\infty} \int_0^T \|(P\mathrm{e}^{-iHt/n}P)^n\psi - \mathrm{e}^{-iH_P t}\psi\|^2\, \mathrm{d}t = 0$$

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Theorem [E.-Neidhardt-Ichinose-Zagrebnov '06]: Under same assumptions, except that *$\mathcal{H}$ need not be separable*

$$\lim_{n\to\infty} (PE_H([0,\pi n/t))\,\mathrm{e}^{-iHt/n}P)^n\psi = \mathrm{e}^{-iH_Pt}\psi\,,$$

uniformly on any compact interval of the variable t, and same for $(P\phi(tH/n)P)^n$ with $|\phi(x)| \le 1$, $\phi(0) = 1$, $\phi'(0) = -i$

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Corollary: Strong convergence holds provided $\|H\| < \infty$

Measurements again: what is anti-Zeno?

Let us now return to “Zeno-type” non-decay probability, $M_n(t) = P_\psi(t/n)P_{\psi_1}(t/n)\cdots P_{\psi_{n-1}}(t/n)$, where ψ_{j+1} are as before, in particular, to the formula

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for $\dim P = 1$.

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for $\dim P = 1$.Since $\lim_{n\to\infty}(f(t/n)^n = \exp\{-\dot{f}(0+)t\}$ if $f(0) = 1$ and the one-sided derivative $\dot{f}(0+)$ exists we see that $M(t) := \lim_{n\to\infty} M_n(t) = 0$ for $\forall t > 0$ if $\dot{P}(0+) = -\infty$, and the same is true if $\dim P > 1$ provided the derivative $\dot{P}_\psi(0+)$ has such a property for *any* $\psi \in P\mathcal{H}$.

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It is idealization, of course, but validity of such idealizations is the heart and soul of theoretical physics and has the same fundamental significance as the reproducibility of experimental data [Bratelli-Robinson’79]

Decay probability estimate

We need to estimate the quantity $1 - P(t)$, in other words $(\psi, P\psi) - (\psi, \mathrm{e}^{iHt} P \mathrm{e}^{-iHt}\psi)$. We rewrite it as

$$1 - P(t) = 2\,\mathrm{Re}\,(\psi, P(I - e^{-iHt})\psi) - \|P(I - e^{-iHt})\psi\|^2$$

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In terms of the spectral measure E_H of H the r.h.s. equals

$$4\int_{-\infty}^{\infty} \sin^2\frac{\lambda t}{2}\, d\|E_\lambda^H\psi\|^2 - 4\left\|\int_{-\infty}^{\infty} \mathrm{e}^{-i\lambda t/2}\,\sin\frac{\lambda t}{2}\, dPE_\lambda^H\psi\right\|^2$$

By Schwarz it is non-negative; our aim is to find tighter upper and lower bounds. In particular, for $\dim P = 1$ we denote $d\omega(\lambda) := d(\psi, E_\lambda^H\psi)$ obtaining

$$4\int_{-\infty}^{\infty} \sin^2\frac{\lambda t}{2}\, d\omega(\lambda) - 4\left|\int_{-\infty}^{\infty} e^{-i\lambda t/2}\,\sin\frac{\lambda t}{2}\, d\omega(\lambda)\right|^2$$

The one-dimensional case

Let first $\dim P = 1$. One can employ the spectral-measure normalization, $\int_{-\infty}^{\infty} d\omega(\lambda) = 1$, to rewrite the decay probability in the following way

$$2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\sin^2\frac{\lambda t}{2}+\sin^2\frac{\mu t}{2}\right)d\omega(\lambda)d\omega(\mu)$$
$$-4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cos\frac{(\lambda-\mu)t}{2}\,\sin\frac{\lambda t}{2}\,\sin\frac{\mu t}{2}\,d\omega(\lambda)d\omega(\mu)\,,$$

or equivalently

$$1-P(t) = 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sin^2\frac{(\lambda-\mu)t}{2}\,d\omega(\lambda)d\omega(\mu)$$

We can thus try to estimate the integrated function

An estimate from above

Take $\alpha \in (0,2]$. Using $|x|^\alpha \geq |\sin x|^\alpha \geq \sin^2 x$ together with $|\lambda-\mu|^\alpha \leq 2^\alpha(|\lambda|^\alpha + |\mu|^\alpha)$ we infer from the above formula

$$\frac{1-P(t)}{t^\alpha} \leq 2^{1-\alpha} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\lambda-\mu|^\alpha d\omega(\lambda) d\omega(\mu)$$

$$\leq 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (|\lambda|^\alpha + |\mu|^\alpha) d\omega(\lambda) d\omega(\mu) \leq 4\langle |H|^\alpha \rangle_\psi$$

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Hence $1 - P(t) = \mathcal{O}(t^\alpha)$ if $\psi \in D(|H|^{\alpha/2})$. If this is true for some $\alpha > 1$ we have *Zeno effect* – which is a slightly weaker sufficient condition than the earlier stated one

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By negation, $\psi \notin D(|H|^{1/2})$ is a *necessary condition* for the *anti-Zeno effect*. Notice that in case $\psi \in \mathcal{H}_{\mathrm{ac}}(H)$ the same follows from *Lipschitz regularity*, since $P(t) = |\hat{\omega}(t)|^2$ and $\hat{\omega}$ is bd and uniformly α-Lipschitz *iff* $\int_{\mathbb{R}} \omega(\lambda)(1 + |\lambda|^\alpha)\, d\lambda < \infty$

An estimate from below

To find a *sufficient condition* note that for $\lambda, \mu \in [-1/t, 1/t]$ there is a positive C independent of t such that

$$\left| \sin \frac{(\lambda - \mu)t}{2} \right| \geq C|\lambda - \mu|t\,;$$

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one can make the constant explicit but it is not necessary. Consequently, we have the estimate

$$1 - P(t) \geq 2C^2t^2 \int_{-1/t}^{1/t} d\omega(\lambda) \int_{-1/t}^{1/t} d\omega(\mu)(\lambda - \mu)^2$$

which in turn implies

$$\frac{1 - P(t)}{t} \geq 4C^2 t \left\{ \int_{-1/t}^{1/t} \lambda^2 \, d\omega(\lambda) \int_{-1/t}^{1/t} d\omega(\lambda) - \left(\int_{-1/t}^{1/t} \lambda \, d\omega(\lambda) \right)^2 \right\}$$

Sufficient conditions

The AZ effect occurs if the r.h.s. diverges as $t \to 0$, e.g., if

$$\int_{-N}^{N} \lambda^2 \, d\omega(\lambda) \int_{-N}^{N} d\omega(\lambda) - \left(\int_{-N}^{N} \lambda \, d\omega(\lambda) \right)^2 \geq c N^\alpha$$

holds for any N and some $c > 0$, $\alpha > 1$

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We can also write it in a more compact form: introduce $H_N^\beta := H^\beta E_H(\Delta_N)$ with the spectral cut-off to the interval $\Delta_N := (-N, N)$, in particular, denote $I_N := E_H(-N, N)$. The sufficient condition then reads

$$\left(\langle H_N^2 \rangle_\psi \langle I_N \rangle_\psi - \langle H_N \rangle_\psi^2 \right)^{-1} = o(N) \quad \text{as} \quad N \to \infty$$

More on the one-dimensional case

Remark: Notice that the condition does *not* require the Hamiltonian H to be unbounded, in contrast to exponential exponential decay; it is enough that the spectral distribution has a slow decay in one direction only

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Example: Consider H bd from below and ψ from $\mathcal{H}_{\rm ac}(H)$ s.t. $\omega(\lambda) \approx c\lambda^{-\beta}$ as $\lambda \to +\infty$ for some $c > 0$ and $\beta \in (1,2)$. While $\int_{-N}^{N} \omega(\lambda)\, d\lambda \to 1$, the other two integrals diverge giving

$$cN^{2-\beta} - c^2 N^{4-2\beta}$$

as the asymptotic behavior of the l.h.s., where the first term is dominating; it gives $\dot{P}(0+) = -\infty$ so AZ effect occurs.

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Remarks: For $\beta > 2$ we have Zeno effect, so *the Z-AZ gap is rather narrow!* Also, $\beta = 2$ with a cut-off may give rapid oscillations around $t = 0$ obscuring existence of Zeno limit

Multiple degrees of freedom

Let $\dim P > 1$ and denote by $\{\chi_j\}$ an orthonormal basis in $P\mathcal{H}$. The second term in the decay-probability formula is

$$-4\sum_m \left| \int_{-\infty}^{\infty} e^{-i\lambda t/2} \sin\frac{\lambda t}{2}\, d(\chi_m, E_\lambda^H \psi) \right|^2$$

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We also expand $\psi = \sum_j c_j \chi_j$ with $\sum_j |c_j|^2 = 1$ and denote $d\omega_{jk}(\lambda) := d(\chi_j, E_\lambda^H \chi_k)$, which is real-valued and symmetric w.r.t. index interchange. Using $d\|E_\lambda^H \psi\|^2 = \sum_{jk} \bar{c}_j c_k d\omega_{jk}(\lambda)$ we can cast the decay-probability into the form

$$1 - P(t) = 4\sum_{jk} \bar{c}_j c_k \bigg\{ \int_{-\infty}^{\infty} \sin^2\frac{\lambda t}{2}\, d\omega_{jk}(\lambda)$$

$$- \sum_m \int_{-\infty}^{\infty} e^{-i\lambda t/2} \sin\frac{\lambda t}{2}\, d\omega_{jm}(\lambda) \int_{-\infty}^{\infty} e^{i\mu t/2} \sin\frac{\mu t}{2}\, d\omega_{km}(\mu) \bigg\} \tag{2}$$

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Next we employ normalization, $\int_{-\infty}^{\infty} d\omega_{jk}(\lambda) = \delta_{jk}$, to derive

$$1 - P(t) = 2 \sum_{jkm} \bar{c}_j c_k \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sin^2 \frac{(\lambda - \mu)t}{2} \, d\omega_{jm}(\lambda) d\omega_{km}(\mu)$$

which can be also written concisely as

$$1 - P(t) = 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sin^2 \frac{(\lambda - \mu)t}{2} \, (\psi, dE_\lambda^H P dE_\mu^H \psi)$$

General sufficient condition

Since $\left|\sin\frac{(\lambda-\mu)t}{2}\right| \ge C|\lambda-\mu|t$ holds for $|\mu t|, |\lambda t| < 1$ we get

$$\begin{aligned} 1-P(t) &\ge 2C^2t^2 \int_{-1/t}^{1/t}\int_{-1/t}^{1/t} (\lambda-\mu)^2\,(\psi, dE_\lambda^H P dE_\mu^H \psi) \\ &= 4C^2t^2 \int_{-1/t}^{1/t}\int_{-1/t}^{1/t} (\lambda^2-\lambda\mu)\,(\psi, dE_\lambda^H P dE_\mu^H \psi) \\ &= 4C^2t^2 \left\{ (\psi, H_{1/t}^2 P I_{1/t}\psi) - \|PH_{1/t}\psi\|^2 \right\} \end{aligned}$$

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Let us summarize the results:

Theorem [E.'05]: In the above notation, suppose that

$$\left(\langle H_N^2 P I_N\rangle_\psi - \|PH_N\psi\|^2\right)^{-1} = o(N)$$

holds as $N\to\infty$ uniformly w.r.t. $\psi \in P\mathcal{H}$, then the permanent observation causes anti-Zeno effect

An interlude: a caricature model

An idealized description of a *quantum wire* and a family of *quantum dots*. Formally Hamiltonian acts in $L^2(\mathbb{R}^2)$ as

$$H_{\alpha,\beta} = -\Delta - \alpha\delta(x-\Sigma) + \sum_{i=1}^{n} \tilde{\beta}_i \delta(x - y^{(i)})\,, \;\; \alpha > 0\,,$$

where $\Sigma := \{(x_1, 0);\, x_1 \in \mathbb{R}^2\}$ and $\Pi := \{y^{(i)}\}_{i=1}^{n} \subset \mathbb{R}^2 \setminus \Sigma$

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Singular interactions defined conventionally through b.c.: we have $\partial_{x_2}\psi(x_1,0+) - \partial_{x_2}\psi(x_1,0-) = -\alpha\psi(x_1,0)$ for the line; around $y^{(i)}$ the wave functions have to behave as $\psi(x) = -\frac{1}{2\pi}\log|x-y^{(i)}|\,L_0(\psi,y^{(i)}) + L_1(\psi,y^{(i)}) + \mathcal{O}(|x-y^{(i)}|)$, where the generalized b.v. $L_j(\psi,y^{(i)}),\, j=0,1,$ satisfy

$$L_1(\psi,y^{(i)}) + 2\pi\beta_i L_0(\psi,y^{(i)}) = 0\,, \quad \beta_i \in \mathbb{R}$$

Resolvent by Krein-type formula

- We introduce auxiliary Hilbert spaces, $\mathcal{H}_0 := L^2(\mathbb{R})$ and $\mathcal{H}_1 := \mathbb{C}^n$, and trace maps $\tau_j : W^{2,2}(\mathbb{R}^2) \to \mathcal{H}_j$ defined by $\tau_0 f := f \restriction_\Sigma$ and $\tau_1 f := f \restriction_\Pi$,

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- canonical embeddings of free resolvent $\mathbf{R}(z)$ to $\mathcal{H}_i$ by $\mathbf{R}_{i,L}(z) := \tau_i R(z) : \ L^2 \to \mathcal{H}_i$, $\mathbf{R}_{L,i}(z) := [\mathbf{R}_{i,L}(z)]^*$, and $\mathbf{R}_{j,i}(z) := \tau_j \mathbf{R}_{L,i}(z) : \mathcal{H}_i \to \mathcal{H}_j$, and

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- operator-valued matrix $\Gamma(z) : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{H}_0 \oplus \mathcal{H}_1$ by

$$\begin{aligned} \Gamma_{ij}(z)g &:= -\mathbf{R}_{i,j}(z)g \quad \text{for} \quad i \neq j \quad \text{and} \quad g \in \mathcal{H}_j \,, \\ \Gamma_{00}(z)f &:= \left[\alpha^{-1} - \mathbf{R}_{0,0}(z)\right] f \quad \text{if} \quad f \in \mathcal{H}_0 \,, \\ \Gamma_{11}(z)\varphi &:= \Big(s_\beta(z)\delta_{kl} - G_z(y^{(k)}, y^{(l)})(1-\delta_{kl}) \Big)\, \varphi \,, \end{aligned}$$

with $s_\beta(z) := \beta + s(z) := \beta + \frac{1}{2\pi}(\ln \frac{\sqrt{z}}{2i} - \psi(1))$

Resolvent by Krein-type formula

To invert it we define the "reduced determinant"

$$D(z) := \Gamma_{11}(z) - \Gamma_{10}(z)\Gamma_{00}(z)^{-1}\Gamma_{01}(z) \ : \ \mathcal{H}_1 \to \mathcal{H}_1 \, ,$$

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then an easy algebra yields expressions for “blocks” of $[\Gamma(z)]^{-1}$ in the form

$$\begin{aligned} [\Gamma(z)]^{-1}_{11} &= D(z)^{-1} \,, \\ [\Gamma(z)]^{-1}_{00} &= \Gamma_{00}(z)^{-1} + \Gamma_{00}(z)^{-1}\Gamma_{01}(z)D(z)^{-1}\Gamma_{10}(z)\Gamma_{00}(z)^{-1} \,, \\ [\Gamma(z)]^{-1}_{01} &= -\Gamma_{00}(z)^{-1}\Gamma_{01}(z)D(z)^{-1} \,, \\ [\Gamma(z)]^{-1}_{10} &= -D(z)^{-1}\Gamma_{10}(z)\Gamma_{00}(z)^{-1} \,; \end{aligned}$$

thus to determine singularities of $[\Gamma(z)]^{-1}$ one has to find the null space of $D(z)$

Resolvent by Krein-type formula

We can write $R_{\alpha,\beta}(z) \equiv (H_{\alpha,\beta} - z)^{-1}$ also as a perturbation of the "line only" Hamiltonian $\tilde{H}_\alpha$ with the resolvent

$$R_\alpha(z) = R(z) + R_{L0}(z)\Gamma_{00}^{-1} R_{0L}(z)$$

We define $\mathbf{R}_{\alpha;L1}(z) : \mathcal{H}_1 \to L^2$ by $\mathbf{R}_{\alpha;1L}(z)\psi := R_\alpha(z)\psi \upharpoonright_\Pi$ for $\psi \in L^2$ and $\mathbf{R}_{\alpha;L1}(z) := \mathbf{R}^*_{\alpha;1L}(z)$. Then we have the result:

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Theorem [E.-Kondej '04]: For $z \in \rho(H_{\alpha,\beta})$ with $\operatorname{Im} z > 0$ the resolvent $R_{\alpha,\beta}(z) := (H_{\alpha,\beta} - z)^{-1}$ equals

$$\begin{aligned} R_{\alpha,\beta}(z) &= R(z) + \sum_{i,j=0}^{1} \mathbf{R}_{L,i}(z)[\Gamma(z)]_{ij}^{-1}\mathbf{R}_{j,L}(z) \\ &= R_\alpha(z) + \mathbf{R}_{\alpha;L1}(z)D(z)^{-1}\mathbf{R}_{\alpha;1L}(z) \end{aligned}$$

Resonance poles

The decay is due to the *tunneling between points and line*. It is absent if the interaction is "switched off" (i.e., line "put to an infinite distance"); the corresponding *free Hamiltonian* is $\tilde{H}_\beta := H_{0,\beta}$. It has m eigenvalues, $1 \le m \le n$; we assume that they satisfy the condition

$$-\frac{1}{4}\alpha^2 < \epsilon_1 < \cdots < \epsilon_m < 0 \quad \text{and} \quad m > 1\,,$$

i.e., the embedded spectrum is simple and non-trivial.

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Let us specify the interactions sites by their Cartesian coordinates, $y^{(i)} = (c_i, a_i)$. We also introduce the notations $a = (a_1, ..., a_n)$ and $d_{ij} = |y^{(i)} - y^{(j)}|$ for the distances in Π

To find resonances in our model we rely on a BS-type argument; our aim is to find zeros of the function $D(\cdot)$

Resonance poles, continued

We seek analytic continuation of $D(\cdot)$ across $(-\frac{1}{4}\alpha^2, 0) \subset \mathbb{R}$ denoting it as $D(\cdot)^{(-1)}$. The first component of $\Gamma_{11}(\cdot)^{(l)}$ is obtained easily. To find the second one let us introduce

$$\mu_{ij}(z,t) := \frac{i\alpha}{2^5\pi} \frac{(\alpha - 2i(z-t)^{1/2})\, \mathrm{e}^{i(z-t)^{1/2}(|a_i|+|a_j|)}}{t^{1/2}(z-t)^{1/2}}\, \mathrm{e}^{it^{1/2}(c_i-c_j)}\,.$$

Then the matrix elements of $(\Gamma_{10}\Gamma_{00}^{-1}\Gamma_{01})^{(\cdot)}(\cdot)$ are

$$\theta_{ij}^{(-1)}(\lambda) = -\int_0^\infty \frac{\mu_{ij}^0(\lambda,t)}{t-\lambda-\alpha^2/4}\, \mathrm{d}t - 2g_{\alpha,ij}(\lambda)$$

where

$$g_{\alpha,ij}(z) := \frac{i\alpha}{(z+\alpha^2/4)^{1/2}}\, \mathrm{e}^{-\alpha(|a_i|+|a_j|)/2}\, \mathrm{e}^{i(z+\alpha^2/4)^{1/2}(c_i-c_j)}\,;$$

the values at the segment and in $\mathbb{C}_+$ are expressed similarly

Resonance poles, continued

Then we can express $\det D^{(-1)}(z)$. To study *weak-coupling asymptotics* it is useful to introduce a *reparametrization*

$$\tilde{b}(a) \equiv (b_1(a), \dots, b_n(a)), \quad b_i(a) = \mathrm{e}^{-|a_i|\sqrt{-\epsilon_i}}$$

denoting the quantity of interest as $\eta(\tilde{b}, z) = \det D^{(-1)}(a, z)$

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If $\tilde{b} = 0$ the zeros are, of course, ev's of the point-interaction Hamiltonian $\tilde{H}_\beta$. Using implicit-function theorem we find the following weak-coupling asymptotic expansion,

$$z_i(b) = \epsilon_i + \mathcal{O}(b) + i\mathcal{O}(b) \quad \text{where} \quad b := \max_{1 \le i \le m} b_i$$

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Remark: This model can exhibit also other long-living resonances due to *weakly violated mirror symmetry*, however, we are not going to consider them here

Dot states

By assumption there is a nontrivial discrete spectrum of $\tilde{H}_\beta$ embedded in $(-\frac{1}{4}\alpha^2, 0)$. Let us denote the corresponding normalized eigenfunctions $\psi_j\,,\ j = 1, \dots, m$, given by

$$\psi_j(x) = \sum_{i=1}^{m} d_i^{(j)} \phi_i^{(j)}(x)\,, \quad \phi_i^{(j)}(x) := \sqrt{-\frac{\epsilon_j}{\pi}}\, K_0(\sqrt{-\epsilon_j}|x - y^{(i)}|),$$

where vectors $d^{(j)} \in \mathbb{C}^m$ solve the equation $\Gamma_{11}(\epsilon_j) d^{(j)} = 0$ and the normalization condition, $\|\phi_i^{(j)}\| = 1$, reads

$$|d^{(j)}|^2 + 2\mathrm{Re} \sum_{i=2}^{m} \sum_{k=1}^{i-1} \overline{d_i^{(j)}} d_k^{(j)} (\phi_i^{(j)}, \phi_k^{(j)}) = 1\,.$$

In particular, if the distances in Π are large (the natural length scale is given by $(-\epsilon_j)^{-1/2}$), the cross terms are small and each $|d^{(j)}|$ is close to one

Decay of the dot states

Now we specify the unstable system identifying its Hilbert space $P\mathcal{H}$ with the span of $\psi_1, \dots, \psi_m$. If it is prepared at $t = 0$ in a state $\psi \in P\mathcal{H}$, then the *undisturbed decay law* is

$$P_\psi(t) = \|P\mathrm{e}^{-iH_{\alpha,\beta}t}\psi\|^2$$

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Our model is similar to (multidimensional) *Friedrichs model*, therefore modifying the standard argument [Demuth'76], cf. [E.-Ichinose-Kondej'05], one can check that in the *weak-coupling situation* the leading term in $P_\psi(t)$ will come from the appropriate semigroup evolution on $P\mathcal{H}$, in particular, for the basis states ψ_j we will have a dominantly exponential decay, $P_{\psi_j}(t) \approx \mathrm{e}^{-\Gamma_j t}$ with $\Gamma_j = 2\,\mathrm{Im}\, z_j(b)$

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Remark: The *long-time behaviour* of $P_{\psi_j}(t)$ is different from Friedrichs model, but this is not important here

Comparison: stable and Zeno dynamics

Suppose now that we perform the Zeno measurement at our system. We have $\dim P < \infty$ and $P\mathcal{H} \subset \mathcal{Q}(H_{\alpha,\beta})$, so $H_P = PH_{\alpha,\beta}P$ with the following matrix representation

$$(\psi_j, H_P\psi_k) = \delta_{jk}\epsilon_j - \alpha \int_\Sigma \bar{\psi}_j(x_1,0)\psi_k(x_1,0)\,\mathrm{d}x_1\,,$$

where the first term corresponds, of course, to $\tilde{H}_\beta$

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Theorem [E.-Ichinose-Kondej'05]: The two dynamics do not differ significantly for times satisfying

$$t \ll C\,\mathrm{e}^{2\sqrt{-\epsilon}|\tilde{a}|}\,,$$

where C is a positive number and $|\tilde{a}| = \min_i |a_i|$, $\epsilon = \max_i \epsilon_i$

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Proof: The norm of $\mathcal{U}_t := (\mathrm{e}^{-i\tilde{H}_\beta t} - \mathrm{e}^{-iH_P t})P$ is small as long as $t\|(\tilde{H}_\beta - H_P)P\| \ll 1$; to see when this is true one has to estimate contribution of the cross-terms. $\square$

There are more possibilities

It can happen that the *two dynamics are identical*. Choose, e.g., $H_0 := -\Delta_{\Omega}^{D} \oplus -\Delta_{\Omega^c}^{D}$, where $\Omega^c := \mathbb{R}^d \setminus \bar{\Omega}$, and suppose that “switching in” the decay means to remove the Dirichlet barrier between the two complementary regions.

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In this case the Zeno-type permanent observation obviously *restores the stable dynamics*

On the other hand, the two dynamics *can be different from the outset*. Replace H_0 above by the *Neumann* direct sum $-\Delta_\Omega^N \oplus -\Delta_{\Omega^c}^N$, so the Zeno and stable time-evolution generators in $P\mathcal{H}$ are $-\Delta_\Omega^N$ and $-\Delta_\Omega^D$, respectively.

If Ω is precompact and ψ is Neumann ground state, $\psi(x) = |\Omega|^{-1/2}$, it is unchanged under the stable dynamics while under Zeno one it can evolve into a function which can be *fractal for almost all times* [Berry'96, Thaller'00]

Back to "unperturbed" decay

The last example inspires to ask how the "unperturbed" decay law can look like, say, when ψ *is not in the domain* of the "stable" Hamiltonian.

Guess: an irregular behaviour expected when the decay is due to a (weak) tunneling through a potential barrier

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Guess: an irregular behaviour expected when the decay is due to a (weak) tunneling through a potential barrier

For definiteness we consider the so-called *Winter model*,

$$H_\alpha = -\Delta + \alpha\delta(|x| - R)\,, \quad \alpha > 0\,,\; R > 0\,;$$

we restrict our attention to the s-wave reducing the task to one-dimensional problem having the Hamiltonian on $L^2(\mathbb{R}_+)$

$$h_\alpha = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \alpha\delta(r - R)$$

with $D(h_\alpha) = \{\phi \in W^{2,2}(\mathbb{R}_+) : \phi(0) = 0,\, \phi'(R+) - \phi'(R-) = \alpha\phi(R)\}$

Decay in Winter model

Using $\psi(\vec{r}, t) = e^{-iH_\alpha t}\psi(\vec{r}, 0)$ and the reduced wave function, $\psi(\vec{r}, t) = \frac{1}{\sqrt{4\pi}} r^{-1}\phi(r, t)$, we can express the decay law as

$$P(t) = \int_0^R |\phi(r, t)|^2 \, \mathrm{d}r$$

with the initial state $\phi(\cdot, 0)$ support contained in $B_R(0)$

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The model is solvable and the time evolution can be expressed through the integral kernel of the resolvent,

$$\mathrm{e}^{-ih_\alpha t} = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_0^\infty \mathrm{e}^{-i\lambda t}\, \mathrm{Im}\, \frac{1}{h_\alpha - \lambda - i\varepsilon}\, \mathrm{d}\lambda\,;$$

recall that $\sigma(h_\alpha) = [0,\infty)$ for $\alpha > 0$

Green's function

The resolvent kernel is given by Krein's formula,

$$\frac{1}{h_\alpha - k^2} = \frac{1}{h_0 - k^2} + \lambda(k)(\Phi_k, \cdot)\Phi_k(r)\,,$$

where $\Phi_k(r) = \frac{1}{k}\sin(kr)\,\mathrm{e}^{ikR}$ holds for $r < R$, and by a direct calculation one finds $\lambda(k) = -\alpha\left(1 + \frac{i\alpha}{2k}(1 - \mathrm{e}^{2ikR})\right)^{-1}$

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This gives $u(t, r, r') = \frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\int_0^\infty \mathrm{e}^{-ik^2t}\,p(k, r, r')\,2k\,\mathrm{d}k$ for the integral kernel of the evolution operator $\mathrm{e}^{-ih_\alpha t}$, where

$$p(k, r, r') = \frac{2k\sin(kr)\sin(kr')}{\pi(2k^2 + 2\alpha^2\sin^2 kR + 2k\alpha\sin 2kR)}$$

Resonance expansion

Singularities of $p(\cdot,\, r,\, r')$ are resonances of the problem and their mirror images, $S = \{k_n,\, -k_n,\, \bar{k}_n,\, -\bar{k}_n\, :\; n\in\mathbb{N}\}$, around which the function behaves as

$$p(k,\, r,\, r') = \frac{i}{2\pi}\,\frac{v_n(r)v_n(r')}{k^2-k_n^2} + \chi(k,\, r,\, r')\,,$$

where v_n solves $h_\alpha v_n(r) = k_n^2 v_n(r)$ and χ is locally analytic

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For $r,\, r' < R$ the function $p(\cdot,\, r,\, r')$ decreases in every direction of the k-plane; thus it can be expressed as sum over the pole singularities

$$p(k,\, r,\, r') = \sum_{\tilde{k} \in S} \frac{1}{k - \tilde{k}}\, \mathrm{Res}_{\tilde{k}}\, p(k,\, r,\, r')$$

and by residue theorem we have $\sum_{\tilde{k} \in S} \mathrm{Res}_{\tilde{k}}\, p(k,\, r,\, r') = 0$

Resonance expansion, continued

Using symmetry of S and $k_{-n} := -\bar{k}_n$ we get

$$p(k,\, r,\, r') = \sum_{n\in\mathbb{Z}} \frac{i}{2\pi}\, \frac{1}{k^2 - k_n^2}\, \frac{k}{k_n}\, v_n(r) v_n(r')\,,$$

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Using symmetry of S and $k_{-n} := -\bar{k}_n$ we get

$$p(k, r, r') = \sum_{n\in\mathbb{Z}} \frac{i}{2\pi} \frac{1}{k^2 - k_n^2} \frac{k}{k_n} v_n(r)v_n(r'),$$

$$\sum_{n\in\mathbb{Z}} \frac{1}{k_n} v_n(r)v_n(r') = 0,$$

and from here the sought kernel is expressed as

$$u(t, r, r') = \sum_{n\in\mathbb{Z}} M(k_n, t)v_n(r)v_n(r')$$

with $M(k_n, t) = \frac{1}{2}\,\mathrm{e}^{u_n^2}\,\mathrm{erfc}\,(u_n)$ and $u_n := -\mathrm{e}^{-i\pi/4}k_n\sqrt{t}$

Resonance expansion, continued

This yields decay law in the form

$$P(t) = \sum_{n,l} C_n \bar{C}_l I_{nl} M(k_n,\, t) \overline{M(k_l,\, t)}$$

with $C_n := \int_0^R \phi(r,\, 0) v_n(r)\, \mathrm{d}r$ and $I_{nl} := \int_0^R v_n(r) \bar{v}_l(r)\, \mathrm{d}r$

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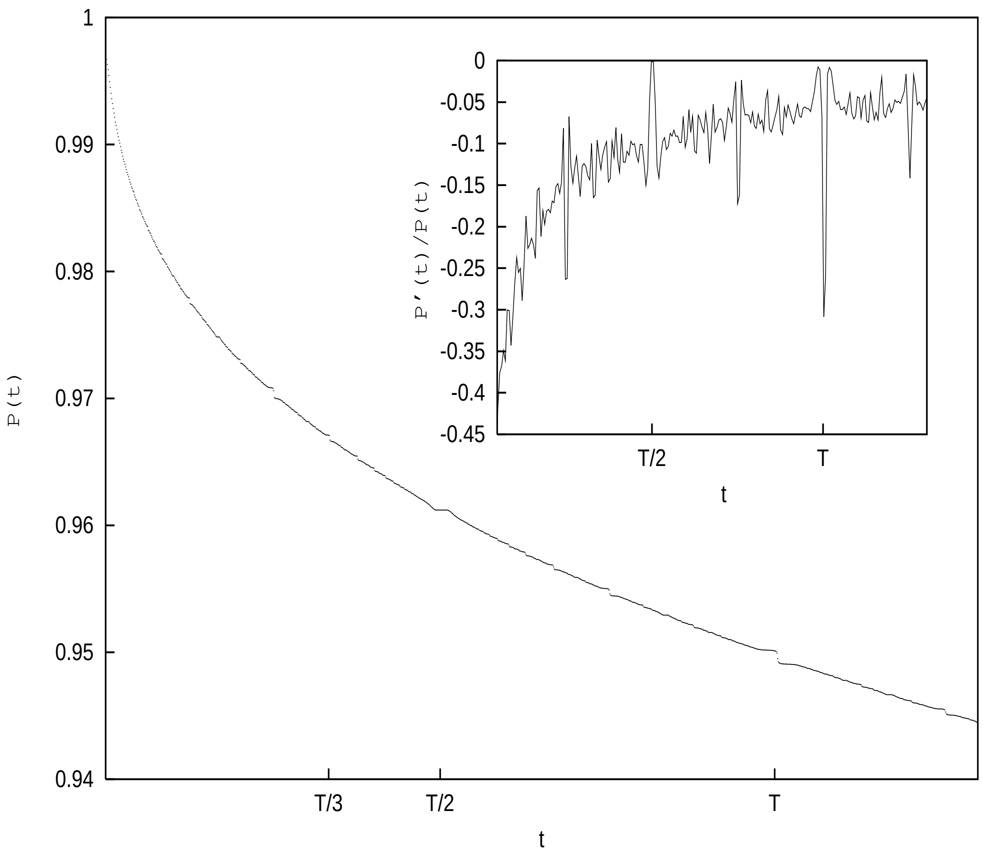
To make use of it we need resonance wave functions which are $v_n(r) = \sqrt{2} Q_n \sin(k_n r)$with the coefficient Q_n equal to

$$\left(\frac{-2ik_n^2}{2k_n + \alpha^2 R \sin 2k_n R + \alpha \sin 2k_n R + 2k_n \alpha R \cos 2k_n R} \right)^{1/2}$$

Now we can pass to numerical examples choosing $\alpha = 500$ using cut-off with $|n| \leq 1000$ for the series evaluation

Example: constant in the ball

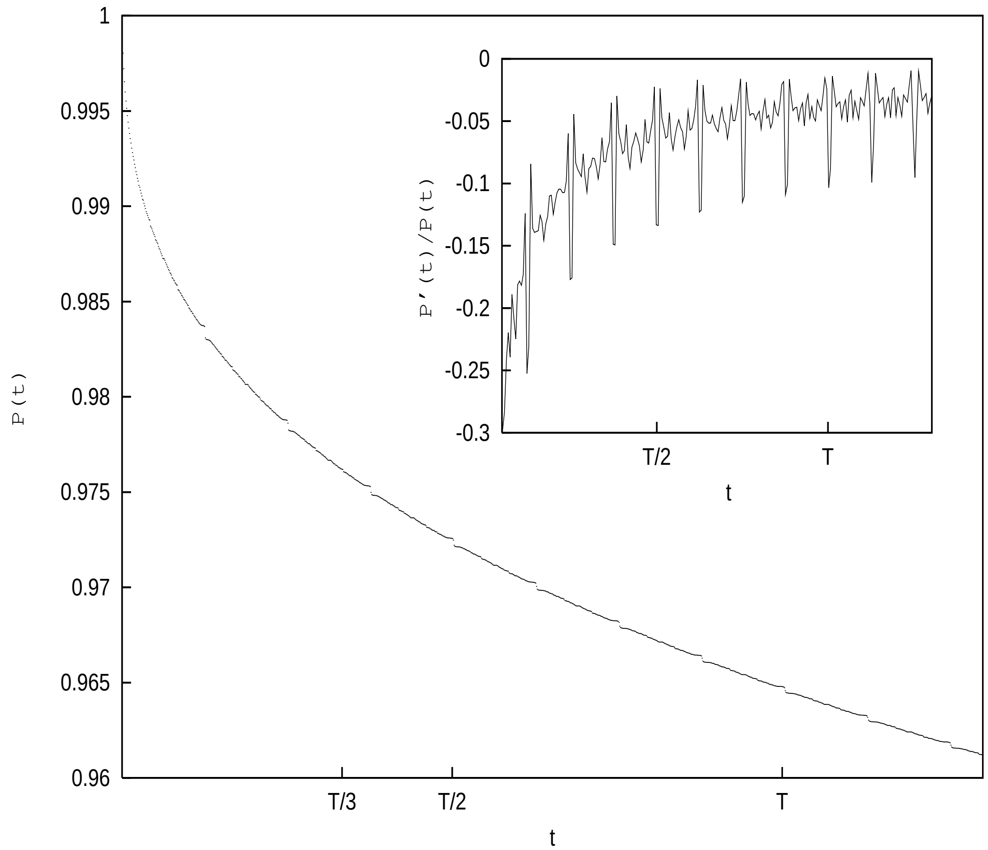
We choose first $\phi(r,\,0) = R^{-3/2}\sqrt{3}r$ for the initial state



The decay law plot; in the inset we show logarithmic derivative averaged over lengths of about $T/200$.

Example: integrable singularity at origin

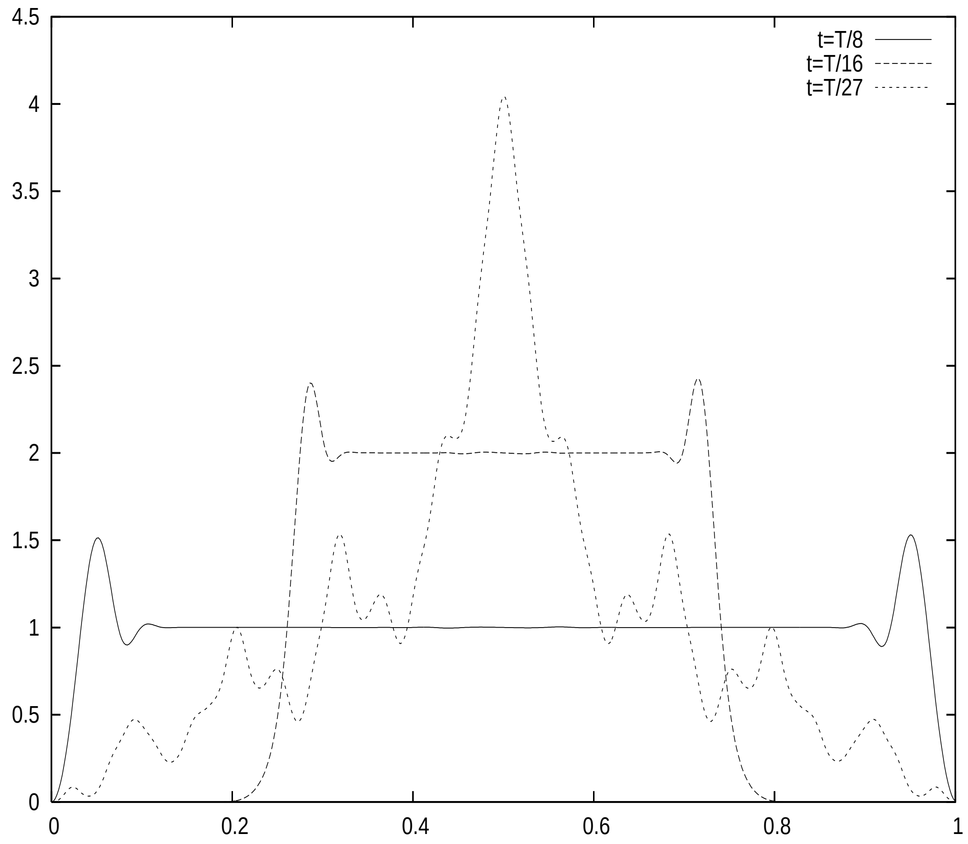
Choose instead $\phi(r, 0) = R^{-1/2}$ for the initial state which means to start from Neumann ground state on $(0, R)$



Plotting the same quantities we see a similar behavior

The wave function plot

For the second example plot the corresponding $|\phi(r,\, t)|^2$



for $t = T/8,\, T/16$, and $T/27$ (the revival time for $\alpha = \infty$ is $T/8$). The decay modifies the step-function form

More on decay law derivatives

A more detailed analysis of $\dot{P}(t) = -2\mathrm{Im}\left(\phi'(R,\, t)\bar{\phi}(R,\, t)\right)$ (equal to flux through the barrier) shows that

- If the coefficients in $\phi(r,\, t) \approx \sum_n C_n \exp(-ik_n^2 t) v_n(r)$ decay as n^{-p} with $p > 1$ we have $\dot{P}(t) \to 0$, uniformly in time, as $\alpha \to \infty$

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- *Slow decay:* take $C_n = (-1)^{n+1} \frac{\sqrt{6}}{Rk_n}$ corresponding to our first example, and limit of $\dot{P}(t_\alpha)$ as $\alpha \to \infty$ at the moving value $t_\alpha := t(1 + 2/\alpha R)$. In this case for *irrational multiples of* T we find that $\dot{P}(t) \to 0$

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- The same is true for $t = \frac{p}{q} T$ with pq *odd*. In contrast, *for* pq *even* we get *nonzero values*, for instance, at the period we have $\lim_{\alpha\to\infty} \dot{P}(T_\alpha) = -\frac{4}{3\sqrt{3}} \approx -0.77$

Some open questions

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- Also, can the formula be valid for *physically interesting* Hamiltonians unbounded from below such as Dirac operators?
- What rigorous claims can be made about “irregular” decays like the one in the Winter model example?

The talk was based on

[EF06] P.E., M. Fraas: The decay law can have an irregular character, `quant-ph/0603067`

[EI04] P.E., T. Ichinose: A product formula related to quantum Zeno dynamics, *Ann. H. Poincaré* **6** (2004), 195–215.

[EIK05] P.E., T. Ichinose, S. Kondej: On relations between stable and Zeno dynamics in a leaky graph decay model, to appear in *Proceedings of the OTAMP 2004 Conference* (Bedlewo 2004); `quant-ph/0504060`

[EINZ06] P.E., T. Ichinose, H. Neidhardt, V. Zagrebnov: Zeno product formula revisited, *Integral Equations and Operator Theory* (2006), to appear; `mp_arc 06-73.`

[E05] P.E.: Sufficient conditions for the anti-Zeno effect, *J. Phys. A: Math. Gen.* **38** (2005), L449–454.

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