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INTEGERS WITH A MAXIMAL NUMBER OF FIBONACCI REPRESENTATIONS

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Abstract. We study the properties of the function $R(n)$ which determines the number of representations of an integer n as a sum of distinct Fibonacci numbers F_k. We determine the maximum and mean values of $R(n)$ for $F_k \leq n < F_{k+1}$.

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1. Introduction

Let $(F_k)_{k\geq 0}$ be the Fibonacci sequence defined by

$$F_0 = F_1 = 1\,, \qquad F_{k+1} = F_k + F_{k-1} \quad \text{for } k \geq 1.$$

Every positive integer n can be written as a sum of distinct Fibonacci numbers, *i.e.* in the form

$$n = F_{m_r} + F_{m_{r-1}} + \cdots + F_{m_1}, \qquad \text{where } \; m_r > m_{r-1} > \cdots > m_1 \geq 1. \tag{1}$$

The expression (1) is called a representation of the number n in the Fibonacci number system. The index of the maximal Fibonacci number that appears in the representation of n is called the length of the representation. Every Fibonacci representation can be written in the form of a finite word $w = w_{m_r} w_{m_r-1} \dots w_1$ in the alphabet $\{0, 1\}$, where $w_i = 1$ for $i = m_1, \dots, m_r$, and $w_i = 0$ otherwise. For example the number $n = 32$ can be represented as

$$32 = 21 + 5 + 3 + 2 + 1 = F_7 + F_4 + F_3 + F_2 + F_1$$

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and this representation corresponds to the word 1001111. Due to the recurrence relation for Fibonacci numbers, different representations of the number n can be obtained by substituting the string 011 by 100 and *vice versa*. All representations of 32 correspond to words

$$1010100, \quad 1010011, \quad 1001111, \quad 111111.$$

The number of different Fibonacci representations of n will be denoted by $R(n)$. Let us enumerate the first twenty values of the sequence $\big(R(n)\big)_{n\geq 1}$,

$$\big(R(n)\big)_{n\geq 1} = 1,1,2,1,2,2,1,3,2,2,3,1,3,3,2,4,2,3,3,1,\ldots \tag{2}$$

For a given positive integer n we can find k such that

$$F_k \leq n < F_{k+1}.$$

It is obvious that every representation of n has length $\leq k$. On the other hand, since

$$F_1 + F_2 + \cdots + F_{k-2} < F_k \leq n,$$

the lengths of every representation of n is at least $k-1$. Thus representations of the number n can be divided into long (having length k) and short (of length $k-1$). Let us denote by $R_1(n)$ the number of long representations of n, and by $R_0(n)$ the number of short representations of n. Clearly

$$R(n) = R_1(n) + R_0(n).$$

If we prefix the short Fibonacci representations of n with the prefix 0, they have the same length as the long representations of n. The lexicographically greatest among all such representations of the number n is called the Zeckendorf representation of n and the corresponding word in the alphabet $\{0,1\}$ is denoted by $\langle n\rangle$. The distinguishing characteristic of this representation is that there are no adjacent 1's. For example, we have $\langle 32\rangle = 1010100$.

The Zeckendorf representation of a number n is a word of the form

$$\langle n\rangle = 10^{r_1}10^{r_2}\ldots 10^{r_l}\,, \qquad \text{where} \;\; r_1,\ldots,r_{l-1} \geq 1\,, \text{ and } r_l \geq 0. \tag{3}$$

The sum $r_1+r_2+\cdots+r_l+l$ determines the length of the Zeckendorf representation of n. Since the relation between the number n and the word (3) is one-to-one, we define for the simplicity of notation

$$\begin{aligned} \varrho(r_1,\ldots,r_l) &:= R(n)\\ \varrho_1(r_1,\ldots,r_l) &:= R_1(n)\\ \varrho_0(r_1,\ldots,r_l) &:= R_0(n) \end{aligned} \tag{4}$$

where $\langle n\rangle = 10^{r_1}10^{r_2}\ldots 10^{r_l}$.

It can be seen easily that $R(n) = 1$ if and only if $n = F_k - 1$ for some $k \geq 2$. The values of $R(n)$ for $n = F_k \pm j$, $j \leq 8$ are given in [2]. The segment of the sequence $R(n)$ between two consecutive occurrences of 1 is a palindrome [2,4], *i.e.*

$$R(F_k - 1 + i) = R(F_{k+1} - 1 - i), \qquad \text{for } \ i = 1, 2, \ldots, F_{k-1} - 1.$$

The aim of this paper is to find the maximal and the mean values of the function $R(n)$ for $F_k - 1 < n < F_{k+1} - 1$, which corresponds to the numbers n whose Zeckendorf representation has a fixed length k. We determine the numbers

$$\begin{aligned}\operatorname{Max}(k) &:= \max\big\{R(n) \mid n \in \mathbb{N}\,,\, F_k \leq n < F_{k+1}\big\} \\ &= \max\left\{\varrho(r_1, \ldots, r_l) \;\middle|\; l \in \mathbb{N},\, r_1, \ldots, r_{l-1} \geq 1,\, r_l \geq 0,\, l + \sum_{i=1}^{l} r_i = k\right\}.\end{aligned}$$

In addition, we classify the arguments of the maxima.

Let us determine several initial values of the sequence $\operatorname{Max}(k)$. It suffices to divide the sequence $(R(n))_{n\geq 1}$ to blocks of length $F_0, F_1, F_2, \ldots$ along the occurrence of consecutive 1's and to find maximal values in these blocks, see (2). We have

$$\begin{aligned}
\operatorname{Max}(1) &= \max\{R(n) \mid 1 \leq n < 2\} &&= R(1) = 1, \\
\operatorname{Max}(2) &= \max\{R(n) \mid 2 \leq n < 3\} &&= R(2) = 1, \\
\operatorname{Max}(3) &= \max\{R(n) \mid 3 \leq n < 5\} &&= R(3) = 2, \\
\operatorname{Max}(4) &= \max\{R(n) \mid 5 \leq n < 8\} &&= R(5) = R(6) = 2, \\
\operatorname{Max}(5) &= \max\{R(n) \mid 8 \leq n < 13\} &&= R(8) = R(11) = 3, \\
\operatorname{Max}(6) &= \max\{R(n) \mid 13 \leq n < 21\} &&= R(16) = 4.
\end{aligned} \tag{5}$$

2. Properties of the functions ϱ, ϱ_0, ϱ_1

Berstel [1] gives an explicit formula for computing the values of functions ϱ, ϱ_1, ϱ_0 defined in (4). Denote the matrix

$$M(r) := \begin{pmatrix} \lceil \frac{r}{2} \rceil & \lfloor \frac{r}{2} \rfloor \\ 1 & 1 \end{pmatrix}.$$

Theorem 2.1 (Berstel)**.** *Let $r_1, \ldots, r_l \in \mathbb{Z}$, $r_1, \ldots, r_{l-1} \geq 1$, $r_l \geq 0$. Then*

$$\begin{pmatrix} \varrho_0(r_1, \ldots, r_l) \\ \varrho_1(r_1, \ldots, r_l) \end{pmatrix} = M(r_1)M(r_2)\ldots M(r_l)\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Since $\varrho(r_1,\dots,r_l) = \varrho_0(r_1,\dots,r_l) + \varrho_1(r_1,\dots,r_l)$, we have explicit formulas for the functions ϱ, ϱ_0, ϱ_1 in the following form

$$\varrho(r_1,\dots,r_l) = \begin{pmatrix}1 & 1\end{pmatrix} M(r_1)M(r_2)\dots M(r_l)\begin{pmatrix}0\\1\end{pmatrix},$$

$$\varrho_0(r_1,\dots,r_l) = \begin{pmatrix}1 & 0\end{pmatrix} M(r_1)M(r_2)\dots M(r_l)\begin{pmatrix}0\\1\end{pmatrix},$$

$$\varrho_1(r_1,\dots,r_l) = \begin{pmatrix}0 & 1\end{pmatrix} M(r_1)M(r_2)\dots M(r_l)\begin{pmatrix}0\\1\end{pmatrix}. \tag{6}$$

Let us now derive some recurrence relations for $\varrho(r_1,\dots,r_l)$ that will be needed for determining the maximal values. If $l=1$ we get directly from (6) that

$$\varrho(r) = \left\lfloor \frac{r}{2} \right\rfloor + 1. \tag{7}$$

Lemma 2.2. *Let $l \in \mathbb{N}$, and let $r_1, r_2, \dots, r_l \in \mathbb{Z}$, $r_1, r_2, \dots, r_{l-1} \geq 1$, $r_l \geq 0$. If r_l is odd, then $\varrho(r_1,\dots,r_l) = \varrho(r_1,\dots,r_l - 1)$.*

Proof. It follows from (6) since for r_l odd we have $M(r_l)\binom{0}{1} = M(r_l - 1)\binom{0}{1}$. □

Lemma 2.3. *Let $l \in \mathbb{N}$, $l \geq 2$ and let $r_1, r_2, \dots, r_l \in \mathbb{Z}$, $r_1, r_2, \dots, r_{l-1} \geq 1$, $r_l \geq 0$. If r_i is even for some $1 \leq i \leq l-1$, then*

$$\varrho(r_1,\dots,r_l) = \varrho(r_1,\dots,r_i)\varrho(r_{i+1},\dots,r_l).$$

Proof. For r_i even we have $M(r_i) = M(r_i)\binom{0}{1}(1\ 1)$. Substituting into (6) we obtain the lemma. □

Lemma 2.4. *Let $l \in \mathbb{N}$, $l \geq 2$, and let $r_1, r_2, \dots, r_l \in \mathbb{Z}$, $r_1, r_2, \dots, r_{l-1} \geq 1$, $r_l \geq 0$. We have*

$$\varrho(r_1, r_2, \dots, r_l) = \frac{r_1+1}{2}\ \varrho(r_2,\dots,r_l) + \varrho_0(r_2,\dots,r_l), \qquad \text{if } r_1 \text{ is odd,}$$
$$\varrho(r_1, r_2, \dots, r_l) = \Big(\frac{r_1}{2} + 1\Big)\ \varrho(r_2,\dots,r_l), \qquad \text{if } r_1 \text{ is even.}$$

Proof. First suppose r_1 is odd. Since

$$\begin{pmatrix}1 & 1\end{pmatrix} M(r_1) = \frac{r_1+1}{2}\begin{pmatrix}1 & 1\end{pmatrix} + \begin{pmatrix}1 & 0\end{pmatrix},$$

substituting into (6) gives the desired result. The statement for r_1 even is a consequence of Lemma 2.3 and the relation (7). □

Lemma 2.5. *Let $l \in \mathbb{N}$, $l \geq 3$, and let $r_1, r_2, \dots, r_l \in \mathbb{Z}$, $r_1, r_2, \dots, r_{l-1} \geq 1$, $r_l \geq 0$. If for some i, $2 \leq i \leq l-1$, the coefficient r_i is odd, then*

$$\varrho(r_1,\dots,r_l) = \varrho(r_1,\dots,r_i - 1)\varrho(r_{i+1},\dots,r_l) + \varrho(r_1,\dots,r_{i-1}+1)\varrho_0(r_{i+1},\dots,r_l).$$

Proof. Again, it suffices to verify the matrix equality

$$M(r_{i-1})M(r_i) = M(r_{i-1})M(r_i-1)\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1 & 1\end{pmatrix} + M(r_{i-1}+1)\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix}$$

for r_i odd and to use (6). $\square$

The following lemma is a direct consequence of the definition of functions ϱ, ϱ_0 and can be found in [4] as Lemma 1.

Lemma 2.6 (Edson, Zamboni)**.** *Let $l \in \mathbb{N}$, $l \geq 2$, and let $r_1, r_2, \ldots, r_l \in \mathbb{Z}$, $r_1, r_2, \ldots, r_{l-1} \geq 1$, $r_l \geq 0$. Then*

- (i) $\varrho_0(r_1, r_2, \ldots, r_l) = \varrho(r_1-2, r_2, \ldots, r_l)$, *for* $r_1 \geq 3$;
- (ii) $\varrho_0(2, r_2, \ldots, r_l) = \varrho(r_2, \ldots, r_l)$;
- (iii) $\varrho_0(1, r_2, \ldots, r_l) = \varrho_0(r_2, \ldots, r_l)$;
- (iv) $\varrho(r_1, \ldots, r_{l-1}, 1, 1, \ldots, 1) = \varrho(r_1, \ldots, r_{l-1})$.

Clearly, $\varrho(r_1, \ldots, r_l) \geq 1$. However, the number of short Fibonacci representations $\varrho_0(r_1, \ldots, r_l)$ can be equal to 0. Using the rules given in Lemma 2.6 we easily deduce that

$$\varrho_0(r_1, \ldots, r_l) = 0 \qquad \Longleftrightarrow \qquad r_1 = r_2 = \cdots = r_{l-1} = 1 \ \text{ and } \ r_l \in \{0, 1\}. \tag{8}$$

3. Lower bound on Max(k)

In order to find the lower estimates of Max(k), let us determine the values $\varrho(r_1, \ldots, r_l)$ on some chosen l-tuples $(r_1, \ldots, r_l)$.

Lemma 3.1.

1) $\varrho(\underbrace{3, 3, \ldots, 3}_{k-1 \text{ times}}, 4) = \varrho(1, \underbrace{3, \ldots, 3}_{k-1 \text{ times}}, 2) = F_{2k+1}$ *for* $k \geq 1$.

2) $\varrho(\underbrace{3, 3, \ldots, 3}_{k \text{ times}}, 2) = \varrho(1, \underbrace{3, \ldots, 3}_{k-1 \text{ times}}, 4) = F_{2k+2}$ *for* $k \geq 1$.

Proof. Let us first show by induction that for the s-th power of the matrix $M(3) = \left(\begin{smallmatrix}2 & 1\\ 1 & 1\end{smallmatrix}\right)$ we have

$$\big(M(3)\big)^s = \begin{pmatrix} F_{2s} & F_{2s-1} \\ F_{2s-1} & F_{2s-2} \end{pmatrix}, \qquad \text{for } \ s \in \mathbb{N}. \tag{9}$$

For $s = 1$ the statement is trivial. For $s \geq 2$ we use the induction hypothesis

$$\big(M(3)\big)^s = \big(M(3)\big)^{s-1}\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix} = \begin{pmatrix} F_{2s-2} & F_{2s-3} \\ F_{2s-3} & F_{2s-4} \end{pmatrix}\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix} = \begin{pmatrix} F_{2s} & F_{2s-1} \\ F_{2s-1} & F_{2s-2} \end{pmatrix}.$$

Note that (9) is valid also for $s = 0$ if we define F_{-1}, F_{-2} in such a way that the recurrence relation is still valid, ($F_{-1} = 0$, $F_{-2} = 1$). It is now easy to use (6) to find

$$\begin{aligned}\varrho(\underbrace{3,3,\ldots,3}_{k-1\text{ times}},4) &= \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix} F_{2k-2} & F_{2k-3} \\ F_{2k-3} & F_{2k-4}\end{pmatrix}\begin{pmatrix}2 & 2\\ 1 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\\ &= \begin{pmatrix} F_{2k-1} & F_{2k-2}\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} = F_{2k+1}\end{aligned}$$

and

$$\begin{aligned}\varrho(1,\underbrace{3,\ldots,3}_{k-1\text{ times}},2) &= \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix} F_{2k-2} & F_{2k-3} \\ F_{2k-3} & F_{2k-4}\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\\ &= \begin{pmatrix} F_{2k} & F_{2k-1}\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = F_{2k+1}.\end{aligned}$$

The relations (2) can be proved similarly. □

As a corollary, we have a lower estimate on the maxima for numbers with Zeckendorf representation of odd length.

Corollary 3.2. $\mathrm{Max}(2k+1) \geq F_{k+1}$ *for* $k \geq 1$.

From the definition of the function ϱ it follows that

$$\varrho(2, r_1, \ldots, r_l) \geq 2\varrho(r_1, \ldots, r_l)$$

and for $r_l > 0$ also

$$\varrho(r_1, \ldots, r_l, 2) \geq 2\varrho(r_1, \ldots, r_l).$$

Therefore we have the following lower estimate on the maxima for numbers with Zeckendorf representation of even length.

Corollary 3.3. $\mathrm{Max}(2k+2) \geq 2\mathrm{Max}(2k-1) \geq 2F_k$ *for* $k \geq 2$.

Our aim is to show that the inequalities in Corollaries 3.2 and 3.3 are in fact equalities.

4. Maxima of the function $R(n)$

Let us now determine the maximum of the function $R(n) = \varrho(r_1, r_2, \ldots, r_l)$, where $F_k \leq n < F_{k+1}$ and $\langle n\rangle = 10^{r_1}10^{r_2}\cdots 10^{r_l}$. The l-tuple $r_1, \ldots, r_l \in \mathbb{Z}$ must satisfy $r_1, r_2, \ldots, r_{l-1} \geq 1$, $r_l \geq 0$ and $\sum_{i=1}^{l} r_i + l = k$. We shall not repeat these assumptions.

Let us show that $\mathrm{Max}(k)$ is not reached on integers n whose Zeckendorf representation has only one 1. More precisely, we have the following proposition.

Proposition 4.1. *Let* $\mathrm{Max}(k) = \varrho(r_1, r_2, \ldots, r_l)$. *Then* $l \geq 2$ *or* $k \leq 5$.

Proof. Suppose by contradiction that $k \geq 6$ and $l = 1$. Then using (7), we have $\mathrm{Max}(k) = \varrho(k-1) = \lfloor \frac{k-1}{2} \rfloor + 1$. For k even we have by Corollary 3.3

$$2F_{\frac{k-2}{2}} \leq \mathrm{Max}(k) = \left\lfloor \frac{k-1}{2} \right\rfloor + 1 = \frac{k}{2},$$

which is in contradiction with $2F_{i-1} > i$ for all $i \geq 3$. For k odd we have by Corollary 3.2

$$F_{\frac{k+1}{2}} \leq \mathrm{Max}(k) = \left\lfloor \frac{k-1}{2} \right\rfloor + 1 = \frac{k+1}{2},$$

which contradicts the fact that $F_i > i$ for all $i \geq 4$. □

In the following several propositions we show that the maximum is reached on l-tuples of a certain specific form. The proofs are done by contradiction. Assuming that the maximal l-tuple does not satisfy the desired properties, we find another l-tuple on which the function ϱ has strictly greater value.

Proposition 4.2. *Let* $\mathrm{Max}(k) = \varrho(r_1, r_2, \ldots, r_l)$ *for* $k \geq 6$. *Then* r_l *is even.*

Proof. Since the above Proposition 4.1 implies that $l \geq 2$, it suffices to prove that for r_l odd we have

$$\varrho(r_1, r_2, \ldots, r_{l-1}, r_l) < \varrho(r_1 + 1, r_2, \ldots, r_{l-1}, r_l - 1). \tag{10}$$

We divide the demonstration of (10) into two cases.

a) Let r_1 be even. Using Lemmas 2.2 and 2.4 we have

$$\begin{aligned} \varrho(r_1, r_2, \ldots, r_{l-1}, r_l) &= \varrho(r_1, r_2, \ldots, r_{l-1}, r_l - 1) \\ &= \left(\frac{r_1}{2} + 1\right) \varrho(r_2, \ldots, r_{l-1}, r_l - 1), \\ \varrho(r_1 + 1, r_2, \ldots, r_{l-1}, r_l - 1) & \\ = \frac{r_1 + 2}{2}\, \varrho(r_2, \ldots, r_{l-1}, r_l - 1) &+ \varrho_0(r_2, \ldots, r_{l-1}, r_l - 1). \end{aligned}$$

In order to obtain (10) we need to show that $\varrho_0(r_2, \ldots, r_{l-1}, r_l - 1) > 0$. Using (8), $\varrho_0(r_2, \ldots, r_{l-1}, r_l - 1) = 0$ with r_l odd implies $r_2 = r_3 = \cdots = r_l = 1$. However, in this case the property (iv) of Lemma 2.6 and Proposition 4.1 give

$$\varrho(r_1, r_2, \ldots, r_{l-1}, r_l) = \varrho(r_1, 1, 1, \ldots, 1) = \varrho(r_1) < \varrho(k-1) < \mathrm{Max}(k),$$

which contradicts the assumption of the proposition. Thus we necessarily have $\varrho_0(r_2, \ldots, r_{l-1}, r_l - 1) > 0$ and (10) is valid.

b) Let r_1 be odd. Again we use Lemmas 2.2 and 2.4 to obtain

$$\varrho(r_1, r_2, \ldots, r_{l-1}, r_l) = \frac{r_1 + 1}{2}\, \varrho(r_2, \ldots, r_{l-1}, r_l) + \varrho_0(r_2, \ldots, r_{l-1}, r_l),$$

$$\varrho(r_1 + 1, r_2, \ldots, r_{l-1}, r_l - 1) = \varrho(r_1 + 1, \ldots, r_l) = \Big(\frac{r_1 + 1}{2} + 1\Big)\varrho(r_2, \ldots, r_l).$$

The validity of (10) is obvious, since $\varrho(r_2, \ldots, r_l) > \varrho_0(r_2, \ldots, r_l)$. $\square$

In order to find the arguments of the maxima of the function ϱ, we use the matrix formula (6). First we introduce a partial ordering on non-negative matrices. Lemma 4.4 then shows that replacing a matrix in (6) by a "bigger" one increases the value of the function ϱ.

Definition 4.3. Let $\mathbb{X} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\tilde{\mathbb{X}} = \left(\begin{smallmatrix} \tilde a & \tilde b \\ \tilde c & \tilde d \end{smallmatrix}\right)$ be integer matrices with non-negative components. We say that $\mathbb{X}$ majores $\tilde{\mathbb{X}}$ (written $\mathbb{X} \succ \tilde{\mathbb{X}}$) if

$$a \geq \tilde a, \quad b \geq \tilde b, \quad a + c \geq \tilde a + \tilde c \quad \text{and} \quad b + d > \tilde b + \tilde d. \tag{11}$$

Lemma 4.4. *Let* $\alpha = (1\ 1)\mathbb{A}\mathbb{X}\mathbb{B}\binom{0}{1}$ *and* $\tilde\alpha = (1\ 1)\mathbb{A}\tilde{\mathbb{X}}\mathbb{B}\binom{0}{1}$, *where*

$$\begin{aligned} \mathbb{A} = \mathbb{I}_2 \quad &\text{or} \quad \mathbb{A} = M(r_1)\ldots M(r_s). \\ \mathbb{B} = \mathbb{I}_2 \quad &\text{or} \quad \mathbb{B} = M(p_1)\ldots M(p_t), \end{aligned}$$

and $\mathbb{X}, \tilde{\mathbb{X}}$ *are non-negative integer matrices. If* $\mathbb{X} \succ \tilde{\mathbb{X}}$, *then* $\alpha > \tilde\alpha$.

Proof. Denote $(\,x\ y\,) = (1\ 1)\mathbb{A}$ and $\binom{z}{u} = \mathbb{B}\binom{0}{1}$. It is easy to see that $x \geq y \geq 1$ and that $z \geq 0$, $u \geq 1$. Let $\mathbb{X} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\tilde{\mathbb{X}} = \left(\begin{smallmatrix} \tilde a & \tilde b \\ \tilde c & \tilde d \end{smallmatrix}\right)$ satisfy (11). Then

$$\begin{aligned} \alpha - \tilde\alpha &= (\,x\ y\,)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z \\ u \end{pmatrix} - (\,x\ y\,)\begin{pmatrix} \tilde a & \tilde b \\ \tilde c & \tilde d \end{pmatrix}\begin{pmatrix} z \\ u \end{pmatrix} \\ &= \Big(\,(a - \tilde a)x + (c - \tilde c)y\ ,\ (b - \tilde b)x + (d - \tilde d)y\,\Big)\begin{pmatrix} z \\ u \end{pmatrix} \\ &\geq \Big(\,(a + c - \tilde a - \tilde c)y\ ,\ (b + d - \tilde b - \tilde d)y\,\Big)\begin{pmatrix} z \\ u \end{pmatrix} \geq (\,0\ 1\,)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1. \end{aligned}$$ $\square$

Proposition 4.5. *Let* $\varrho(r_1, r_2, \ldots, r_l) = \mathrm{Max}(k)$. *Then* $r_i \leq 5$ *for all* $i = 1, 2, \ldots, l$.

Proof. Let $\langle n\rangle = 10^{r_1}10^{r_2}\cdots 10^{r_l}$, and assume that there exists an index i such that $r_i \geq 6$. Denote by m the number with Zeckendorf representation $\langle m\rangle = 10^{r_1}\cdots 10^{r_{i-1}}10^{r_i-3}10^2 10^{r_{i+1}}\cdots 10^{r_l}$. Zeckendorf representations $\langle n\rangle$ and $\langle m\rangle$

have the same length. Since

$$M(r_i) = \begin{pmatrix} \lceil \frac{r_i}{2} \rceil & \lfloor \frac{r_i}{2} \rfloor \\ 1 & 1 \end{pmatrix} \prec \begin{pmatrix} \lceil \frac{r_i-3}{2} \rceil & \lfloor \frac{r_i-3}{2} \rfloor \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} r_i - 3 & r_i - 3 \\ 2 & 2 \end{pmatrix} = M(r_i - 3)M(2),$$

we have according to Lemma 4.4

$$R(n) = \varrho(r_1, r_2, \ldots, r_l) < \varrho(r_1, \ldots, r_{i-1}, r_i - 3, 2, r_{i+1}, \ldots, r_l) = R(m),$$

which contradicts the assumption of the proposition. □

Proposition 4.6. *Let* $\varrho(r_1, r_2, \ldots, r_{l-1}, r_l) = \mathrm{Max}(k)$, *where* $k \geq 6$ *and the* r_i *are odd for* $i = 1, 2, \ldots, l-1$. *Then* $r_1 \in \{1, 3\}$, $r_2, \ldots, r_{l-1} = 3$, *and* $r_l \in \{2, 4\}$.

Proof. As a consequence of Proposition 4.2, the final coefficient r_l is even, and due to Proposition 4.5 it can take only values $\{0, 2, 4\}$. Assumption of the present proposition with Proposition 4.5 implies that $r_1, r_2, \ldots, r_{l-1} \in \{1, 3, 5\}$. First let us show by contradiction that 5 does not occur. Suppose the opposite, *i.e.* that there exists an index $1 \leq i \leq l-1$ such that $r_i = 5$. Let i be the maximal index with this property. Let s be the minimal non-negative integer, such that $r_{i+s} \neq 3$. Then $r_{i+s} = 1$ or $i + s = l$ and $r_{i+s} \in \{0, 2, 4\}$.

1) Let $r_{i+s} = 1$. We verify that

$$\tilde{\mathbb{X}} = M(5)\big(M(3)\big)^{s-1}M(1) \prec \big(M(3)\big)^{s+1} = \mathbb{X}.$$

According to (9), we obtain

$$\tilde{\mathbb{X}} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} F_{2s-2} & F_{2s-3} \\ F_{2s-3} & F_{2s-4} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} F_{2s+2} & F_{2s} \\ F_{2s} & F_{2s-2} \end{pmatrix}, \quad \mathbb{X} = \begin{pmatrix} F_{2s+2} & F_{2s+1} \\ F_{2s+1} & F_{2s} \end{pmatrix}.$$

Obviously $\tilde{\mathbb{X}} \prec \mathbb{X}$ and using Lemma 4.4 we obtain

$$\mathrm{Max}(k) = \varrho(r_1, \ldots, r_{i-1}, 5, \underbrace{3, \ldots, 3}_{s-1\,\text{times}}, 1, r_{i+s+1}, \ldots, r_l)$$
$$< \varrho(r_1, \ldots, r_{i-1}, \underbrace{3, \ldots, 3}_{s+1\,\text{times}}, r_{i+s+1}, \ldots, r_l),$$

which is a contradiction.

2) Let $r_{i+s} = 2$. Similarly as in (1) we use matrices and Lemma 4.4 to obtain the contradiction

$$\mathrm{Max}(k) = \varrho(r_1, \ldots, r_{i-1}, 5, \underbrace{3, \ldots, 3}_{s-1\,\text{times}}, 2) < \varrho(r_1, \ldots, r_{i-1}, \underbrace{3, \ldots, 3}_{s\,\text{times}}, 4).$$

3) Let $r_{i+s} = 4$. Similarly as in (1) we use matrices and Lemma 4.4 to obtain the contradiction

$$\mathrm{Max}(k) = \varrho(r_1, \dots, r_{i-1}, 5, \underbrace{3, \dots, 3}_{s-1 \text{ times}}, 4) < \varrho(r_1, \dots, r_{i-1}, \underbrace{3, \dots, 3}_{s+1 \text{ times}}, 2).$$

4) Let $r_{i+s} = 0$. Similarly as in (1) we use matrices and Lemma 4.4 to obtain the contradiction

$$\mathrm{Max}(k) = \varrho(r_1, \dots, r_{i-1}, 5, \underbrace{3, \dots, 3}_{s-1 \text{ times}}, 0) < \varrho(r_1, \dots, r_{i-1}, \underbrace{3, \dots, 3}_{s \text{ times}}, 2).$$

Thus we have shown that $r_1, \dots, r_{l-1} \le 3$, *i.e.* all take values in $\{1, 3\}$.

Let us now prove by contradiction that at most one of the coefficients $r_1, \dots, r_{l-1}$ is equal to 1. Assume that there exist indices $i, i+s$, $1 \le i < i+s \le l-1$ such that $r_i = r_{i+s} = 1$ and $r_{i+1} = r_{i+2} = \dots = r_{i+s-1} = 3$. Denote

$$\begin{aligned}
\tilde{\mathbb{X}} &= M(1)\big(M(3)\big)^{s-1} M(1) = \begin{pmatrix} F_{2s-1} & F_{2s-3} \\ F_{2s} & F_{2s-2} \end{pmatrix}, \\
\mathbb{X} &= \big(M(3)\big)^{s} = \begin{pmatrix} F_{2s} & F_{2s-1} \\ F_{2s-1} & F_{2s-2} \end{pmatrix}.
\end{aligned}$$

Since $\tilde{\mathbb{X}} \prec \mathbb{X}$, we derive that

$$\begin{aligned}
\mathrm{Max}(k) &= \varrho(r_1, \dots, r_{i-1}, 1, \underbrace{3, \dots, 3}_{s-1 \text{ times}}, 1, r_{i+s+1, \dots, r_l}) \\
&< \varrho(r_1, \dots, r_{i-1}, \underbrace{3, \dots, 3}_{s \text{ times}}, r_{i+s+1}, \dots, r_l),
\end{aligned}$$

which contradicts the maximality of $\varrho(r_1, \dots, r_l)$. Thus at most one of the coefficients $r_1, \dots, r_{l-1}$ is equal to 1 and the others are equal to 3.

If $l = 2$, the proposition is proved. For $l \ge 3$ we show by contradiction that $r_2 = \dots = r_{l-1} = 3$. Suppose that $r_i = 1$ for some $2 \le i \le l-1$. Since

$$\begin{aligned}
\big(1\ 1\big)\big(M(3)\big)^{i-1} M(1) &= \big(F_{2i}\ F_{2i-2}\big), \\
\big(1\ 1\big) M(1) \big(M(3)\big)^{i-1} &= \big(F_{2i}\ F_{2i-1}\big),
\end{aligned}$$

it follows that

$$\mathrm{Max}(k) = \varrho(\underbrace{3, \dots, 3}_{i-1 \text{ times}}, 1, r_{i+1}, \dots, r_l) < \varrho(1, \underbrace{3, \dots, 3}_{i-1 \text{ times}}, r_{i+1}, \dots, r_l),$$

which is a contradiction.

It remains to show that $r_l \ne 0$. But if $r_l = 0$, then $r_{l-1} = 3$. Relation $M(3)M(0) \prec M(4)$ implies a contradiction. $\square$

We are now in position to state the theorem about the maximal values of $R(n)$.

Theorem 4.7.

$$\begin{aligned}
&\max\{R(n) \mid F_{2k+1} \le n < F_{2k+2}\} = \mathrm{Max}(2k+1) = F_{k+1} \qquad \textit{for} \;\; k \ge 0,\\
&\max\{R(n) \mid F_{2k+2} \le n < F_{2k+3}\} = \mathrm{Max}(2k+2) = 2F_k \qquad \textit{for} \;\; k \ge 1.
\end{aligned}$$

Proof. In the proof we shall make use of the following inequalities for Fibonacci numbers, which are not difficult to demonstrate.

$$F_{x+1}F_{y+1} \le 2F_{x+y} \qquad \text{for} \;\; x, y \ge 0, \tag{12}$$

where the equality holds only if $x = 1$ or $y = 1$.

$$2F_xF_y \le F_{x+y+1} \qquad \text{for} \;\; x, y \ge 1, \tag{13}$$

where the equality holds only if $x = y = 2$.

Since the lower bounds on the maxima of the function $R(n)$ are known from Corollaries 3.2 and 3.3, it suffices to prove inequalities

$$\mathrm{Max}(2k+1) \le F_{k+1} \qquad \text{and} \qquad \mathrm{Max}(2k+2) \le 2F_k. \tag{14}$$

Let us show it by induction on k. For initial values of k the validity of the theorem follows from (5). Now assume that

$$\mathrm{Max}(2j+1) \le F_{j+1} \qquad \text{and} \qquad \mathrm{Max}(2j+2) \le 2F_j, \qquad \text{for} \;\; j < k.$$

With this induction hypothesis we want to show (14).

• Let us first show that $\mathrm{Max}(2k+2) \le 2F_k$.
Let $r_1, r_2, \ldots, r_l$ be an l-tuple such that $\varrho(r_1, r_2, \ldots, r_l) = \mathrm{Max}(2k+2)$ where $k \ge 2$. Proposition 4.2 implies that r_l is even. Since $r_1 + r_2 + \cdots + r_l + l = 2k+2$, there must exist an $i < l$ such that r_i is even. Let i be the maximal $i < l$ with this property. The number $r_{i+1} + \cdots + r_l + (l-i)$ is odd, say $2m+1$. Then $r_1 + \cdots + r_i + i = 2k + 2 - (2m+1)$. Lemma 2.3, the induction hypothesis and inequality (12) implies

$$\begin{aligned}
\mathrm{Max}(2k+2) &= \varrho(r_1, \ldots, r_l) = \varrho(r_1, \ldots, r_i)\; \varrho(r_{i+1}, \ldots, r_l)\\
&\le \; \mathrm{Max}(2k-2m+1)\; \mathrm{Max}(2m+1) \; = \; F_{k-m+1}F_{m+1} \; \le \; 2F_k.
\end{aligned} \tag{15}$$

• Now let us show the inequality $\mathrm{Max}(2k+1) \le F_{k+1}$.
Let $r_1, r_2, \ldots, r_l$ be an l-tuple such that $\varrho(r_1, r_2, \ldots, r_l) = \mathrm{Max}(2k+1)$ where $k \ge 2$. Suppose that besides r_l there exist another $i < l$ such that r_i is even and let i be the maximal index $i < l$ with this property. Let us denote $r_{i+1} + \cdots + r_l + (l-i) =$

$2m+1$. Then $r_1+\cdots+r_2+i = 2k+1-(2m+1) = 2k-2m$. Lemma 2.3, the induction hypothesis and inequality (13) implies

$$\begin{aligned}\mathrm{Max}(2k+1) &= \varrho(r_1,\dots,r_l) = \varrho(r_1,\dots,r_i)\ \varrho(r_{i+1},\dots,r_l)\\ &\le\ \mathrm{Max}(2k-2m)\ \mathrm{Max}(2m+1)\ =\ 2F_{k-m-1}F_{m+1}\ \le\ F_{k+1}. \quad (16)\end{aligned}$$

It remains to consider the case that the l-tuple $r_1,r_2,\dots,r_l$ which satisfies $\varrho(r_1,r_2,\dots,r_l) = \mathrm{Max}(2k+1)$ contains all r_i odd for $1\le i\le l-1$. According to Proposition 4.6 the maximal l-tuple is of the form $(1,3,\dots,3,4)$, $(3,\dots,3,4)$, $(1,3,\dots,3,2)$, or $(3,\dots,3,2)$. Note that for fixed length of the Zeckendorf representation only two of these are possible, namely $(1,3,\dots,3,2)$, or $(3,\dots,3,4)$ for length $1 \bmod 4$, and $(1,3,\dots,3,4)$, $(3,\dots,3,2)$ for length $3 \bmod 4$. The values of the function ϱ for these l-tuples was determined in Lemma 3.1. Therefore the statement of the theorem is proved. □

5. Argument of Max(k)

In this section we determine the integers on which the maximum of the function $R(n)$ is reached for a fixed length $\sum_{i=1}^{l} r_i + l$ of the Zeckendorf representation $\langle n\rangle = 10^{r_1}\cdots 10^{r_l}$. The proof of Theorem 4.7 allows us to determine the l-tuples $r_1,\dots,r_l$ representing such integers n.

Suppose first that the Zeckendorf representation of n has odd length. In this case the proof of Theorem 4.7 indicates that unless equality holds in (16), all the coefficients $r_1,\dots,r_{l-1}$ are odd and therefore the l-tuples $r_1,\dots,r_{l-1},r_l$ are of very specific form (as a consequence of Prop. 4.6).

Equality in (16) provides an exceptional l-tuple. In order to make (16) true, the relation (13) necessitates that $k=4$ (hence $m=1$) and

$$\varrho(r_1,\dots,r_i) = \mathrm{Max}(6) \qquad \text{and} \qquad \varrho(r_{i+1},\dots,r_l) = \mathrm{Max}(3).$$

Since according to the table (5) we have $\mathrm{Max}(6) = R(16) = \varrho(2,2)$ and $\mathrm{Max}(3) = R(3) = \varrho(2)$ necessarily $l=3$ and $r_1=r_2=r_3=2$.

Corollary 5.1.

(i) $\mathrm{Max}(4k+3)$ *is reached precisely on two arguments for* $k\ge 1$ *and on one argument for* $k=0$. *We have* $\mathrm{Max}(3)=\varrho(2)$, *and for* $k\ge1$

$$\mathrm{Max}(4k+3) = \varrho(1,\underbrace{3,\dots,3}_{k-1\ times},4) = \varrho(\underbrace{3,\dots,3}_{k\ times},2).$$

(ii) $\mathrm{Max}(4k+1)$ *is reached precisely on two arguments for* $k\ge3$ *or* $k=1$, *on three arguments for* $k=2$, *and on one argument for* $k=0$. *We have* $\mathrm{Max}(1)=\varrho(0)$, $\mathrm{Max}(9)=\varrho(1,3,2)=\varrho(3,4)=\varrho(2,2,2)$, *and for* $k=1$

and $k \geq 3$

$$\mathrm{Max}(4k+1) = \varrho(1, \underbrace{3, \dots, 3}_{k-1\ times}, 2) = \varrho(\underbrace{3, \dots, 3}_{k-1\ \ times}, 4).$$

As for integers with even length of their Zeckendorf representation, proof of Theorem 4.7 requires that an l-tuple $r_1, \dots, r_l$ on which the maximum of ϱ is reached must satisfy equality in (15). Relation (12) for Fibonacci numbers implies that $m-k=1$ or $m=1$. This can be true only if $i=1$ or $i=l-1$ respectively. Equality in (15) further requires that either $r_1 = 2$, $r_2, \dots, r_l$ are odd and $\varrho(r_2, \dots, r_l)$ is maximal, or $r_l = 2$ and $\varrho(r_2, \dots, r_l)$ is maximal, respectively.

Corollary 5.2. *Let* $k \geq 3$ *and let* $r_1, \dots, r_l$ *satisfy* $\sum_{i=1}^{l} r_i + l = 2k$. *Then* $\varrho(r_1, \dots, r_l) = \mathrm{Max}(2k)$ *if and only if*

$$\begin{array}{lllcl} & r_1 = 2 & \textit{and} & \varrho(r_2, \dots, r_l) & = & \mathrm{Max}(2k-3) \\ \textit{or} & & & & & \\ & r_l = 2 & \textit{and} & \varrho(r_1, \dots, r_{l-1}) & = & \mathrm{Max}(2k-3). \end{array}$$

Recall that the elements of the sequence $\big(R(n)\big)_{n\in\mathbb{N}}$ can be grouped into palindromes $R(F_k), \dots, R(F_{k+1}-2)$ separated by values $R(F_{k+1}-1) = 1$. Corollaries 5.1 and 5.2 show that up to the exceptional initial cases, the maximal value in each palindrome occurs twice (for k odd) and four times (for k even). The description of arguments of the maxima of $R(n)$ in the palindrome, *i.e.* for n with fixed length of Zeckendorf representation, is given in Theorem 5.3. We need to introduce the following notation,

$$i_{2k+1} = \begin{cases} F_{k+1}F_{k-3} + 1 & \text{for } k \text{ even,} \\ F_k F_{k-2} + 1 & \text{for } k \text{ odd,} \end{cases}$$

$$i_{2k} = \begin{cases} F_{k+2}F_{k-5} + F_3 + 1 & \text{for } k \text{ even,} \\ F_{k+1}F_{k-4} + F_3 + 1 & \text{for } k \text{ odd,} \end{cases}$$

$$j_{2k} = \begin{cases} F_{k+1}F_{k-3} + 1 & \text{for } k \text{ even,} \\ F_k F_{k-2} + 1 & \text{for } k \text{ odd.} \end{cases}$$

Theorem 5.3.

(i) $\mathrm{Max}(2k+1)$ *for* $k \geq 1$, $k \neq 4$ *is reached precisely on the integers*

$$F_{2k+1} - 1 + i_{2k+1}, \quad F_{2k+2} - 1 - i_{2k+1}.$$

For $k=4$, $\mathrm{Max}(2k+1) = \mathrm{Max}(9)$ *is reached precisely on three integers, namely*

$$F_9 - 1 + i_9 = 63, \quad F_{10} - 1 - i_9 = 79, \quad \textit{and their average } 71.$$

(ii) Max$(2k)$ *for* $k \geq 3$, $k \neq 6$, *is reached precisely on the integers*

$$F_{2k} - 1 + i_{2k}, \quad F_{2k+1} - 1 - i_{2k},$$
$$F_{2k} - 1 + j_{2k}, \quad F_{2k+1} - 1 - j_{2k}.$$

For $k = 6$, Max$(2k)$ = Max(12) *is reached precisely on five integers, namely*

$$F_{12} - 1 + i_{12} = 270, \quad F_{13} - 1 - i_{12} = 338,$$
$$F_{12} - 1 + j_{12} = 296, \quad F_{13} - 1 - j_{12} = 312,$$
$$\textit{and their avarage } \ 304.$$

Proof. Corollaries 5.1 and 5.2 show that up to the exceptional initial cases, the maximal value in the palindrome $R(F_k), \dots, R(F_{k+1} - 2)$ occurs twice for k odd and four times for k even. From the symmetry of the palindrome, for k odd there is an integer $i_k \in \{1, 2, \dots, F_{k-1} - 1\}$ such that

$$R(F_k - 1 + i_k) = R(F_{k+1} - 1 - i_k) = \mathrm{Max}(k).$$

Without loss of generality i_k is in our considerations the smaller of the two integers satisfying it. Similarly, for k even we have $i_k, j_k \in \{1, 2, \dots, F_{k-1} - 1\}$ such that

$$R(F_k - 1 + i_k) = R(F_{k+1} - 1 - i_k) = R(F_k - 1 + j_k) = R(F_{k+1} - 1 - j_k) = \mathrm{Max}(k).$$

We consider $i_k < j_k$ to be the two smallest of the four integers satisfying it.

We derive the compact form of i_k and j_k from arguments of maxima given in Corollaries 5.1 and 5.2. For that we use the relation

$$F_i + F_{i+4} + F_{i+8} + \cdots + F_{i+4(k-1)} = F_{2k+i-2} F_{2k-1}\,, \qquad \text{for } \ i, k \geq 1,$$

which can be shown using $F_k = \frac{1}{\sqrt{5}}(\tau^{k+1} - {\tau'}^{k+1})$, where $\tau = \frac{1}{2}(1 + \sqrt{5})$ is the golden ratio and $\tau' = \frac{1}{2}(1 - \sqrt{5})$ its algebraic conjugate. □

It is interesting to study the position of the maximal values in the palindrome $R(F_k - 1), R(F_k), \dots, R(F_{k+1} - 1)$, *i.e.* the position of integers i_k, (i_k and j_k) in the set $1, 2, \dots, F_{k-1}$. This is described by Proposition 5.4 and illustrated in Figure 1.

Proposition 5.4. *Let* $k \geq 1$. *Then*

$$\lim_{k\to\infty} \frac{i_{2k+1}}{F_{2k}} = \lim_{k\to\infty} \frac{i_{2k}}{F_{2k-1}} = \frac{1}{\tau + 2}, \qquad \left| i_{2k+1} - \frac{F_{2k}}{\tau + 2} \right| < \frac{1}{2},$$
$$\lim_{k\to\infty} \frac{j_{2k}}{F_{2k-1}} = \frac{\tau}{\tau + 2}, \qquad \left| j_{2k} - \frac{\tau F_{2k-1}}{\tau + 2} \right| < \frac{1}{2}.$$

The proposition shows that the numbers i_{2k+1} and j_{2k} are the closest integers to the asymptotic position of the maximal value. Let us mention that it is slightly more complicated in case of i_{2k}.

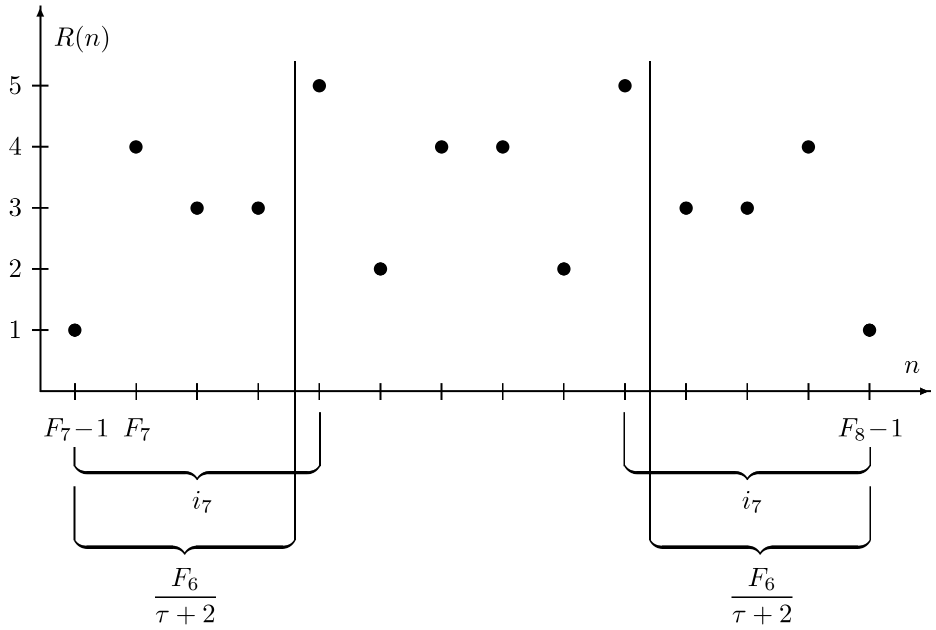

FIGURE 1. Illustration of the function $R(n)$ for $n \in [20, 33]$. The values $R(F_k - 1), R(F_k), \ldots, R(F_{k+1} - 1)$ for $k = 7$ form a palindrome. Since $k \equiv 3 \bmod 4$, the maximal value $\mathrm{Max}(7)$ appears twice and these local maxima are at the integers nearest to the asymptotical position, which is marked by the vertical lines.

6. Mean value of $R(n)$

Berstel in his article [1] states an open question about the mean value of the function $R(n)$. In this section we answer his question. In particular, we determine the mean value of $R(n)$ for integers with fixed length k of their Zeckendorf representation, *i.e.* the value

$$\frac{1}{F_{k-1}} \sum_{n=F_k}^{F_{k+1}-1} R(n).$$

Proposition 6.1. *Let $k \geq 1$. Then*

$$\sum_{n=F_k}^{F_{k+1}-1} R(n) = \frac{1}{3}\left(2^k - (-1)^k\right).$$

Proof. Consider the word $w = w_l w_{l-1} \ldots w_1$ in the alphabet $\{0, 1\}$ where $w_l = 1$. The word w is a representation of the number $n = \sum_{i=1}^{l} w_i F_i$. Also w is a long representation of n, if $\sum_{i=1}^{l} w_i F_i < F_{l+1}$, and w is a short representation of n, if

$\sum_{i=1}^{l} w_iF_i \geq F_{l+1}$. It can be easily shown that the latter occurs if and only if the word w has the prefix $1010\cdots1011$. More precisely,

$$\sum_{i=1}^{l} w_iF_i \geq F_{l+1}$$

if and only if

$$w_lw_{l-1}\cdots w_1 = (10)^i 11\, w_{l-2i-2}\cdots w_1, \qquad \text{for some } \ i \geq 0,\ i \leq \left\lfloor \frac{l-2}{2} \right\rfloor .$$

Therefore the number of words $w_l\cdots w_1$ with $w_l = 1$ that represent an integer $n \geq F_{l+1}$ is equal to the number of distinct suffixes $w_{l-2i-2}\cdots w_1$, *i.e.*

$$\sum_{i=0}^{\lfloor\frac{l-2}{2}\rfloor} 2^{l-2i-2} = \left\lfloor \frac{2^l-1}{3} \right\rfloor . \tag{17}$$

Consequently, the number of words $w_l\cdots w_1$ with $w_l = 1$ which represent an integer $n < F_{l+1}$ is equal to

$$2^{l-1} - \left\lfloor \frac{2^l-1}{3} \right\rfloor . \tag{18}$$

Since the sets of Fibonacci representations of distinct integers n are disjoint, we obtain

$$\begin{aligned}
\sum_{n=F_k}^{F_{k+1}-1} R_0(n) &= \#\left\{ w_{k-1}\cdots w_1 \in \{0,1\}^* \;\middle|\; w_{k-1}=1,\ \sum_{i=1}^{k-1} w_iF_i \geq F_k \right\} \\
&= \left\lfloor \frac{2^{k-1}-1}{3} \right\rfloor , \\
\sum_{n=F_k}^{F_{k+1}-1} R_1(n) &= \#\left\{ w_k\cdots w_1 \in \{0,1\}^* \;\middle|\; w_k=1,\ \sum_{i=1}^{k} w_iF_i < F_{k+1} \right\} \\
&= 2^{k-1} - \left\lfloor \frac{2^k-1}{3} \right\rfloor .
\end{aligned}$$

Together we obtain

$$\sum_{n=F_k}^{F_{k+1}-1} R(n) = 2^{k-1} - \left\lfloor \frac{2^k-1}{3} \right\rfloor + \left\lfloor \frac{2^{k-1}-1}{3} \right\rfloor = \frac{1}{3}\left(2^k - (-1)^k\right). \qquad \square$$

Since $F_k = \frac{1}{\sqrt{5}}\left(\tau^{k+1} - \tau'^{k+1}\right)$, the mean value of the function $R(n)$ for $F_k \leq n < F_{k+1}$ is equal to

$$\frac{\frac{1}{3}\left(2^k - (-1)^k\right)}{\frac{1}{\sqrt{5}}\left(\tau^k - \tau'^k\right)} \sim \frac{\sqrt{5}}{3}\left(\frac{2}{\tau}\right)^k.$$

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