MODEL OF A PROFESSOR'S DIDACTICAL ACTION IN MATHEMATICS EDUCATION

PROFESSOR'S VARIABILITY AND STUDENTS' ALGORITHMIC FLEXIBILITY IN SOLVING ARITHMETICAL PROBLEMS[1]

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Abstract: *The paper deals with the issue of problem solving. This was a common theme of two independent projects, which complement each other. One study detects phenomena in graphical models of word problem assignments; the pre-algebraic features of models are discussed. The other gives these phenomena precision by an action model of the problem, focusing on the variability in word problems. Common aspects of both studies are presented.*

Keywords: problem solving, graphical models, variability of teachers, psychological perspective, theory of didactical situations, coding of word problem, reference language, model.

1. INTRODUCTION

In the paper we are presenting two studies originally executed as independent entities; both are dealing with the same topic: problem solving. The first one (J. Novotná) belongs more to the psychological perspective than the purely didactical one, although the didactical concern is not absent. The second one (B. Sarrazy) examines the effects of variability in the formulation of problem assignments on students' flexibility when using taught algorithms in new situations; the research was developed in the framework of the theory of didactical situations starting from various results in the psychological domain. These two studies, although at the beginning carried out separately and on different levels of education, showed themselves to be perfectly complementary. The first one allows the detection of a set of phenomena, whereas the second one gives them precision through an action model of the problem focusing on the variability in word problems. We believe that connecting these two approaches allows us to open interesting perspectives for a better understanding of the role of problem solving in teaching and learning mathematics by giving precision to certain conditions of their use.

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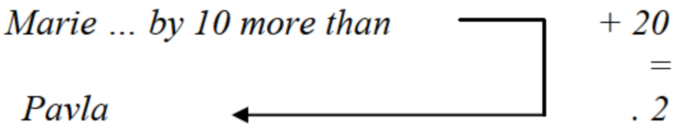
2. MODELS OF WORD PROBLEM ASSIGNMENTS

In this part of the article we investigate the ways that students are modelling word problem assignments when grasping the problems' structure -see e.g. (Novotná, 1999). The following terminology is used: *Coding of word problem* is the transformation of the word problem text into a suitable system (*reference language*) in which data, conditions and unknowns can be recorded in a more clearly organized and/or more economical form. The result of this process is called a *model*[2] (in both cases – models taught by teachers or models as results of the inner need of the solver). The reference language contains basic symbols and rules for creating a model. There exist different reference languages for any one type of word problem.

From a student's solution (Jakub, 13 years, individual experiment)

<u>*Problem to be solved*</u>: Marie and Pavla each had some money but Marie had 10 CZK more than Pavla. Pavla managed to double the amount of money she had and Marie added 20 CZK more to her original amount. They now found that both of them had the same amount. How many crowns did each of them have at the beginning?

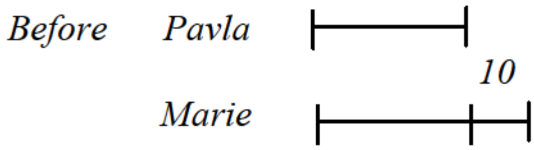
Jakub writes:



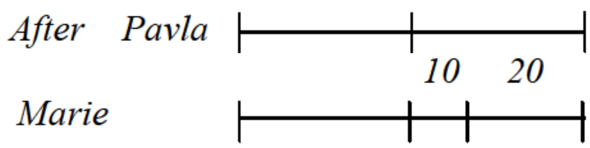
This record of the assignment did not allow him to find a suitable solving strategy. The experimenter recommends him to use the visualisation with the help of line segments, see (Novotná, 1998):

Experimenter (E): *Try to record the situation at the beginning.*

Jakub (J):



Experimenter E: *And after the change?*
J starts to draw a new line segment.
E: *Would it not be better to record it in the same schema?*
After a short discussion, J's graphical representation is



J: *Aha … I do not need to construct an equation!*

This simple example illustrates the influence of an appropriately chosen reference language on the quality of grasping the text and on constructing a mathematical

[2] Our use of the word model corresponds with the representation in (Pierce, 1987).

model of the word problem.[3] In the example, two reference languages were used: verbal and graphical ones.

The solver's choice of one of the reference languages is influenced by several factors: internal ones, e.g. by his/her previous experience, preferred information processing style, personal preferences, and external ones, mainly the demands of the teacher or school system.

From the point of view of impulses for creating a model we distinguish three types: *spontaneous independent* model creation, *externally managed* model creation, and creation of a figure in the *role of a signal* (Novotná, 1999). In real life situations, it is rare to have to solve standard problems with the help of known solving algorithms. To prepare children for dealing with life situations in a successful way, the spontaneous case of model creation is crucial.

The solver's goal when creating a model is to get a better understanding of the problem structure (except in the cases when the reasons for model creation are fully external, e.g. teacher's demands, no intrinsic motivation). A model can have different forms: from detailed rewriting of the assignment to more clearly organized forms, from a verbal description of the assigned conditions to their symbolic record. In this context we can speak of non-algebraic, pre-algebraic or algebraic model forms. In this perspective, a spontaneously created model can indicate the level of pre-algebraic/algebraic thinking of its author.

2.1 Our research

The original aim of our experiment (Novotná, Kubínová, 1999) was to analyse and classify spontaneously created graphical models of word problems. The experiment was conducted with pupils from the 3rd (age 9-10) to 8th grades (age 14-15) in basic schools in both Prague and České Budějovice. The sample consisted of 25 3rd graders, 21 4th graders, 24 5th graders, 22 6th graders, 28 7th graders and 23 8th graders, all of them were from non-specialised classes. The word problem dealt with was a non-standard problem that is not presented in currently used Czech textbooks and the participating pupils have not solved similar ones before. It had the following structure:

A packing case full of ceramic vases was delivered to a shop. In each case there were b boxes, each of the boxes contained k smaller boxes with p presentation packs in each of the smaller boxes, each presentation pack contained m parcels and in each parcel were v vases. How many vases were there altogether in the packing case?[4]

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[3] Duval (1995) states that without distinguishing between object and its representation it is not possible to understand mathematics. In order to separate object from its representation, the student must be able to represent a mathematical concept at least in two semiotic systems. Duval (1999) studies *auxiliary representations* as a tool that helps the solver understand formulations and reformulations in mathematics.

[4] The number of "unpacking levels" and the numbers labeled *b, k, p, m, v* were modified according to the age of solvers. For the 7th/8th grades, the mixed arithmetic-algebraic assignment was used (*v/v* and *k* were not substituted by numbers).

The traditional reference language in Czech schools used already at the primary level is a verbal one (Fig. 1a). The use of grahical models (Fig 1b) was the spontaneous decision of solvers. Spontaneously and independently created reference languages were used by solvers of 6th to 7th grades. The younger solvers tried to apply the verbal model copying the one presented by the teacher for other word problem types. Only one of the 3rd to 5th graders used any form of pictorial representation. The probable reason is that at this age, pictures are always connected with the real situation that they represent. The lack of pictures in the solutions indicates that children do not connect the word problem presented to them in school with real objects/situations or found the creation of diagrams for the problem too difficult.

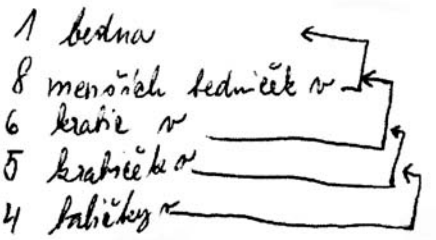

Fig. 1a

Fig. 1b

The larger amount of figures and schemes in the solutions of 6th graders and older students is connected with the use of teaching strategies supporting students' development of the ability to visualise situations. There occured a rich variability in graphical models used spontaneously by individual students. Graphical models were of two main types: procedural and conceptual. We call a model: *procedural* when it clearly expresses the process in time how it is described in the assignment (Fig. 2a); *conceptual* when all pieces of information are recorded as a whole not showing the changes in time (Fig 2b). As to the shape similarity, both *iconic* (consisting of real shape record, Fig. 2b) and *symbolic* (keeps the structural similarity only, Fig. 2a) models were found. Students used various types of accompanying explanatory means (arrows, words, …, Fig. 2b). Big differences were identified also in the completeness of the records.

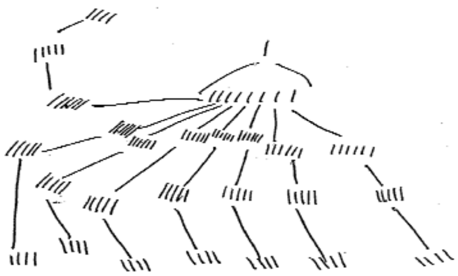

Fig. 2a

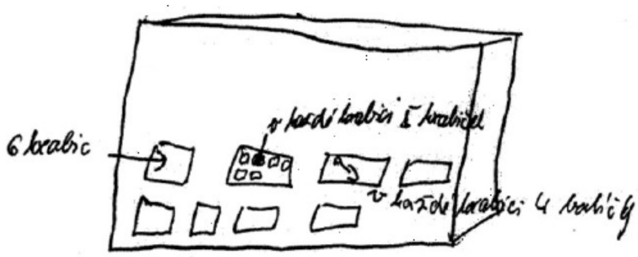

Fig. 2b

When analyzing the use of assignment models in our sample of solutions of 6th to 8th graders we identified difference in the performances of the groups of students from different classes (taught by different teachers): Either the majority of students kept using the externally managed model (mostly verbal, non-algebraic) or most of models were individualized spontaneously created models (of all three "algebraic levels").

Traditionally, the level of pre-algebraic/algebraic thinking is characterized by use of letters (or other symbols, e.g. *, □) when solving mathematical problems. Students' ability to operate with algebraic symbols in a systematic way is often based on very formalistic knowledge and most often it is taught transmissively. Visualization is seldom used (Novotná, Kubínová, 2001). In the (spontaneous) use of individualized models, we can observe echoes of a conscious transition towards pre-algebraic/algebraic thinking. There are several variables that might be considered as indicators of pre-algebraic/algebraic level of a model from which we present here those that are clearly present in the models in Fig. 1, 2:

- Level of *revealing the structure*[5]: Higher for the model in Fig. 1b than in Fig. 1a.
- Level of *thinking in symbolic language*[6]: Higher for the model in Fig. 2a than in Fig 2b.
- Level of *the reference language abstractness*: Higher for the model in Fig. 2a (small lines in the model represent different real objects even if they have the same form – for the solver it is an abstract "universal" symbol) than in Fig. 2b (the solver tried to distinguish symbols for various real objects).

2.2 Results

As mentioned in (Malara, Navarra, 2001), "… the difficulties in the approach to algebra are rooted in the scarce attention paid to the relational or structural aspects of arithmetics which constitute the basis of elementary algebra. A longitudinal study of the use of different reference languages when solving word problems indicates that the spontaneously created models produced by students have pre-algebraic features. Referring to (Drouhard, 2001), it is closely related to the fact that "(students) have to learn how to read, write, understand, speak, and above all how to use this particular language in order to solve problems and to 'think algebraically' ".

The individual differences in the form of graphical models could be explained by the internal students' cognitive processes, (Novotná, 1999), (Novotná, Kubínová, 1999). By this approach we were not able to explain the striking difference "spontaneity versus copying" in the students groups. The psychological perspective did not offer any explanation of the observed fact. It was necessary to search for it outside the psychological approach. We found a suitable tool for the explanation in the scope of the Theory of didactical situations (Brousseau, 1997), namely in the notion of variability of teachers introduced by B. Sarrazy (Sarrazy, 2002).

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[5] (Arcavi, 1994): "Many students who manage to handle the algebraic techniques successfully, often fail to see algebra as a tool for understanding, expressing and communicating generalizations, for revealing structure, …"

[6] (Crawford, 2001): „… three broad indicators are defined as essentials to algebraic thinking:
1. Ability to think in symbolic language, to understand algebra as generalized arithmetics and to understand algebra as study of mathematical structures. …"

3. VARIABILITY OF TEACHERS

To place the question studied by the second research it is necessary to return to an episode well known among French educationalists: In 1979, the researchers from IREM in Grenoble presented to students of 9-10 years the following problem: "On the boat, there are 26 sheep and 10 goats. What is the age of the captain?" More than three quarters of students used addition of numerical data from the assignment. This phenomenon is well known to educationalists as one of the apparent effects of the "didactical contract" (Brousseau, 1997; Chevallard, 1988; Sarrazy, 1995). But there remains one question and this is the explanation why, at the same level of competence, students from certain classes are more sensitive to the formal aspects of the problem assignment (answering that the age of the captain is 26) and that others are more flexible in the solving process - rejecting the validity of such problem type (answering: "It is impossible to find the answer to this question.").

3.1 Our field of interest

Learning mathematics is not restricted to learning algorithms only but it is manifested by identifying conditions for their use in new situations: by this criterion it is possible to admit that the child learned something. But these conditions are not present in the algorithms itself and cannot be explicated by the teachers. This is one of the reasons why the "didactical contract" is in large part implicit: the teacher cannot tell students what he is expecting from them without relinquishing the ability to determine what the students learned (Brousseau, 1997). How could it be explained that certain students show that they are able to use the taught knowledge in new contexts, while others, although "knowing" the taught algorithms, are not able to re-contextualise their knowledge? If there is no satisfactory explicative model, these differences are attributed to the charisma of individuals, to their cognitive skills …, or simply to the mysterious mental properties for which teachers do not have any didactical tool for transforming them or letting them develop. The central hypothesis of our research is to consider these inter-individual differences of the sensibility on the didactical contract (measured by an index), as an effect of the teachers' didactical variability in the domain of setting arithmetical problems.

The obtained model is based on the following idea which can be formulated simply: the more the *same* form of didactical organisation presents the modalities of different realisations, the more uncertainty attached is added to this form. To satisfy the teacher's expectations, such a student has to 'examine' the domain of validity of his knowledge much more than a student who is exposed to a strongly ritualised (repetitive) teaching and therefore a much reduced variability. In other words, a strongly ritualised teaching would allow the student to know in advance what he has to do and thus, to adopt a behaviour *ad hoc* (adapted). On the other hand, by the interruptions of introduced routines, a strong variability makes the following strategies futile (controversial): the students cannot rely only on the indicators of introduced routines (semantic indicators, triggers …) and therefore cannot either anticipate or master the liaison of sequences which allow him to discover the behaviours expected by the teacher. This model was not refuted by our results.

3.2 Theoretical model: degree of variability

We will study the problem assignments invented by the teachers.

We asked 7 teachers of the 4th year of elementary school (age of students 9-10), all with the same length of experience at this level, to write down 6 arithmetic problems (without consulting any documents such as textbooks): 3 with a solution requiring addition and the other 3 requiring subtraction. The problems were to be different from the point of view of their difficulty. The 42 obtained problems were analysed using the classification presented bellow.

Consulting a certain number of researches allowed us to identify 14 variables; each of them could explain the small or larger difficulties of a problem. We grouped them into 3 categories:

A. ***Numerical***, grouping the variables which relate to the numerical values of the problem: the type of numerals used; presence of irrelevant data;

B. ***Rhetorical***, relating to the organisation forms of the presentation of the problem (story): organisation of the assignment of the problem: presence/absence of an semantic indicator in the assignment; theme of the assignment; presence/absence of a trigger in the assignment; syntagmatic organisation and temporal organisation; position of the question; vocabulary used; type of formulation: classic and written forms.

C. ***Semantic-conceptual***: This last cluster groups together at the same time, the variables connected with certain rhetorical aspects (the presence of a trigger in the question as for e.g. "altogether") and certain logical-mathematical variables (the operation that should be used): type of the additive structure; nature of the unknown; correspondence between the syntagmatic order and the operative order; correspondence between the trigger and mathematical operator; correspondence between semantic indictor and mathematical operator.

3.3 Results and conclusions

The procedure for calculating the variety index for additive problems (*IVa*) consisted in counting, for each of the 14 variables, the variations observed (*Vo*) over the 3 problems as a whole in relation to the number of possible variations (*Vp*) (*IVa* = *Vo*/*Vp*). If we only calculated the sum of the variations (*Vo*) we would underestimate the value of the index in cases where certain variations are formally excluded by the choice made on other variables. The same calculation procedure was used to measure the variety index in relation to the subtractive problems (*IVs*). The arithmetic average of *IVa* and *IVs* was retained as the measure of variety index.

The 42 problems were assigned to 27 8-9-year-old students. We observed a strong correlation[7] (r = -.97; s.; p. < .001) between the average of success in the problems solved by students and the variety index calculated for these problems. Thus, it can be affirmed that the higher the variety index, the more difficult and contrasted the problems (although this is not significant, we nonetheless observe a positive correlation between the dispersion of success and the variety index). In other words, teachers with a high variety index produce variations that have highly significant effects on the difficulty of the presentations of the problems. As a result, the variety index constitutes a faithful summary of the "ability" of the teachers to make relevant variations in the wording of the problem – indeed, variability greatly reduces the average score (by half), which shows that the pupils are responsive to the contract and that their success does not resist variation. We can thus assume that variability is a variable that might explain the phenomenon of responsiveness to the contract.

The observed correlation (r = -.74; s.; p. < .05) between the values of the variety index of the 7 teachers and the average scores of the flexibility (formalism) degree allows us to validate our initial hypothesis: the more the teacher shows an important variability, the more the students show the flexibility in the solution; vice versa, the weaker the variability of the teacher, the more the students are formalists and rely more on the formal aspects of the assignment than on their comprehension when producing the answer.

So, repetitive teaching can guide the students more easily to adapt themselves to the educational situations by determination of indicators (e.g. triggers) only to answer the situations; thus, students may adopt an appropriate behaviour without being in need to understand the meaning of the mathematic knowledge mobilised by the situation. A developed variability invalidates these strategies: the student cannot rely only on these indices any more, and correlatively, the student's involvement is more probable.

4. CONCLUDING REMARKS

In our experiments presented in Part 1, the variability of teachers proved to be the variable explaining the significant differences in the number of spontaneously created models by students in some groups. It confirms our conviction that students' results differ when they are asked to reproduce only the reference language presented by the teacher or when they get acquainted with several reference languages or may even use their own reference languages. In the last two cases their results are better. In addition to that, these cases support the development of the student's personality, mainly his/her ability of critical analysis and consciousness of their responsibility for their own activity.

Moreover, we believe that analysis of models created by students enables the teacher to help them in case that their effort to solve the problem correctly is not successful

[7] The linear coefficient of Bravais-Pearson was used.

(mainly in determining the type of obstacles the student faced). This point is elaborated in more details in (Novotná, 2003).

If the teacher decides to get his/her students acquainted with several types of reference languages, he should be aware that there are not only positive consequences, but also negative ones. One of the most important dangers is the increased uncertainty in less able students who, besides the uncertainty concerning their ability to solve the problem correctly they are also facing the uncertainty, which reference language enables them to solve the problem.

One question remains to be examined in the following work: the origin of teachers' variability. The theoretical framework currently worked on in DAEST (Laboratory of Didactics and Anthropology in the Teaching Sciences and Techniques in Bordeaux) promises to provide a consistent framework for examining such a question, which is understandably interesting primarily for teacher training.

REFERENCES

Arcavi, A. (1994). Symbol Sense: Informal sense-making in formal mathematics. *For the Learning of mathematics*, 14, 24-35.

Brousseau, G. (1997). *Theory of Didactical situations in mathematics 1970-1990*, [Edited and translated by M. Cooper, N. Balacheff, R. Sutherland & V. Warfield], Dordrecht: Kluwer Academic Publishers. (French version: Brousseau, G. (1998). *Théorie des situations didactiques*. [Textes rassemblés et préparés par N. Balacheff, M. Cooper, R. Sutherland, V. Warfield]. Grenoble: La pensée sauvage.).

Chevallard, Y. (1988). *Sur l'analyse didactique: Deux études sur els notions de contrat et de situation* (doc. Ronéo), Aix Marseille: IREM *14*.

Crawford, A.R. (2001). Developing algebraic thinking: Past, present, and future. In: *The Future of the Teaching and learning of Algebra*, Proceedings of the 12th ICMI Study Conference. Eds. H. Chick, K, Stacey, J. Vincent, Jill and J. Vincent. Melbourne: University of Melbourne, 192-198.

Drouhard, J.-P. (2001). Researches in language aspects of algebra: a turning point? In: *The Future of the Teaching and learning of Algebra*, Proceedings of the 12th ICMI Study Conference. Eds. H. Chick, K, Stacey, J. Vincent, Jill and J. Vincent. Melbourne: University of Melbourne, 238-242.

Duval, R. (1995). *Sémiosis et pensée humaine: Registres sémiotiques et apprentissages intellectuels*. Bern: Peter Lang.

Duval, R. (1999) *Conversion et articulation des représentations analogiques*, IUFM Nord-Pas de Calais.

Malara, N., Navarra, G. (2001). „Brioshi" and other mediation tools employed ina teaching of arithmetic from a relational point of view with the aim of approaching algebra as a language. In: *The Future of the Teaching and learning of Algebra*, Proceedings of the 12th ICMI Study Conference. Eds. H. Chick, K, Stacey, J. Vincent, Jill and J. Vincent. Melbourne: University of Melbourne, 412-419.

Novotná, J. (1999). Pictorial representation as a means of grasping word problem structures. *Psychology of Mathematical Education*, 12, http://www.ex.ac.uk/~Pernst/.

Novotná, J., Kubínová, M. (1999). Wie beeinflusst eine Visualisierung der Aufgabenstellung den Prozess der Lösung einer Textaufgabe. In: *Beiträge zum Mathematikunterricht 1999*, Vorträge auf 33. Tagung für Didaktik der Mathematik vom 1. bis 5.3.1999, Bern. Ed. M. Neubrand. Hildesheim, Berlin: Verlag Franzbecker, 397-400.

Novotná, J., Kubínová, M. (2001). The influence of symbolic algebraic sescriptions in word problem assignments on grasping processes and on solving strategies. In: *The Future of the Teaching and learning of Algebra*, Proceedings of the 12th ICMI Study Conference. Eds. H. Chick, K, Stacey, J. Vincent, Jill and J. Vincent. Melbourne: University of Melbourne, 494-500.

Novotná, J. (2003). *Etude de la résolution des «problèmes verbaux» dans l'enseignement des mathématiques. De l'analyse atomique à l'analyse des situations*, Bordeaux: Université Victor Segalen Bordeaux 2.

Peirce, C. S. *Collected Papers of Charles Sanders Peirce* (Volumes I-VI, ed. by Charles Hartshorne and Paul Weiss, 1931-1935, Volumes VII-VIII, ed. by Arthur W. Burks, 1958, quotations according to volume and paragraph). Cambridge: Harvard University Press.

Sarrazy, B. (1995). Le contrat didactique. *Revue Française de Pédagogie, 112*, 85-118.

Sarrazy, B. (2002). Effects of variability on responsiveness to the didactic contract in problem-solving among pupils of 9-10 years, *European Journal of Psychology of education*, XVII. 4, 321-241.