Critical Exponent of Balanced Sequences

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Program

Program

Combinatorics on Words

- a general tool applicable in many fields
- examples of studied problems:
  - Random number generators (shuffling of generators using Sturmian sequences)
  - Hash functions (dithering method)
  - Quasicrystal models
  - Non-standard numeration systems
  - Discrete Schrödinger operators
    $(H\psi)(n) := \psi(n+1) + \psi(n-1) + u_n\psi(n) \quad \text{for } \psi \in \ell^2(\mathbb{Z})$
  - Formalization of CoW (a generic proof assistant Isabelle)

Birth of CoW

Axel Thue (1863 – 1922) answered in 1906 the following questions:

- **Question 1**: Is there a binary sequence without cubes?
- **Question 2**: Is there a ternary sequence without squares?

Example

$$\mathtt{ababbabab}\,\underbrace{\mathtt{abbbabbbabbb}}_{(\mathtt{abbb})^3}\,\mathtt{abababbba}\ldots$$

$$\mathtt{a}\,\underbrace{\mathtt{bb}}_{\mathtt{b}^2}\,\mathtt{a}\,\underbrace{\mathtt{cabcab}}_{(\mathtt{cab})^2}\,\mathtt{b}\,\underbrace{\mathtt{abbabb}}_{(\mathtt{abb})^2}\,\mathtt{cccabababa}\ldots$$

No binary sequence without squares:

No binary sequence without squares:

a

No binary sequence without squares:

$$\mathtt{aa}$$

No binary sequence without squares:

$$\mathtt{ab}$$

No binary sequence without squares:

abb

No binary sequence without squares:

aba

No binary sequence without squares:

abaa

No binary sequence without squares:

abab

Thue-Morse sequence

Definition

The Thue-Morse sequence $\mathbf{u}_{TM} = u_0 u_1 u_2 \ldots$ *is defined recursively*

$$u_0 = 0; \quad u_{2n} = u_n; \quad u_{2n+1} = 1 - u_n \quad \textit{for all } n \geq 0.$$

$$\mathbf{u}_{TM} = 0110100110010110100101100110100110010110011010 0\ldots$$

Theorem

Define a morphism φ_{TM} *on* $\{0,1\}$ *by* $\varphi_{TM}(0) = 01; \varphi_{TM}(1) = 10$. *Then* $\varphi_{TM}(\mathbf{u}_{TM}) = \mathbf{u}_{TM}$.

$$\underbrace{\mathbf{u}_{TM}}_{\mathbf{v}} = 0\underbrace{11}_{2}0\underbrace{1}_{1}0\underbrace{}_{0}0\underbrace{11}_{2}0\underbrace{}_{0}0\underbrace{1}_{1}0\underbrace{11}_{2}0\underbrace{1}_{1}0\underbrace{}_{0}0\ldots$$

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Rational powers

Definition

Let $e \in \mathbb{Q}$. A word z is an e-th power of a word u if z is a prefix of $u^\omega = uuuuu\ldots$ and $e = \frac{|z|}{|u|}$. We write $z = u^e$.

Example

$$\mathtt{abbabb} = (\mathtt{abb})^2$$
$$\mathtt{abbcabbcabbc} = (\mathtt{abbc})^3$$
$$\mathtt{abbabbab} = (\mathtt{abb})^{8/3} = (\mathtt{abbabb})^{4/3}$$
$$\mathtt{starosta} = (\mathtt{staro})^{8/5}$$

Critical exponent

Definition

Let $\mathbf{u}$ *be a sequence. The* critical exponent *of* $\mathbf{u}$
$E(\mathbf{u}) = \sup\{e \in \mathbb{Q} : u^e$ *is a non-empty factor of* $\mathbf{u}\}$.

Example

Not only $\mathbf{u}_{TM}$ does not contain cubes, but $\mathbf{u}_{TM}$ does not contain overlaps: *awawa*, where w is a factor and a is a letter. Therefore $E(\mathbf{u}_{TM}) = 2$.

Program

Introduction & Motivation

Dejean's theorem (conjecture), 1972 – 2011:
(proven by Currie, Rampersad; Rao)
the least critical exponent of sequences over an alphabet of size d:

- 2 for $d = 2$;
- $7/4$ for $d = 3$;
- $7/5$ for $d = 4$;
- $\frac{d}{d-1}$ for $d \geq 5$.

Conjecture for balanced sequences

- **Rampersad, Shallit, Vandomme**, 2019:
  the least critical exponent of balanced sequences over an alphabet of size d equals $\frac{d-2}{d-3}$ for $d \geq 5$
- proven for $5 \leq d \leq 8$
- **Dolce, Dvořáková, Pelantová**, 2021:
- proven for $9 \leq d \leq 10$
- disproven: new bound $\frac{d-1}{d-2}$ for $11 \leq d \leq 12$

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- new conjecture: $\frac{d-1}{d-2}$ for $d \geq 11$

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$$\frac{d}{d-1} < \frac{d-1}{d-2} < \frac{d-2}{d-3}$$

One World Combinatorics on Words Seminar

- Arseny Shur: *Abelian repetition threshold revisited*
- Pascal Ochem: *Avoiding large squares in trees and planar graphs*
- Szymon Stankiewicz: *Square-free reducts of words*
- Robert Mercas: *(Trying to do a) Counting of distinct repetitions in words*
- Eric Rowland: *Avoiding fractional powers on an infinite alphabet*
- Matthieu Rosenfeld: *A simple proof technique to study avoidability of repetitions*
- Jarkko Peltomäki: *Avoiding abelian powers cyclically*
- Lucas Mol: *Extremal square-free words and variations*

Definitions CoW

- *sequence* $\mathbf{u} = u_0 u_1 u_2 \dots$ over $\mathcal{A}$
- *bispecial factor* of $\mathbf{u}$
- *return word* v to a factor u of $\mathbf{u}$

Example

$\mathbf{u}_F = \mathtt{ab}\underline{\mathtt{aaba}}\mathtt{b}\underline{\mathtt{aaba}}\mathtt{ababaa} \dots$, $\mathcal{A} = \{\mathtt{a}, \mathtt{b}\}$
$\mathbf{u}_F = \varphi(\mathbf{u}_F)$, *where* $\varphi : \mathtt{a} \to \mathtt{ab}, \quad \mathtt{b} \to \mathtt{a}$
aba *is a bispecial factor since* aaba, baba *and* abab, abaa *are factors of* $\mathbf{u}_F$
$v = \mathtt{aabab}$ *is a return word to* $u = \mathtt{aaba}$

(Asymptotic) critical exponent

- *critical exponent* of $\mathbf{u}$
  $E(\mathbf{u}) = \sup\{e \in \mathbb{Q} : u^e \text{ is a non-empty factor of } \mathbf{u}\}$
- *asymptotic critical exponent* of $\mathbf{u}$
  $E^*(\mathbf{u}) = \lim_{n\to\infty} \sup\{e \in \mathbb{Q} : u^e \text{ is a factor of } \mathbf{u} \text{ and } |u| \geq n\}$

Evidently, $E^*(\mathbf{u}) \leq E(\mathbf{u})$.

Proposition (Dvořáková, Pelantová)

Let $\mathbf{u}$ be a uniformly recurrent aperiodic sequence. Let w_n be the n-th bispecial of $\mathbf{u}$ and v_n a shortest return word to w_n. Then
$E(\mathbf{u}) = 1 + \sup\{\frac{|w_n|}{|v_n|} : n \in \mathbb{N}\}$ *and* $E^*(\mathbf{u}) = 1 + \limsup_{n\to\infty} \frac{|w_n|}{|v_n|}$.

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Example

$|w_n| = F_{n+2} + F_{n+1} - 2$ and $|v_n| = F_{n+1}$ with $F_0 = 0, F_1 = 1$
$E(\mathbf{u}_F) = 2 + \tau = 2 + \frac{1+\sqrt{5}}{2} = E^*(\mathbf{u}_F)$ – minimal for Sturmian

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Balanced sequences

Definition

$\mathbf{u}$ *over* $\mathcal{A}$ *balanced if* $|u| = |v| \Rightarrow |u|_a - |v|_a \leq 1$ *for all* $a \in \mathcal{A}$

Theorem (Graham 1973, Hubert 2000)

$\mathbf{v}$ *recurrent aperiodic is balanced iff* $\mathbf{v}$ *obtained from a Sturmian sequence* $\mathbf{u}$ *over* $\{\mathtt{a}, \mathtt{b}\}$ *by replacing*

- $\mathtt{a}$ *with a constant gap sequence* $\mathbf{y}$ *over* $\mathcal{A}$,
- $\mathtt{b}$ *with a constant gap sequence* $\mathbf{y}'$ *over* $\mathcal{B}$,

where $\mathcal{A}$ *and* $\mathcal{B}$ *disjoint. We write* $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$.

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Example

$\mathbf{v} = \mathrm{colour}(\mathbf{u}_F, \mathbf{y}, \mathbf{y}')$, where $\mathbf{y} = (0102)^\omega$ and $\mathbf{y}' = (34)^\omega$

$$\begin{aligned} \mathbf{u}_F &= \mathtt{abaababaabaabab}\ldots \\ \mathbf{v} &= 03104230140230 4\ldots \end{aligned}$$

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Computation of asymptotic critical exponent

Program implemented by Daniela Opočenská
Recall $E^*(\mathbf{v}) = 1 + \limsup_{n\to\infty} \frac{|w_n|}{|v_n|}$

Program:
Input: slope θ quadratic irrational, $\mathrm{Per}(\mathbf{y})$, $\mathrm{Per}(\mathbf{y}')$
Output: $E^*(\mathbf{v})$, where $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$

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Program

Minimal critical exponent

d	θ	$\mathbf{y}$	$\mathbf{y}'$	$E(\mathbf{v})$	$E^*(\mathbf{v})$
3	$[0,1,\overline{2}]$	$(01)^\omega$	2^ω	$2+\frac{1}{\sqrt{2}}$	$2+\frac{1}{\sqrt{2}}$
4	$[0,\overline{1}]$	$(01)^\omega$	$(23)^\omega$	$1+\frac{1+\sqrt{5}}{4}$	$1+\frac{1+\sqrt{5}}{4}$
5	$[0,1,\overline{2}]$	$(0102)^\omega$	$(34)^\omega$	$\frac{3}{2}$	$\frac{3}{2}$
6	$[0,2,1,1,\overline{1,1,1,2}]$	0^ω	$(123415321435)^\omega$	$\frac{4}{3}$	$\frac{4}{3}$
7	$[0,1,3,\overline{1,2,1}]$	$(01)^\omega$	$(234526432546)^\omega$	$\frac{5}{4}$	$\frac{5}{4}$
8	$[0,3,1,\overline{2}]$	$(01)^\omega$	$(23452673254623752643257 6)^\omega$	$\frac{6}{5}=1.2$	$\frac{12+3\sqrt{2}}{14}\doteq 1.16$
9	$[0,2,3,\overline{2}]$	$(01)^\omega$	$(234567284365274863254768)^\omega$	$\frac{7}{6}\doteq 1.167$	$1+\frac{2\sqrt{2}-1}{14}\doteq 1.13$
10	$[0,4,2,\overline{3}]$	$(01)^\omega$	$(23456728496325476829436527 4869)^\omega$	$\frac{8}{7}\doteq 1.14$	$1+\frac{\sqrt{13}}{26}\doteq 1.139$
11	$[0,6,3,1,\overline{2}]$	$(01)^\omega$	$(234567892A436587294A638527496A832547698A)^\omega$	$\frac{10}{9}\doteq 1.11$	$\frac{32-\sqrt{2}}{28}\doteq 1.092$
12	$[0,1,3,\overline{2}]$	$(012345)^\omega$	$(6789AB)^\omega$	$\frac{11}{10}=1.1$	$\frac{8-\sqrt{2}}{6}\doteq 1.098$
$d\geq 14$ even	$[0,1,\lfloor d/4\rfloor,\overline{1}]$	$(12\ldots d/2)^\omega$	$(1'2'\ldots d/2')^\omega$	$\frac{d-1}{d-2}$	$1+\frac{2}{d\tau^{N-1}}$, where $\tau^{N+1}<d/2<\tau^{N+2}$

Table: Baranwal, Rampersad, Shallit, Vandomme, Dolce, Dvořáková, Opočenská, Pelantová: balanced sequences with the least critical exponent with alphabet size d.

$$\frac{d}{d-1}<\frac{d-1}{d-2}<\frac{d-2}{d-3}$$

Minimal asymptotic critical exponent

d	θ	$\mathrm{Per}(\mathbf{y})$	$\mathrm{Per}(\mathbf{y}')$	$E^*(\mathbf{v})$
3	$[0,1,\overline{2}]$	1	2	$2+\frac{1}{\sqrt{2}}$
4	$[0,\overline{1}]$	2	2	$1+\frac{\tau}{2}$
5	$[0,1,\overline{2}]$	2	4	$\frac{3}{2}$
6	$[0,\overline{1}]$	4	4	$\frac{5}{4}<\frac{4}{3}=\min E$
7	$[0,5,1,\overline{1,1,1,5,2}]$	1	32	$1.14095<\frac{5}{4}=\min E$
8	$[0,\overline{1}]$	8	8	$1+\frac{1}{8\tau^2}<\frac{6}{5}=\min E$
9	$[0,1,\overline{4}]$	8	16	$1.03299<\frac{7}{6}=\min E$

Table: Balanced sequences with the least asymptotic critical exponent over alphabets of size d.

Summary and open problems

Minimal critical exponent of balanced sequences

- program for computation of critical exponent
- conjecture by Rampersad, Shallit, Vandomme refuted
- new bound $\frac{d-1}{d-2}$ proven for $d \geq 12, d$ even and also for $d = 11, 13, 15, 17, 19, 21, 23$
- working on its proof for $d \geq 11$

Minimal asymptotic critical exponent of balanced sequences

- program for computation of asymptotic critical exponent
- open problem: minimality

References

- N. Rampersad, J. Shallit, and É. Vandomme, *Critical exponents of infinite balanced words*, Theoretical Computer Science, 777 (2019), 454–463.
- F. Dolce, Ľ. Dvořáková, and E. Pelantová, *On balanced sequences and their asymptotic critical exponent*, Proceedings LATA 2021, LNCS, 12638 (2021), 293–304.
- F. Dolce, Ľ. Dvořáková, and E. Pelantová, *Computation of critical exponent in balanced sequences*, Proceedings WORDS 2021, LNCS, 12847 (2021), 78–90.
- F. Dolce, Ľ. Dvořáková, and E. Pelantová, *On balanced sequences and their critical exponent*, arXiv:2108.07503 (2021)
- Ľ. Dvořáková, D. Opočenská, E. Pelantová, A. Shur, *Minimal critical exponent of balanced sequences*, accepted in TCS

Thank you for attention