STRUCTURE SENSE FOR UNIVERSITY ALGEBRA

Jarmila Novotná*), Naďa Stehlíková*), Maureen Hoch**)

*) Charles University in Prague, Faculty of Education, **) Tel Aviv University, Israel

Building on some research on structure sense in school algebra, this contribution focuses on structure sense in university algebra, namely on students' understanding of algebraic operations and their properties. Two basic stages of this understanding are distinguished and described in detail. Some examples are given on student teachers' insufficient structure sense and interpreted in terms of various stages of structure sense.

INTRODUCTION

Many researchers report that the transition from secondary schools to university is often a painful process for students. When learning a new idea, the old idea does not disappear. Thus in the transition to advanced mathematical thinking, there exist simultaneously in a person's mind concept images formed earlier and new ideas based on definitions and deductions. The abstract algebra course usually presents the first "obstacle" university students, future mathematics teachers, meet.

Many researches have focused on students' coming to understand abstract algebra concepts such as groups (Asiala et al., 1997, Dubinsky et al, 1994, Hazzan, 1999, Zazkis et al., 1996). Simpson & Stehlikova (in press) suggest that the transition from working with an example structure to working abstractly involves an intricate sequence of shifts of attention:

1. Seeing the elements in the set as objects upon which the operations act.
2. Attending to the interrelationships between elements in the set which are consequences of the operations.
3. Seeing the signs used by the teacher in defining the abstract structure as abstractions of the objects and operations, and seeing the names of the relationships amongst signs as the names for the relationships amongst the objects and operations.
4. Seeing other sets and operations as *examples* of the general structure and as *prototypical* of the general structure.
5. Using the formal system of symbols and definitional properties to derive consequences and seeing that the properties inherent in the theorems are properties of all examples.

Obviously, students must first understand how each operation works and what the objects in the set are; this is not necessarily straightforward. In this paper we will focus on the first two stages only.

STRUCTURE SENSE

Structure sense has been defined and examined in several papers describing students' difficulties when applying knowledge in an algebraic context. In Linchevski & Livneh (1999) structure sense is defined and used for describing students' difficulties when using arithmetic knowledge in the early algebra. In Hoch (2003) and Hoch &

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Dreyfus (2006) structure sense is used to analyse students' use of previously learned algebraic techniques.

The authors (Hoch & Dreyfus, 2006) define structure sense for high school algebra as follows:

> A student is said to display structure sense (SS) if s/he can:
> - Recognise a familiar structure in its simplest form.
> - Deal with a compound term as a single entity and through an appropriate substitution recognise a familiar structure in a more complex form.
> - Choose appropriate manipulations to make best use of the structure.

The above definition inspired us to attempt to define structure sense for one aspect of abstract algebra, namely binary operations and their properties.

METHODOLOGY

This study is based on the first two authors' longitudinal observation of students, future mathematics teachers, during the course Theoretical Arithmetic and Algebra. Students enter the course with rich experience with building number sets and with linear and polynomial algebra (Novotna, 2000). Still, they often have problems with basic algebraic concepts. During the last three years, we systematically collected students' works, especially those which contained mistakes. There were about 40 students in each year.

First, we only chose work with mistakes which we attributed to students' insufficient understanding of binary operations and their properties and the notion of identity and inverse. Initially, taking mistakes as developmental stages of students' understanding, we tried to organise them in a way to fit the scheme for the development of understanding the binary operation presented in (Dubinsky et al., 1994). Then we classified them according to our perception of how abstract students' understanding of an operation/an object was. For instance, whether he/she based his/her considerations on his/her concept image of the object (Tall & Vinner, 1981) or on the definition introduced in the course. Finally, we were inspired by Simpson & Stehlikova's scheme presented above which we combined with Hoch & Dreyfus's structure sense definition. As the mathematics we are dealing with is more complex than the mathematics Hoch & Dreyfus investigated, the model we propose below is more complicated and multi-levelled.

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We distinguish two main stages of the developing structure sense each of which is further subdivided.

SSE: Structure sense as applied to elements of sets and the notion of binary operation

A student is said to display structure sense if he/she can:

(SSE-1) Recognise a binary operation in familiar structures.

(SSE-2) See elements of the set as objects to be manipulated / understand the closure property.

(SSE-3) Recognise a binary operation in "non-familiar" structures.

(SSE-4) See similarities and differences of the forms of defining the operations (formula, table, other).

SSP: Structure sense as applied to properties of binary operations

A student is said to display structure sense if he/she can:

(SSP-1) Understand ID in terms of its definition (abstractly).

(SSP-2) See the relationship between ID and IN: ID → IN.

(SSP-3) Use one property for another: C → ID, C → IN, C → A.

(SSP-4) Keep the quality and order of quantifiers.
(SSP-5) Apply the knowledge of ID and IN spontaneously.

Abbreviations ID, IN, C, A stand for identity, inverse, commutative property, associative property.

For the sake of clarity, we will explain individual aspects of structure sense on particular examples.

SSE: Elements of sets and the notion of binary operation

The first stage concerns the notion of binary operation (and its recognition in a certain set) and understanding elements of sets as objects to be used in the operation.

A student is said to display structure sense for elements of sets and binary operations (SSE) for algebraic structures with one binary operation if he/she can:

(SSE-1) Recognise a binary operation in familiar structures

By recognise, we mean that a student is able to determine whether something is a binary operation. By familiar structures, we mean structures which a student meets prior to university such as number sets with numerical operations and set functions **R** → **R** with the composition of functions (see also below). Non-familiar structures will be loosely characterised as those which are not familiar to a student.

<u>Example:</u> A student displays SSE-1 if he/she can determine whether the following are binary operations (**N** is the set of natural numbers, **Z** is the set of integers, **R** is the set of real numbers):

$(N,\circ)$: $x \circ y = x + y$ $\quad (N,\rhd)$: $x \rhd y = x - y$ $\quad (Z,\oplus)$: $x \oplus y = x + y$ $\quad (Z,*)$: $x * y = x - y$

$(Z,\otimes)$: $x \otimes y = x \cdot y$ $\quad (R,\bullet)$: $x \bullet y = x \div y$ $\quad (R,\succ)$: $x \succ y \Leftrightarrow \exists k \in R : x = y + k$

(SSE-2) See elements of the set as objects to be manipulated / understands the closure property

<u>Example:</u> A student lacks SSE-2 when given a set of congruences and asked to find the identity and he/she starts working with numbers. Later he/she answers that

identity is 1 without taking into consideration the nature of objects in the set he/she is dealing with.

(SSE-3) Recognise a binary operation in "non-familiar" structures

Example 1: A student displays SSE-3 if he/she can determine whether the following are binary operations:

$(Z,\oplus)$: $x \oplus y = x + y - 4$ $(R,*)$: $x * y = x \cdot y - 2$ $(Z,\otimes)$: $x \otimes y = 5x - 6y$

$(Z,\bullet)$: $x \bullet y = 3x + xy$ $(R,\circ)$: $x \circ y = x^y$

Example 2: A student lacks SSE-3, if he/she says that the operation in the following structure is associative because + and · are associative: (**R**, •), where **R** is the set of real numbers and $x \bullet y = 3x + xy$ (the operation is not associative). As the operation • is composed of + and ·, he/she puts together its properties to get the properties of •.

(SSE-4) See similarities and differences of the forms of defining the operations

Example 1: A student displays SSE-4 if he/she can see that the two definitions of operation * in (Z_4, ∗) are the same (Z_p is the set of integers 0, …, p - 1):

Definition 1: $x, y \in Z_4$, $x * y$ is the remainder when dividing the sum $x + y$ by 4.

Definition 2:

∗	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Example 2: A student displays SSE-4 if he/she can see that the operations * in (Z_4, ∗) with Definition 2 of ∗ and the operation ○ in (M, ○), where M = {e, a, b, c} and ○ is defined by the table, are the same (isomorphic).

○	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

Example 3: A student displays SSE-4 if he/she can see that the operations in the following structures are not isomorphic: (M, ○) where M = {e, a, b, c} and ○ is defined as above, and (K, ∗) where K = {X, Y, XY, N} and ∗ is defined as follows:

*	N	X	Y	XY
N	N	X	Y	XY
X	X	N	XY	Y
Y	Y	XY	N	X
XY	XY	Y	X	N

Note on examples 2 and 3: A student understanding examples 2 and 3 displays a higher degree of SSE-4 than is the case with the first example as he/she has to see letters and combinations of letters (not only numbers) as objects to be manipulated (SSE-2).

SSP: Properties of binary operations

The second stage of SS involves attending to the interrelationships between objects which are the consequences of the operations. We, as teachers, "would like our students to attend not to the particular objects and operation, but to the fact that imposing the operation on the set of objects creates interrelationships which are important, such as associativity, inverses etc." (Simpson & Stehlikova, in press). The second stage can be analysed only for students who have at least partial understanding of SSE.

The situation is more complicated here as we have objects of two types: properties (commutative, associative, distributive in case of 2 operations) and important objects (identities, inverses). Moreover, we can distinguish two standpoints. The first focuses on individual properties and objects, the second concerns understanding the role of quantifiers in the definition (their type and order).

For the subdivision of SSP, we looked into mutual relationships among objects.

A student is said to display structure sense for properties of binary operations (SSP) for algebraic structures with one binary operation if he/she can:

(SSP-1) Understand ID in terms of its definition (abstractly)

Example 1: A student lacks (SSP-1) if he/she answers that there is no identity in $(Z_{99},+)$, where $Z_{99}=\{1,2,...99\}$ and + is addition in congruence modulo 99, because there is no 0 in the set.

Example 2: Consider the following structure: $(\mathbf{Z}, \bullet)$, where **Z** is the set of integers, $x \bullet y = x + y - 4$ (correct answer for ID: $n = 4$).

A student lacks SSP-1, if he/she answers (1) n does not exist because for $n = 0$ it holds $x \bullet n = x + 0 - 4 \neq x$; or (2) $n = 4$ because $x + n - 4 = x + 4 - 4 = x$; but later when he/she calculates the inverse element, he/she gives the answer $x^{-1} = 4 - x$ because $x \bullet (4 - x) = x + (4 - x) - 4 = 0$. (See also the comment below.)

(SSP-2) See the relationship between ID and IN (the latter does not exist without the former): ID → IN

Example: A student lacks SSP-2 if he/she makes the following mistake: Given (F, +), where F is the set of odd numbers and + is the addition of integers. The student says that the inverse to 3 is –3 as both are odd (however, identity $0 \in Z$ is not element of F).

Comment: This mistake can also be interpreted in terms of the student's concept image of inverse. Number –3 could have simply been chosen because his/her concept image of inverse is a negative number. It is widely accepted that students tend to rely on their images from number theory when studying and applying group theory (e.g., Hazzan, 1999, Stehlikova, 2004). They often hold a deeply rooted image of the additive identity in numerical contexts necessarily being 0 and the additive inverse a negative number.

(SSP-3) Use one property for another: C → ID, C → IN, C → A

Example 1: A student lacks SSP-3 if he/she makes the following mistake: ($P(M)$, –), where $P(M)$ is the set of all subsets of the set M, – is the difference of sets $X - Y = \{x \in M;\ x \in X \wedge x \notin Y\}$ and the student says $n = \emptyset$ because $X - \emptyset = X$ (correct answer: n does not exist).

Example 2: A student lacks SSP-3 if he/she makes the following mistake: (R^+, ○), where R^+ is the set of positive real numbers and $x \circ y = x^y$ and the student says that it is $n = 1$ because $x^1 = x$ (correct answer: except for $x = 1$, the inverse does not exist).

Example 3: A student displays SSP-3, if he/she understands that he/she does not have to investigate all possibilities for A if the operation is C and is given by a table (e.g. at (M, ○) above).

(SSP-4) Keep the quality and order of quantifiers

Example: A student lacks SSP-4 if he/she makes the following mistake: Given (L, ∗), where L is the set of all positive rational numbers, $x * y = x/2 + y/2 + xy$ (it does not have an identity) and the student answers $n = \frac{x}{1+2x}$ with the following justification:

We will get n by solving the equation $\frac{x}{2} + \frac{n}{2} + xn = x$. Then $n \in$ L as the denominator does not equal 0 for $x \in$ L and the quotient of two positive rational numbers is a positive rational number. As the operation is commutative, it is sufficient to check one equality from the definition: $x * n = \frac{x}{2} + \frac{1}{2}.\frac{x}{1+2x} + x.\frac{x}{1+2x} = x$.

The student does not understand quantifiers. Instead of "there exists n such that for all x ...", he/she uses "for all x there exists n such that ...". On the other hand, the student has SSP-3 (he/she uses C for IN).

(SSP-5) Apply the knowledge of ID and IN spontaneously

By that we mean that in a certain context, without being specifically asked to, a student is able to use the knowledge of ID and IN to find the solution to a problem.

Example 1: A student displays SSP-5 if he/she applies the knowledge of ID and IN in (Z_p,+, ·) when dividing two polynomials with coefficients from Z_p. For example, in

(Z_5,+, .), where $-0 = 0$, $-1 = 4$, $-2 = 3$, $-3 = 2$, $-4 = 1$; $1^{-1} = 1$, $2^{-1} = 3$, $3^{-1} = 2$, $4^{-1} = 4$, when dividing $(3x^5 + 4x^4 + 2x^3 + x^2 + 4x + 3):(2x^3 + 3x^2 + 4x + 1)$, he/she is able to calculate $3 : 2 = 3 . 2^{-1} = 3 . 3 = 4$. On the other hand, he/she lacks SSP-5 if the answer for 3 : 2 is 3/2.

Example 2: A student displays SSP-5 if he/she is solving an equation $x + 50 = 5$ in structure (Z_{99},+) (see above) and he/she says: "I will subtract 50 from both sides of the equation which means that I will add the additive inverse of 50, that is 49, to both sides." (Stehlikova, 2004)

DISCUSSION AND CONCLUSIONS

The vague terms "familiar and non-familiar structures" can be specified to a certain extent by saying that they must be "conceptual entities in the student's eyes; that is to say, the student has procedures that can take these objects as inputs" (Harel & Tall, 1989). What will be "familiar" depends on individual students and the way abstract algebra was introduced to him/her. We can distinguish at least three paths (V means a property or an object, A in index means a familiar structure, B in index means a non-familiar structure, D stands for a formal definition):

V_A	V_A	D
↓ abstraction	↓ analogy	↓ construction
D	V_B	V_A, V_B
↓ construction	↓ abstraction	
V_B	D	

The first two paths represent the abstraction of specific properties of one or more mathematical objects to form the basis of the definition of the new abstract mathematical object, the third is the process of construction of the abstract concept through logical deduction from definition (Harel & Tall, 1989).

There is another way of interpreting some problems students have with understanding binary operations, their properties and objects (identity, inverse). Stehlikova (2004) in her research on structuring mathematical knowledge in advanced mathematics described a student coming to know a particular arithmetic structure as a process of development from dependence of the new structure on ordinary arithmetic to gradual independence.

> In general, there were either students who started reasoning inside [the new structure] quite early during their work spontaneously and these were able to find the additive identity easily and on the other hand, there were students who relied more on their [ordinary arithmetic] knowledge and their attention had to be specifically drawn to number 99 in order for them to notice its properties. These students mostly said that there was no additive identity because there was no 0. (p. 140)

The image of 0 as the additive identity does not always have to function as an obstacle. For some students, it serves as a generic model of additive identity and they can reconstruct its properties in ordinary arithmetic and use them as a tool for finding out the identity in another structure (Stehlikova, 2004).

The presented model only accounts for binary operations and their properties. A model for the student's understanding of, say, groups would have to be far more complex (see e.g. Dubinsky et al., 1994).

If we attribute students' difficulties to their lack of structure sense, we can concentrate on developing their structure sense. The above model can serve as a basis for a teaching programme explicitly addressing the problematic issues.

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