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Bateman's dual system revisited: quantization, geometric phase and relation with the ground-state energy of the linear harmonic oscillator

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Abstract

By using the Feynman–Hibbs prescription for the evolution amplitude, we quantize the system of a damped harmonic oscillator coupled to its time-reversed image, known as Bateman's dual system. The time-dependent quantum states of such a system are constructed and discussed entirely in the framework of the classical theory. The corresponding geometric (Pancharatnam) phase is calculated and found to be directly related to the ground-state energy of the 1D linear harmonic oscillator to which the 2D system reduces under appropriate constraint. © 2004 Elsevier Inc. All rights reserved.

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1. Introduction

From the very outset of quantum theory a tremendous effort has been devoted to answering the question where is the boundary between the classical (macroscopic) and the quantum (microscopic) world, or in other words, when a quantum system starts to behave classically. The correspondence principle introduced by Bohr in the end of 1920s served as a heuristic prescription to construct quantum mechanics. Roughly speaking, the quantum theory should approach the classical theory in the limit of large quantum numbers. However, in general, this limit is quite subtle [1,2]. On the other hand, a typical statement in majority of standard textbooks is that classical mechanics applies in the limit $\hbar \rightarrow 0$, and this paradigm has become over the years a starting point for a host of semi-classical asymptotic treatments [3–5]. Yet, still many precautions should be taken when it comes to non-physical infinities at caustics, boundary layers analysis or to connection rules [6,7]. In terms of canonically conjugated variables, say p and q, the connection between the classical and quantum descriptions has been established by Ehrenfest's theorems [1,4,8] which give the law of motion for the mean values of the operators $\hat{p}$ and $\hat{q}$ in a form emulating the classical equations of motion. However, only for systems with quadratic Hamiltonians is this correspondence exact, i.e., the mean values $\langle \hat{p} \rangle$ and $\langle \hat{q} \rangle$ follow classical phase-space trajectories [4]. In particular, the system of quantum harmonic oscillators with bilinear coupling can be completely characterized using classical trajectories because the quantum Wigner function satisfies the classical Liouville equation [9].

In the present article, the first of the series, we attempt to shed some further light on the quantum–classical relation by studying the quantization of the 2D system of a damped harmonic oscillator coupled to its time-reversed image, first introduced by Bateman [10]. Bateman's dual system received considerable attention in the past as it represents a simple explicit example of a dissipative system which could be tackled by means of canonical quantization [11–13]. However, the quantum mechanics (QM) of this system is plagued with many conceptual problems [11,12] (e.g., the wave functions cannot be normalized in the usual manner, the Hamiltonian is not self-adjoint and represents the energy only for a restricted set of dynamical solutions) and it was shown [13] that a consistent quantization can only be achieved in quantum field theory (QFT). More recently, Bateman's system has been studied in connection with (Chern-Simons) gauge theories [14], as an example of an exactly decoherent system [15] and as a toy model for the recent proposal by G.'t Hooft about deterministic QM [16,17]. These aspects as well as the QFT of the Bateman system will be the object of future papers. The aim of this paper is to present a thorough analysis of the QM of this system which will be the basis for the next papers: our results include the time-dependent wave functions and the geometric phases associated to them. We also study the reduction of the 2D Bateman system to the 1D linear harmonic oscillator (l.h.o.).

An outline of the paper is as follows:

In Section 2 we quantize Bateman's system by using the Feynman–Hibbs prescription for the time-evolution amplitude (kernel) [18–20]: this allows us to avoid

the pitfalls of canonical quantization. We show that the kernel is fully expressible in terms of solutions of the classical equations of motion and that it is invariant with respect to the choice of the fundamental system of those solutions. This might be viewed as a two-dimensional extension of an existing one-dimensional result [20]. An important ingredient of the kernel calculation is the fluctuation factor [19]. This is calculated by employing the Van Vleck–Pauli–Morette determinant technique which allows us to avoid a direct manipulation of the Schrödinger equation. Instead, only the phase-space structure of the classical solutions (e.g., Lagrangian manifold) is used.

In Section 3 we use Mehler's formula [21,22] for a spectral decomposition of the kernel in order to obtain the time-dependent wave functions in hyperbolic radial coordinates (r, u). They are expressed in terms of generalized Laguerre polynomials and are shown to satisfy the correct time-dependent Schrödinger equation. The explicit form of the wave functions uncovers the root of the difficulties connected with the canonical quantization—the unboundedness of Bateman's system in the hyperbolic angle u. Use of the radial kernel [19] allows us to factorize away the explicit u dependence. The "radial" wave functions then correctly fulfill both orthonormalization and completeness relations. In addition, the radial wave functions satisfy the radial, time-dependent Schrödinger equation with the Hamiltonian $\hat{H}_l$. For the "azimuthal" quantum number $l = \pm \frac{1}{2}$, the latter turns out to be formally identical with the Hamiltonian of the 1D l.h.o..

To better understand the structure of the wave functions, we focus our attention on the algebraic setting of Bateman's Hamiltonian. Identifying the dynamical group as $SU(1, 1)$, we are able to pinpoint the structure of the ground state, which turns out to be a (squeezed) coherent state. The aforementioned peculiar behavior of the wave functions is then attributed to the remarkable properties of the $SU(1, 1)$ group representations: the unboundedness of Bateman's system in the variable u could be seen as a consequence of the non-existence of a unitary irreducible representation of $SU(1, 1)$ in which the generator J_2 would have at the same time a real and discrete spectrum [23]. The requirement of discreteness of the J_2 spectrum then leads to an *effective* non-hermiticity and oddness of J_2 under time reversal. To accommodate this point in the Feynman–Hibbs kernel prescription, a new inner product has to be defined. In fact, one of the merits of the presented method is that it naturally provides the consistent inner product for wave functions (the rather artificial and involved method of Racah [11] is not needed). In the case when we restrict our attention to the stationary quantum states, the connection with existing canonical quantization results [11–13] is readily established.

With the (full) time-dependent wave functions at hand, we are able to calculate in Section 4 the exact geometric (Pancharatnam) phase for Bateman's dual system. We find that Pancharatnam's phase is explicitly $\hbar$ independent and consists of three autonomous contributions: overall ground-state fluctuations of $\hat{p}$ and $\hat{x}$ gathered during the period of evolution and the Morse index.

We then show that the (full) wave functions become periodic in configuration space when the hyperbolic angle u solves the classical equations of motion. In this case the period of the wave functions matches the inverse of the reduced frequency

of the original Bateman dual system and Pancharatnam's phase boils down to the ordinary Berry–Anandan phase. In Section 5 we take this observation over to the radial wave functions and then, by setting $l = \pm \frac{1}{2}$, to the 1D l.h.o. Because the harmonic oscillator wave functions obtained in this way are constructed entirely from the fundamental system of solutions of Bateman's dual system, the Berry–Anandan phase bears an imprint (or memory) of the original 2D system even after the reduction to the 1D l.h.o. is performed. The geometric phase thus obtained can be directly identified with the zero point energy of the 1D l.h.o., and, in general, it is different from the usual $E_0 = \hbar\Omega/2$. This is in line with the results obtained in [17].

2. Time-evolution amplitude (kernel) for Bateman's dual system

In the following a decisive rôle will be played by the matrix elements of the time evolution operator $U(t_b, t_a)$ in the localized state basis

$$\langle \mathbf{x}_b; t_b | \mathbf{x}_a; t_a \rangle \equiv \langle \mathbf{x}_b | U(t_b, t_a) | \mathbf{x}_a \rangle. \tag{1}$$

They are referred as *time-evolution amplitudes*, or simply *kernels*. Due to the fact that $U(t_b, t_a)$ fulfills the time-dependent Schrödinger equations

$$i\hbar \frac{\partial}{\partial t_b} U(t_b, t_a) = \hat{H} \, U(t_b, t_a),$$
$$i\hbar \frac{\partial}{\partial t_b} U(t_a, t_b) = -U(t_a, t_b) \hat{H}, \quad t_b > t_a, \tag{2}$$

the kernel satisfies the equations [24]:

$$i\hbar \frac{\partial}{\partial t_b} \langle \mathbf{x}_b; t_b | \mathbf{x}_a; t_a \rangle = \hat{H}(-i\hbar \partial_{\mathbf{x}_b}, \mathbf{x}_b) \langle \mathbf{x}_b; t_b | \mathbf{x}_a; t_a \rangle,$$
$$i\hbar \frac{\partial}{\partial t_b} \langle \mathbf{x}_a; t_a | \mathbf{x}_b; t_b \rangle = -\mathcal{T}\hat{H}^\dagger(-i\hbar \partial_{\mathbf{x}_b}, \mathbf{x}_b)\mathcal{T}^{-1} \langle \mathbf{x}_a; t_a | \mathbf{x}_b; t_b \rangle, \quad t_b > t_a, \tag{3}$$

with the initial condition

$$\lim_{t_b \to t_a} \langle \mathbf{x}_a; t_a | \mathbf{x}_b; t_b \rangle = \delta(\mathbf{x}_a - \mathbf{x}_b). \tag{4}$$

Here $\mathcal{T}$ is the (anti-unitary) time reversal operator and $\hat{H}^\dagger$ is the Hermitian-adjoint Hamiltonian (in most applications $\hat{H}$ is both Hermitian and even under time reversal so $\dagger$ and $\mathcal{T}$ are usually omitted). An important observation is that for quadratic Hamiltonians the kernel has a very simple form, namely

$$\langle \mathbf{x}_b; t_b | \mathbf{x}_a; t_a \rangle = F[t_a, t_b] \exp\left(\frac{i}{\hbar} S_{cl}[\mathbf{x}]\right), \quad t_b > t_a. \tag{5}$$

The function $F[t_a, t_b]$ is the so called *fluctuation factor* [19] and is independent of both $\mathbf{x}_a$ and $\mathbf{x}_b$ [18,19]. The form (5) is usually attributed to Feynman and Hibbs [19,20], but one may readily see that it is nothing but the kernel version of the celebrated WKB approximation (often referred to as the Van Vleck [7] formula), which turns out to be an exact relation for quadratic Hamiltonians.

For a system with a time-independent Hamiltonian, the kernel reads

$$\langle \mathbf{x}_b; t_b | \mathbf{x}_a; t_a \rangle = \langle \mathbf{x}_b | \exp\left(-\frac{i}{\hbar} \hat{H}(t_b - t_a) \right) | \mathbf{x}_a \rangle. \tag{6}$$

Inserting the resolution of unity (completeness relation)

$$\sum_m | \psi_m \rangle \langle \psi_m | = 1.$$

(here $| \psi_m \rangle$ are orthonormal base kets at $t = 0$ spanning the Hilbert space) into (6), we obtain that

$$\langle \mathbf{x}_b; t_b | \mathbf{x}_a; t_a \rangle = \sum_m \psi_m(\mathbf{x}_b, t_b) \psi_m^*(\mathbf{x}_a, t_a). \tag{7}$$

Here we have identified $\psi_m(\mathbf{x}, t) = \langle \mathbf{x} | \psi_m(t) \rangle$. The symbol $*$ denotes usual complex conjugation. Note that $\psi_m(\mathbf{x}, t)$ and $\psi_m^*(\mathbf{x}, t)$, obey first and second equation in (3), respectively.

In the following we shall use the Feynman–Hibbs prescription (5) for the the quantization of Bateman's dual system [11,13]. Our analysis will also reveal a host of subtleties which are hidden in the seemingly clear relation (5).

2.1. Lagrangian and classical equations of motion

Bateman's dual model describes a 2D interacting system of damped-amplified harmonic oscillators. The corresponding Lagrangian reads [10–14,22]

$$L = m\dot{x}\dot{y} + \frac{\gamma}{2}(x\dot{y} - \dot{x}y) - \kappa xy \tag{8}$$

giving the (classical) equations of motion:

$$
\begin{aligned}
m\ddot{x}_{\mathrm{cl}} + \gamma \dot{x}_{\mathrm{cl}} + \kappa x_{\mathrm{cl}} &= 0, \\
m\ddot{y}_{\mathrm{cl}} - \gamma \dot{y}_{\mathrm{cl}} + \kappa y_{\mathrm{cl}} &= 0.
\end{aligned}
\tag{9}
$$

It is interesting to observe that the equation for x describes the damped harmonic oscillator, while the equation for y characterizes the amplified oscillator. In addition, with appropriate initial conditions both systems are mutual mirror images. In this sense it may be sometimes helpful to think of y as describing an effective degree of freedom for the reservoir to which system with the x degree of freedom is coupled [11,13,14].

In the following it will be useful to work with the rotated variables [14]: $x_1 = (x + y)/\sqrt{2}$, $x_2 = (x - y)/\sqrt{2}$. Then

$$L = \frac{m}{2}(\dot{x}_1^2 - \dot{x}_2^2) + \frac{\gamma}{2}(\dot{x}_1 x_2 - \dot{x}_2 x_1) - \frac{\kappa}{2}(x_1^2 - x_2^2) = \frac{m}{2}\dot{\mathbf{x}}\dot{\mathbf{x}} + \frac{\gamma}{2}\mathbf{x} \wedge \dot{\mathbf{x}} - \frac{\kappa}{2}\mathbf{x}\mathbf{x}, \tag{10}$$

where we introduced the notation $\mathbf{ab} = g_{\alpha\beta} a^\alpha b^\beta$, $\mathbf{a} \wedge \mathbf{b} = \varepsilon^{\alpha\beta} a_\alpha b_\beta$ and $x^\alpha = (x_1, x_2)$ with the metric tensor $g_{\alpha\beta} = (\sigma_3)_{\alpha\beta}$ (note also that $\varepsilon^{\alpha\beta} = -\varepsilon_{\alpha\beta}$). The corresponding conjugate momenta read

$$\mathbf{p} = m\dot{\mathbf{x}} - \frac{1}{2}\gamma\sigma_1\mathbf{x}. \tag{11}$$

In (x_1, x_2) coordinates the equations of motion read

$$m\ddot{\mathbf{x}}_{\mathrm{cl}} + \gamma\sigma_1\dot{\mathbf{x}}_{\mathrm{cl}} + \kappa\mathbf{x}_{\mathrm{cl}} = \mathbf{0}. \tag{12}$$

Notice that if $\mathbf{u}(t)$ is a solution of (12) so are $\sigma_1\mathbf{u}(t)$, $\sigma_3\mathbf{u}(-t)$, and $i\sigma_2\mathbf{u}(-t)$. For the future reference it is useful to realize that the Wronskian is t independent (i.e., it is a time invariant of the system). Indeed, in our case the Wronskian has the form (t_0 is arbitrary):

$$W(t) = W(t_0)\exp\left(-\int_{t_0}^{t}\mathrm{d}t\,\mathrm{Tr}\Big(\frac{\gamma}{m}\sigma_1\Big)\right), \tag{13}$$

Eq. (13) is nothing but Liouville's theorem of a differential calculus applied to (12).

2.2. Classical action

Using the usual definition for the action:

$$S[\mathbf{x}] = \int_{t_a}^{t_b}\mathrm{d}t\,L,$$

we can write

$$\begin{aligned}
S_{\mathrm{cl}}[\mathbf{x}] &= \int_{t_a}^{t_b}\mathrm{d}t\left[\frac{m}{2}\left(\frac{\mathrm{d}}{\mathrm{d}t}(x_1\dot{x}_1 - x_2\dot{x}_2) - x_1\ddot{x}_1 + x_2\ddot{x}_2\right) - \frac{\gamma}{2}(x_1\dot{x}_2 - x_2\dot{x}_1)\right. \\
&\qquad\left. -\frac{\kappa}{2}(x_1^2 - x_2^2)\right] \\
&= \frac{m}{2}(x_1\dot{x}_1 - x_2\dot{x}_2)\big|_{t_a}^{t_b} - \int_{t_a}^{t_b}\mathrm{d}t\,\frac{\mathbf{x}}{2}(m\ddot{\mathbf{x}} + \gamma\sigma_1\dot{\mathbf{x}} + \kappa\mathbf{x}) \\
&= \frac{m}{2}[\mathbf{x}_{\mathrm{cl}}(t_b)\dot{\mathbf{x}}_{\mathrm{cl}}(t_b) - \mathbf{x}_{\mathrm{cl}}(t_a)\dot{\mathbf{x}}_{\mathrm{cl}}(t_a)]. \tag{14}
\end{aligned}$$

2.3. Fundamental system of solutions

A fundamental system of solutions (i.e., a maximal system of linearly independent solutions) for Eq. (12) consists of *four* real 1×2 vectors $\mathbf{u}_i$ ($i = 1, 2, 3, 4$). The reason why there are four independent solutions is, roughly speaking, a result of the fact that we have two boundary conditions for each index. Independence of solutions may be checked via the Wronskian, which has to be non-zero at least at one time t (actually the Wronskian is time independent here). In our case the Wronskian is the determinant of a 4×4 matrix:

$$W(t) = W(t_0) = \begin{vmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 \\ \dot{\mathbf{u}}_1 & \dot{\mathbf{u}}_2 & \dot{\mathbf{u}}_3 & \dot{\mathbf{u}}_4 \end{vmatrix}. \tag{15}$$

An important technical simplification may be achieved by realizing that we may always find such a fundamental system where two arbitrary solutions (say, $\mathbf{u}_3$ and $\mathbf{u}_4$) are set to zero at t_a. This is due to the fact that in order to fulfill the boundary

condition on $\mathbf{x}(t_a)$ we need only two linearly independent vectors. Let us fix the following convention: $\mathbf{u}_3 \equiv \mathbf{v}_1$ and $\mathbf{u}_4 \equiv \mathbf{v}_2$. Then the condition on the fundamental system may be rephrased as

$$W(t) = W(t_a) = |\mathbf{u}_1(t_a)\mathbf{u}_2(t_a)| \times |\dot{\mathbf{v}}_1(t_a)\dot{\mathbf{v}}_2(t_a)| \neq 0.$$

Note that if we had assumed the existence of a fundamental system having three linearly independent solutions being zero at t_a, the Wronskian would vanish identically. Any real solution of (12) might thus be written as

$$\mathbf{x}_{\mathrm{cl}}(t) = \alpha_1 \mathbf{u}_1(t) + \alpha_2 \mathbf{u}_2(t) + \beta_1 \mathbf{v}_1(t) + \beta_2 \mathbf{v}_2(t),$$

with α_i and β_i being real numbers. Applying Cramer's rule, the solution $\mathbf{x}_{\mathrm{cl}}(t)$ with two fixed points $\mathbf{x}_{\mathrm{cl}}(t_a) \equiv \mathbf{x}_a$ and $\mathbf{x}_{\mathrm{cl}}(t_b) \equiv \mathbf{x}_b$ reads

$$\mathbf{x}_{\mathrm{cl}}(t) = \frac{[\mathbf{u}_1(t)D_1 + \mathbf{u}_2(t)D_2 + \mathbf{v}_1(t)D_3 + \mathbf{v}_2(t)D_4]}{U_a V_b}, \tag{16}$$

where $U_a = |\mathbf{u}_1(t_a)\mathbf{u}_2(t_a)|$, $V_b = |\mathbf{v}_1(t_b)\mathbf{v}_2(t_b)|$ and

$$
\begin{aligned}
D_1 &= \begin{vmatrix} \mathbf{x}_a & \mathbf{u}_2(t_a) & \mathbf{0} & \mathbf{0} \\ \mathbf{x}_b & \mathbf{u}_2(t_b) & \mathbf{v}_1(t_b) & \mathbf{v}_2(t_b) \end{vmatrix} = \mathbf{x}_a \wedge \mathbf{u}_2(t_a) \times V_b, \\
D_2 &= \begin{vmatrix} \mathbf{u}_1(t_a) & \mathbf{x}_a & \mathbf{0} & \mathbf{0} \\ \mathbf{u}_1(t_b) & \mathbf{x}_b & \mathbf{v}_1(t_b) & \mathbf{v}_2(t_b) \end{vmatrix} = -\mathbf{x}_a \wedge \mathbf{u}_1(t_a) \times V_b, \\
D_3 &= \begin{vmatrix} \mathbf{u}_1(t_a) & \mathbf{u}_2(t_a) & \mathbf{x}_a & \mathbf{0} \\ \mathbf{u}_1(t_b) & \mathbf{u}_2(t_b) & \mathbf{x}_b & \mathbf{v}_2(t_b) \end{vmatrix}, \\
D_4 &= \begin{vmatrix} \mathbf{u}_1(t_a) & \mathbf{u}_2(t_a) & \mathbf{0} & \mathbf{x}_a \\ \mathbf{u}_1(t_b) & \mathbf{u}_2(t_b) & \mathbf{v}_1(t_b) & \mathbf{x}_b \end{vmatrix}.
\end{aligned} \tag{17}
$$

An equivalent, and more useful, way of writing $\mathbf{x}_{\mathrm{cl}}(t)$ is to expand it in terms of $\mathbf{x}_a$ and $\mathbf{x}_b$. After some algebra we get

$$\mathbf{x}_{\mathrm{cl}}(t) = \frac{[x_a^1 \mathbf{B}_1(t) + x_a^2 \mathbf{B}_2(t) + x_b^1 \mathbf{B}_3(t) + x_b^2 \mathbf{B}_4(t)]}{U_a V_b}, \tag{18}$$

where

$$\mathbf{B}_i(t) = \begin{pmatrix} B_i^1(t) \\ B_i^2(t) \end{pmatrix}. \tag{19}$$

B_i^1 and B_i^2 are given by the determinant D,

$$D = \begin{vmatrix} \mathbf{u}_1(t_a) & \mathbf{u}_2(t_a) & \mathbf{0} & \mathbf{0} \\ \mathbf{u}_2(t_b) & \mathbf{u}_2(t_b) & \mathbf{v}_1(t_b) & \mathbf{v}_2(t_b) \end{vmatrix} = U_a V_b, \tag{20}$$

with ith row substituted by $(u_1^1(t), u_2^1(t), v_1^1(t), v_2^1(t))$ or $(u_1^2(t), u_2^2(t), v_1^2(t), v_2^2(t))$, respectively. So for example:

$$B_3^1(t) = \begin{vmatrix} \mathbf{u}_1(t_a) & \mathbf{u}_2(t_a) & \mathbf{0} & \mathbf{0} \\ u_1^1(t) & u_2^1(t) & v_1^1(t) & v_2^1(t) \\ u_1^2(t_b) & u_2^2(t_b) & v_1^2(t_b) & v_2^2(t_b) \end{vmatrix}. \tag{21}$$

As a result, the classical action $S_{cl}[\mathbf{x}]$ might be written as

$$S_{cl}[\mathbf{x}] = \frac{m}{2D}\left[x_a^1\,\mathbf{x}_b\dot{\mathbf{B}}_1(t_b) + x_a^2\,\mathbf{x}_b\dot{\mathbf{B}}_2(t_b) + x_b^1\,\mathbf{x}_b\dot{\mathbf{B}}_3(t_b) + x_b^2\,\mathbf{x}_b\dot{\mathbf{B}}_4(t_b)\right.$$
$$\left. - x_a^1\,\mathbf{x}_a\dot{\mathbf{B}}_1(t_a) - x_a^2\,\mathbf{x}_a\dot{\mathbf{B}}_2(t_a) - x_b^1\,\mathbf{x}_a\dot{\mathbf{B}}_3(t_a) - x_b^2\,\mathbf{x}_a\dot{\mathbf{B}}_4(t_a)\right]. \tag{22}$$

An explicit representation of the action is

$$S_{cl}[\mathbf{x}] = \frac{m}{2D}\left[-\left(x_a^1\right)^2\dot{B}_1^1(t_a) + \left(x_a^2\right)^2\dot{B}_2^2(t_a) + \left(x_b^1\right)^2\dot{B}_3^1(t_b) - \left(x_b^2\right)^2\dot{B}_4^2(t_b)\right.$$
$$+ x_a^1 x_a^2\left(\dot{B}_1^2(t_a) - \dot{B}_2^1(t_a)\right) + x_b^1 x_b^2\left(\dot{B}_4^1(t_b) - \dot{B}_3^2(t_b)\right)$$
$$+ x_a^1 x_b^1\left(\dot{B}_1^1(t_b) - \dot{B}_3^1(t_a)\right) - x_a^1 x_b^2\left(\dot{B}_1^2(t_b) + \dot{B}_4^1(t_a)\right)$$
$$\left. + x_a^2 x_b^2\left(\dot{B}_4^2(t_a) - \dot{B}_2^2(t_b)\right) + x_a^2 x_b^1\left(\dot{B}_2^1(t_b) + \dot{B}_3^2(t_a)\right)\right].$$

Using the basic properties of determinants it is possible to show now that both $S_{cl}[\mathbf{x}]$ and $\mathbf{x}_{cl}(t)$ are independent of the choice of the fundamental system of solutions. We show this in Appendix B.

2.4. Fluctuation factor

We can now take advantage of the Feynman–Hibbs observation [18,19] about the time-evolution amplitude (kernel) for systems governed by quadratic Hamiltonians:

$$\langle \mathbf{x}_b; t_b | \mathbf{x}_a; t_a \rangle = F[t_a, t_b]\exp\left(\frac{i}{\hbar}S_{cl}[\mathbf{x}]\right), \quad t_b > t_a. \tag{23}$$

As remarked, the fluctuation factor $F[t_a, t_b]$ is independent of $\mathbf{x}_a$ and $\mathbf{x}_b$. In addition, from (23) follows that:

$$F[t_a, t_b] = \langle 0; t_b | 0; t_a \rangle = \langle 0 | U(t_b, t_a) | 0 \rangle,$$

and so for time-independent $\hat{H}$ one has $F[t_a, t_b] = F[t_b - t_a]$.

The most usual way of calculating the fluctuation factor is via the Van Vleck–Pauli–Morette determinant [25–27]

$$F[t_a, t_b] = \sqrt{\det_2\left(\frac{i}{2\pi\hbar}\frac{\partial^2 S_{cl}}{\partial \mathbf{x}_a^\alpha \partial \mathbf{x}_b^\beta}\right)} = \sqrt{\det_2\left(\frac{i}{2\pi\hbar}\frac{\partial \mathbf{p}_{a\,\alpha}}{\partial \mathbf{x}_b^\beta}\right)}$$
$$= \sqrt{\det_2\left(\frac{i}{2\pi\hbar}\frac{\partial \mathbf{p}_{b\,\alpha}}{\partial \mathbf{x}_a^\beta}\right)}. \tag{24}$$

The symbol $\det_2(\ldots)$ denotes 2×2 determinant. In Eq. (24) we have also used the identities $\mathbf{p}_a = -(\partial S_{cl}/\partial \mathbf{x}_a)$ and $\mathbf{p}_b = (\partial S_{cl}/\partial \mathbf{x}_b)$ (one should take a little care when using the covariant and contravariant indices). Indices a and b are kept fixed throughout calculation.

Actually (24) is correct only for sufficiently short elapsed times $t_b - t_a$ as it was proved by Pauli [28]. In the general case, the determinant on the RHS of (24) will become infinite every time the classical (position space) orbit touches (or crosses) a caustic. A detailed examination of quadratic systems reveals [7,19] that (23) remains valid even after passing through the caustic, provided we write the fluctuation factor as

$$F[t_a, t_b] = \sqrt{\left| \det_2 \left(\frac{\mathrm{i}}{2\pi\hbar} \frac{\partial \mathbf{p}_{b\,\alpha}}{\partial \mathbf{x}_a^\beta} \right) \right|}, \tag{25}$$

and insert a factor $\exp(-\mathrm{i}\pi/2)$ for every reduction of the rank of $1/\det_2 \left(\partial^2 S_{\mathrm{cl}}/\partial \mathbf{x}_a^\alpha \partial \mathbf{x}_b^\beta \right)$ at the caustic. Thus, we have $(t_b > t_a)$:

$$\langle \mathbf{x}_b; t_b | \mathbf{x}_a; t_a \rangle = \exp\left(-\mathrm{i}\frac{\pi}{2} n_{a,b} \right) F[t_a, t_b] \exp\left(\frac{\mathrm{i}}{\hbar} S_{\mathrm{cl}}[\mathbf{x}] \right). \tag{26}$$

Here $n_{a,b}$ is the Morse (or Maslov) index [3,7,19,29–31] of the classical path running from $\mathbf{x}_a$ to $\mathbf{x}_b$.[1] The form (26) is due to Gutzwiller [32] and the prescription (26) is nothing but the connection formula for relating the kernels on both sides of the caustic in a continuous way [3]. To simplify the discussion we omit for a while the delicate issue of caustics assuming that the determinant in (24) is positive. We shall, however, return to it in Sections 4 and 5.

Now we are ready to calculate the fluctuation factor. A little algebra gives us

$$\begin{aligned}
F[t_a, t_b] &= \frac{m}{4\pi\hbar D} \Big[- \Big(\dot{B}_1^1(t_b)\dot{B}_2^1(t_b) - \dot{B}_3^1(t_a)\dot{B}_2^1(t_b) + \dot{B}_1^1(t_b)\dot{B}_3^1(t_a) - \dot{B}_3^1(t_a)\dot{B}_3^1(t_a) \\
&\quad + \dot{B}_4^2(t_a)\dot{B}_1^2(t_b) + \dot{B}_4^2(t_a)\dot{B}_4^1(t_a) - \dot{B}_2^2(t_b)\dot{B}_1^2(t_b) - \dot{B}_2^2(t_b)\dot{B}_4^1(t_a) \Big) \Big]^{\frac{1}{2}} \\
&= \frac{m}{2\pi\hbar} \sqrt{\frac{W}{D}}.
\end{aligned} \tag{27}$$

Here we have used the equations of the motion and the fact that the Wronskian is time independent. Since the kernel is uniquely determined from the classical action, our argument on the uniqueness of the classical action implies that $\langle \mathbf{x}_b; t_b | \mathbf{x}_a; t_a \rangle$ does not depend on the choice of a fundamental system. Note also that due to the fact that $F[t_a, t_b] = F[t_b - t_a]$, it follows from (27) that $D(t_a, t_b) = D(t_b - t_a)$.

---

[1] The set of all points where the inverse of the Van Vleck–Pauli–Morette determinant vanishes is called a caustic. The Morse index then counts how many times the classical orbit crosses (or touches) the caustic when passing from the initial to the final position. In the literature, crossing points are often called focal or conjugate points.

3. Wave functions for Bateman's dual system

3.1. Wave functions $\psi_{n,l}(r, u, t)$ and $\psi_{n,l}(r, t)$

In order to calculate the wave function it is useful to rewrite the kernel in hyperbolic polar coordinates (r, u), with $x_1 = r \cosh u$ and $x_2 = r \sinh u$, and then apply the defining relation [18]:

$$\langle r_b, u_b; t_b | r_a, u_a; t_a \rangle = \sum_{n,l} \psi_{n,l}(r_b, u_b, t_b) \psi_{n,l}^{(*)}(r_a, u_a, t_a), \quad t_b > t_a. \tag{28}$$

Here we have used the symbol $(*)$ instead of the usual complex conjugation symbol $*$—the need for this refinement will show up in the following. Invoking (5), (22), and (27) (see also Appendix C) we obtain:

$$
\begin{aligned}
\langle r_b, u_b; t_b | r_a, u_a; t_a \rangle &= \frac{m}{2\pi\hbar} \sqrt{\frac{W}{D}} \exp\left[\frac{\mathrm{i}m}{2D\hbar} \left(- r_a^2 \dot{B}_1^1(t_a) + r_b^2 \dot{B}_3^1(t_b) \right. \right. \\
&\quad \left. \left. + 2r_a r_b \dot{B}_1^1(t_b) \cosh(\Delta u) - 2r_a r_b \dot{B}_1^2(t_b) \sinh(\Delta u) \right) \right] \\
&= \frac{m}{2\pi\hbar} \sqrt{\frac{W}{D}} \exp\left[\frac{\mathrm{i}m}{2\hbar} \left(- \left[r_a^2 + r_b^2 \right] \frac{\dot{B}_1^1(t_a)}{D} \right. \right. \\
&\quad \left. \left. + 2r_a r_b \left[\frac{\dot{B}_1^1(t_b)}{D} \cosh(\Delta u) - \frac{\dot{B}_1^2(t_b)}{D} \sinh(\Delta u) \right] \right) \right],
\end{aligned}
\tag{29}
$$

$$\Delta u = u_b - u_a; \quad t_b > t_a.$$

By observing that

$$[\dot{B}_1^1(t_b)]^2 - [\dot{B}_1^2(t_b)]^2 = WD, \tag{30}$$

we can put

$$\frac{\dot{B}_1^1(t_b)}{D} = \sqrt{\frac{W}{D}} \cosh \alpha,$$

$$\frac{\dot{B}_1^2(t_b)}{D} = \sqrt{\frac{W}{D}} \sinh \alpha.$$

In Appendix D we show that

$$\alpha(t_a, t_b) = \Gamma(t_a - t_b) + \beta. \tag{31}$$

Here $\Gamma = \gamma/2m$ and β is a *complex* constant. It is also useful to denote the reduced oscillators frequency as $\Omega = \sqrt{(1/m)(\kappa - (\gamma^2/4m))}$. If not indicated otherwise, Ω will be assumed to be real throughout. That is, we shall mostly be concerned with the under-damped case although occasionally a result can be taken over to the over-damped case.

Eq. (31) allows to rewrite the kernel (29) in the following form:

$$\langle r_b, u_b; t_b | r_a, u_a; t_a \rangle = \frac{m}{2\pi\hbar} \sqrt{\frac{W}{D}} \exp\left[-\frac{\mathrm{i}m}{4D\hbar} \left(\frac{\mathrm{d}D}{\mathrm{d}t_a} r_a^2 - \frac{\mathrm{d}D}{\mathrm{d}t_b} r_b^2 \right) \right]$$

$$\times \exp\left[\frac{\mathrm{i}m}{\hbar} \sqrt{\frac{W}{D}} r_a r_b \cosh(\Delta u - \alpha) \right]. \tag{32}$$

This expression may be recast into a more suitable form if we apply the Laurent expansion [21,22][2]

$$\exp(\mathrm{i}a\cosh(u)) = \sum_{l=-\infty}^{\infty} (-1)^l I_l(-\mathrm{i}a) \mathrm{e}^{-lu}, \tag{33}$$

($I_l(\ldots)$ are the modified (or hyperbolic) Bessel functions) together with the addition theorem for the generalized Laguerre polynomials L_n^l (Mehler's formula [21,22]):

$$\sum_{n=0}^{\infty} n! \frac{L_n^l(z_1) L_n^l(z_2) b^n}{\Gamma(n+l+1)} = \frac{(z_1 z_2 b)^{-\frac{1}{2}l}}{1-b} \exp\left[-b\frac{z_1+z_2}{1-b} \right] I_l\left(2\frac{\sqrt{z_1 z_2 b}}{1-b} \right). \tag{34}$$

For this purpose we set

$$z_1 = \frac{m}{\hbar} \sqrt{W} \frac{r_a^2}{\rho(t_a)}, \quad z_2 = \frac{m}{\hbar} \sqrt{W} \frac{r_b^2}{\rho(t_b)},$$

with $r_a^2; r_b^2 \geqslant 0$[3] and

$$\rho(t) = V(t) W \left(\int \frac{\mathrm{d}t}{V(t)} \right)^2 + V(t) = \sqrt{\sum_{i<j}^{4} \left(\mathbf{u}_i(t) \wedge \mathbf{u}_j(t) \right)^2}. \tag{35}$$

This allows to identify the parameter b appearing in (34) with $b(t_b)$, where $b(t)$ reads

$$b(t) = \frac{-\mathrm{i}\sqrt{V(t)} + \sqrt{\rho(t) - V(t)}}{\mathrm{i}\sqrt{V(t)} + \sqrt{\rho(t) - V(t)}} = \frac{\left(-\mathrm{i}\sqrt{V(t)} + \sqrt{\rho(t) - V(t)} \right)^2}{\rho(t)}$$

$$= \exp\left(-\mathrm{i}2 \arcsin\sqrt{\frac{V(t)}{\rho(t)}} \right),$$

where $V(t) = \mathbf{u}_3(t) \wedge \mathbf{u}_4(t) = \mathbf{v}_1(t) \wedge \mathbf{v}_2(t)$. Note that $\rho(t_a) = U_a$ and that $b(t_a) = 1$.

The previous manipulations permit us to formulate the kernel in the desired form

---

[2] Because $\exp(\mathrm{i}a\cosh(u))$ is an analytic function of u—the only essential singularities are in $u = \pm\infty$—the Laurent expansion (33) is well defined for any complex u.

[3] Because $\cosh(u) \geqslant \sinh(u)$ then $|x_1| \geqslant |x_2|$ and so we have automatically that $r^2 \geqslant 0$ is a kinematic invariant.

$$\langle r_b, u_b; t_b | r_a, u_a; t_a \rangle = \frac{\mathrm{i}}{\pi} \sum_{n,l} \frac{n!}{\Gamma(n+l+1)} \left[b^*(t_a) b(t_b) \right]^{n+\frac{l+1}{2}} \left(\frac{m}{\hbar} \sqrt{\frac{W}{\rho(t_a)\rho(t_b)}} \right)^{l+1}$$

$$\times L_n^l \left(\frac{m}{\hbar} \sqrt{W} \frac{r_a^2}{\rho(t_a)} \right) L_n^l \left(\frac{m}{\hbar} \sqrt{W} \frac{r_b^2}{\rho(t_b)} \right) (r_a r_b)^l$$

$$\times \exp \left(\frac{m}{2\hbar} \left[\frac{\mathrm{i}}{2} \frac{\dot{\rho}(t_b)}{\rho(t_b)} - \frac{\sqrt{W}}{\rho(t_b)} \right] r_b^2 - \frac{m}{2\hbar} \left[\frac{\mathrm{i}}{2} \frac{\dot{\rho}(t_a)}{\rho(t_a)} + \frac{\sqrt{W}}{\rho(t_a)} \right] r_a^2 \right)$$

$$\times \exp \left\{ l(u_a - u_b + \alpha(t_a, t_b)) \right\}. \tag{36}$$

Let us identify the wave function $\psi_{n,l}(r,u,t) = \langle r, u | \psi_{n,l}(t) \rangle$. Note that Eq. (36) immediately implies that $\psi_{n,l}^{(*)}(r,u,t)$ cannot be associated with $\psi_{n,l}^*(r,u,t)$. It is not difficult to see that this peculiar behavior goes into account of the seemingly harmless expansion (33). The point is that we have tacitly used the discrete (Laurent) expansion even if an alternative integral (continuous) expansion was available [21]. This favoritism towards discrete l's was deliberate (see also next Section). A careful analysis will reveal that the discreteness of l is not compatible with a unitary representation of the dynamic symmetry group of the theory. The remedy will be found in a self-adjoint extension of $\hat{H}$ and it will turn out that $\psi_{n,l}^{(*)}(r,u,t) = \psi_{n,l}(r,-u,-t)$.

Now, from (28) and (36) we may deduce the wave functions

$$\psi_{n,l}(r,u,t) = \sqrt{\frac{1}{\pi}} \sqrt{\frac{n!}{\Gamma(n+l+1)}} \left(\sqrt{\frac{m}{\hbar \rho(t)}} W^{1/4} \right)^{l+1} [b(t)]^{n+\frac{l+1}{2}} L_n^l \left(\frac{m}{\hbar} \sqrt{W} \frac{r^2}{\rho(t)} \right) r^l$$

$$\times \exp \left(\frac{m}{2\hbar} \left[\frac{\mathrm{i}}{2} \frac{\dot{\rho}(t)}{\rho(t)} - \frac{\sqrt{W}}{\rho(t)} \right] r^2 \right) \exp \left\{ - l \left(u + \Gamma t - \frac{\beta}{2} \right) \right\},$$

$$\psi_{n,l}^{(*)}(r,u,t) = \sqrt{\frac{1}{\pi}} \sqrt{\frac{n!}{\Gamma(n+l+1)}} \left(\sqrt{\frac{m}{\hbar \rho(t)}} W^{1/4} \right)^{l+1} [b^*(t)]^{n+\frac{l+1}{2}} L_n^l \left(\frac{m}{\hbar} \sqrt{W} \frac{r^2}{\rho(t)} \right) r^l$$

$$\times \exp \left(-\frac{m}{2\hbar} \left[\frac{\mathrm{i}}{2} \frac{\dot{\rho}(t)}{\rho(t)} + \frac{\sqrt{W}}{\rho(t)} \right] r^2 \right) \exp \left\{ l \left(u + \Gamma t - \frac{\beta}{2} \right) \right\}. \tag{37}$$

Obviously, neither $\psi_{n,l}(r,u,t)$ nor $\psi_{n,l}^{(*)}(r,u,t)$ belong to ordinary Hilbert space because they cannot be normalized in the usual manner (they do not belong to the space of square integrable functions ℓ^2). The latter observation is in agreement with [11,13], and we shall comment more on this point in the next subsection. We note that the kernel (29) (and consequently the wave functions (37)) satisfies the time-dependent Schrödinger equation:[4]

---

[4] Actually $\psi_{n,l}^{(*)}(r,u,t)$ fulfills the time-reversed (time-dependent) Schrödinger equation, see Eq. (49).

$$\left(i\hbar \frac{\partial}{\partial t_b} - \hat{H}(r_b, u_b) \right) \langle r_b, u_b; t_b | r_a, u_a; t_a \rangle = 0, \quad t_b > t_a,$$

where

$$\hat{H} = \frac{1}{2m} \left[\hat{p}_r^2 - \frac{1}{r^2} \hat{p}_u^2 + m^2 \Omega^2 r^2 \right] - \Gamma \hat{p}_u$$

$$= \frac{1}{2m} \left[-\hbar^2 \frac{\partial^2}{\partial r^2} - \frac{\hbar^2}{r} \frac{\partial}{\partial r} + \frac{\hbar^2}{r^2} \frac{\partial^2}{\partial u^2} + m^2 \Omega^2 r^2 \right] + i\hbar \Gamma \frac{\partial}{\partial u}. \tag{38}$$

The Hamiltonian (38) is the so called Bateman Hamiltonian [11]. We now define the *radial* kernel $\langle r_b; t_b | r_a; t_a \rangle_{n,l}$ as [19]

$$\langle r_b, u_b; t_b | r_a, u_a; t_a \rangle = \sum_{n,l} \frac{\langle r_b; t_b | r_a; t_a \rangle_{n,l}}{\pi \sqrt{r_a r_b}} \, e^{l(\alpha(t) - \Delta u)}.$$

The corresponding wave function $\psi_{n,l}(r,t) = \langle r | \psi_{n,l}(t) \rangle$ reads

$$\psi_{n,l}(r,t) = \sqrt{\frac{n!}{\Gamma(n+l+1)}} \left(\sqrt{\frac{m}{\hbar \rho(t)}} W^{1/4} \right)^{l+1} [b(t)]^{n+\frac{l+1}{2}} r^{l+\frac{1}{2}} L_n^l \left(\frac{m}{\hbar} \sqrt{W} r^2 / \rho(t) \right)$$

$$\times \exp\left(\frac{m}{2\hbar} \left[\frac{i}{2} \frac{\dot{\rho}(t)}{\rho(t)} - \frac{\sqrt{W}}{\rho(t)} \right] r^2 \right). \tag{39}$$

It is simple to persuade oneself that $\psi_{n,l}(r,t)$ fulfills both the orthonormalization condition

$$\int_{-\infty}^{\infty} dr \, \psi_{n,l}^{(*)}(r,t) \, \psi_{n',l}(r,t) = \delta_{nn'},$$

and the resolution of unity

$$\sum_{n=0}^{\infty} \psi_{n,l}^{(*)}(r,t) \, \psi_{n,l}(r',t) = \delta(r - r').$$

Note that both the radial kernel and the wave function Eq. (39) satisfy the time-dependent Schrödinger equation

$$\left(i\hbar \frac{\partial}{\partial t_b} - \hat{H}_l(r_b) \right) \langle r_b; t_b | r_a; t_a \rangle_{n,l} = 0, \quad t_b > t_a,$$

where

$$\hat{H}_l = \frac{1}{2m} \left[-\hbar^2 \frac{\partial^2}{\partial r^2} + \frac{\hbar^2}{r^2} \left(l^2 - \frac{1}{4} \right) + m^2 \Omega^2 r^2 \right] - i\hbar \dot{\alpha}(t)\, l - i\hbar \Gamma l. \tag{40}$$

The term proportional to $1/r^2$ is analogous to the *centrifugal barrier* known from rotationally invariant systems and so the quantum number l can be viewed as analog of the *azimuthal* quantum number. Note that, due to the structure of $\alpha(t_a, t_b)$, the term $\dot{\alpha}(t) + \Gamma$ must be zero.

Since the generalized Laguerre polynomials L_n^l are defined for all $l \in \mathbb{C}$ indices,[5] the wave functions (39) satisfy the time-dependent Schrödinger equation with the Hamiltonian (40) also for non-integer l's. The key observation then is that if we continue l to the values $\pm\frac{1}{2}$, the Hamiltonian $\hat{H}_l$ describes the 1D l.h.o.. If we make use of the rules connecting Hermite polynomials with the $l = \pm\frac{1}{2}$ Laguerre polynomials [21], we may rewrite the continued radial wave functions in a simple form

$$\psi_{n,\frac{1}{2}}(r,t) = \frac{1}{2^{2n+1}} \sqrt{\frac{1}{n!\,\Gamma\left(n+\frac{3}{2}\right)}} \left(\sqrt{\frac{m}{\hbar\,\rho(t)}} W^{\frac{1}{4}}\right)^{\frac{1}{2}} [b(t)]^{n+\frac{3}{4}} H_{2n+1}\left(\sqrt{\frac{m}{\hbar\,\rho(t)}} W^{\frac{1}{4}} r\right)$$
$$\times \exp\left(\frac{m}{2\hbar}\left[\frac{\mathrm{i}}{2}\frac{\dot{\rho}(t)}{\rho(t)} - \frac{\sqrt{W}}{\rho(t)}\right]r^2\right),$$

$$\psi_{n,-\frac{1}{2}}(r,t) = \frac{1}{2^{2n}} \sqrt{\frac{1}{n!\,\Gamma\left(n+\frac{1}{2}\right)}} \left(\sqrt{\frac{m}{\hbar\,\rho(t)}} W^{\frac{1}{4}}\right)^{\frac{1}{2}} [b(t)]^{n+\frac{1}{4}} H_{2n}\left(\sqrt{\frac{m}{\hbar\,\rho(t)}} W^{\frac{1}{4}} r\right)$$
$$\times \exp\left(\frac{m}{2\hbar}\left[\frac{\mathrm{i}}{2}\frac{\dot{\rho}(t)}{\rho(t)} - \frac{\sqrt{W}}{\rho(t)}\right]r^2\right). \tag{41}$$

In passing we mention that the quantum numbers n and l appearing in (36) and (37) have been seemingly independent. So far the only obvious restriction was that $n \geqslant 0$ integers. However, for a consistent probabilistic interpretation (in the r variable) and analytical continuation (41), the wave function $\psi_{n,l}(r,u,t)$ is required to be bounded for $|r| < \infty$. Using the asymptotic expansion for $L_n^l(z)$ (see e.g. [21,22]):

$$L_n^l(z) = \frac{[\Gamma(n+l+1)]^2}{n!\,\Gamma(l+1)} {}_1F_1(-n,l+1;z) \approx \frac{\Gamma(n+l+1)}{n!}(-z)^n; \quad z \to \infty$$
$$\approx 1 - \frac{n}{l}z; \quad z \to 0,$$

where ${}_1F_1(-n,l+1;z)$ is the confluent hypergeometric series [22]. It is not difficult to see that the only allowed values of n and l at which $\psi_{n,l}(r,u,t)$ fulfills above requirements are those where:

$$2n + l + 1 = 0, \pm 1, \pm 2, \ldots, \quad l \neq 0. \tag{42}$$

So $|l| \geqslant 1$ and $n \geqslant 0$.

3.2. Meaning of quantum numbers n and l

Let us now consider the meaning of the quantum numbers n and l. To do this we must first understand the algebraic structure of the Hamiltonian (38). In [11,13] the following ladder operators were introduced:

---

[5] Analytic continuation for l with $\Re(l) = -1, -2, -3, \ldots$ is however required, see e.g. [33].

$$A = \frac{1}{\sqrt{2\hbar m \Omega}} \left[\hat{p}_1 - im\Omega x_1 \right],$$

$$B = \frac{1}{\sqrt{2\hbar m \Omega}} \left[\hat{p}_2 - im\Omega x_2 \right], \tag{43}$$

with

$$[A, A^\dagger] = [B, B^\dagger] = 1, \quad [A, B] = [A, B^\dagger] = 0. \tag{44}$$

The Hamiltonian (38) can be then rewritten as

$$\hat{H} = \hbar\Omega(A^\dagger A - B^\dagger B) + i\hbar\Gamma(A^\dagger B^\dagger - AB) = 2\hbar(\Omega\mathcal{C} - \Gamma J_2), \tag{45}$$

where we have made explicit the associated $SO(2,1) \equiv SU(1,1)$ algebraic structure:

$$\mathcal{C}^2 = \frac{1}{4}(A^\dagger A - B^\dagger B)^2,$$

$$J_+ = A^\dagger B^\dagger, \quad J_- = AB,$$

$$J_3 = \frac{1}{2}(A^\dagger A + B^\dagger B + 1), \tag{46}$$

$$[J_+, J_-] = -2J_3, \quad [J_3, J_\pm] = \pm J_\pm.$$

Here $\mathcal{C}$ is the only Casimir operator ($SU(1,1)$ has rank 1). In addition, $[\mathcal{C}, \hat{H}] = [J_2, \hat{H}] = 0$. If one defines

$$J_1 = \frac{1}{2}(J_+ + J_-), \quad J_2 = -\frac{i}{2}(J_+ - J_-), \tag{47}$$

the more familiar $SU(1,1)$ algebraic structure appears:

$$[J_1, J_2] = -iJ_3, \quad [J_3, J_2] = -iJ_1, \quad [J_1, J_3] = -iJ_2,$$

with

$$\mathcal{C}^2 = J_3^2 - J_2^2 - J_1^2 + \frac{1}{4}. \tag{48}$$

It is simple to check that

$$\mathcal{C} = \frac{1}{4\hbar\Omega m} \left[\hat{p}_r^2 - \frac{1}{r^2}\hat{p}_u^2 + m^2\Omega^2 r^2 \right],$$

$$J_2 = \frac{1}{2\hbar}\hat{p}_u.$$

In this connection it is important to recognize that the system described by the Hamiltonian (45) is both conservative and invariant under time reversal. Because the latter point has been treated in the literature in a somehow ambiguous fashion (cf. [11]), we discuss it in detail in Appendix E. We prove there that $\mathcal{T}\mathcal{C}\mathcal{T}^{-1} = \mathcal{C}$ and $\mathcal{T}J_2\mathcal{T}^{-1} = J_2$ from which it follows that $\mathcal{T}\hat{H}\mathcal{T}^{-1} = \hat{H}$. It is precisely this time-reversal issue which obscures the quantization of Bateman's system, bringing about many subtleties which are difficult to grasp without an explicit knowledge of the time-dependent wave functions (37).

Now, the crucial observation is that although from (47), i.e., from the very definition, J_2 appears to be Hermitian, (37) implies that it has a purely imaginary spectrum in $|\psi_{n,l}(t)\rangle$ (and this holds for all t). The root of this "pathological" behavior is in the non-existence of a unitary irreducible representation of $SU(1,1)$ in which J_2 would have at the same time a real and discrete spectrum [23]. Nevertheless, both the discreteness and complexness of J_2 spectra are vital in our analysis since they bring dissipative features in the dynamics: this was also the case considered in [11–13], to which we are going to compare our results. Thus the usual unitary representations of $SU(1,1)$ are clearly not useful for our purpose. On the other hand, by resorting to non-unitary representations of $SU(1,1)$ (known as non-unitary principal series [35,36]) we lose the hermiticity[6] of J_2 and hence the spectral theorem along with the resolution of unity.

Actually the situation is not so hopeless. One may indeed redefine the inner product [13,36] to get a unitary irreducible representation (known as complementary series [35,36]) out of non-unitary principal series. This may be easily done when we notice, using (28), (37) and Schwinger's prescription (3), that the states $\psi_{n,l}^{(*)}$ are not simple complex conjugates of $\psi_{n,l}$ because they fulfill the time-dependent Schrödinger equation

$$\left(i\hbar \frac{\partial}{\partial t} + \mathcal{T}\hat{H}^{\dagger}(r,u)\mathcal{T}^{-1} \right) \psi_{n,l}^{(*)}(r,u,t) = 0, \tag{49}$$

with the (effectively) non-Hermitian Hamiltonian. Accordingly, what we have loosely denoted in (28) as $\psi_{n,l}^{(*)}(r,u,t)$ is actually

$$\langle \mathcal{T}\psi_{n,l}(-t)|r,u \rangle = \langle \mathcal{T}\psi_{n,l}(-t)|r,-u \rangle^{*} = \psi_{n,l}(r,-u,-t), \tag{50}$$

as can be also double-checked from the explicit form (37). For the sake of simplicity we use $[\mathcal{T}|\psi_{n,l}(t)\rangle]^{\dagger} = \langle \mathcal{T}\psi_{n,l}(t)|$. Clearly, if J_2 were Hermitian then $\psi_{n,l}^{(*)} = \psi_{n,l}^{*}$ as one would expect.

The above considerations have some important implications. To see this, let us rewrite the kernel (36) by means of the states $|\psi_{n,l}(t)\rangle$:

$$\langle r_b, u_b; t_b | r_a, u_a; t_a \rangle = \sum_{n,l} \psi_{n,l}(r_b, u_b, t_b) \, \psi_{n,l}^{(*)}(r_a, u_a, t_a)$$

$$= \sum_{n,l} \langle r_b, u_b | \psi_{n,l}(t_b) \rangle \langle \mathcal{T}\psi_{n,l}(-t_a) | r_a, u_a \rangle. \tag{51}$$

We can formally introduce the conjugation operation ("bra vector") as $\langle \psi_{n,l}(t)| \equiv [\mathcal{T}|\psi_{n,l}(-t)\rangle]^{\dagger}$. Then the resolution of unity can be written in a deceptively simple form

$$\sum_{n,l} |\psi_{n,l}(t)\rangle \langle \psi_{n,l}(t)| = 1. \tag{52}$$

---

[6] To be precise, we should talk about self-adjointness rather than hermiticity, but we shall assume here and throughout that this ambiguity does not cause any harm in the present context.

The price which has been paid for this simplicity is that we have endowed the Hilbert space with a new inner product. In this context two points should be stressed. First, under the new inner product $|\psi_{n,l}(t)\rangle$ has a finite (and positive) norm. Second, J_2 is Hermitian with respect to this inner product. Indeed, integrating by parts we get

$$
\begin{aligned}
\langle \psi_{n,l}(t)|J_2\psi_{m,k}(t)\rangle &= \int \mathrm{d}r\,\mathrm{d}u\; r\,\psi_{n,l}^{(*)}(r,u,t)\left(-\frac{\mathrm{i}}{2}\frac{\partial}{\partial u}\right)\psi_{m,k}(r,u,t)\\
&= \int \mathrm{d}r\,\mathrm{d}u\; r\left[\left(-\frac{\mathrm{i}}{2}\frac{\partial}{\partial u}\right)\psi_{n,l}(r,u,t)\right]^{(*)}\psi_{m,k}(r,u,t)\\
&= \langle J_2\psi_{n,l}(t)|\psi_{m,k}(t)\rangle.
\end{aligned}
\tag{53}
$$

In (53) we have applied (52) and (50) together with the fact that the "surface" term is zero (if $k \neq l$ then integration w.r.t. the variable r gives zero [21], if $k = l$ then the product $\psi_{n,l}^{(*)}\psi_{m,l}$ is u independent).

Note that the (53) implies that $[\mathcal{T}J_2\mathcal{T}^{-1}]^\dagger = J_2$, or equivalently: $\mathcal{T}J_2^\dagger\mathcal{T}^{-1} = J_2$. Indeed,

$$
\langle J_2\psi_{n,l}(t)| = [\mathcal{T}J_2|\psi_{n,l}(-t)\rangle]^\dagger = \langle\psi_{n,l}(t)|[\mathcal{T}J_2\mathcal{T}^{-1}]^\dagger.
\tag{54}
$$

The spurious time irreversibility of J_2 apparent in (54) is an obvious consequence of dealing with the non-unitary representation of $SU(1,1)$. This is understandable, since a mechanism which does not preserve the norm (dissipation) is inherently connected with time irreversibility. From a mathematical point of view, we can interpret the relation $\mathcal{T}J_2\mathcal{T}^{-1} = J_2^\dagger$ as a self-adjoint extension of J_2 in the space spanned by the $|\psi_{n,l}(t)\rangle$ vectors. It should be also clear that when J_2 is time reversible (e.g., in the usual Hilbert space ℓ^2) it is also automatically Hermitian.

A pivotal consequence of the above is that

$$
\langle\psi_{n,l}(t)| = \left[\mathcal{T}|\psi_{n,l}(-t)\rangle\right]^\dagger = \left[\mathcal{T}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\hat{H}}|\psi_{n,l}(0)\rangle\right]^\dagger = \langle\psi_{n,l}(0)|\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\hat{H}}.
\tag{55}
$$

Thus, the time-evolution operator is unitary under the new inner product. It is this unitarity condition, intrinsically built in the kernel formula (6) (and successively taken over by the Feynman–Hibbs prescription), which naturally leads to a "consistent" inner product introduced in a somehow intuitive manner in [11,13]. From now on the modified inner product will be always tacitly assumed.

So far we have dealt with the peculiar structure of the Hilbert space. To interpret the quantum numbers n, l labelling the constituent states, we start with the observation that from the explicit form (37) one can readily construct the Hermitian operator $\tilde{\mathcal{C}}$ (commuting with J_2) which is diagonalized by $\psi_{n,l}(r,u,t)$. Indeed one may check that

$$
\begin{aligned}
J_2\psi_{n,l}(r,u,t) &= -\frac{\mathrm{i}}{2}\frac{\partial}{\partial u}\psi_{n,l}(r,u,t) = \mathrm{i}\frac{l}{2}\psi_{n,l}(r,u,t),\\
\tilde{\mathcal{C}}\psi_{n,l}(r,u,t) &= \frac{\sqrt{W}}{2\Omega\rho}(2n+l+1)\,\psi_{n,l}(r,u,t).
\end{aligned}
\tag{56}
$$

Here

$$\tilde{\mathcal{C}} = \mathcal{C} - \frac{m}{4\Omega\hbar}\left(\Omega^2 - \frac{W}{\rho^2} - \frac{\dot{\rho}^2}{4\rho^2}\right)r^2 + \frac{\mathrm{i}}{4\Omega}\frac{\dot{\rho}}{\rho}\left(r\frac{\partial}{\partial r} + 1\right)$$
$$= \mathrm{e}^{2\zeta}\,\hat{R}(t)\,\mathcal{C}\,\hat{R}^{-1}(t). \tag{57}$$

The unitary operator $\hat{R}(t)$ has the form

$$\hat{R}(t) = \hat{S}(\xi;t)\exp(\mathrm{i}\zeta\,G_A)\exp(\mathrm{i}\zeta\,G_B),$$
$$\hat{S}(\xi;t) = \exp(\mathrm{i}\,\xi\,r^2),$$

with

$$G_A = \mathrm{i}\frac{1}{2}\Big(A^2 - (A^\dagger)^2\Big), \quad G_B = \mathrm{i}\frac{1}{2}\Big(B^2 - (B^\dagger)^2\Big),$$

and

$$\zeta = \frac{1}{4}\ln\left(\frac{W}{\Omega^2\rho^2}\right), \quad \xi = \frac{m}{4\hbar}\frac{\dot{\rho}}{\rho} \quad \Rightarrow \quad \xi = -\frac{m}{2\hbar}\dot{\zeta}.$$

The reader may recognize in G_A and G_B the $SU(1,1)$ displacement operators (i.e., generalized coherent states generators) [35]. Alternatively, one may view G_A and G_B as the single-mode squeeze operators [37]. One may also notice that J_2 is nothing but the generator of two mode-squeeze [37]. Actually, the fact that there should be a close connection between $SU(1,1)$ squeezed states and damped oscillators was firstly proposed in [38]. Finally we should point out that in deriving (57) the relation $\hat{p}_r = -\mathrm{i}\hbar\,(\partial/\partial r + 1/2r)$ was used [19].

From Eq. (57) we find that $\tilde{\mathcal{C}}$ is Hermitian whenever ρ is a real function and $\mathcal{T}\tilde{\mathcal{C}}(t)\mathcal{T}^{-1} = \tilde{\mathcal{C}}(-t)$. The latter together with Eq. (56) implies that

$$\mathcal{T}\,|\psi_{n,l}(t)\rangle = |\psi_{n+l,-l}(-t)\rangle, \tag{58}$$

and so in the static case (i.e., when $\rho(t) = $ const. and $V(t) = \rho\,\sin^2(\Omega t)$, see also Section 5) we have

$$J_2\,|\psi^s_{n,l}\rangle = \mathrm{i}\frac{l}{2}\,|\psi^s_{n,l}\rangle,$$
$$\mathcal{C}\,|\psi^s_{n,l}\rangle = \frac{1}{2}(2n + l + 1)|\psi^s_{n,l}\rangle. \tag{59}$$

Here (and throughout) the convention $|\psi^s_{n,l}\rangle \equiv |\psi^s_{n,l}(0)\rangle$ is employed. Notice that in view of relations (E.13) and (58), the time reversed stationary states fulfill the condition:

$$\mathcal{T}\,|\psi^s_{n,l}\rangle = |\psi^s_{n+l,-l}\rangle. \tag{60}$$

Using (56), (57), and (59), we can find the relation between $|\psi_{n,l}(t)\rangle$ and the stationary states $|\psi^s_{n,l}\rangle$. The following relation holds:

$$|\psi_{n,l}(t)\rangle = \hat{R}(t)\,|\psi^s_{n,l}\rangle. \tag{61}$$

Thus the vectors $|\psi_{n,l}(t)\rangle$ appearing in the spectral decomposition of the kernel (28) have a tight connection with $SU(1,1)$ coherent states and, as we shall see soon, they describe indeed coherent states which (if expressed in the $|r,u\rangle$ representation) rotate in their position spread and/or pulsate in their width ("breathers").

However, before discussing other algebraic properties of the time-dependent states $|\psi_{n,l}(t)\rangle$, it is convenient to establish a connection with the results presented in [11,13]. Let us first denote by $\{|n_A, n_B\rangle\}$ the set of eigenstates of $A^\dagger A$ and $B^\dagger B$. From (44) follows that n_A and n_B are non-negative integers. Defining

$$j = \frac{1}{2}(n_A - n_B), \quad m = \frac{1}{2}(n_A + n_B), \tag{62}$$

we can label the eigenstates of $\mathcal{C}$ and $(J_3 - \frac{1}{2})$ as $|j, m\rangle$ rather than $|n_A, n_B\rangle$. As a result one may write

$$\mathcal{C}|j, m\rangle = j|j, m\rangle, \quad J_3|j, m\rangle = \left(m + \frac{1}{2}\right)|j, m\rangle, \tag{63}$$

with two obvious conditions: $|j| = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ and $m = |j|, |j| + \frac{1}{2}, |j| + 1, \ldots$ The latter is compatible with (42).

Defining the vacuum state $|0, 0\rangle$ as the state fulfilling $A|0, 0\rangle = B|0, 0\rangle = 0$, then the discrete states $|j, m\rangle$ can be explicitly written as

$$|j, m\rangle = c_{j,m} (A^\dagger)^{j+m} (B^\dagger)^{m-j} |0, 0\rangle, \tag{64}$$

with

$$c_{j,m} = [(j + m)! \, (m - j)!]^{-1/2}.$$

To relate the eigenstates of J_2 with those of J_3 above constructed, we may employ the following relation

$$\exp(\theta J_1) J_2 \exp(-\theta J_1) = -J_2 \cos \theta - \mathrm{i} J_3 \sin \theta,$$

with $\theta \in \mathbb{C}$, to obtain that

$$J_2 = \pm \mathrm{i} \exp\left(\pm \frac{\pi}{2} J_1\right) J_3 \exp\left(\mp \frac{\pi}{2} J_1\right). \tag{65}$$

The former implies that

$$\begin{aligned}
J_2 |\Psi_{j,m}^{(\pm)}\rangle &= \pm \mathrm{i} \left(m + \frac{1}{2}\right) |\Psi_{j,m}^{(\pm)}\rangle, \\
|\Psi_{j,m}^{(\pm)}\rangle &= \exp\left(\pm \frac{\pi}{2} J_1\right) |j, m\rangle.
\end{aligned} \tag{66}$$

Because $\mathcal{C}$ commutes both with J_2 and J_1, we also have

$$\mathcal{C}|\Psi_{j,m}^{(\pm)}\rangle = j|\Psi_{j,m}^{(\pm)}\rangle. \tag{67}$$

The relation (67) coincides with the result found in [11]. Similar identity was also realized in [13].

Comparing (66) and (67) with (59) we can identify

$$|\psi_{n,l}^s\rangle = \left|\Psi_{n+\frac{l}{2}+\frac{1}{2}, \frac{l}{2}-\frac{1}{2}}^{(+)}\right\rangle, \tag{68}$$

(note, as $m \geqslant 0$; $l \geqslant 1$) or equivalently

$$|\Psi_{j,m}^{(+)}\rangle = \left|\psi_{j-m-1, 2m+1}^s\right\rangle. \tag{69}$$

Similar identification holds for $|\Psi_{j,m}^{(-)}\rangle$;

$$|\psi_{n,l}^s\rangle = \left|\Psi_{n+\frac{l}{2}+\frac{1}{2},\, -\frac{l}{2}-\frac{1}{2}}^{(-)}\right\rangle, \tag{70}$$

$(l \leqslant -1)$ or equivalently

$$|\Psi_{j,m}^{(-)}\rangle = \left|\psi_{j+m,\, -2m-1}^s\right\rangle. \tag{71}$$

Matching (60) with ()(68)–(71), the states $|\Psi_{j,m}^{(+)}\rangle$ and $|\Psi_{j,m}^{(-)}\rangle$ can be related in a simple way, namely

$$\mathcal{T}\,|\Psi_{j,m}^{(+)}\rangle = |\Psi_{j,m}^{(-)}\rangle = |\Psi_{j,-(m+1)}^{(+)}\rangle, \tag{72}$$

which can be interpreted as a continuation of $|\Psi_{j,m}^{(+)}\rangle$ to negative m's. Eq. (72) we can check the consistency of the inner product defined above. Indeed, using (66) we have

$$\langle\psi_{n,l}^s|\psi_{n',l'}^s\rangle = \left\langle\Psi_{n+\frac{l}{2}+\frac{1}{2},\frac{l}{2}-\frac{1}{2}}^{(-)}\middle|\Psi_{n'+\frac{l'}{2}+\frac{1}{2},\frac{l'}{2}-\frac{1}{2}}^{(+)}\right\rangle$$

$$= \left\langle n+\frac{l}{2}+\frac{1}{2},\frac{l}{2}-\frac{1}{2}\middle|n'+\frac{l'}{2}+\frac{1}{2},\frac{l'}{2}-\frac{1}{2}\right\rangle = \delta_{nn',\, ll'}. \tag{73}$$

Now, to understand better the physical nature of $|\psi_{n,l}(t)\rangle$ let us first note that

$$\begin{pmatrix} A^\dagger \\ B \end{pmatrix} \quad\text{and}\quad \begin{pmatrix} A \\ -B^\dagger \end{pmatrix}^t$$

are right and left $SU(1,1)$ doublets, respectively. For instance, under an $SU(1,1)$ rotation

$$\mathrm{e}^{\mathrm{i}\eta J_1}\begin{pmatrix} A^\dagger \\ B \end{pmatrix}\mathrm{e}^{-\mathrm{i}\eta J_1} = \begin{pmatrix} \cosh\frac{\eta}{2} & \mathrm{i}\sinh\frac{\eta}{2} \\ -\mathrm{i}\sinh\frac{\eta}{2} & \cosh\frac{\eta}{2} \end{pmatrix}\begin{pmatrix} A^\dagger \\ B \end{pmatrix} = \mathbf{M}\begin{pmatrix} A^\dagger \\ B \end{pmatrix}. \tag{74}$$

The transformation matrix $\mathbf{M}$ is then clearly an element of the $SU(1,1)$ group as $\mathbf{M}^\dagger\mathbf{g} = \mathbf{g}\mathbf{M}^{-1}$; $\mathbf{g} = \mathrm{diag}(1,-1)$. Analogous transformation rules hold also for "rotations" with respect to J_2 and J_3.

Thus, in terms of the ladder operators A and B, the states $|\Psi_{j,m}^{(\pm)}\rangle$ read

$$|\Psi_{j,m}^{(\pm)}\rangle = c_{j,m}\left(\frac{A^\dagger\pm B}{\sqrt{2}}\right)^{j+m}\left(\frac{B^\dagger\pm A}{\sqrt{2}}\right)^{m-j}\exp[\pm\pi/4(A^\dagger B^\dagger+AB)]\,|0,0\rangle,$$

and so

$$|\psi_{n,l}^s\rangle = \tilde{c}_{n,l}\,(A^\dagger+B)^n\,(B^\dagger+A)^{|l|-n-1}\exp[\pi/4\,(A^\dagger B^\dagger+AB)]\,|0,0\rangle, \tag{75}$$

with

$$\tilde{c}_{n,l} = \left[n!\,(|l|-n-1)!\,2^{|l|-1}\right]^{-1/2}.$$

It follows from (68) and (70) that (75) is true both for $l \geqslant 1$ and $l \leqslant -1$. Using the Baker–Campbell–Hausdorff relation one can find that

$$\exp[\theta J_1] = \exp\left[\tan\left(\frac{\theta}{2}\right)J_+\right]\exp\left[2\lg\left(\cos\frac{\theta}{2}\right)J_3\right]\exp\left[\tan\left(\frac{\theta}{2}\right)J_-\right], \tag{76}$$

(the formula (76) is an analog of the Gaussian decomposition well known from $SO(3)$ group) and so (75) can be recast into a simple form

$$|\psi_{n,l}^s\rangle = \frac{\tilde{c}_{n,l}}{\sqrt{2}}\,(A^\dagger + B)^n\,(B^\dagger + A)^{|l|-n-1}\,\exp(J_+)\,|0,0\rangle$$
$$= c_{n,l}\,(\mathsf{A}^\dagger)^n\,(\mathsf{B}^\dagger)^{|l|-n-1}\,|0\rangle\rangle, \tag{77}$$

with

$$\mathsf{A} = \frac{1}{\sqrt{2}}(A - B^\dagger), \quad \mathsf{B} = \frac{1}{\sqrt{2}}(B - A^\dagger),$$
$$\mathsf{A}^\dagger = \frac{1}{\sqrt{2}}(A^\dagger + B), \quad \mathsf{B}^\dagger = \frac{1}{\sqrt{2}}(B^\dagger + A). \tag{78}$$

Since, the canonical commutation relations are conserved by a similarity transformation (74), A and $\mathsf{A}^\dagger$ are new annihilation and creation operators, respectively (the same holds true for B and $\mathsf{B}^\dagger$), with a new vacuum state $|0\rangle\rangle$ However, because the similarity transformation (74) for $\eta = -i\pi/2$ is not a unitary transformation, A and $\mathsf{A}^\dagger$ (and B, $\mathsf{B}^\dagger$) are not Hermitian conjugates. This should not be surprising: we have already observed that the state $|\psi_{n,l}^s\rangle$ does transform under a non-unitary representation of $SU(1,1)$ and recognized this point as the origin of the "anomalous" behavior of $\psi_{n,l}$ and $\psi_{n,l}^{(*)}$. However, it should be born in mind that under the inner product above introduced the A, $\mathsf{A}^\dagger$ (and B, $\mathsf{B}^\dagger$) are Hermitian conjugates. Indeed, one may readily check that, for instance, $\langle\langle 0|\mathsf{A} = \langle\langle \mathsf{A}^\dagger\,0|$.

In connection with Eq. (77) we should mention that $|0\rangle\rangle = 1/\sqrt{2}[\exp(J_+)]|0,0\rangle$ is a two-mode Glauber coherent state [37]. The coherent state $|0\rangle\rangle$ can be physically visualized as a boson condensate or as a new vacuum associated with A and B operators. This suggests to us that the state $|\psi_{n,l}^s\rangle$ can be interpreted as an excited state of n type-A oscillators, and $(|l| - n - 1)$ type-B oscillators with the coherent state type vacuum $|0\rangle\rangle = |\psi_{0,-1}^s\rangle$.

Yet another, interesting interpretation of $|\psi_{n,l}^s\rangle$ can be obtained if we employ the $SU(1,1)$ coherent states (i.e., generalized coherent states [35]). We demonstrate this in Appendix F.

Using (77) it is now simple to interpret the quantum numbers n and l appearing in the time-dependent states $|\psi_{n,l}(t)\rangle$. Let us first observe that

$$\tilde{\mathsf{A}}^\dagger(t) \equiv \hat{R}(t)\,\mathsf{A}^\dagger\,\hat{R}^{-1}(t) = a\mathsf{A}^\dagger - b\mathsf{B}^\dagger$$
$$\tilde{\mathsf{B}}^\dagger(t) \equiv \hat{R}(t)\,\mathsf{B}^\dagger\,\hat{R}^{-1}(t) = a^*\mathsf{B}^\dagger - b^*\mathsf{A}^\dagger, \tag{79}$$

('$*$' denotes a complex conjugation) where

$$a = \left(1 + \frac{i}{4\Omega}\frac{\dot{\rho}}{\rho}\right)\,\cosh\zeta - \frac{i}{4\Omega}\frac{\dot{\rho}}{\rho}\,\sinh\zeta$$

$$b = \left(1 - \frac{i}{4\Omega}\frac{\dot{\rho}}{\rho}\right)\,\sinh\zeta + \frac{i}{4\Omega}\frac{\dot{\rho}}{\rho}\,\cosh\zeta.$$

and $|a|^2 - |b|^2 = 1$. In a similar manner, we also find

$$\tilde{A}(t) \equiv \hat{R}(t)\,A\,\hat{R}^{-1}(t) = a^*\,A + b^*\,B$$
$$\tilde{B}(t) \equiv \hat{R}(t)\,B\,\hat{R}^{-1}(t) = a\,B + b\,A. \tag{80}$$

Inverting Eqs. (79) and (80) we immediately get the following useful relations:

$$\sqrt{2}A = a\left(\tilde{B}^\dagger + \tilde{A}\right) + b^*\left(\tilde{A}^\dagger - \tilde{B}\right),$$
$$\sqrt{2}A^\dagger = a^*\left(\tilde{A}^\dagger - \tilde{B}\right) + b\left(\tilde{B}^\dagger + \tilde{A}\right),$$
$$\sqrt{2}B = a^*\left(\tilde{A}^\dagger + \tilde{B}\right) + b\left(\tilde{B}^\dagger - \tilde{A}\right),$$
$$\sqrt{2}B^\dagger = a\left(\tilde{B}^\dagger - \tilde{A}\right) + b^*\left(\tilde{A}^\dagger + \tilde{B}\right). \tag{81}$$

By virtue of unitarity of $\hat{R}(t)$ the $\tilde{A}(t)$, $\tilde{A}^\dagger(t)$ (and $\tilde{B}(t)$, $\tilde{B}^\dagger(t)$) are new annihilation and creation operators (albeit not Hermitian conjugates) with the vacuum state

$$|[\zeta, \xi, t]\rangle\rangle = \hat{R}(t)\,|0\rangle\rangle = \hat{S}(\xi; t)|[\zeta, t]\rangle\rangle.$$

Hence, from (77) we have

$$|\psi_{n,l}(t)\rangle = c_{n,l}\,(\tilde{A}^\dagger(t))^n\,(\tilde{B}^\dagger(t))^{|l|-n-1}\,|[\zeta, \xi, t]\rangle\rangle, \tag{82}$$

As a matter of fact, the state

$$|[\zeta, t]\rangle\rangle = \exp(i\zeta G_A)\exp(i\zeta G_B)\,|0\rangle\rangle,$$

is also a coherent state (it saturates the uncertainty relations) called two-mode squeezed state [37] or (from rather historical reasons) (two-mode) two-photon coherent state [40]. Term squeeze (or squeezing) coined in [40] reminds that although the dispersions of canonical variables saturate the uncertainty relations their distribution over the phase space is distorted (or "squeezed") in such a way that the dispersion of one canonical variable is reduced at the cost of an increase in the dispersion of the canonically conjugated one. The concept of squeezing can be extended also to $SU(1,1)$ coherent states [41], however, the actual interpretation is in this case somehow less clear and states thus obtained do not seem to be particularly relevant in the present context.

The physical meaning of $|[\zeta, \xi, t]\rangle\rangle$ can be understood from the corresponding dispersions of $\hat{x}_i$ and $\hat{p}_i$;

$$\left\langle (\Delta\hat{x}_1)^2 \right\rangle = -\frac{\hbar}{2m\Omega}\left[\left\langle (A - A^\dagger)^2 \right\rangle - \left\langle (A - A^\dagger) \right\rangle^2\right] = \frac{\hbar}{2m\Omega}\exp(-2\zeta),$$
$$\left\langle (\Delta\hat{x}_2)^2 \right\rangle = \left\langle (\Delta\hat{x}_1)^2 \right\rangle,$$
$$\left\langle (\Delta\hat{p}_1)^2 \right\rangle = \frac{\hbar m\Omega}{2}\left[\left\langle (A + A^\dagger)^2 \right\rangle - \left\langle (A + A^\dagger) \right\rangle^2\right]$$
$$= \frac{\hbar m\Omega}{2}\left[\exp(2\zeta) + \left(\dot{\zeta}/\Omega\right)^2\exp(-2\zeta)\right], \tag{83}$$
$$\left\langle (\Delta\hat{p}_2)^2 \right\rangle = \left\langle (\Delta\hat{p}_1)^2 \right\rangle.$$

It is important to realize that (83) are obtained using (81) together with the modified inner product. So, for example, the relation

$$\langle\langle[\mathrm{i}\tilde{\mathsf{A}}^\dagger[\zeta,\xi,t]| = \{\mathcal{T}[\mathrm{i}\tilde{\mathsf{A}}^\dagger|[\zeta,\xi,-t]\rangle)]\}^\dagger = \mathrm{i}\langle\langle[\zeta,\xi,t]|\tilde{\mathsf{B}}^\dagger,$$

has to be employed. We may easily observe that if $\rho = $ const. (i.e., $\hat{S}(\xi;t) = 1$), the dispersions (83) saturate the Heisenberg uncertainty relations. As a result, the state $|[\zeta,t]\rangle\rangle$ is indeed the squeezed coherent state with the *squeeze parameter* ζ. If we evaluate the coherent state in the position representation we get a (minimum) wave packet specified by its half-width (stipulated via the $\hat{x}_i$ dispersion) and by the mean position (stipulated via mean of $\hat{x}_i$) [42]. If $\rho \neq$ const. (i.e., $\hat{S}(\xi;t) \neq 1$) we do not have any more a minimum uncertainty packet—uncertainty product does not stay $\hbar^2/4$ any more. Apart from the time dependence gained through ζ (which would still allow for the minimum uncertainty wave packet) there is an additional contribution in the dispersion of $\hat{p}_i$ as it can be directly observed from (83). So depending on the time behavior of ζ, the wave packet width now oscillates and/or spreads and this, in turn, "deflects" the dispersion of $\hat{p}_i$ from that of minimum uncertainty product.

It follows that Eq. (82) allows for a simple physical explication of $\langle r, u|\psi_{n,l}(t)\rangle$: the state $\langle r, u|\psi_{n,l}(t)\rangle$ describes the excited state of n type-$\tilde{\mathsf{A}}$ oscillators, and $(|l| - n - 1)$ type-$\tilde{\mathsf{B}}$ oscillators with the vacuum state $\langle r, u|[\zeta,\xi,t]\rangle\rangle$. The vacuum state itself is represented by the static wave packet which pulsates (and/or spreads) in its width. The excitations then may be understood as 2D Galilean boosts combined with $SU(1,1)$ rotations.

4. Geometric phase for Bateman's Dual System

4.1. Intermezzo: geometric phase

Since in the following an important rôle will be played by geometric phases, we give here a very brief introduction to this subject.

One has first to recall that a Hilbert space $\mathcal{H}$ is a line bundle over the projective space $\mathcal{P}$, i.e., the equivalence class of all vectors that differ by a multiplication with a complex number. We shall denote a generic element of $\mathcal{P}$ as $|\tilde{\psi}\rangle$. The inner product on $\mathcal{H}$ naturally endows $\mathcal{P}$ with two important geometric structures: a metric [43,44]

$$\mathrm{d}s^2 = ||\mathrm{d}|\tilde{\psi}\rangle||^2 - |\langle\tilde{\psi}|\mathrm{d}|\tilde{\psi}\rangle|^2, \tag{84}$$

and a $U(1)$ connection (Berry connection [44,45])

$$\mathcal{A} = \mathrm{i}\langle\tilde{\psi}|\mathrm{d}|\tilde{\psi}\rangle. \tag{85}$$

When a point evolves on $\mathcal{P}$ along a closed loop, say γ, the total phase change ϕ_{tot} of $|\psi\rangle$ on $\mathcal{H}$ consist of two contributions: the dynamical part

$$\phi_{\mathrm{dyn}} = -\hbar^{-1}\int_0^\tau \mathrm{d}t\,\langle\psi(t)|\hat{H}|\psi(t)\rangle, \tag{86}$$

(with τ being the time period at which the system traverses the whole loop γ), and the geometric part (Berry–Anandan phase)[7]

$$
\begin{aligned}
\exp(\mathrm{i}\phi_{\mathrm{BA}}) &= \exp(\mathrm{i}\phi_{\mathrm{tot}} - \mathrm{i}\phi_{\mathrm{dyn}}) \\
&= \langle\psi(0)|\psi(\tau)\rangle \exp\left(\mathrm{i}\int_0^\tau \mathrm{d}t \langle\psi(t)|\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle\right) \\
&= \exp\left(\mathrm{i}\int_0^\tau \mathrm{dt}\,\langle\tilde{\psi}(t)|\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\tilde{\psi}(t)\rangle\right) = \exp\left(\mathrm{i}\int_\gamma \mathcal{A}\right).
\end{aligned}
\tag{87}
$$

So the Berry–Anandan phase ϕ_{BA} may be geometrically understood as (an)holonomy with respect to the natural (Berry's) connection on the projective space $\mathcal{P}$ [43]. It is needless to say that all above relations are meant to be valid only for states $|\psi(t)\rangle$ being normalized to unity. In general, corresponding division of the state norm must be invoked. As our states $\psi_{n,l}(r,u,t)$ are not normalized, this should be kept in mind in the following where, for clarity of notation, we shall omit this normalization factor.

Actually, the geometric phase can also be defined for open paths. In this case the phase is usually referred to as Pancharatnam's phase [47]. The trick is that any path on $\mathcal{P}$ can be closed by joining endpoints with a geodesic constructed with respect to the metric (84). The geometric phase of the loop thus constructed is then defined to be equal to the geometric phase associated to the open path. So, if γ_{o} is an open path on $\mathcal{P}$ and γ_{g} is the corresponding geodesic on $\mathcal{P}$, then the associated Pancharatnam phase ϕ_{P} reads

$$
\begin{aligned}
\exp(\mathrm{i}\phi_{\mathrm{P}}) &= \exp\left(\mathrm{i}\int_{\gamma_{\mathrm{o}}+\gamma_{\mathrm{g}}} \mathcal{A}\right) = \exp\left(\mathrm{i}\int_{\gamma_{\mathrm{o}}} \mathcal{A}\right) \\
&= \exp\left(\mathrm{i}\int_{t_i}^{t_f} \mathrm{d}t\,\langle\tilde{\psi}(t)|\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\tilde{\psi}(t)\rangle\right) \\
&= \langle\psi(t_i)|\psi(t_f)\rangle \exp\left(\mathrm{i}\int_{t_i}^{t_f} \mathrm{d}t\,\langle\psi(t)|\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle\right).
\end{aligned}
\tag{88}
$$

Here we have used the fact that parallel transport along a geodesic does not bring any anholonomy. Note that ϕ_{P} is well defined only if the endpoints are not orthogonal. It should be also clear that ϕ_{P} is defined only modulo 2π.

4.2. Exact geometric phase in Bateman's system

To find the geometric phase for Bateman's system we firstly compute the dynamical phase

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[7] It would be perhaps more correct to call the geometric phase presented in this paper as Anandan's phase [44] or Aharonov–Anandan's phase [43]. However, due to a historical reasons, the Berry phase [46] is usually taken as a synonym for any cyclic geometric phase. We feel that the name Berry–Anandan phase is a suitable compromise between rigor and tradition.

$$\phi_{\text{dyn}} = -\int_{t_i}^{t_f} dt \, \langle \psi_{n,l}(t) | i \frac{d}{dt} | \psi_{n,l}(t) \rangle = -\hbar^{-1} \int_{t_i}^{t_f} dt \, \langle \psi_{n,l}(t) | \hat{H} | \psi_{n,l}(t) \rangle.$$

Using (37), (56), and (57) we find that

$$\phi_{\text{dyn}} = -(2n+l+1) \int_{t_i}^{t_f} dt \, \frac{\sqrt{W}}{2\rho} \left(\frac{\dot{\rho}^2}{4W} + \frac{\Omega^2 \rho^2}{W} + 1 \right) + i\Gamma l(t_f - t_i). \tag{89}$$

In order to proceed further it is convenient to define the complex number

$$z(t) = i \sqrt{\frac{V(t)}{\rho(t)}} + \sqrt{1 - \frac{V(t)}{\rho(t)}}.$$

Notice that $z\bar{z} = 1$. When evolving in the time interval (t_i, t_f), $z(t)$ traverses a curve γ in the Gaussian plane. The index of the curve γ (i.e., the number of revolutions around the origin) is then defined as [48]:

$$\text{ind}\,\gamma = \frac{1}{2\pi i} \oint_\gamma \frac{dz}{z}. \tag{90}$$

Consequently the dynamic phase can be rewritten in the form

$$\phi_{\text{dyn}} = -(2n+l+1) \int_{t_i}^{t_f} dt \left(\frac{\dot{\rho}^2}{8\rho\sqrt{W}} + \frac{\Omega^2 \rho}{2\sqrt{W}} \right) - (2n+l+1)(\pi\,\text{ind}\,\gamma)$$
$$+ i\Gamma l(t_f - t_i). \tag{91}$$

Using (90) we may write also the total phase in a fairly compact form, indeed

$$\phi_{\text{tot}} = \arg\left\{ \langle \psi_{n,l}(t_i) | \psi_{n,l}(t_f) \rangle \right\}$$
$$= -(2n+l+1) \left(\arcsin\sqrt{\frac{V(t_f)}{\rho(t_f)}} - \arcsin\sqrt{\frac{V(t_i)}{\rho(t_i)}} \right) - \frac{\pi}{2} n_{i,f} + i\Gamma l(t_f - t_i)$$
$$= -(2n+l+1)(2\pi\,\text{ind}\,\gamma) - \frac{\pi}{2} n_{i,f} + i\Gamma l(t_f - t_i), \tag{92}$$

With $n_{i,f}$ being the Morse index of the classical trajectory running between $\mathbf{x}_i$ and $\mathbf{x}_f$.

At this stage a remark should be added. The fact that we get an imaginary piece both in ϕ_{tot} and ϕ_{dyn} should not be surprising because we work with the modified inner product. Notice that the "troublesome" contribution in ϕ_{tot} and ϕ_{dyn} correctly flips sign if one passes from $\psi(\ldots)$ to $\psi^{(*)}(\ldots)$.

Substituting (91) and (92) into the equation Eq. (88), we get Pancharatnam's phase

$$\phi_{\text{P}} = (2n+l+1) \int_{t_i}^{t_f} dt \left(\frac{\dot{\rho}^2}{8\rho\sqrt{W}} + \frac{\Omega^2 \rho}{2\sqrt{W}} \right) - (2n+l+1)(\pi\,\text{ind}\,\gamma) - \frac{\pi}{2} n_{i,f}. \tag{93}$$

Note that because ρ, W, and V are solely constructed out of solutions of the classical equations of motion, ϕ_{P} is manifestly $\hbar$ independent.

The meaning of the terms on the RHS of (93) can be easily understood. Let us assemble the first two pieces in Eq. (93) together. Using the fact that

$$\rho = \frac{\sqrt{W}}{\Omega} \exp(-2\zeta),$$

(see explanation below Eq. (57)) together with (35) we obtain

$$\phi_P = (2n + l + 1) \frac{\Omega}{2} \int_{t_i}^{t_f} dt \Big(\exp(-2\zeta) + \exp(2\zeta) \exp(-2\zeta) (\dot{\zeta}/\Omega) \Big) + \frac{\pi}{2} n_{i,f}$$

$$= (2n + l + 1) \int_{t_i}^{t_f} dt \left(\frac{\left\langle (\Delta \hat{p}_i)^2 \right\rangle}{\hbar m} + \frac{m\Omega \left\langle (\Delta \hat{x}_i)^2 \right\rangle}{\hbar} \right) + \frac{\pi}{2} n_{i,f}. \quad (94)$$

The index "i" is either 1 or 2 (it really does not matter as dispersions are symmetric, one may write also the symmetrized version with the prefactor $\frac{1}{2}$, however, some care should be taken as we do not have Euclidean scalar product and so, for instance, $(\hat{p}_2)^2 = -\hat{p}_2\hat{p}^2$). It is important to recognize that $\langle \ldots \rangle$ in (94) represents the mean value with respect to the *ground* state.

Thus ϕ_P is a collection of three contributions; overall ground-state fluctuations of $\hat{p}$ and $\hat{x}$ gathered during the time period $t_f - t_i$ and the Morse index. While the first two are basic characteristics of the ground-state wave packet, the Morse index contribution, on the other hand, reflects the geometrical features of the path traversed by the ground-state wave packet in the configuration-space. As explained before, its presence is inevitable for providing a correct analytical continuation of the Feynman–Hibbs kernel prescription around the focal points.

The above considerations show that the properties of ϕ_P are basically encoded in the structure and in the time dependence of the ground state. This intertwining of the ground state with geometric phase will be of a crucial importance in the following.

An interesting question which one can raise in the present context is how the non-abelian (Wilczek–Zee) geometric phase [49] looks in Bateman's system and to what extent it influences the presented results. This is definitely a challenging task as there does not exist at present any formulation of non-abelian geometric phases outside of the scope of adiabatic approximation (i.e., geometric phases pioneered by Berry). As the geometric phases (87), (88), and (93) are rather Aharonov–Anandan type, such an extension would be of a particular interest. We intend to investigate this question in the future work.

5. The ground state of the 1D linear harmonic oscillator

5.1. Berry–Anandan phase

An interesting implication of (93) arises when we turn our attention to the special case of $l = -\frac{1}{2}$. From Section 3 we know that such a choice corresponds to the 1D l.h.o. In addition, from (37) and (41) we see that the following relation between the

1D l.h.o. wave function $\psi_n^{\text{lho}}(r,t) \equiv \psi_{n,-\frac{1}{2}}(r,t)$ and Bateman's system wave function $\psi_{n,l}(r,u,t)$ holds:

$$\psi_n^{\text{lho}}(r,t) = \sqrt{\pi r}\, \psi_{n,-\frac{1}{2}}(r, -\Gamma t + \beta/2, t). \tag{95}$$

The important point about $\psi_n^{\text{lho}}(r,t)$ is that it is constructed exclusively from the fundamental system of solutions corresponding to Bateman's dual system, and not, as one could expect, from fundamental system of solutions of the 1D l.h.o. In addition, it should be noted that the value $u = -\Gamma t + \beta/2$ entering (95) is nothing but the solution of the *classical* equations of the motion: $\dot{u} = \partial H/\partial p_u$, $\dot{p}_u = -\partial H/\partial u$ with the classical $p_u = J_2 = 0$ and Bateman's dual Hamiltonian H. One may naturally wonder then whether some measurable information about the original system could be tracked down in $\psi_n^{\text{lho}}(r,t)$. We shall see that this is indeed the case: a memory of the underlying Bateman's system is imprinted in the ground state energy of the reduced system.

Let us look at the geometric phase of the 1D l.h.o. "inherited" through the reduction (95). Being τ the mutual period of both ρ and V, then the following chain of reasonings holds:

$$\begin{aligned}
\psi_n^{\text{lho}}(r,\tau) &= \sqrt{\pi r}\, \psi_{n,-\frac{1}{2}}(r,u,\tau)|_{u=\beta/2-\tau\Gamma} \\
&= \sqrt{\pi r}\left(\exp\{i\phi_{\text{tot}}\}\, \psi_{n,-\frac{1}{2}}(r,u,0)|_{u=\beta/2-\tau\Gamma} \right) \\
&= \sqrt{\pi r}\left(\exp\{i[\phi_{\text{tot}} - \tau\Gamma p_u/\hbar]\}\, \psi_{n,-\frac{1}{2}}(r,u+\tau\Gamma,0) \right)|_{u=\beta/2-\tau\Gamma} \\
&= \exp\left\{ i\left[\phi_{\text{BA}} + \frac{1}{\hbar}\int_0^\tau \langle \psi_n^{\text{lho}}(t)|\hat{H}_{-\frac{1}{2}}|\psi_n^{\text{lho}}(t)\rangle\, dt \right] \right\} \psi_n^{\text{lho}}(r,0). \tag{96}
\end{aligned}$$

On the other hand, because of the periodicity of ρ and V, the wave function $\psi_n^{\text{lho}}(r,t)$ must be τ-periodic as well. This implies that $\psi_n^{\text{lho}}(r,0) = e^{i2\pi m}\psi_n^{\text{lho}}(r,\tau)$, where m is an arbitrary integer. Using the fact that ϕ_{BA} is defined modulo 2π we can write

$$\int_0^\tau \langle \psi_n^{\text{lho}}(t)|\hat{H}_{-\frac{1}{2}}|\psi_n^{\text{lho}}(t)\rangle\, dt = \hbar\,(2\pi n - \phi_{\text{BA}}). \tag{97}$$

In particular, if $\psi_n^{\text{lho}}(r,t)$ are eigenstates of $\hat{H}_{-\frac{1}{2}}$, then we get the quantized energy spectrum:

$$E_n^{\text{lho}} = \frac{\hbar}{\tau}(2\pi n - \phi_{\text{AB}}). \tag{98}$$

In the usual semiclassical treatment the presence of the Berry–Anandan phase modifies the energy spectrum via the Bohr–Sommerfeld quantization condition[30,50]. In the case of the simple 1D l.h.o. (not the one obtained from the Bateman's system after reduction), the Berry–Anandan phase materializes only due to the Morse index contribution, i.e., $\phi_{\text{BA}} = -\pi n_{a,b}/2$. When $(t_b - t_a) = \tau = 2\pi/\Omega$, the Morse index is simple [19]: $n_{a,b} = 2$ and one recovers the standard relation:

$$\tilde{E}_n^{\text{lho}} = \hbar\Omega\left(n + \frac{1}{2}\right). \tag{99}$$

In this respect the result (98) might seem rather peculiar, especially in the case when $\phi_{BA} \neq -\pi$. The fact that ϕ_{BA} can indeed be different from $-\pi$ will be explicitly illustrated in the following subsection. However, the basic reason for this to happen is not difficult to understand: the Morse index of the underlying Bateman's system (at $u = -\Gamma\tau$) is not necessarily equal to the Morse index of the corresponding 1D l.h.o. at the same elapsed time τ.

Let us add two more comments at the end. Firstly, the foregoing analysis can be naturally extended on the case $l = +\frac{1}{2}$, but instead of doing that we may directly observe from (41) that

$$\tilde{\psi}_n^{\text{lho}}(r,t) \equiv \psi_{n,\frac{1}{2}}(r,t) = \psi_{n+\frac{1}{2},-\frac{1}{2}}(r,t), \tag{100}$$

so the 1D l.h.o. states obtained from $\psi_{n,\frac{1}{2}}(r,t)$ describe the higher energy states than $\psi_n^{\text{lho}}(r,t)$, and thus, for instance, the ground state comes entirely from the $l = -\frac{1}{2}$ case. Secondly, it may happen that ρ and V can have more than one common period. Such periods can give rise to (generally) different (not $\bmod(2\pi)$) Berry–Anandan phases. This situation is fairly standard in many systems (see e.g., [20]) and the corresponding phases have as a rule different physical consequences.

5.2. Practical example—stationary states

To elucidate the previous analysis, we consider here an explicit example in which the following fundamental system of solutions is chosen:

$$
\begin{aligned}
u_1^1(t) &= \sqrt{2}\cos(\Omega t)\,\cosh(\Gamma t), \\
u_2^1(t) &= -\sqrt{2}\cos(\Omega t)\,\sinh(\Gamma t), \\
u_1^2(t) &= \sqrt{2}\cos(\Omega t)\,\sinh(\Gamma t), \\
u_2^2(t) &= -\sqrt{2}\cos(\Omega t)\,\cosh(\Gamma t), \\
v_1^1(t) &= \sqrt{2}\sin(\Omega t)\,\cosh(\Gamma t), \\
v_2^1(t) &= -\sqrt{2}\sin(\Omega t)\,\sinh(\Gamma t), \\
v_1^2(t) &= \sqrt{2}\sin(\Omega t)\,\sinh(\Gamma t), \\
v_2^2(t) &= -\sqrt{2}\sin(\Omega t)\,\cosh(\Gamma t).
\end{aligned}
\tag{101}
$$

The Wronskian is

$$
W = \begin{vmatrix}
\sqrt{2} & 0 & 0 & 0 \\
0 & -\sqrt{2} & 0 & 0 \\
0 & \sqrt{2}\Gamma & \sqrt{2}\Omega & 0 \\
-\sqrt{2}\Gamma & 0 & 0 & -\sqrt{2}\Omega
\end{vmatrix} = 4\Omega^2,
\tag{102}
$$

and the determinant D

$$D = 4\sin^2\Omega(t_b - t_a).$$

As a result we have

$$B_1^1(t) = 4 \cosh[\Gamma(t - t_a)] \sin[\Omega(t_b - t)] \sin[\Omega(t_b - t_a)],$$
$$B_3^1(t) = 4 \cosh[\Gamma(t_b - t)] \sin[\Omega(t - t_a)] \sin[\Omega(t_b - t_a)],$$
$$B_1^2(t) = 4 \sinh[\Gamma(t_a - t)] \sin[\Omega(t_b - t)] \sin[\Omega(t_b - t_a)].$$

The classical action has the form

$$S_{\mathrm{cl}} = \frac{m\Omega}{2\sin[\Omega(t_b - t_a)]} \left\{ (r_a^2 + r_b^2) \cos[\Omega(t_b - t_a)] - 2r_a r_b \cosh[u_b - u_a - \Gamma(t_b - t_a)] \right\}, \tag{103}$$

and we find that the fluctuation factor reads

$$F[t_a, t_b] = \frac{m}{2\pi\hbar} \frac{\Omega}{|\sin[\Omega(t_b - t_a)]|}. \tag{104}$$

Eqs. (103) and (104) lead to the kernel:

$$\langle r_b, u_b; t_b | r_a, u_a; t_a \rangle = \frac{m}{2\pi\hbar} \frac{\Omega}{|\sin[\Omega(t_b - t_a)]|} \exp\left[\frac{im\Omega}{2\hbar \sin(\Omega(t_b - t_a))} \right.$$
$$\left. \times \left\{ (r_a^2 + r_b^2) \cos(\Omega(t_b - t_a)) - 2r_a r_b \cosh(\Delta u - \Gamma(t_b - t_a)) \right\} \right]. \tag{105}$$

Note that the kernel is indeed independent of the fundamental system of solutions. One may check that the kernel (105) satisfies the time-dependent Schrödinger equation (38).

We now rewrite the kernel applying the expansion (36). Remembering that there is an absolute value in (105) and employing the fact that

$$V(t) = 2\sin^2(\Omega t),$$
$$\rho(t) = 2,$$
$$b(t) = \mathrm{e}^{-2i\Omega t},$$

we get

$$\langle r_b, u_b; t_b | r_a, u_a; t_a \rangle = \frac{i}{\pi} \sum_{n,l} \frac{n!}{\Gamma(n + l + 1)} \left(\frac{m\Omega}{\hbar} \right)^{l+1} L_n^l\left(\frac{m\Omega}{\hbar} r_a^2 \right) L_n^l\left(\frac{m\Omega}{\hbar} r_b^2 \right)$$
$$\times (r_a r_b)^l \, \mathrm{e}^{-\frac{m\Omega}{2\hbar}(r_a^2 + r_b^2)} \, \mathrm{e}^{-i\Omega(2n + l + 1)(t_b - t_a)} \, \mathrm{e}^{-l\Delta u - l\Gamma(t_b - t_a)}. \tag{106}$$

The explicit form of the wave function is then

$$\psi_{n,l}(r, u, t) = \sqrt{\frac{n!}{\pi\Gamma(n + l + 1)}} \left(\frac{m\sqrt{\Omega}}{\hbar} \right)^{l+\frac{1}{2}} r^l \, \mathrm{e}^{-\frac{m\Omega}{2\hbar} r^2} L_n^l\left(\frac{m\Omega}{\hbar} r^2 \right) \mathrm{e}^{-l(u + \Gamma t)} \, \mathrm{e}^{-i\Omega(2n + l + 1)t}. \tag{107}$$

The radial wave function is on the other hand:

$$\psi_{n,l}(r, t) = \sqrt{\frac{n!}{\pi\Gamma(n + l + 1)}} \left(\frac{m\sqrt{\Omega}}{\hbar} \right)^{l+\frac{1}{2}} r^{l+\frac{1}{2}} \mathrm{e}^{-\frac{m\Omega}{2\hbar} r^2} L_n^l\left(\frac{m\Omega}{\hbar} r^2 \right) \mathrm{e}^{-i\Omega(2n + l + 1)t}. \tag{108}$$

Note that $\psi_{n,l}(r,t)$ is an eigenstate of $\hat{H}_l$. In passing it is also interesting to consider the reduced wave function $|\psi\rangle_{H_I}$ which is obtained from the full one through the formula $|\psi\rangle_{H_I} = \exp[-\Gamma t \partial_u]|\psi\rangle_H$. This amounts to substitute u with $(u - \Gamma t)$ into the total wave function (107), which then becomes

$$\psi_{n,l}(r, u - \Gamma t, t) = \sqrt{\frac{n!}{\pi \Gamma(n+l+1)}} \left(\frac{m\sqrt{\Omega}}{\hbar}\right)^{l+\frac{1}{2}} r^l \, e^{-\frac{m\Omega}{2\hbar}r^2}$$

$$\times L_n^l\left(\frac{m\Omega}{\hbar}r^2\right) e^{-lu} e^{-i\Omega(2n+l+1)t}. \tag{109}$$

It is easy to verify that it satisfies the reduced Schrödinger equation:

$$i\partial_t \psi_{n,l}(r, u - \Gamma t, t) = 2\Omega\mathcal{C}\,\psi_{n,l}(r, u - \Gamma t, t), \tag{110}$$

Because V is periodic with *fundamental period* $\tau = \pi/\Omega$ and because $\psi_{n,-\frac{1}{2}}$ is an eigenstate of $\hat{H}_{-\frac{1}{2}}$, the energy spectrum of the related 1D l.h.o. is done by the prescription (98). The corresponding ground-state energy can be calculated from (93) and (98). We obtain

$$E_0^{\text{lho}} = -\hbar\frac{\phi_{\text{BA}}}{\tau} = \hbar\pi\frac{n_{a,b}}{2\tau}. \tag{111}$$

We see therefore that the fundamental system (101) reflects the dynamics of the underlying Bateman's dual system in E_0^{lho} only via the Morse index $n_{a,b}$. To find $n_{a,b}$, we first define a Lagrangian manifold [3,7]: L = $\{\mathbf{x}, \mathbf{p} = \partial S_{\text{cl}}/\partial \mathbf{x}\}$, where S_{cl} is the action taken as a function of the end point $\mathbf{x}$ (= $\mathbf{x}_b$). For quadratic actions the Lagrangian manifold is clearly *n*-dimensional plane in the 2*n*-dimensional phase space. Because the starting point of our analysis was the (configuration-space) kernel we are primarily interested in orbits for which the initial and final positions are given. Of course, if the initial position and momenta were given, then we would have a unique orbit, but since instead we have initial and final positions, it is not clear that any such orbit exists, and if does, whether it is unique. Let us therefore consider the time evolution of L. As time progresses, the Lagrangian manifold evolves (foliates) the phase space.[8] If we denote a point of the initial-time Lagrangian manifold as $\mathbf{z}_a = (\mathbf{p}_a, \mathbf{x}_a)$ (note that set of all such point forms the Cauchy data for the Hamiltonian dynamics) then due to quadratic nature of H the final-time point $\mathbf{z}_b$ is related with $\mathbf{z}_a$ via linear canonical transformation (symplectic matrix): $\mathbf{z}_b = \mathbb{S}\mathbf{z}_a$, where $\mathbb{S} = \mathbb{S}(t_a, t_b)$. Taking only the $\mathbf{x}$ part of $\mathbf{z}_b$ we may write

$$\mathbf{x}_b = \mathbb{S}_1\mathbf{p}_a + \mathbb{S}_2\mathbf{x}_a \iff \mathbb{S}_1\mathbf{p}_a = -\mathbb{S}_2\mathbf{x}_a + \mathbf{x}_b. \tag{112}$$

Clearly, if $\mathbb{S}_1(t_a, t_b)$ were invertible (i.e., if $\det(\mathbb{S}_1) \neq 0$) then for given points $\mathbf{x}_b$ and $\mathbf{x}_a$ would exist only one $\mathbf{p}_a$ and consequently only one classical orbit would run between $\mathbf{x}_b$ and $\mathbf{x}_b$. If however $\det(\mathbb{S}_1) = 0$, then either none or infinitely many solutions may be

---

[8] Due to its very definition, Lagrangian manifolds transform under canonical transformations into other Lagrangian manifolds [3]. So namely Lagrangian manifolds evolve into other Lagrangian manifolds under (Hamiltonian) time evolution.

obtained, depending on the ranks of $\mathbb{S}_1$ and the corresponding augment matrix. So if $\det(\mathbb{S}_1) = 0$ and $\mathbf{x}_a$ and $\mathbf{x}_b$ are such that Eq. (112) is not satisfied, then there are no orbits arriving at $\mathbf{x}_b$. If, however, $\mathbf{x}_a$ and $\mathbf{x}_b$ satisfy Eq. (112), then there is an n-dimensional infinity of initial momenta which maps onto $\mathbf{x}_b$ and thus density of particles is infinite at $\mathbf{x}_b$—orbits are focused in the configuration space (the situation is schematically depicted in Fig. 1).It is not difficult to see that the precarious points $\mathbf{x}_b$ in which $\det(\mathbb{S}_1) = 0$ are precisely the focal (conjugate) points. Indeed,

$$0 = \det(\mathbb{S}_1) = \det_2\left(\frac{\partial \mathbf{x}_b^\alpha}{\partial \mathbf{p}_a^\beta}\right) = \det_2\left[\left(\frac{\partial^2 S_{\mathrm{cl}}}{\partial \mathbf{x}_a^\alpha \mathbf{x}_b^\beta}\right)^{-1}\right].$$

When passing through a caustic, $\det(\mathbb{S}_1)$ may change the sign depending on the rank of $\det(\mathbb{S}_1(t_a, t_b))$ at the caustic. The caustic is said to have multiplicity k if the rank of $\det(\mathbb{S}_1(t_a, t_b))$ is k. The change of sign will directly influence the form of the fluctuation factor $F[t_a, t_b]$ (which is basically the square root of $1/\det[\mathbb{S}_1(t_a, t_b)]$) as the correct branch cut must be chosen. The phase of $F[t_a, t_b]$ can be consistently prescribed demanding continuity in the kernel [3]. It turns out that a phase factor $\exp(-ik\pi/2)$ must appear when passing a caustic of multiplicity k. As more caustics are passed along an orbit, the phase factor will accumulate. The Morse index $n_{a,b}$ appearing in (26) then simply counts caustics (including their multiplicity) encountered by an orbit passing from the initial to the final Lagrangian manifold.

Let us analyze our particular situation. Using the Hamilton equations of motion we easily get the following solution for $\mathbf{z} \equiv (p_1, p_2, x^1, x^2)$ (to be specific we work here in (x^1, x^2, p_1, p_2) phase-space coordinates)
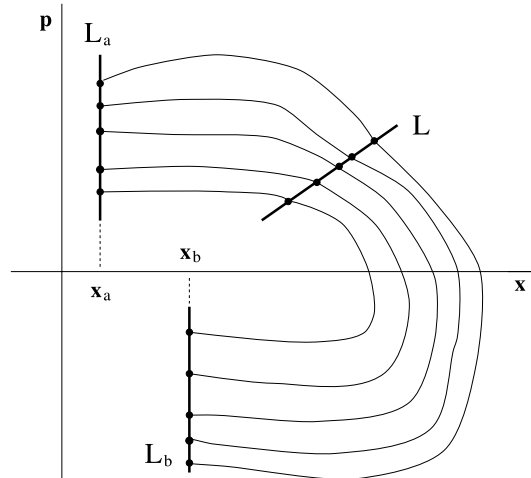

Fig. 1. Evolution of a Lagrangian manifold schematically depicted at three distinct times. Note that due to the initial condition (4) we have $L_a = \{\mathbf{x}_a, \mathbf{p}_a = \text{anything}\}$, note also that the Lagrangian manifold L_b is responsible for a caustic at $\mathbf{x}_b$. Five possible orbits are shown.

$$\mathbf{z} = \exp\left[(t - t_a)\begin{pmatrix} \frac{\gamma}{2m}\sigma_1 & -m\Omega^2\sigma_3 \\ \frac{1}{m}\sigma_3 & -\frac{\gamma}{2m}\sigma_1 \end{pmatrix}\right]\mathbf{z}_a$$

$$= \exp\left[(t - t_a)\begin{pmatrix} 0 & -m\Omega^2\sigma_3 \\ \frac{1}{m}\sigma_3 & 0 \end{pmatrix}\right]\exp\left[(t - t_a)\begin{pmatrix} \frac{\gamma}{2m}\sigma_1 & 0 \\ 0 & -\frac{\gamma}{2m}\sigma_1 \end{pmatrix}\right]\mathbf{z}_a, \quad (113)$$

and so

$$\mathbb{S}_1(t_a, t_b) = \frac{\sin[\Omega(t_b - t_a)]}{\Omega m}\ \sigma_3. \qquad (114)$$

To get the Morse index for the Berry–Anandan phase (111), we simply notice that during the time interval $(0, \tau)$ orbits pass one caustic at the conjugate time $\tau = \pi/\Omega$, with multiplicity 2 (see Figs. 2–4). The Morse index is $n_{a,b} = 2$ and the ground state of the 1D l.h.o. $E_0^{\mathrm{lho}} = \hbar\Omega$. Of course, if τ is a period for V, 2τ is also a period and any integer (positive or negative) times τ is likewise. It is clear then that the Morse index based on any such period should be the same because V (and hence the wave function) cannot distinguish whether the fundamental period alone or its integer multiples are in use. Indeed, for instance, for the time interval $(0, 2\tau)$ orbits pass two caustics, at the conjugate times $\tau = \pi/\Omega$ and $\tau = 2\pi/\Omega$, both with multiplicity 2 (see Figs. 2–4). The Morse index is in such a case $n_{a,b} = 4$ and we get again $E_0^{\mathrm{lho}} = \hbar\Omega$.

We see then that although our prescription does not allow to pinpoint E_0^{lho} precisely (the ground states are defined $\mathrm{mod}(\hbar\Omega)$), there is no way how to bring the usual fraction $1/2$ into the result. The factor 2 is a "memory" of the underlying 2D system. The fact that this may happen is not difficult to understand. The usual derivations of the ground state energy hinge either on the Heisenberg–Weyl algebra or directly on the Schrödin-
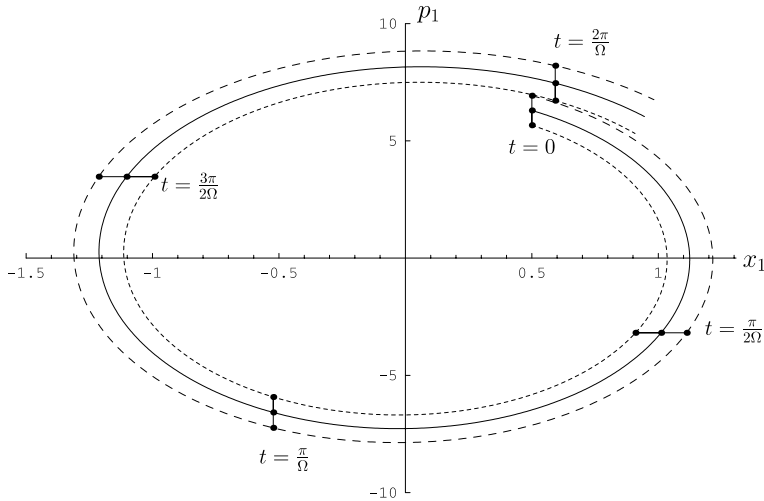

Fig. 2. Evolution of a Lagrangian manifold in (x_1, p_1) phase space. Three orbits with different values of $\mathbf{p}_a$ are plotted. The Lagrangian manifold is shown at five different times. Note that times $t = \pi/\Omega$ and $t = 2\pi/\Omega$ correspond to the conjugate times. This and the following plots are done for $\kappa = 40$, $\gamma = 1.2$, $m = 1$.
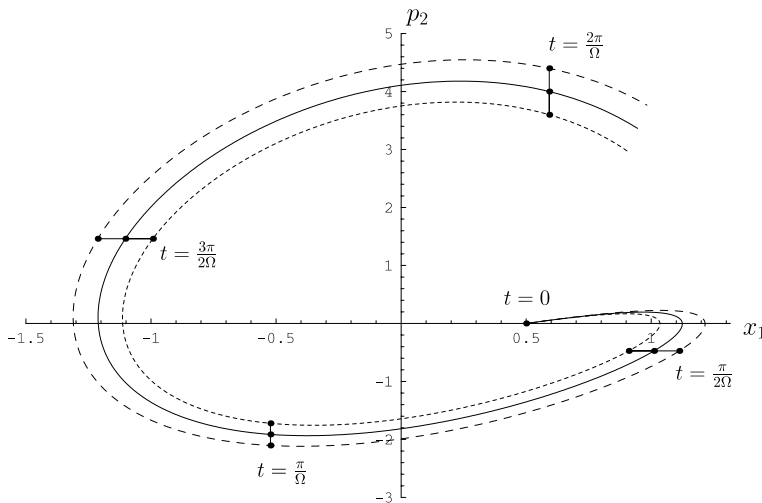
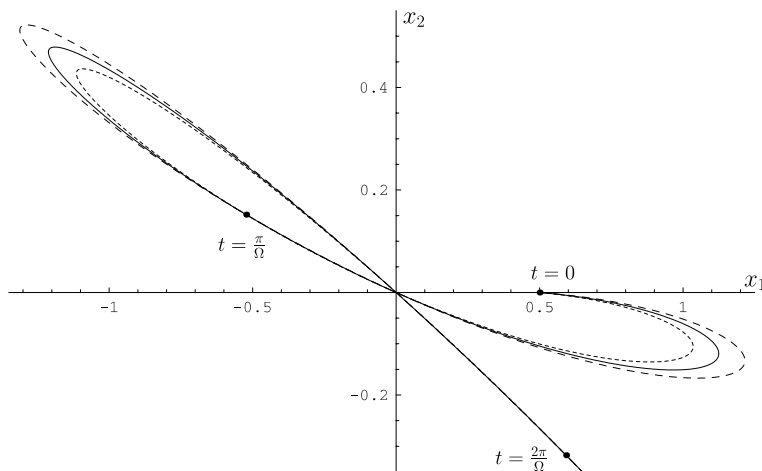Fig. 3. The same plot as Fig. 2 in the (x_1, p_2) phase space.

Fig. 4. The corresponding (x_1, x_2) configuration-space orbits. Focal points occur in $\mathbf{x}_a$ and $\mathbf{x}_b$ at times $t = \pi/\Omega$ and $t = 2\pi/\Omega$, respectively. Irrespective of the initial-time momenta, all orbits reach the focal points at the same time.

ger equation and thus have only local character while Berry–Anandan phase is (non-local) global characteristics of a system. We may thus conclude that the ground state of the reduced system can generally change according to the global properties of the original underlying system on which the reduction is performed.

Actually the above analysis is still not the whole story. In the paper to follow we will show that there is yet another—not so far considered—contribution to the ground state energy, reflecting the dissipative nature of the system when working

with the $SU(1, 1)$ non-unitary representation. Such a contribution will manifest itself in the form of an additional phase factor—"dissipative" phase (see also [17]).

6. Conclusions

In this paper we have studied the quantization of Bateman's dual system of damped–antidamped harmonic oscillators [10,11] by using the Feynman–Hibbs kernel formula. It has been known for some time that Bateman's system is difficult to quantize due to a multitude of conceptual problems [11–13]. In order to address some of these problems and to improve our intuition for the complications involved, we have found it very convenient that classical mechanics lies *manifestly* at the heart of the Feynman–Hibbs prescription. Thanks to the fact that Bateman's dual Hamiltonian (Lagrangian) is quadratic, the kernel is *fully* expressible in terms of the fundamental system of solutions of classical equations of motion and in addition, it is independent of the choice of such classical solutions.

Using the spectral decomposition of the time-evolution amplitude we have been able to calculate the full wave functions, fulfilling the time-dependent Scrödinger equation. This helped us to understand the reported controversy in the quantization of Bateman's system. We have shown that the above inconsistency has its origin in two interrelated issues: apparent non-hermiticity of the Hamiltonian and oddness of J_2 under time reversal (and thus time irreversibility of the Hamiltonian). We have argued that both these "pathologies" are the consequence of a single fact, namely that what has been invariably used in the literature (mostly implicitly) was the non-unitary irreducible representation of the $SU(1, 1)$ dynamic group. The fact is that from the representation theory it follows that there is no $SU(1, 1)$ unitary irreducible representation in which J_2 would have at the same time a real and discrete spectrum [23]. There are then two possible solutions. We may work with the unitary representation of $SU(1, 1)$—Bateman's dual system will then be free of pathologies but not particularly interesting from the physical point of view. The interesting features, e.g., dissipation along with time irreversibility, enter the scene precisely when the non-unitary irreducible representation of $SU(1, 1)$ is used. To treat the latter situation mathematically we have to face the non-hermiticity of J_2. The remedy naturally arises from the Feynman–Hibbs prescription and is based on a redefinition of the inner product: we have illustrated the mathematical and logical consistency of such a procedure. The relation with existing results [11–13] was established and the corresponding underlying $SU(1, 1)$ coherent state structure of quantum states was discussed in detail. The reader may contrast our approach with a somewhat more customary treatment of non-Hermitian quantum systems by means of resonant or Gamow states [51].

Although the kernel is invariant under the choice of the fundamental system of classical solutions, this is not the case for the wave functions. For one-dimensional quadratic systems it has been argued [20] that various choices of fundamental solutions correspond merely to different unitary transformations. However, this shows up not to be correct in our case. In fact, the basic feature of Bateman's dual system is that states are unitary inequivalent under $SU(1, 1)$ symmetry. As a result, the wave

functions and thus the geometric (Pancharatnam) phases are found to depend on the choice of the fundamental system of classical solutions in a non-trivial way. It is also worthwhile stressing that the geometric phases here obtained are given only in terms of the parameters of classical solutions and are thus manifestly $\hbar$ independent (as should be expected from phases with entirely geometric origin).

The crucial observation made in the course of our analysis is that when we analytically continue the "azimuthal" quantum number l to $\pm 1/2$, the reduced (or radial) wave functions formally fulfill the time-dependent Schrödinger equation for the one-dimensional harmonic oscillator. We have found that the geometric phase of the 1D l.h.o. obtained via such a reduction is not equal to the expected Berry phase [20], but bears a memory of the classical motion of the original Bateman dual system. This "shadow" of the underlying 2D system originates from three sources: overall ground-state dispersions of $\hat{p}$ and $\hat{x}$ gathered during the period of evolution and the Morse index contribution. It should be noted that the built-in memory in the geometric phase is not a new idea, having been used in various contexts as the Born–Oppenheimer approximation [52] and the dynamic quantum Zeno effect [53].

A remarkable feature of the reduction procedure is that it allows us to find a relation between the ground-state energy of the 1D l.h.o. obtained after reduction and the Pancharatnam phase. To put some flesh on the bones and to demonstrate the mechanism at hand, we have resorted to a specific fundamental system of solutions. The corresponding wave functions then proved to be, after reduction to those of a simple 1D l.h.o., energy eigenstates and periodic with the period τ, the latter being connected with the reduced frequency of the original system. The reduced geometric phase—Berry–Anandan phase—was then found to be directly proportional to the ground-state energy of the 1D l.h.o. It was shown that the ground-state energy is controlled by the Morse index affiliated with Bateman's dual system (not with the 1D l.h.o. itself !). Finding a Lagrangian manifold and following its evolution in phase-space, we were able to track down the number of focal points in the interval $(0, \tau)$ and hence to identify the Morse index. It turned out that the ground-state energy thus acquired is different from the usual $E_0 = \hbar \Omega / 2$.

Finally, we remark that the reader may find some resonance of the method presented here with the results [54–56] which suggest that the quantum mechanical energy spectrum can be determined from purely classical quantities such as lengths and stability indices of the periodic orbits alone. Whether or not this formal similarity can go any further is definitely a challenging question which is being investigated by the authors [17].

Appendix A

It is useful to list some of the expressions of Section 2 as they look in (x, y) coordinates. The Lagrangian reads:

$$L = m\dot{x}\dot{y} + \frac{\gamma}{2}(x\dot{y} - \dot{x}y) - \kappa xy = \frac{m}{2}\dot{\mathbf{x}}\dot{\mathbf{x}} + \frac{\gamma}{2}\dot{\mathbf{x}} \wedge \mathbf{x} - \frac{\kappa}{2}\mathbf{x}\mathbf{x}, \tag{A.1}$$

where $x^\alpha = (x, y)$ with the metric tensor $g_{\alpha\beta} = (\sigma_1)_{\alpha\beta}$. The canonical momenta are

$$\mathbf{p} = m\dot{\mathbf{x}} - \frac{1}{2}\gamma\sigma_3\mathbf{x}. \tag{A.2}$$

and the classical equations of motion can be written as

$$m\ddot{\mathbf{x}}_{\mathrm{cl}} + \gamma\sigma_3\dot{\mathbf{x}}_{\mathrm{cl}} + \kappa\mathbf{x}_{\mathrm{cl}} = \mathbf{0}. \tag{A.3}$$

Notice that if $\mathbf{u}(t)$ is a solution of (A.3) then $\sigma_3\mathbf{u}(t)$, $\sigma_1\mathbf{u}(-t)$, and $i\sigma_2\mathbf{u}(-t)$ are also solutions. The Wronskian for the (x, y) system is

$$W(t) = W(t_0)\exp\left(-\int_{t_0}^{t} dt\, \mathrm{Tr}\left(\frac{\gamma}{m}\sigma_3\right)\right), \tag{A.4}$$

and the action reads

$$\begin{aligned}
S_{\mathrm{cl}}[\mathbf{x}] &= \int_{t_a}^{t_b} dt\left[\frac{m}{2}\left(\frac{d}{dt}(x\dot{y} + \dot{x}y) - x\ddot{y} - \ddot{x}y\right) + \frac{\gamma}{2}(x\dot{y} - \dot{x}y) - \frac{\kappa}{2}xy - \frac{\kappa}{2}xy\right] \\
&= \frac{m}{2}(x\dot{y} + \dot{x}y)\Big|_{t_a}^{t_b} - \int_{t_a}^{t_b} dt\, \frac{\mathbf{x}}{2}(m\ddot{\mathbf{x}} + \gamma\sigma_3\dot{\mathbf{x}} + \kappa\mathbf{x}) \\
&= \frac{m}{2}[\mathbf{x}_{\mathrm{cl}}(t_b)\dot{\mathbf{x}}_{\mathrm{cl}}(t_b) - \mathbf{x}_{\mathrm{cl}}(t_a)\dot{\mathbf{x}}_{\mathrm{cl}}(t_a)]
\end{aligned} \tag{A.5}$$

and more explicitly

$$\begin{aligned}
S_{\mathrm{cl}}[\mathbf{x}] = \frac{m}{2D}\Big[&- (x_a^1)^2 \dot{B}_1^2(t_a) - (x_a^2)^2 \dot{B}_2^1(t_a) + (x_b^1)^2 \dot{B}_3^2(t_b) \\
&+ (x_b^2)^2 \dot{B}_4^1(t_b) - x_a^1 x_a^2 \left(\dot{B}_1^1(t_a) + \dot{B}_2^2(t_a)\right) + x_b^1 x_b^2 \left(\dot{B}_4^2(t_b) + \dot{B}_3^1(t_b)\right) \\
&+ x_a^1 x_b^1 \left(\dot{B}_1^2(t_b) - \dot{B}_3^2(t_a)\right) + x_a^1 x_b^2 \left(\dot{B}_1^1(t_b) - \dot{B}_4^2(t_a)\right) \\
&- x_a^2 x_b^2 \left(\dot{B}_4^1(t_a) - \dot{B}_2^1(t_b)\right) + x_a^2 x_b^1 \left(\dot{B}_2^2(t_b) - \dot{B}_3^1(t_a)\right)\Big].
\end{aligned}$$

Finally, the fluctuation factor in (x, y) coordinates is

$$\begin{aligned}
F[t_a, t_b] &= \frac{m}{4\pi\hbar D}\Big[-\left(\dot{B}_1^2(t_b)\dot{B}_2^2(t_b) - \dot{B}_3^2(t_a)\dot{B}_2^2(t_b) - \dot{B}_1^2(t_b)\dot{B}_3^1(t_a)\right. \\
&\qquad + \dot{B}_3^2(t_a)\dot{B}_3^1(t_a) + \dot{B}_4^1(t_a)\dot{B}_1^2(t_b) - \dot{B}_4^1(t_a)\dot{B}_4^2(t_a) \\
&\qquad \left.- \dot{B}_2^1(t_b)\dot{B}_1^1(t_b) + \dot{B}_2^1(t_b)\dot{B}_4^2(t_a)\right)\Big]^{\frac{1}{2}} \\
&= \frac{m}{2\pi\hbar}\sqrt{\frac{W}{D}}.
\end{aligned} \tag{A.6}$$

Notice that under transformation $(x_1, x_2) \rightarrow (x, y)$ both W and D do not change.

Appendix B

We prove here that both $S_{\mathrm{cl}}[\mathbf{x}]$ and $\mathbf{x}_{\mathrm{cl}}(t)$ are independent of the choice of the fundamental system. The proof can be done in two steps. First we show that both $S_{\mathrm{cl}}[\mathbf{x}]$ and $\mathbf{x}_{\mathrm{cl}}$ do not depend on the scaling of $\mathbf{u}_i$ and $\mathbf{v}_i$. Indeed, if we rescale, for example $\mathbf{u}_1 \rightarrow \alpha \mathbf{u}_1$, then

$$U_a \rightarrow \alpha U_a, \quad V_b \rightarrow V_b,$$
$$D_1 \rightarrow D_1, \quad D_2 \rightarrow \alpha D_2,$$
$$D_3 \rightarrow \alpha D_3, \quad D_4 \rightarrow \alpha D_4.$$

Analogous relations are valid for other vectors. It is now simple to see that both S_{cl} and $\mathbf{x}_{\mathrm{cl}}$ remain unchanged under such a rescaling.

Second, we show that S_{cl} and $\mathbf{x}_{\mathrm{cl}}$ remain unchanged under the substitution

$$\mathbf{u}_1 \rightarrow \mathbf{u}_1 + \alpha \mathbf{u}_2 + \beta \mathbf{v}_1 + \gamma \mathbf{v}_2, \tag{B.1}$$

where α, β, and γ are arbitrary real constants. The previous substitution is possible to achieve, for example, in successive steps:

$$\mathbf{u}_1 \rightarrow \mathbf{u}_1 + \beta \mathbf{v}_1,$$
$$\mathbf{v}_1 \rightarrow \mathbf{v}_1 + \frac{\gamma}{\beta} \mathbf{v}_2,$$
$$\mathbf{v}_2 \rightarrow \mathbf{v}_2 + \frac{\alpha}{\gamma} \mathbf{u}_2.$$

It may be directly seen that both S_{cl} and $\mathbf{x}_{\mathrm{cl}}$ are invariant under each of the former substitutions and so they are invariant with respect to (B.1), too. To see it more explicitly let us perform, for instance, the substitution $\mathbf{u}_1 \rightarrow \mathbf{u}_1 + \beta \mathbf{v}_1$, then

$$U_a \rightarrow U_a, \quad V_b \rightarrow V_b,$$
$$D_1 \rightarrow D_1, \quad D_2 \rightarrow D_2,$$
$$D_4 \rightarrow D_4, \quad D_3 \rightarrow D_3 - \beta D_1.$$

Plugging the previous substitution into (16), we get that $\mathbf{x}_{\mathrm{cl}}(t) \rightarrow \mathbf{x}_{\mathrm{cl}}(t)$. The same is true for S_{cl} as it might be directly seen from relations (14) and (A.5).

The previous two observations therefore lead us to the conclusion that S_{cl} does not depend on the particular choice of the fundamental system of solutions (expressible as a linear combination of $\mathbf{u}_i$ and $\mathbf{v}_i$).

Appendix C

We derive here some relations used in Section 3.1. As we have mentioned in Section 2.2, having one solution, say for (x_1, x_2) coordinates, we can get another one if

we multiply the original one by σ_3. So namely if one has the fundamental system of solutions $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2)$, one can generate another fundamental system $(\sigma_3\mathbf{u}_1, \sigma_3\mathbf{u}_2, \sigma_3\mathbf{v}_1, \sigma_3\mathbf{v}_2)$ (it is simple to see that this is indeed a fundamental system by looking at the Wronskian). Identical reasonings as in the Section 2.4 will lead us to the result

$$\mathbf{x}_{\text{cl}}(t) = \frac{1}{D}\left[x_a^1\begin{pmatrix}B_2^2(t)\\B_2^1(t)\end{pmatrix} + x_a^2\begin{pmatrix}B_1^2(t)\\B_1^1(t)\end{pmatrix} + x_b^1\begin{pmatrix}B_4^2(t)\\B_4^1(t)\end{pmatrix} + x_b^2\begin{pmatrix}B_3^2(t)\\B_3^1(t)\end{pmatrix}\right].\tag{C.1}$$

Comparing (C.1) with (18) we get the following useful identities:

$$\begin{aligned}B_1^1(t) &= B_2^2(t), \quad B_1^2(t) = B_2^1(t),\\ B_3^1(t) &= B_4^2(t), \quad B_3^2(t) = B_4^1(t).\end{aligned}\tag{C.2}$$

Similar analysis may be done with the fundamental systems $(\sigma_3\mathbf{u}_1(-t), \sigma_3\mathbf{u}_2(-t), \sigma_3\mathbf{v}_1(-t), \sigma_3\mathbf{v}_2(-t))$ and $(i\sigma_2\mathbf{u}_1(-t), i\sigma_2\mathbf{u}_2(-t), i\sigma_2\mathbf{v}_1(-t), i\sigma_2\mathbf{v}_2(-t))$. In this case it can be directly checked that we have

$$B_i^j(t) = (-1)^{i+j} B_i^j(-t).\tag{C.3}$$

From the definition (20) and relations (C.2) we may observe that

$$\frac{\mathrm{d}}{\mathrm{d}t_a}D = 2\dot{B}_1^1(t_a), \quad \frac{\mathrm{d}}{\mathrm{d}t_b}D = 2\dot{B}_3^1(t_b),\tag{C.4}$$

so namely

$$2\frac{\dot{B}_1^1(t_a)}{D} = \frac{\mathrm{d}}{\mathrm{d}t_a}\ln D = \text{Tr}\left(\mathbf{D}^{-1}\frac{\mathrm{d}}{\mathrm{d}t_a}\mathbf{D}\right),\tag{C.5}$$

$$2\frac{\dot{B}_3^1(t_b)}{D} = \frac{\mathrm{d}}{\mathrm{d}t_b}\ln D = \text{Tr}\left(\mathbf{D}^{-1}\frac{\mathrm{d}}{\mathrm{d}t_b}\mathbf{D}\right).\tag{C.6}$$

Here $D = \det\mathbf{D}$. Because $D(t_a, t_b) = D(t_b - t_a)$ it fulfills the equation

$$\frac{\mathrm{d}D(t_a, t_b)}{\mathrm{d}t_a} + \frac{\mathrm{d}D(t_a, t_b)}{\mathrm{d}t_b} = 0,$$

and so we have the identity

$$\dot{B}_1^1(t_a) = -\dot{B}_3^1(t_b).\tag{C.7}$$

Appendix D

We prove here the relation (31). For this purpose it is simpler to work in the (x, y) coordinates. The kernel can be then constructed in analogous way as in (x_1, x_2) coordinate. A simple calculation shows that

$$S_{\text{cl}}[r, u] = \frac{m}{2D}\left[-\frac{r_a^2}{2}\frac{\mathrm{d}D}{\mathrm{d}t_a} + \frac{r_b^2}{2}\frac{\mathrm{d}D}{\mathrm{d}t_b} + r_a r_b\left(e^{u_a - u_b}\dot{B}_1^1(t_b) + e^{u_b - u_a}\dot{B}_2^2(t_b)\right)\right],\tag{D.1}$$

and that

$$\dot{B}_1^1(t_b)\dot{B}_2^2(t_b) = WD.$$

The latter allows to identify

$$\frac{\dot{B}_1^1(t_b)}{D} = \sqrt{\frac{W}{D}}e^{\alpha}; \quad \frac{\dot{B}_2^2(t_b)}{D} = \sqrt{\frac{W}{D}}e^{-\alpha}.$$

This identification fixes α modulo $i\pi$ and leads to the equation

$$\alpha = \frac{1}{2}\ln\left(\frac{\dot{B}_1^1(t_b)}{\dot{B}_2^2(t_b)}\right). \tag{D.2}$$

In addition, from the symmetry reasonings result the following useful relations:

$$\dot{B}_1^1(t_b) = -\dot{B}_4^2(t_a), \quad \dot{B}_2^2(t_b) = -\dot{B}_3^1(t_a).$$

Let us now consider $\dot{B}_1^1(t_b)$. Its explicit structure reads

$$\dot{B}_1^1(t_b) = \begin{vmatrix} \dot{u}_1^1(t_b) & \dot{u}_2^1(t_b) & \dot{v}_1^1(t_b) & \dot{v}_2^1(t_b) \\ u_1^2(t_a) & u_2^2(t_a) & v_1^2(t_a) & v_2^2(t_a) \\ \mathbf{u}_1(t_b) & \mathbf{u}_2(t_b) & \mathbf{v}_1(t_b) & \mathbf{v}_2(t_b) \end{vmatrix}. \tag{D.3}$$

Rules for differentiation of determinants tell that $\dot{B}_1^1(t_b)$ fulfills the equation

$$\frac{d^2}{dt_a^2}\dot{B}_1^1(t_b) - \frac{\gamma}{m}\frac{d}{dt_a}\dot{B}_1^1(t_b) + \frac{\kappa}{m}\dot{B}_1^1(t_b) = 0, \tag{D.4}$$

with the boundary condition $\dot{B}_1^1(t_b)|_{t_a=t_b} = 0$. The general solution of (D.4) reads

$$\dot{B}_1^1(t_b) = e^{\Gamma(t_a)}\left(e^{i\Omega t_a}f(t_b) + e^{-i\Omega t_a}\tilde{f}(t_b)\right). \tag{D.5}$$

Here f and $\tilde{f}$ are some functions of t_b. Applying the boundary condition we get

$$\dot{B}_1^1(t_b) = Ce^{-\Gamma(t_b-t_a)}\sin\Omega(t_b - t_a), \tag{D.6}$$

with C being a constant. The result is clearly the only one which is compatible with the differential equation for $\dot{B}_4^2(t_a)$ ($= -\dot{B}_1^1(t_b)$):

$$\frac{d^2}{dt_b^2}\dot{B}_4^2(t_a) + \frac{\gamma}{m}\frac{d}{dt_b}\dot{B}_4^2(t_a) + \frac{\kappa}{m}\dot{B}_4^2(t_a) = 0, \tag{D.7}$$

fulfilling the boundary condition $\dot{B}_4^2(t_a)|_{t_b=t_a} = 0$.

The same reasonings can be now applied on $\dot{B}_2^2(t_b)$. The latter fulfills the differential equation

$$\frac{d^2}{dt_a^2}\dot{B}_2^2(t_b) + \frac{\gamma}{m}\frac{d}{dt_a}\dot{B}_2^2(t_b) + \frac{\kappa}{m}\dot{B}_2^2(t_b) = 0, \tag{D.8}$$

with the boundary condition $\dot{B}_2^2(t_b)|_{t_a=t_b} = 0$. The solution is

$$\dot{B}_2^2(t_b) = \tilde{C}e^{\Gamma(t_b-t_a)}\sin\Omega(t_b - t_a), \tag{D.9}$$

with $\tilde{C}$ being a constant. The result is the only one which is compatible with the differential equation for $\dot{B}_3^1(t_a)$ $(= -\dot{B}_2^2(t_b))$:

$$\frac{d^2}{dt_b^2}\dot{B}_3^1(t_a) - \frac{\gamma}{m}\frac{d}{dt_b}\dot{B}_3^1(t_a) + \frac{\kappa}{m}\dot{B}_3^1(t_a) = 0, \tag{D.10}$$

fulfilling the boundary condition $\dot{B}_3^1(t_a)|_{t_b=t_a} = 0$.

Gathering the results (D.2), (D.6), and (D.9) together we obtain

$$\alpha(t_a, t_b) = \Gamma(t_a - t_b) + \ln C - \ln \tilde{C}, \tag{D.11}$$

modulo $i\pi$. Using l'Hopital rule one may persuade oneself that $\lim_{t_a \to t_b} |\dot{B}_2^2(t_b)/\dot{B}_1^1(t_b)| = |C/\tilde{C}| = 1$. Thus, $\ln C - \ln \tilde{C}$ is either zero or purely imaginary.

Appendix E

Let us first emphasize that the formulation of a time reversal transformation must avoid using properties of the forces or interactions that determine the dynamics, because it is the transformation properties of the dynamic equations which we seek to determine. The latter is the crux often overlooked by many authors. Since the kinematics are those properties of the motion that are independent of the dynamics, we require that the "admissible" time-reversal transformation should be formulated in kinematic terms. This means that the time-reversal transformation must be consistent with the algebraic structure of the operators representing the (kinematic) observables and that in the absence of forces or interactions (i.e., in the absence of causal effects), the dynamic equations must be left invariant.

In our case the kinematic observables may be taken to be x^α and P^α (note that $\mathbf{P} = m\dot{\mathbf{x}}$ are the kinetic momenta and not the full canonical momenta (11)). Working in (x_1, x_2) coordinates, the algebraic structure is then determined by the Heisenberg–Weyl group

$$[x^\alpha, P^\beta] = i\hbar(\sigma_3)^{\alpha\beta}, \quad [\mathbf{x}, \sigma_3] = [\mathbf{P}, \sigma_3] = 0, \tag{E.1}$$

On the other hand, if $\hat{H}_0$ is the Hamiltonian in the absence of interaction ($\gamma = 0$), the dynamic (Schrödinger) equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}_0|\psi(t)\rangle, \tag{E.2}$$

must transform under time reversal $\mathcal{T}$ into

$$i\hbar\frac{d}{dt'}|\psi'(t')\rangle = \hat{H}_0|\psi'(t')\rangle, \tag{E.3}$$

where $t' = -t$. Applying $\mathcal{T}$ to both sides of Eq. (E.2) we obtain

$$\mathcal{T}i\mathcal{T}^{-1}\hbar\frac{d}{dt}|\psi'(t')\rangle = \mathcal{T}\hat{H}_0\mathcal{T}^{-1}|\psi'(t')\rangle. \tag{E.4}$$

Comparing (E.4) with (E.3) and using the requirement that $\hat{H}_0$ is invariant under time reversal we obtain the relation: $\mathcal{T}i\mathcal{T}^{-1} = -i$. Invoking Wigner's theorem [34],

the latter implies that $\mathcal{T}$ must be antiunitary and thus may be written as $\mathcal{T} = \mathcal{U}\mathcal{K}$ with $\mathcal{K}$ being the complex conjugation operator and $\mathcal{U}$ being some unitary operator.

At the same time, in accordance with the classical conditions, time reversal requires that

$$\mathbf{x}_{T} = \mathbb{B}\mathbf{x}, \quad \mathbf{P}_{T} = -\mathbb{B}\mathbf{P}. \tag{E.5}$$

The matrix $\mathbb{B}$ is supposed to leave $\hat{H}_0$ invariant under the time reversal. This means that $\mathbb{B}^t(\sigma_3)\mathbb{B} = \sigma_3$ and $\mathbb{B}^2 = 1$ (i.e., time reversal when repeated must restore the original situation). So $\mathbb{B}$ must be part of a discrete (two-element) subgroup of $O(1, 1; R)$ (i.e., the real Lorentz group in the plane). It is well known that the only matrices fulfilling the above conditions are ± 1 and $\pm\sigma_3$.

To decide the form of $\mathbb{B}$ we use the fact that $\hat{H}_0$ is invariant under $O(1, 1; R)$. This means that $\mathbf{x}$ and $\mathbf{P}$ transform under $O(1, 1; R)$ in the usual manner, i.e.,

$$\begin{aligned} \mathbf{x}' &= U(\varepsilon)\mathbf{x}U^{-1}(\varepsilon) = \mathbb{G}(\varepsilon)\mathbf{x}, \\ \mathbf{P}' &= U(\varepsilon)\mathbf{P}U^{-1}(\varepsilon) = \mathbb{G}(\varepsilon)\mathbf{P}, \end{aligned} \tag{E.6}$$

where $U(\varepsilon)$ is a unitary representation of $O(1, 1; R)$ in the state space and $\mathbb{G}(\varepsilon)$ is an element of $O(1, 1; R)$ in 2-dimensional vector space. As a result, the following relation must hold for any $\mathbb{G} \in O(1, 1; R)$:

$$U\mathcal{T}\mathbf{x}\mathcal{T}^{-1}U^{-1} = U\mathbf{x}_{T}U^{-1} = \mathbb{G}\mathbf{x}_{T} = \mathbb{G}\mathbb{B}\mathbf{x}. \tag{E.7}$$

However, the very same relation may be recast in a slightly different form, namely

$$U\mathcal{T}\mathbf{x}\mathcal{T}^{-1}U^{-1} = U\mathbb{B}\mathbf{x}U^{-1} = U\mathbb{B}U^{-1}\mathbb{G}\mathbf{x} = \mathbb{G}'\mathbb{B}\mathbb{G}^2\mathbf{x}. \tag{E.8}$$

Comparing both (E.7) and (E.8) we get that $\mathbb{G}\mathbb{B} = \mathbb{G}'\mathbb{B}\mathbb{G}^2$. As this must be true for all $\mathbb{G} \in O(1, 1; R)$, we can choose

$$\mathbb{G}(\varepsilon) = \exp(\varepsilon\sigma_1) = \cosh(\varepsilon) + \sinh(\varepsilon)\sigma_1. \tag{E.9}$$

It is then obvious that the case $\mathbb{B} = \pm 1$ is ruled out and we are left with $\mathbb{B} = \pm\sigma_3$. However, the "+" sign is the only plausible one. This is because the signature of the time reversal should be preserved under continuous change of coordinates and so namely when we shrink the x_2 coordinate into the origin (i.e., perform a dimensional reduction) x_1 coordinate must behave under time reversal as in ordinary 1D l.h.o.

As a upshot of the performed analysis we have the following transformations:

$$\begin{aligned} \mathbf{x} &\to \mathbf{x}_{T} = \mathcal{T}\mathbf{x}\mathcal{T}^{-1} = \sigma_3\mathbf{x}, \\ \mathbf{p} &\to \mathbf{p}_{T} = \mathcal{T}\mathbf{p}\mathcal{T}^{-1} = -\sigma_3\mathbf{p}, \\ r &\to r_{T} = \mathcal{T}r\mathcal{T}^{-1} = r, \\ u &\to u_{T} = \mathcal{T}u\mathcal{T}^{-1} = -u, \end{aligned} \tag{E.10}$$

(here $\mathbf{p}$ are full, i.e., canonical momenta). Similarly, we find that

$$\begin{aligned} \mathcal{T}A\mathcal{T}^{-1} &= -A, \quad \mathcal{T}B\mathcal{T}^{-1} = B, \\ \mathcal{T}J_+\mathcal{T}^{-1} &= -J_+, \quad \mathcal{T}J_-\mathcal{T}^{-1} = -J_-, \\ \mathcal{T}J_1\mathcal{T}^{-1} &= -J_1, \quad \mathcal{T}J_3\mathcal{T}^{-1} = J_3, \end{aligned} \tag{E.11}$$

and the time-reversed commutation relations

$$\left[x_{_T}^{\alpha}, p_{_T}^{\beta}\right] = -\mathrm{i}\hbar(\sigma_3)^{\alpha\beta}, \quad [\mathbf{x}_{_T}, \sigma_3] = [\mathbf{p}_{_T}, \sigma_3] = 0. \tag{E.12}$$

Transformation rules (E.10) or (E.11) assert that

$$\mathcal{T}\mathcal{C}\mathcal{T}^{-1} = \mathcal{C}, \quad \mathcal{T}J_2\mathcal{T}^{-1} = J_2. \tag{E.13}$$

We thus finally arrive at the conclusion that $\mathcal{T}\hat{H}\mathcal{T}^{-1} = \hat{H}$. The latter is not actually compatible with the time reversal presented in [11], where $\mathbb{B} = 1$ was incorrectly assumed.

Appendix F

Using the fact that $SU(1,1)$ ladder operators are J_+ and J_- and the $SU(1,1)$ vacuum state is the state $|j,|j|\rangle$ (i.e., $J_-|j,|j|\rangle = 0$), we may write [35]:

$$|\psi_{n,l}^s\rangle = \left(\frac{1}{\sqrt{2}}\right)^{2|j|+1} \sqrt{\frac{(2|j|)!(m-|j|)!}{(m+|j|)!}} L_{m-|j|}^{2|j|}\left(-\frac{J_+}{2}\right)|1\rangle\rangle,$$

$$j = n + \frac{l}{2} + \frac{1}{2}; \quad m = \frac{l}{2} - \frac{1}{2}. \tag{F.1}$$

Here $|z\rangle\rangle = \exp(zJ_+)|j,|j|\rangle$ is the (unnormalized) coherent state of $SU(1,1)$ [35] (which can be identified with the Gelfand–Neimark z-basis [39]).

In deriving Eq. (F.1) we have employed (76) together with the "annihilation" and "creation" relations

$$(J_-)^k|j,m\rangle = a_{j,m,k}|j,m-k\rangle, \quad m-k \geqslant |j|$$
$$= 0, \quad m-k \leqslant |j|, \tag{F.2}$$
$$(J_+)^k|j,m\rangle = b_{j,m,k}|j,m+k\rangle,$$

where

$$a_{j,m,k} = \sqrt{\frac{(m-j)!(m+j)!}{(m-j-k)!(m+j-k)!}},$$

$$b_{j,m,k} = \sqrt{\frac{(m+k+j)!(m+k-j)!}{(m+j)!(m-j)!}}.$$

Note that $(J_+J_-)|j,m\rangle = (m^2 - j^2)|j,m\rangle$. Form (F.1) suggests that the state $|\psi_{n,l}^s\rangle$ can be alternatively interpreted as an excited $SU(1,1)$ coherent state $|1\rangle\rangle$. Because $L_n^\alpha(x)$ is a polynomial of n-th order in x, the relation (F.1) asserts that there is up to $(m-|j|)$ new "$SU(1,1)$ excitations" condensed into the coherent state $|1\rangle\rangle$.

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