Superpositions of probability distributions

Petr Jizba* and Hagen Kleinert†
ITP, Freie Universität Berlin, Arnimallee 14 D-14195 Berlin, Germany
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Probability distributions which can be obtained from superpositions of Gaussian distributions of different variances $v=\sigma^2$ play a favored role in quantum theory and financial markets. Such superpositions need not necessarily obey the Chapman-Kolmogorov semigroup relation for Markovian processes because they may introduce memory effects. We derive the general form of the smearing distributions in v which do not destroy the semigroup property. The smearing technique has two immediate applications. It permits simplifying the system of Kramers-Moyal equations for smeared and unsmeared conditional probabilities, and can be conveniently implemented in the path integral calculus. In many cases, the superposition of path integrals can be evaluated much easier than the initial path integral. Three simple examples are presented, and it is shown how the technique is extended to quantum mechanics.

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I. INTRODUCTION

Path integrals are a powerful tool in diverse areas of physics, both computationally and conceptually. They often provide the easiest route to derivation of perturbative expansions as well as an excellent framework for nonperturbative analysis [1]. One of the key properties of path integrals in statistical physics is that the related time evolution of the conditional probabilities $P(x_b,t_b|x_a,t_a)$ fulfills the Chapman-Kolmogorov (CK) equation for continuous Markovian processes,

$$P(x_b,t_b|x_a,t_a) = \int dx P(x_b,t_b|x,t)P(x,t|x_a,t_a). \qquad (1)$$

Conversely, any probability satisfying this equation possesses a path integral representation, as shown by Kac and Feynman [2–4]. Equation (1) also serves as a basis for deriving a Fokker-Planck time evolution equation [1,5,6] for $P(x_b,t_b|x_a,t_a)$ from either the stochastic differential equation obeyed by the variable x or the Hamiltonian driving the time evolution of $P(x_b,t_b|x_a,t_a)$. Such equations are used to explain many different physical phenomena, for example turbulence [7] or epitaxial growth [8]. In information theory they serve as a tool for modeling various queueing processes [9], while in mathematical finance they are conveniently applied in theory of option pricing for efficient markets [1,10–12].

A trivial property of $P(x_b,t_b|x_a,t_a)$ satisfying the CK equation (1) is the initial condition

$$P(x_b,t_a|x_a,t_a) = \delta(x_b - x_a). \qquad (2)$$

The right-hand side can be written as a scalar product of Dirac's bra and ket states $\langle x_b|$ and $|x_a\rangle$ as

$$P(x_b,t_a|x_a,t_a) = \langle x_b|x_a\rangle. \qquad (3)$$

In many practical applications one encounters conditional probabilities formulated as a superposition of path integrals in which the Hamiltonians H are rescaled by a factor v, i.e.,

$$\bar{P}(x_b,t_b|x_a,t_a) = \int_0^\infty dv\, \omega(v,t_{ba}) \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x\mathcal{D}p\, e^{\int_{t_a}^{t_b} d\tau [ip\dot{x} - vH(p,x)]}, \qquad (4)$$

where $\omega(v,t_{ba})$ is some positive, continuous and normalizable smearing function of $v \geqslant 0$ and $t_{ba} \equiv t_b - t_a \geqslant 0$. For instance, probability distributions which can be obtained from superpositions of Gaussian distributions of different volatilities $v=\sigma^2$ play an important role in financial markets [1,10]. Such smearing distributions show up also in nonperturbative approximations to quantum statistical partition functions [13], in systems with time reparametrization invariance [1,14], in polymer physics [1,15], in superstatistics [16], etc. Whenever smeared path integrals fulfill the CK equation, the Feynman-Kac formula ensures that such superpositions can themselves be written as a path integral without smearing,

$$\bar{P}(x_b,t_b|x_a,t_a) = \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x\mathcal{D}p\, e^{\int_{t_a}^{t_b} d\tau [ip\dot{x} - \bar{H}(p,x)]}, \qquad (5)$$

with a new Hamiltonian $\bar{H}(p,x)$ given by

$$e^{-\int_{t_a}^{t_b} d\tau \bar{H}(p,x)} = \int_0^\infty dv\, \omega(v,t_{ba}) e^{-\int_{t_a}^{t_b} d\tau v H(p,x)}. \qquad (6)$$

In general, the smeared path integral (4) does not conserve the CK equation. Physically this implies that the superposition (4) introduces memory into the system. The aim of this paper is to find conditions for the smearing distributions where this is avoided. Various physical consequences will be derived from this.

The paper is organized as follows. Section II starts with a warm up example of smearing distributions for a Gaussian conditional probability. In Sec. III we derive the most general class of continuous smearing distributions fulfilling the CK equation. Section IV is devoted to a construction of the Hamiltonian $\bar{H}$ and to a discussion of related issues. In Sec.

*jizba@physik.fu-berlin.de. On leave from FNSPE, Czech Technical University, Břehová 7, 115 19 Praha 1, Czech Republic.
†kleinert@physik.fu-berlin.de

V we show how the outlined path integral representation can be physically interpreted in terms of two coupled stochastic processes. Section VI discusses three specific smeared systems without memory. In particular we show how the explicit knowledge of $\bar{H}(p,x)$ can streamline practical calculations of numerous path integrals. Various remarks and generalizations are proposed in the concluding Sec. VII. For the reader's convenience, we include two appendices where we perform some finer mathematical manipulations needed in Sec. V.

II. SMEARING OF A GAUSSIAN DISTRIBUTION

Our goal is to find the most general form of $\omega(v,t)$ fulfilling the CK relation (1). Let us first illustrate what we want to achieve by smearing out a simple Gaussian system whose Hamiltonian is $H=vp^2/2$ leading to a conditional probability

$$P_v(x_b,t_b|x_a,t_a) = \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x\mathcal{D}p e^{\int_{t_a}^{t_b}d\tau[ip\dot{x}-vp^2/2]} = \frac{1}{\sqrt{2\pi v t_{ba}}}e^{-(x_b-x_a)^2/(2vt_{ba})}, \tag{7}$$

where $t_{ba}\equiv t_b-t_a$. This obeys the *Fokker-Planck equation*

$$\partial_{t_b}P_v(x_b,t_b|x_a,t_a) = \frac{v}{2}\partial_{x_b}^2 P_v(x_b,t_b|x_a,t_a), \tag{8}$$

which can be solved explicitly, including the initial condition (2) with the help of the differential operator for the momentum $\hat{p}\equiv -i\partial_x$ as

$$P_v(x_b,t_b|x_a,t_a) = e^{-t_{ba}v\hat{p}^2/2}\delta(x_b-x_a), \tag{9}$$

or, using the Dirac *bra* and *ket* states in (3), as

$$P_v(x_b,t_b|x_a,t_a) = \langle x_b|e^{-t_{ba}v\hat{p}^2/2}|x_a\rangle. \tag{10}$$

Due to the completeness relation $\int dx|x\rangle\langle x|=1$, this expression obviously satisfies the CK equation (1).

Let us now assume that $\omega(v,t)$ can be written as a Fourier transform of the form (compare also Ref. [17]):

$$\omega(v,t_{ba}) = \int_{-i\infty}^{i\infty}\frac{d\xi}{2\pi i}e^{\xi v - H_\omega(\xi/t_{ba})t_{ba}}, \tag{11}$$

where $H_\omega(\xi)$ can be viewed as the Hamiltonian affiliated with distribution ω. Then the smeared transition probability has the integral representation

$$\bar{P}(x_b,t_b|x_a,t_a) = \int_{-i\infty}^{i\infty}\frac{d\xi}{2\pi i}\left\langle x_b\left|\frac{e^{-H_\omega(\xi/t_{ba})t_{ba}}}{\xi - t_{ba}\hat{p}^2/2}\right|x_a\right\rangle. \tag{12}$$

Assuming that $H_\omega(\xi)$ is a regular function which becomes infinite on a large semicircle in the upper half plane we can use the residue theorem to evaluate this as

$$\bar{P}(x_b,t_b|x_a,t_a) = \langle x_b|e^{-H_\omega(i\hat{p}^2/2)t_{ba}}|x_a\rangle. \tag{13}$$

This can be written as a path integral

$$\bar{P}(x_b,t_b|x_a,t_a) = \int_{x(t_a)=x_a}^{x(t_b)=x_b}\mathcal{D}x\mathcal{D}p e^{\int_{t_a}^{t_b}d\tau[ip\dot{x}-H_\omega(p^2/2)]}. \tag{14}$$

Thus the Hamiltonian $\bar{H}(p,x)$ of the smeared system in Eq. (5) is simply equal to $H_\omega(p^2/2)$.

In the following we shall generalize this treatment to non-Gaussian Hamiltonians in Eq. (7).

III. SMEARING OF A GENERAL DISTRIBUTION

We now embark on finding the most general smearing function $\omega(v,t_{ab})$ to guarantee the CK relation for the smeared expression (4). Replacing the probabilities in (1) by (4), we bring the right-hand side to the explicit form

$$\int_0^\infty dv'\omega(v',t')\int_0^\infty dv''\omega(v'',t'')\int_{-\infty}^\infty dx\int_{x(t_c)=x}^{x(t_b)=x_b}\mathcal{D}x\mathcal{D}p \times e^{\int_{t_c}^{t_b}d\tau(ip\dot{x}-v''H)}\int_{x(t_a)=x_a}^{x(t_c)=x}\mathcal{D}x\mathcal{D}p e^{\int_{t_a}^{t_c}d\tau(ip\dot{x}-v'H)}$$
$$= \frac{t}{2t't''}\int_0^\infty dv\int_{-tv}^{tv}d\zeta\,\omega\left(\frac{(tv-\zeta)}{2t'},t'\right)\omega\left(\frac{(tv+\zeta)}{2t''},t''\right) \times\int_{x(t_a)=x_a}^{x(t_b)=x_b}\mathcal{D}x\mathcal{D}p e^{\int_{t_a}^{t_b}d\tau(ip\dot{x}-vH)}, \tag{15}$$

where $t'\equiv t_{ca}=t_c-t_a$ and $t''\equiv t_{bc}=t_b-t_c$. In the second line we have used the substitution $t''v''+t'v'=tv$ and $t''v''-t'v'=\zeta$. Comparing the right-hand side of (15) with the left-hand side of (4) expressed in the smeared form (4), we obtain an integral equation for the smearing function $\omega(v,t)$:

$$\int_{-tv}^{tv}d\zeta\,\omega\left(\frac{(tv-\zeta)}{2t'},t'\right)\omega\left(\frac{(tv+\zeta)}{2t''},t''\right) = \frac{2t't''\omega(v,t)}{t}. \tag{16}$$

Setting $z\equiv tv/t''$ and going over to an integration variable $z'=(tv+\zeta)/2t''$, this becomes

$$\int_0^z dz'\omega(z',t'')\omega\left(\frac{t''}{t'}(z-z'),t'\right) = \frac{t'}{t}\omega\left(\frac{t''}{t}z,t\right), \tag{17}$$

or equivalently

$$\int_0^z dz'\omega(z',t)a\omega\left(a(z-z'),\frac{t}{a}\right) = b\omega\left(bz,\frac{t}{b}\right), \tag{18}$$

with positive real a,b satisfying $1+1/a=1/b$. Since the left-hand side is a convolution integral, the solution of this equation is found by a Laplace transformation. Defining

$$\tilde{\omega}(\xi,t) = \int_0^\infty dz e^{-\xi z}\omega(z,t), \quad \text{Re}\,\xi>0, \tag{19}$$

we may reduce Eq. (18) to the functional equation

$$\tilde{\omega}(\xi,t)\tilde{\omega}\left(\frac{\xi}{a},\frac{t}{a}\right) = \tilde{\omega}\left(\frac{\xi}{b},\frac{t}{b}\right). \tag{20}$$

It should be stressed that due to assumed normalizability and positivity of $\omega(v,t)$, the smearing distribution is always

Laplace transformable. The substitution $\alpha = 1/a$ transforms (20) to

$$\tilde{\omega}(\xi,t)\tilde{\omega}(\alpha\xi,\alpha t) = \tilde{\omega}(\xi + \alpha\xi, t + \alpha t). \tag{21}$$

By considering only real ξ, Eq. (21) can be solved by method of iterations familiar from the theory of functional equations [18]. Assume for a moment that α is a positive integer, say n, then successive iterations of Eq. (21) give

$$\tilde{\omega}(n\xi, nt) = [\tilde{\omega}(\xi,t)]^n. \tag{22}$$

Now let $r = m/n$ be a positive rational number (m and n positive integers) and ζ and τ arbitrary positive real numbers. Then, for $\xi = r\zeta = (m/n)\zeta$ and $t = r\tau = (m/n)\tau$, we have $n\xi = m\zeta$ and $nt = m\tau$, so that Eq. (22) yields

$$\tilde{\omega}(\xi,t) = \tilde{\omega}(r\zeta, r\tau) = [\tilde{\omega}(\zeta,\tau)]^r, \tag{23}$$

for all positive ζ and τ and all positive rationals r. Assuming that $\tilde{\omega}$ is continuous, we may extend the Eq. (23) to all positive real r. It is then solved by

$$\tilde{\omega}(\xi,t) = \tilde{\omega}(t\xi/t, t1) = [\tilde{\omega}(\xi/t,1)]^t \equiv [G(\xi/t)]^t, \quad t > 0, \tag{24}$$

where $G(x)$ is any continuous function of x. The above derivation is meaningless for $t=0$. In this case we must instead of (24) consider

$$\tilde{\omega}(\xi,0) = \tilde{\omega}(\xi 1, \xi 0) = [\tilde{\omega}(1,0)]^\xi \equiv \kappa^\xi. \tag{25}$$

The constant κ is determined by the initial value of the smearing distribution $\omega(v,t)$. Thus Eq. (25) implies that

$$\lim_{t\to+0} \omega(v,t) = \theta(v + \log\kappa)\delta(v + \log\kappa) = \delta^+(v + \log\kappa). \tag{26}$$

Note that Eq. (24) implies positivity of $\tilde{\omega}(\xi,t)$ for all t and ξ, hence $G(x)$ must also be positive for all x. This allows us to write

$$[G(\xi/t)]^t = e^{-F(\xi/t)t}, \tag{27}$$

where $F(x)$ is some continuous function of x. The associated inverse Laplace transform gives then the complete solution for $\omega(v,t)$. Usually, the inverse Laplace transform is expressed as as a complex Bromwich integral [19] [compare (11)]:

$$\omega(v,t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} d\xi e^{\xi v}\tilde{\omega}(\xi,t), \tag{28}$$

where the real constant γ is such that it exceeds the real part of all the singularities of $\tilde{\omega}(\xi,t)$. For our purpose it will be preferable to use, instead of (28), the inversion formula due to Post [20]:

$$\omega(v,t) = \lim_{k\to\infty} \frac{(-1)^k}{k!}\left(\frac{k}{v}\right)^{k+1} \partial_\xi^k \tilde{\omega}(\xi,t)|_{\xi=k/v}. \tag{29}$$

For practical calculations this formula is rarely used due to the need to evaluate derivatives of arbitrary high orders. For our purpose, however, it has the advantage that it shows that a real Laplace transform $\tilde{\omega}(\xi,t)$ leads to a real smearing function $\omega(v,t_{ab})$. Moreover, one does not need to know the pole structure of $\tilde{\omega}(\xi,t)$ in the complex ξ plane, which is in general not known.

The result (24) can be mildly generalized to the smearing of composite Hamiltonians $vH(p,x) = vH_1(p,x) + H_2(p,x)$ as long as $[\hat{H}_1, \hat{H}_2] = 0$. At the same time it should be stressed that the entire approach fails for time-dependent Hamiltonians.

To end this section we note that if the smearing distribution $\omega(v,t)$ would have included negative v values, the integral in Eq. (18) would have been replaced with $\int_0^z \mapsto \int_\infty^{-\infty}$. Then a two-sided Laplace transformation would have brought us to the same equations (23) and (24) as before, and the associated general $\omega(v,t)$ would been recovered via the inverse of the two-sided Laplace transform. By restricting ourselves to $v \geqslant 0$, all calculations become simpler.

IV. EXPLICIT REPRESENTATION OF $\bar{H}(p,x)$

We now determine the Hamiltonian $\bar{H}(p,x)$ explicitly in the general case. For this we must first make sure that the path integral for the initial distribution in (4),

$$P_v(x_b,t_b|x_a,t_a) = \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x\mathcal{D}p e^{\int_{t_a}^{t_b} d\tau [ip\dot{x} - vH(p,x)]}, \tag{30}$$

is properly defined. The classical Hamiltonian $H(p,x)$ must be set up in such a way that the Hamiltonian operator driving the time evolution

$$\partial_t P_v(x,t|x_a,t_a) = -v\hat{H}(\hat{p},x)P_v(x,t|x_a,t_a), \tag{31}$$

has all momentum operators $\hat{p} = -i\partial_x$ to the left of the x variables. Only then can one guarantee the probability conservation law for $P(x_b,t_b|x_a,t_a)$:

$$\int dx \partial_t P_v(x,t|x_a,t_a) = -\int dx v\hat{H}(\hat{p},x)P_v(x,t|x_a,t_a). \tag{32}$$

The right-hand side vanishes after a partial integration. The relation between $\hat{H}(\hat{p},x)$ and $H(p,x)$ is explained in Ref. [1].

With the help of Post's inversion formula (29) we now rewrite the smeared conditional probability (4) in the form

$$\bar{P}(x_b,t_b;x_a,t_a) = \lim_{k\to\infty} \frac{(-1)^{k-1}}{(k-1)!}\int_0^\infty d\xi \xi^{k-1}\partial_\xi^k \tilde{\omega}(\xi,t_{ba}) \times \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x\mathcal{D}p e^{\int_{t_a}^{t_b} d\tau (ip\dot{x} - kH/\xi)}. \tag{33}$$

Inserting here Eq. (19), the integration over ξ turns into

$$\int_0^\infty dv v^k \omega(v,t_{ba}) \int_0^\infty d\xi \xi^{k-1} e^{-v\xi} \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x\mathcal{D}p e^{\int_{t_a}^{t_b} d\tau [ip\dot{x} - kH/\xi]}. \tag{34}$$

The path integral can be written in Dirac operator form (10) as

$$\langle x_b | e^{-kt_{ba}\hat{H}/\xi} | x_a \rangle, \tag{35}$$

so that (34) becomes

$$\int_0^\infty dv v^k \omega(v,t_{ba}) \left\langle x_b \left| \int_0^\infty d\xi \xi^{k-1} e^{-v\xi} e^{-kt_{ba}\hat{H}/\xi} \right| x_a \right\rangle. \tag{36}$$

The ξ integral yields a Bessel function of the second type

$$2(kt_{ba}\hat{H}/y)^{k/2} K_k(2\sqrt{kyt_{ba}\hat{H}}). \tag{37}$$

Applying the limiting form of the Bessel function for large index [21]

$$K_k(\sqrt{k}x) \sim \frac{1}{2} k^{-k/2} \Gamma(k) \left(\frac{x}{2}\right)^{-k} e^{-x^2/4}, \tag{38}$$

we can cast (33) to

$$\bar{P}(x_b,t_b|x_a,t_a) = \left\langle x_b \left| \int_0^\infty dv\, \omega(v,t_{ba}) e^{-vt_{ba}\hat{H}} \right| x_a \right\rangle. \tag{39}$$

We now use Eqs. (19) and (24) to rewrite this as

$$\langle x_b | \tilde{\omega}(t_{ba}\hat{H},t_{ba}) | x_a \rangle = \langle x_b | \tilde{\omega}(\hat{H},1)^{t_{ba}} | x_a \rangle. \tag{40}$$

With Eq. (27), this becomes

$$\bar{P}(x_b,t_b|x_a,t_a) = \langle x_b | e^{-t_{ba}F(\hat{H})} | x_a \rangle, \tag{41}$$

which can be expressed as a path integral

$$\bar{P}(x_b,t_b|x_a,t_a) = \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x\mathcal{D}p\, e^{\int_{t_a}^{t_b} dt[ip\dot{x} - F_{\rm cl}(H)]}. \tag{42}$$

Here $F_{\rm cl}(H)$ denotes the classical function of the energy which makes the path integral (42) equal to the operator expression $F(\hat{H})$ in (41). The construction of (42) is highly nontrivial since one must ensure that the path integral leads to the correct operator order in $\tilde{\omega}(\hat{H},1)^{t_{ba}}$. The task is simple only if $\hat{H}$ depends only on $\hat{p}$. Then the ordering problem disappears and $F_{\rm cl}[H(p)] = F[H(p)]$. In general, the ordering problem has a unique solution [22] in the perturbative definition of path integrals (see Sec. 10.6 in Ref. [1]), in which they are expanded around harmonic path integrals with the help of Feynman diagrams and a specific extension of the theory of distribution. The result is equivalent to performing the expansion in $1+\varepsilon$ time dimensions and taking the limit $\varepsilon \to 0$ at the end.

There are other instances in which the operator ordering can be uniquely assigned. In Appendix B we discuss one such example.

A few observations are useful concerning he nature of $F(H)$. First, the normalization condition

$$1 = \int_0^\infty dv\, \omega(v,t) = \tilde{\omega}(0,t), \tag{43}$$

implies that $F(0)=0$. Second, the necessary positivity of $\omega(v,t)$ implies, via Post's formula, that all derivatives $(-1)^k \partial^k \tilde{\omega}(\xi,t)/\partial \xi^k$ are positive for large ξ and k. Taking into account that $\tilde{\omega}(x,t) > 0$ we conclude that $\tilde{\omega}(\xi,t)$ must be decreasing and convex for large ξ and any t. Asymptotic decrease and convexity of $e^{-F(x/t)t}$ ensure that $F(\hat{H})$ must be a monotonically increasing function of $\hat{H}$ for large spectral values of $\hat{H}$. Third, the real nature of $\tilde{\omega}(x,t)$ makes $F(\hat{H})$ a real function of $\hat{H}$. This is in contrast to quantum-mechanical path integrals where the smearing function $\omega(v,t)$ is not necessarily real.

V. KRAMERS-MOYAL EXPANSIONS

Let us study the implications of the above smearing procedure upon the time evolution equations for the conditional probabilities $P_v(x_b,t_b|x_a,t_a)$ and $\bar{P}(x_b,t_b|x_a,t_a)$ which have the general form (31). In statistical physics these are called Kramers-Moyal (KM) equations [1,6,23]. The negative time evolution operator $-v\hat{H}(\hat{p},x)$ is called *Kramers-Moyal operators* $\mathbb{L}_v(-\partial_x,x)$. Thus Eq. (31) for $P_v(x_b,t_b|x_a,t_a)$ and an analogous equation for $\bar{P}(x_b,t_b|x_a,t_a)$ are written as

$$\partial_{t_b} P_v(x_b,t_b|x_a,t_a) = \mathbb{L}_v P_v(x_b,t_b|x_a,t_a), \tag{44}$$

$$\partial_{t_b} \bar{P}(x_b,t_b|x_a,t_a) = \bar{\mathbb{L}} \bar{P}(x_b,t_b|x_a,t_a). \tag{45}$$

The Kramers-Moyal operator $\mathbb{L}_v$ has the expansion

$$\mathbb{L}_v(-\partial_{x_b},x_b) = \sum_{n=1}^{\infty} (-\partial_{x_b})^n D_v^{(n)}(x_b,t_b), \tag{46}$$

whose coefficients $D_v^{(n)}(x,t)$ are equal to the moments of the short-time transition probabilities:

$$D_v^{(n)}(x,t) = \frac{1}{n!} \lim_{\tau\to 0} \frac{1}{\tau} \int_{-\infty}^{\infty} dy (y-x)^n P_v(y,t+\tau|x,t). \tag{47}$$

Inserting Eq. (31), these can also be calculated from the formula

$$D_v^{(n)}(x,t) = \frac{1}{n!} \lim_{\tau\to 0} \frac{1}{\tau} \int_{-\infty}^{\infty} dy (y-x)^n \langle y | e^{-v\hat{H}\tau} | x \rangle. \tag{48}$$

The same equation holds for $\bar{P}(x_b,t_b|x_a,t_a)$ with the replacement $v\hat{H} \to \hat{\bar{H}}$, $\mathbb{L}_v \to \bar{\mathbb{L}}$, and $D_v^{(n)}(x,t) \to \bar{D}^{(n)}(x,t)$.

Our smearing procedure can be recast in the language of KM equations. We show that the two equations (44) and (45) can be replaced by an equivalent pair of KM equations. First we observe that Eq. (17) can be rewritten as

$$\omega(z,t) = \int_0^\infty dz' P_\omega(z,t|z',t') \omega(z',t'), \tag{49}$$

with the conditional probability

$$P_\omega(z,t|z',t') = \frac{t}{t-t'} \theta(tz - t'z') \omega\left(\frac{tz - t'z'}{t-t'}, t-t'\right), \tag{50}$$

which satisfies the initial condition

$$\lim_{\tau\to 0} P_\omega(z,t+\tau|z',t) = \delta^+(z-z'). \tag{51}$$

Moreover, expressing $\omega(z,t)$ in Eq. (49) in terms of $P_\omega(x_b,t_b|x_a,t_a)$ via (50), we find that $P_\omega(x_b,t_b|x_a,t_a)$ satisfies a CK equation

$$P_\omega(x_b,t_b|x_a,t_a) = \int dx P_\omega(x_b,t_b|x,t)P_\omega(x,t|x_a,t_a). \quad (52)$$

This implies that $P_\omega(x_b,t_b|x_a,t_a)$ obeys a KM time evolution equation

$$\partial_{t_{ba}} P_\omega(v_b,t_b|v_a,t_a) = \mathbb{L}_\omega P_\omega(v_b,t_b|v_a,t_a), \quad (53)$$

with the KM operator

$$\mathbb{L}_\omega = \sum_{n=1}^{\infty} (-\partial_v)^n K^{(n)}(v,t_{ba}). \quad (54)$$

The expansion coefficients are obtained by analogy with (47) from the short-time limits

$$\begin{aligned} K^{(n)}(v,t) &= \lim_{\tau\to 0} \frac{1}{n!\tau}\int_{-\infty}^{\infty} dv'(v'-v)^n P_\omega(v',t+\tau|v,t) \\ &= \lim_{\tau\to 0} \frac{1}{n!\tau}\left(\frac{\tau}{t_{ba}+\tau}\right)^n \int_0^{\infty} dv'(v'-v)^n \omega(v',\tau). \end{aligned} \quad (55)$$

A crucial observation for the further development is that due to the equality of the Kernel in the time evolution equations (53) and (49), the same KM operator governs the time evolution of $\omega(v,t_{ba})$:

$$\partial_{t_{ba}} \omega(v,t_{ba}) = \mathbb{L}_\omega \omega(v,t_{ba}), \quad (56)$$

In Appendix A we show that this equation can also be derived directly from the KM equations (44) and (45).

The smearing procedure of the path integral is equivalent to replacing the pair of KM equations (44) and (45) by the pair (44) and (56). This separates the dynamics of the smearing distribution from the dynamics of the transition amplitude. This may serve as a convenient starting point, for instance in quantum optics [24], in superstatistics [16], or in numerous option pricing models (see, e.g., Ref. [1] and citations therein).

For applications it is useful to remember Pawula's theorem [25], according to which the coefficients of the expansions of KM operators are either all nonzero or, if there is a finite number of them, they can be nonzero only up to $n=2$. This follows from the necessary positivity of the probabilities. If one artificially truncates $D_v^{(n)}(x,t)$ in (46) or $K^{(n)}(x,t)$ at some $n \geq 3$, then the ensuing transition probabilities *always* develop negative values, at least for sufficiently short times. This is the basic reason why phenomenological models for KM operators go usually only up to $n=2$.

Consider such a truncated model. Then the KM equations (44) and (56) reduce to the Fokker-Planck equations

$$\partial_t \omega(v,t) = \mathbb{L}_\omega^{\mathrm{FP}} \omega(v,t), \quad (57)$$

$$\partial_{t_b} P_v(x_b,t_b|x_a,t_a) = \mathbb{L}_v^{\mathrm{FP}} P_v(x_b,t_b|x_a,t_a), \quad (58)$$

with

$$\mathbb{L}_\omega^{\mathrm{FP}} = -\partial_v K^{(1)}(v,t) + \partial_v^2 K^{(2)}(v,t), \quad (59)$$

$$\mathbb{L}_v^{\mathrm{FP}} = -\partial_{x_b} D_v^{(1)}(x_b,t_b) + \partial_{x_b}^2 D_v^{(2)}(x_b,t_b). \quad (60)$$

Furthermore, Eqs. (57) and (58) allow us to find two coupled stochastic processes described by the two coupled Itō stochastic differential equations

$$dx_b = D_v^{(1)}(x_b,t_b)dt_b + \sqrt{2D_v^{(2)}(x_b,t_b)}dW_1, \quad (61)$$

$$dv = K^{(1)}(v,t_{ba})dt_{ba} + \sqrt{2K^{(2)}(v,t_{ba})}dW_2. \quad (62)$$

Here $W_1(t_b)$ and $W_2(t_{ba})$ are Wiener processes, i.e., Gaussian random walks.

VI. SIMPLE EXAMPLES

To demonstrate the usefulness of the superposition procedure we now discuss some important classes of $G(x)$ functions in Eq. (24).

(i) We start with the trivial choice

$$G(x) = e^{-ax+b}, \quad b \in \mathbb{R};\ a \in \mathbb{R}_0^+, \quad (63)$$

which gives

$$\tilde{\omega}(\zeta,t) = e^{-a\zeta+bt}, \quad (64)$$

and consequently

$$\omega(v,t) = e^{bt}\delta(v-a). \quad (65)$$

By requiring that ω is normalized to 1 for any t we have $b=0$, i.e., no smearing distribution. In this case the Hamiltonian $\bar{H}(p,x)=aH(p,x)$. Note that a is basically the averaged value of v over the δ function distribution.

(ii) A less trivial choice of $G(x)$ is

$$G(x) = \left(\frac{a}{x+b}\right)^c, \quad a \in \mathbb{R}^+;\ b,c \in \mathbb{R}_0^+, \quad (66)$$

which gives

$$\tilde{\omega}(\zeta,t) = \left(\frac{at}{\zeta+bt}\right)^{ct}, \quad (67)$$

leading thus to

$$\omega(v,t) = \frac{1}{\Gamma(ct)}(at)^{ct}e^{-btv}v^{ct-1}. \quad (68)$$

Further restriction on the coefficients is obtained by requiring normalizability of $\omega(v,t)$. The normalization condition $F(0)=0$ can be fulfilled in two ways. Either we set $c=0$, in which case $\omega(v,t)=\delta(v)$ and $\bar{H}(p,x)=0$, or we assume that $a=b$. In the latter case

$$\omega(v,t) = \frac{(bt)^{ct}v^{ct-1}}{\Gamma(ct)}e^{-btv}, \quad (69)$$

i.e., it corresponds to the *Gamma* distribution [4,10] $f_{bt,ct}(v)$ which is of a particular importance in financial data analysis [1,10,26] and superstatistics [27,28]. In the special case when $c=db=da$ and $b\to\infty$ we obtain that $\omega(v,t)=\delta(v-d)$. The Hamiltonian $\bar{H}$ associated with the distribution (69) reads

$$\bar{H}(p,x) = \bar{v}b\left[\log\left(\frac{H(p,x)}{b}+1\right)\right]_{\rm cl}, \tag{70}$$

where $\bar{v}=c/b$ is the mean of $\omega(v,t)$. In particular, for $H=p^2/2$ we obtain the smearing relation between path integrals

$$\int_{x(t_a)=x_a}^{x(t_b)=x_b}\mathcal{D}x\mathcal{D}p e^{\int_{t_a}^{t_b}d\tau[ip\dot{x}-\bar{v}b\,\log(p^2/2b+1)]}$$
$$= \int_0^\infty dv\,\omega(v,t_{ba})\int_{x(t_a)=x_a}^{x(t_b)=x_b}\mathcal{D}x\mathcal{D}p e^{\int_{t_a}^{t_b}d\tau(ip\dot{x}-vp^2/2)}. \tag{71}$$

Inserting for the path integral on the right-hand side the result (15), the integral over v reads explicitly

$$\int_0^\infty dv\frac{(bt_{ba})^{ct_{ba}}v^{ct_{ba}-1}}{\Gamma(ct_{ba})}\sqrt{\frac{1}{2\pi t_{ba}v}}e^{-bt_{ba}v}e^{-x_{ba}^2/2vt_{ba}}, \tag{72}$$

and yields

$$K_{1/2-ct_{ba}}(2\sqrt{2b}|x_{ba}|)\frac{t_{ba}^{-3/2}}{\sqrt{\pi}\Gamma(ct_{ba})}\left(\frac{2\sqrt{2b}t_{ba}}{|x_{ba}|}\right)^{1/2-ct_{ba}}. \tag{73}$$

In Fourier space, the superposition (71) leads to a Tsallis distribution [10,16,27].

Another interesting consequence arises when we consider the ensuing Itō stochastic equations. To this end we use the Hamiltonian from Refs. [1,10] which has the form $p^2/2+ip(r/v-1/2)$, with r being a constant. The corresponding drift and diffusion coefficients $D_v^{(1)}$ and $D_v^{(2)}$ are easily calculable giving

$$D_v^{(1)}(x,t_b) = \left(r-\frac{v}{2}\right),\quad D_v^{(2)}(x,t_b) = \frac{v}{2}. \tag{74}$$

The coefficients $K^{(n)}$ are

$$K^{(1)}(v,t_{ba}) = \frac{1}{t_{ba}}\left(\frac{c}{b}-v\right) = \frac{1}{t_{ba}}(\bar{v}-v),$$

$$K^{(n)}(v,t_{ba}) = \frac{1}{t_{ba}^n}\frac{c}{nb^n},\quad n\geq 2. \tag{75}$$

Hence we find the two coupled Itō equations (61) and (62) in the form

$$dx_b = \left(r-\frac{v}{2}\right)dt_b + \sqrt{v}dW_1,$$

$$dv = \frac{1}{t_{ba}}(\bar{v}-v)dt_{ba} + \frac{1}{t_{ba}}\sqrt{\frac{\bar{v}}{b}}dW_2. \tag{76}$$

One may now view x_b as a logarithm of a stock price S, and v and r as the corresponding variance and drift. If, in addition, we replace for large t_{ab} the quantity $\sqrt{\bar{v}}$ with $\sqrt{v}$, the systems (76) reduces to

$$dS = rSdt_b + \sqrt{v}SdW_1,$$

$$dv = \gamma(\bar{v}-v)dt_{ba} + \varepsilon\sqrt{v}dW_2, \tag{77}$$

where $\gamma=1/t_{ba}$ and $\varepsilon=1/(bt_{ba})$. The system of equations (77) constitute Heston's stochastic volatility model [26]. The parameters $\bar{v}$, γ, and ε can be then interpreted as the long-time average, the drift of the variance and the volatility of the variance, respectively. Heston's model may be used in quantitative finance to evaluate, for instance, the price of options (for further reference on this model see, e.g., Ref. [1] and citations therein).

(iii) As a third example we consider

$$G(x) = e^{-a\sqrt{x}},\quad a\in\mathbb{R}^+. \tag{78}$$

This implies

$$\tilde{\omega}(\zeta,t) = e^{-a\sqrt{t\zeta}}, \tag{79}$$

and

$$\omega(v,t) = \frac{ae^{-a^2t/4v}}{2\sqrt{\pi}\sqrt{\dfrac{v^3}{t}}}. \tag{80}$$

Image function $\tilde{\omega}(\zeta,t)$ already fulfils the normalized condition $F(0)=0$. In the literature, (80) is known as the *Weibull* distribution of order 1 (e.g., Refs. [4,32]) $\omega(v,t)=w(v;1,a^2t/2)$. The smearing distribution (80) has also important applications in the so-called inverse χ^2 superstatistics [27]. Because integral over $v^\alpha\omega(v,t)$ does not exist for $\alpha>1/2$, the distribution (80) does not have moments. The Hamiltonian $\bar{H}(p,x)=a[\sqrt{H(p,x)}]_{cl}$. In particular, this implies an identity [29]

$$\int_0^\infty dv\,w\left(v;1,\frac{t}{2}\right)\int_{\mathbf{x}(t_a)=\mathbf{x}_a}^{\mathbf{x}(t_b)=\mathbf{x}_b}\mathcal{D}\mathbf{x}\mathcal{D}\mathbf{p}e^{\int_{t_a}^{t_b}d\tau[i\mathbf{p}\dot{\mathbf{x}}-v(\mathbf{p}^2c^2+m^2c^4)]}$$
$$= \int_{\mathbf{x}(t_a)=\mathbf{x}_a}^{\mathbf{x}(t_b)=\mathbf{x}_b}\mathcal{D}\mathbf{x}\mathcal{D}\mathbf{p}e^{\int_{t_a}^{t_b}d\tau(i\mathbf{p}\dot{\mathbf{x}}-c\sqrt{\mathbf{p}^2+m^2c^2})}. \tag{81}$$

The right-hand side represents the path integral for the free relativistic particle in the Newton-Wigner representation [30,31]. Previously, this has been evaluated by group path integration [33]. With our smearing method, we can obtain the same result much faster by a direct calculation of the left-hand side of (81). Due the the quadratic nature of the Hamiltonian we obtain immediately in D space dimensions:

$$\int_{\mathbf{x}(t_a)=\mathbf{x}_a}^{\mathbf{x}(t_b)=\mathbf{x}_b}\mathcal{D}\mathbf{x}\mathcal{D}\mathbf{p}e^{\int_{t_a}^{t_b}d\tau(i\mathbf{p}\dot{\mathbf{x}}-c\sqrt{\mathbf{p}^2+m^2c^2})},$$
$$=\int_0^\infty dv\frac{e^{-t_{ba}/4v}}{2\sqrt{\pi}\sqrt{\dfrac{v^3}{t_{ba}}}}e^{-vm^2c^4t_{ba}-\mathbf{x}_{ba}^2/4vc^2t_{ba}}\left(\frac{1}{4vc^2\pi t_{ba}}\right)^{D/2},$$
$$=2ct_{ba}\left(\frac{m\gamma}{2\pi t_{ba}}\right)^{D+1/2}K_{D+1/2}\left(\frac{mc^2t_{ba}}{\gamma}\right), \tag{82}$$

with $\gamma=(1+\mathbf{x}_{ba}^2/c^2t_{ba}^2)^{-1/2}$, $\mathbf{x}_{ba}\equiv\mathbf{x}_b-\mathbf{x}_a$. The result in Eq. (82) agrees, of course, with those of earlier authors [33].

If Eq. (82) is taken to the limit $m=0$, it reads

$$\int_{\mathbf{x}(t_a)=\mathbf{x}_a}^{\mathbf{x}(t_b)=\mathbf{x}_b}\mathcal{D}\mathbf{x}\mathcal{D}\mathbf{p}e^{\int_{t_a}^{t_b}d\tau(i\mathbf{p}\dot{\mathbf{x}}-c|\mathbf{p}|)}=ct\Gamma\left(\frac{D+1}{2}\right)\left(\frac{\gamma^2}{\pi t^2c^2}\right)^{(D+1)/2}. \quad (83)$$

This is of a particular importance in econophysics as it can be directly related to a Lévy noise distribution of order 1, see, e.g. Ref. [1].

VII. CONCLUSIONS AND OUTLOOKS

In this paper we have identified the most general class of continuous smearing distributions $\omega(v,t)$ defined on $\mathbb{R}^+\times\mathbb{R}^+$ for which superpositions of Marcovian processes remain Markovian. This insight was used to rephrase the original problem of computing a complicated conditional probability $P(x_b,t_b|x_a,t_a)$ as a problem solving two coupled Kramers-Moyal equations, one for a simple basic distribution, and the other for the smearing distribution $\omega(v,t)$. If the associated Kramers-Moyal expansions are truncated after $n=2$, they become Fokker-Planck equations whose respective sample paths follow two coupled stochastic equations à la Itō. As an application of this decomposition procedure we have demonstrated how the log-normal fluctuations of a stock price with a Gamma smearing distribution of the variances lead to the celebrated Heston model of stochastic volatility.

As a second application we have shown that the ensuing relation between smeared path integrals and nonsmeared ones permits one to solve exactly (and often quite fast) relatively large classes of path integrals. In this connection we have discussed in some detail two important situations: path integrals arising in the framework of generalized statistics of Tsallis [28] and a world-line representation of the euclidean Newton-Wigner propagator. The latter situation can easily accommodate relativistic systems with gauge potentials via minimal coupling. Useful applications are expected in the field of world-line representations of effective actions [14,34,35].

For simplicity we have considered smearing distributions for path integrals with only bosonic degrees of freedom. There is no problem in extending our procedure to path integrals with Grassmann (i.e., fermionic) variables, and to more general initial functional integrals. Such extensions may be useful in polymer physics, in particular in the theory of self-avoiding chains, and in their field-theoretic treatments [1,36].

Our procedure can also be applied to quantum mechanics with only small modifications. There the CK relation is the composition law for time translation amplitudes reflecting the semigroup property of the evolution operator $e^{-i\int dtH}$. The smearing functions acts then upon probability amplitudes rather than conditional probabilities, in which case they may be complex rather than positive.

Note added. We recently became aware of Refs. [17,27], devoted to superstatistics. There exists a conceptual overlap between the present paper and the above mentioned ones. In fact, practitioners in superstatistics can quickly benefit from our paper by substituting for our v their intensive parameter β and for our smearing distribution $\omega(v,t)$ their probability density $f(\beta)$. In superstatistical setting one could also call our smearing procedure as "the superstatistical average″ (though our distributions are generally time dependent). In any case, it seems to us that important things are worth repeating several times using different framework and different words. For a more detailed discussion of the superstatistics paradigm, the reader is referred to Refs. [16,17,27].

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APPENDIX A

It is instructive to check that Eq. (58) can be derived directly from the original KM equations (44) and (45) for $P_v(x_b,t_b|x_a,t_a)$ and $\bar{P}(x_b,t_b|x_a,t_a)$. This provides an important cross check for the complete equivalence between both systems of KM equations. To this end, we multiply (44) by $\omega(v,t_{ba})$ and integrate over v. This leads to

$$\partial_{t_b}\bar{P}(x_b,t_b|x_a,t_a)-\int_0^\infty dv P_v(x_b,t_b|x_a,t_a)\partial_{t_{ba}}\omega(v,t_{ba})$$
$$=\int_0^\infty dv\,\omega(v,t_{ba})\mathbb{L}_vP_v(x_b,t_b|x_a,t_a). \quad (A1)$$

The first line has been rewritten using the chain rule for derivatives, while in the second line we have used the fact that $P_v(x_b,t_b|x_a,t_a)$ fulfills the KM equation (44). We now insert in the second line a trivial unit integral $\int_0^\infty dv'\,\omega(v',\tau)=1$, where τ is a positive infinitesimal time. Then we subtract and add a term $\bar{\mathbb{L}}\bar{P}(x_b,t_b|x_a,t_a)$, i.e., the right-hand side of the KM equation (45). Thus the second line in (A1) becomes

$$\int_0^\infty dv\int_0^\infty dv'\,\omega(v,t_{ba})\omega(v',\tau)[\mathbb{L}_v-\mathbb{L}_{v'}]P_v(x_b,t_b|x_a,t_a)$$
$$+\bar{\mathbb{L}}\bar{P}(x_b,t_b|x_a,t_a). \quad (A2)$$

Equation (A2) is a convenient short-hand version of more involved form where the $\tau\to 0$ limit acts both on $\omega(v,\tau)$ and $\mathbb{L}_v$. For instance,

$$\ldots\int_0^\infty dv'\,\omega(v',\tau)\mathbb{L}_{v'}\ldots, \quad (A3)$$

actually means

$$\dots \sum_{k=1}^{\infty} \frac{1}{\tau k!} \int_0^{\infty} dv' \omega(v',\tau)(-\partial_{x_b})^k \times \int_{-\infty}^{\infty} dy (y-x_b)^k P_{v'}(y,t_b+\tau,x_b,t_b) \dots , \quad (A4)$$

and similarly for the $\mathbb{L}_v$ term. With this in mind we now Taylor expand $\mathbb{L}_{v'}$ around v and obtain

$$-\sum_{n=1}^{\infty} \frac{1}{n!} \int_0^{\infty} dv \int_0^{\infty} dv' \omega(v,t_{ba})\omega(v',\tau) \times (v'-v)^n [\partial_v^n \mathbb{L}_v] P_v(x_b,t_b|x_a,t_a) + \bar{\mathbb{L}} \bar{P}(x_b,t_b|x_a,t_a). \quad (A5)$$

To compute the derivatives $\partial_v^n \mathbb{L}_v$ we use the representation (9) for a more general unsmeared Hamiltonian operator $H(\hat{p},x)$:

$$P_v(x_b,t_b|x_a,t_a) = e^{-vt_{ba}\hat{H}(\hat{p}_b,x_b)}\delta(x_b-x_a) = e^{-vt_{ba}[\hat{H}^{\dagger}(\hat{p}_a,x_a)]^*}\delta(x_b-x_a), \quad (A6)$$

from which directly follows that

$$\partial_v^n P_v(y,t_b+\tau|x_b,t_b) = (-\tau)^n \hat{H}^n(\hat{p},y) e^{-v\tau\hat{H}(\hat{p},y)}\delta(y-x_b) = (-\tau)^n [(\hat{H}^{\dagger})^n(\hat{p}_b,x_b)]^* \times e^{-v\tau[\hat{H}^{\dagger}(\hat{p}_b,x_b)]^*}\delta(y-x_b). \quad (A7)$$

Appearance of the term $[\hat{H}^{\dagger}(\hat{p},x)]^*$ in the second lines is a consequence of the identity $\langle x_b|\hat{A}(\hat{p},\hat{x})|x_a\rangle = \langle x_a|\hat{A}^{\dagger}(\hat{p},\hat{x})|x_b\rangle^*$ which is valid for any operator $\hat{A}$. We can now simplify (A5) with some algebra involving product rules of derivatives and δ functions. Abbreviating $\hat{H}(\hat{p},x)$ by $\hat{H}_x$, this gives

$$-\sum_{n=1}^{\infty} \frac{1}{n!\tau}\left(\frac{\tau}{t_{ba}+\tau}\right)^n \int_0^{\infty} dv \int_0^{\infty} dv' \omega(v,t_{ba})\omega(v',\tau) \times (v'-v)^n \partial_v^n P_v(x_b,t_b+\tau|x_a,t_a) + \bar{\mathbb{L}} \bar{P}(x_b,t_b|x_a,t_a). \quad (A8)$$

In deriving the latter we have utilized the identity

$$\int dy (y-x_b)^k [(\hat{H}_y^{\dagger})^n]^* e^{-v\tau(\hat{H}_y^{\dagger})^*}\delta(y-x_b) = \int dy\, \delta(y-x_b) \hat{H}_y^n e^{-v\tau\hat{H}_y}(y-x_b)^k, \quad (A9)$$

and the fact that

$$\hat{H}_y^n e^{-\tau v\hat{H}_y} = \left(\frac{-\partial_\tau}{v}\right)^n e^{-\tau v\hat{H}_y}. \quad (A10)$$

Extending for a moment the sum over n in (A5) to start from $n=0$, the relevant terms entering the calculation of $[\partial_v^n \mathbb{L}_v] P_v(x_b,t_b|x_a,t_a)$ in Eq. (A8) have the form

$$\sum_{k=0}^{\infty} \frac{(-\partial_{x_b})^k}{k!} \delta(y-x_b) e^{-\tau v H_y}(y-x_b)^k e^{-t_{ba} v H_{x_b}} \delta(x_b-x_a) = \sum_{k=0}^{\infty} \frac{(-\partial_{x_b})^k}{k!} \delta(y-x_b)[y(-i\tau)-y]^k e^{-t_{ba}vH_y}\delta(y-x_a) = e^{[y-y(-i\tau)]\partial_{x_b}} \delta(y-x_b) e^{-t_{ba}vH_y}\delta(y-x_a). \quad (A11)$$

We now apply the shift operator $e^{[y-y(-i\tau)]\partial_{x_b}}$ to the x_b variable in the δ function and cast the resulting $\delta[x_b - y(-i\tau)]$ into Dirac's bra-ket form

$$\delta[x_b - y(-i\tau)] = \langle x_b|y(-i\tau)\rangle = \langle x_b|e^{-\tau v\hat{H}}|y\rangle. \quad (A12)$$

If we now perform the integral over y in (A9) and use the fact that

$$\left(\frac{-\partial_\tau}{v}\right)^n \langle x_b|e^{-v(\tau+t_{ba})\hat{H}}|x_a\rangle = \left(\frac{-\partial_v}{t_{ba}+\tau}\right)^n \langle x_b|e^{-v(\tau+t_{ba})\hat{H}}|x_a\rangle,$$

we obtain the announced result (A8). We now use the definition of the moments in the second line of Eq. (55), and the fact that the surface terms in the partial integrations over v' vanish due to the positivity of the real part of the Hamiltonian spectrum, to rewrite the first term in (A8) as

$$-\sum_{n=1}^{\infty} \int_0^{\infty} dv (-\partial_v)^n [K^{(n)}(v,t_{ba})\omega(v,t_{ba})] P_v(x_b,t_b|x_a,t_a).$$

Remembering Eq. (54), this can be written as

$$-\int dv P_v(x_b,t_b|x_a,t_a) \mathbb{L}_\omega \omega(v,t_{ba}). \quad (A13)$$

If we now take into account the evolution equation (44), we obtain from Eq. (A1) the equation

$$\int_0^{\infty} dv P_v(x_b,t_b|x_a,t_a) \partial_{t_{ba}} \omega(v,t_{ba}) = \int dv P_v(x_b,t_b|x_a,t_a) \mathbb{L}_\omega \omega(v,t_{ba}). \quad (A14)$$

This result must hold for any distribution $P_v(x_b,t_b|x_a,t_a)$, thus proving the KM equation (56) for $\omega(v,t)$.

APPENDIX B

Here we show how to define uniquely the classical Hamiltonian $F_{\rm cl}[H(p,x)]$ if the initial Hamiltonian $H(p,x)$ depends not only on p but also on x, This will be done with the help of the KM equations.

We start with an observation that when the unsmeared KM operator has no explicit time dependence, i.e., when

$$\mathbb{L}_v(-\partial_x,x) = \sum_{n=1}^{\infty} (-\partial_x)^n D_v^{(n)}(x), \quad (B1)$$

then the *unsmeared* classical Hamiltonian $H_{\rm cl}(p,x)$ has the form

$$H_{\rm cl}(p,x) = -\sum_{n=1}^{\infty} (-ip)^n D_v^{(n)}(x), \tag{B2}$$

provided we define the path integral by time slicing in the post-point form. This is done by rewriting the short-time matrix elements $\langle x_n|\exp(-\hat{H}\tau)|x_{n-1}\rangle$ as integrals $\int (dp_n/2\pi)\langle x_n|p_n\rangle\langle p_n|\exp(-\hat{H}\tau)|x_{n-1}\rangle$ and defining the classical Hamiltonian via the identity

$$\langle p_n|\exp(-\hat{H}\tau)|x_{n-1}\rangle \equiv e^{-\tau H_{\rm cl}(p_n,x_{n-1})}e^{-ip_n x_{n-1}}. \tag{B3}$$

Then to order $\mathcal{O}(\tau^2)$ the Hamiltonian $H_{\rm cl}(p,x)$ coincides with (B2). Full discussion of this relation can be found in Ref. [1]. The KM equation is basically a Schrödinger-type equation with the non-hermitian Hamiltonian $\hat{H}(p,x) = \Sigma_n(-i\hat{p})^n D_v^{(n)}(\hat{x}) = -\mathbb{L}_v(-\partial_x, x)$.

Let us now see how the KM equation looks for the smeared distribution. This will allow us to identify the *smeared* classical Hamiltonian $F_{\rm cl}(H)$. To this end we note that from the definition (47) the smeared KM coefficients can be written as

$$D^{(n)}(x) = \frac{1}{n!}\lim_{\tau\to 0}\frac{1}{\tau}\int_0^{\infty} dv\,\omega(v,\tau)\int_{-\infty}^{\infty} dy(y-x)^n\langle y|e^{-v\hat{H}\tau}|x\rangle$$
$$= \frac{1}{n!}\lim_{\tau\to 0}\frac{1}{\tau}\int_{-\infty}^{\infty} dy(y-x)^n\langle y|e^{-F(\hat{H})\tau}|x\rangle. \tag{B4}$$

Furthermore, we can insert in front of $e^{-F(\hat{H})\tau}$ a completeness relation $\int dp|p\rangle\langle p|=\hat{1}$, and make use of the fact that only terms up to order $\mathcal{O}(\tau)$ are relevant in $e^{-F(\hat{H})\tau}$. This leads directly to

$$\frac{1}{n!}\lim_{\tau\to 0}\frac{1}{\tau}\int_{-\infty}^{\infty} dpdy(y-x)^n\langle y|p\rangle\langle p|e^{-F(\hat{H})\tau}|x\rangle$$
$$= \frac{1}{n!}\lim_{\tau\to 0}\frac{1}{\tau}\int_{-\infty}^{\infty}\frac{dpdy}{2\pi}y^n e^{ipy}e^{-F_{\rm cl}(H)\tau}. \tag{B5}$$

Here $F_{\rm cl}(H)$ is obtained from $F(\hat{H})$ by using the commutator $[\hat{p},x]=-i$ to move all $\hat{p}$'s to the left of all x's and dropping the hats. Equation (B5) can be further simplified if we write $F_{\rm cl}(H)=\Sigma_n p^n f_n(x)$ and the y integral as the nth derivative of δ function, i.e.

$$D^{(n)}(x) = \frac{(-i)^n}{n!}\lim_{\tau\to 0}\frac{1}{\tau}\int_{-\infty}^{\infty} dp e^{-\tau F_{\rm cl}(H)}\partial_p^n\delta(p)$$
$$= \frac{i^n}{n!}\lim_{\tau\to 0}\frac{1}{\tau}\partial_p^n[e^{-\tau F_{\rm cl}(H)}]|_{p=0} = (-i)^n f_n(x). \tag{B6}$$

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