Poisson-Lie T-plurality and invariant D-branes

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Outline

1. Gluing matrix

2. Poisson–Lie T–plurality

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Gluing matrix

Gluing matrix (operator) is one possible formulation of boundary conditions for 2–dimensional sigma models.

Let us consider a sigma model on $\Sigma = \mathbb{R} \times \langle 0, \pi \rangle$ with oppositely charged endpoints given by the action

$$\begin{aligned} S_{\mathcal{F}}[\phi] &= \int_{\Sigma} d^2x \, \partial_- \phi \cdot \mathcal{F}(\phi) \cdot \partial_+ \phi^t + \\ &+ \int_{\sigma=0} A \cdot \frac{\partial \phi^t}{\partial \tau} \mathrm{d}\tau - \int_{\sigma=\pi} A \cdot \frac{\partial \phi^t}{\partial \tau} \mathrm{d}\tau \end{aligned} \tag{1}$$

together with the boundary conditions of the form

$$\partial_- \phi|_{\sigma=0,\pi} = \mathcal{R} \, \partial_+ \phi|_{\sigma=0,\pi} , \qquad \sigma \equiv x_+ - x_- . \tag{2}$$

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together with the boundary conditions of the form

$$\partial_- \phi|_{\sigma=0,\pi} = \mathcal{R} \, \partial_+ \phi|_{\sigma=0,\pi} , \qquad \sigma \equiv x_+ - x_- . \quad (2)$$

The gluing operator $\mathcal{R}$ in principle encodes the configuration of D–branes through the corresponding projector $\mathcal{N}$ onto the tangent space of the branes. By inspection of Eq. (2) one finds that

$$\partial_\tau \phi|_{\sigma=0,\pi} \in \mathrm{Ran}\,(\mathcal{R} + id), \qquad \tau \equiv x_+ + x_-.$$

That means that the projector $\mathcal{N} = \mathcal{N}^2$ has a prescribed range

$$\mathcal{N} \circ (\mathcal{R} + id) = (\mathcal{R} + id),\ \mathrm{Ran}\,\mathcal{N} = \mathrm{Ran}\,(\mathcal{R} + id),$$

but its kernel may be not necessarily unique. In particular, when the target space does not possess a positive definite metric we may not be allowed to require that $\mathcal{N}$ is a symmetric operator with respect to the metric.

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The $U(1)$ potential A

In general, the potential A need not to be a smooth function on the whole target space. Instead, it may be smooth only along D–branes. Therefore, its associated field strength 2–form $\Delta = \mathrm{d}A$ is necessarily closed only along D–branes, i.e. satisfies

$$N_\kappa{}^\nu N_\lambda{}^\rho N_\mu{}^\sigma \partial_{[\nu} \Delta_{\rho\sigma]} = 0. \tag{3}$$

Essential requirements on $\mathcal{R}$: Conformality

If we want to have string solutions of the sigma model which are defined by the condition of vanishing stress tensor, the stress tensor must vanish also on the boundary as a consequence of boundary conditions. This requirement leads to a condition relating the gluing operator and the metric on the target space[1]

$$R \cdot (\mathcal{F} + \mathcal{F}^t) \cdot R^t = (\mathcal{F} + \mathcal{F}^t). \tag{4}$$

[1] non–script letters = matrices of operators acting from the right

Essential requirements on $\mathcal{R}$: D–branes are submanifolds

We expect our D–branes to be submanifolds of the target space. Therefore, by Frobenius theorem, $\mathrm{Ran}\,\mathcal{N}$ must be an integrable distribution

$$[\mathrm{Ran}\,\mathcal{N}, \mathrm{Ran}\,\mathcal{N}] \subset \mathrm{Ran}\,\mathcal{N}.$$

In terms of the matrix of the projector $\mathcal{N}$ we can equivalently write

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Essential requirements on $\mathcal{R}$: Boundary equation of motion

When computing the variations of the action we obtain also boundary terms. When we require them to vanish whenever the boundary conditions are satisfied we arrive at the restriction

$$N \cdot ((\mathcal{F} + \Delta) - (\mathcal{F} + \Delta)^t \cdot R^t) = 0 \tag{6}$$

where we recall that $\Delta = \mathrm{d}A$ satisfies

$$N_\kappa{}^\nu N_\lambda{}^\rho N_\mu{}^\sigma \partial_{[\nu} \Delta_{\rho\sigma]} = 0. \tag{7}$$

Essential requirements on $\mathcal{R}$

The requirements postulated here are the minimal ones. In particular, when transformation of gluing matrices under ordinary T–duality was considered in C. Albertsson, U. Lindström and M. Zabzine, Nucl. Phys. B678(2004)295, [hep–th/0202069] and C. Albertsson, U. Lindström and M. Zabzine, J. High. Energy Phys. 12(2004)056, [hep–th/0410217] another condition was imposed.
This condition in our notation reads

$$\mathcal{N}\mathcal{R} = \mathcal{R}\mathcal{N}.$$

Although this condition has some useful consequences, we prefer not to impose it. The reason is that it is not preserved under Poisson–Lie T–duality or T–plurality, as we have shown in C. Albertsson, L. Hlavatý and L. Šnobl, J. Math. Phys. 49(20080)032301, [arXiv:0706.0820].

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Equivalently, but without $\mathcal{N}$

If we assume that the electric field strength Δ is defined by (6) (its non–uniqueness doesn't affect the sigma model dynamics) we can find an equivalent version of (7) in the form[2]

$$0 = \frac{\partial}{\partial y^\vartheta} \left(\mathcal{F}^t \cdot R^t - \mathcal{F}\right)_{\rho[\nu} (R+\mathbf{1})_\mu{}^\rho (R+\mathbf{1})_{\lambda]}{}^\vartheta - \left(\mathcal{F}^t \cdot R^t - \mathcal{F}\right)_{\rho[\nu} \frac{\partial}{\partial y^\vartheta} (R+\mathbf{1})_\mu{}^\rho (R+\mathbf{1})_{\lambda]}{}^\vartheta . \quad (8)$$

This together with (5) written as Frobenius integrability condition

$$[\mathrm{Ran}\,(\mathcal{R}+id), \mathrm{Ran}\,(\mathcal{R}+id)] \subset \mathrm{Ran}\,(\mathcal{R}+id) \quad (9)$$

allows us to study Eq. (4-7) without knowing $\mathcal{N}$ explicitly.

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Poisson–Lie T–plurality

C.Klimčík and P.Ševera, PLB351(1995)455,[hep-th/9502122],
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The target space of the sigma model becomes an isotropic subgroup G of a Drinfel'd double $(G|\tilde{G})$.

Drinfel'd double $(G|\tilde{G})$ is a Lie group D whose Lie algebra $\mathfrak{d}$ is equipped with a symmetric ad–invariant nondegenerate bilinear form $\langle\,.\,,.\,\rangle$ and admits a decomposition $\mathfrak{d} = \mathfrak{g} \dot{+} \tilde{\mathfrak{g}}$ into a pair of subalgebras maximally isotropic with respect to the form $\langle\,.\,,.\,\rangle$.

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Manin triple

Manin triple $(\mathfrak{d}, \mathfrak{g}, \tilde{\mathfrak{g}})$ is any such decomposition of the Lie algebra of a given Drinfel'd double. Any Drinfel'd double defines at least two nonisomorphic Manin triples, namely $(\mathfrak{d}, \mathfrak{g}, \tilde{\mathfrak{g}})$ and $(\mathfrak{d}, \tilde{\mathfrak{g}}, \mathfrak{g})$ provided $\mathfrak{g} \not\cong \tilde{\mathfrak{g}}$.

When working with the algebras $\mathfrak{g}, \tilde{\mathfrak{g}}$ we shall assume that their structure is expressed in a pair of bases $T, \tilde{T}$ mutually dual with respect to $\langle \, . \, , . \, \rangle$, i.e. their elements T_a and $\tilde{T}^a$ satisfy

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For notational simplicity, we write the bases $T, \tilde{T}$ as column vectors of their elements.

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The action for Poisson–Lie T–dual sigma models

$\mathcal{F}$ in (1) is written as $\mathcal{F} = e(g) \cdot F(g) \cdot e(g)^t$ where $e(g)$ are components of the right–invariant Maurer–Cartan form $\mathrm{d}gg^{-1}$ and

$$F(g) = (E_0^{-1} + \Pi)^{-1}, \; \Pi = b(g) \cdot a(g)^{-1}. \qquad (10)$$

E_0 is an invertible constant matrix such that its symmetric part is also invertible. $a(g), b(g)$ are submatrices of the adjoint representation of the group G on $\mathfrak{d}$,

$$\begin{aligned} gTg^{-1} &\equiv Ad(g) \triangleright T = a^{-1}(g) \cdot T, \qquad (11) \\ g\tilde{T}g^{-1} &\equiv Ad(g) \triangleright \tilde{T} = b^t(g) \cdot T + a^t(g) \cdot \tilde{T}. \end{aligned}$$

Lie bialgebra

Manin triples are in the following $1-1$ correspondence with Lie bialgebras. The Lie bracket on $\mathfrak{g}$ is retained and the Lie cobracket on $\mathfrak{g}$ is defined using the Lie bracket on $\tilde{\mathfrak{g}}$ in the following way

$$\delta(T_a) = \widetilde{f}_a^{bc} T_b \otimes T_c.$$

The mixed Jacobi identities $(T_a, T_b, \tilde{T}^c), (T_a, \tilde{T}^b, \tilde{T}^c)$ are equivalent to the compatibility condition for the Lie bracket and Lie cobracket, namely that δ is a 1–cocycle.

Poisson–Lie group

It is well–known that Lie bialgebras give rise to Poisson–Lie groups, i.e. to Lie groups equipped with a Poisson bivector $\pi \in \Gamma(\wedge^2 TG)$ which is multiplicative with respect to the group multiplication

$$\pi(gh) = L_{g*}\pi(h) + R_{h*}\pi(g).$$

The matrix Π now becomes the matrix of the Poisson bivector π on the Poisson–Lie group G when expressed in terms of right–invariant basis vector fields.

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Poisson–Lie T–plurality II

Equations of motion of Poisson–Lie sigma models (10) allow the following lift $l : \Sigma \to D$ of the solution $g : \Sigma \to G$ into the Drinfel'd double.

The bulk equations of motion of the dualizable σ–models can be written as Bianchi identities for the $\tilde{\mathfrak{g}}$–valued fields

$$(\rho_+)_a = -\rho_+(g)^b F(g)_{cb}(a(g)^{-1})_a^c, \; (\rho_-)_a = \rho_-(g)^b F(g)_{bc}(a(g)^{-1})_a^c.$$

These fields can be (locally) integrated in terms of $\tilde{h} : \Sigma \to \tilde{G}$,

$$\begin{aligned} \tilde{\rho}_+(\tilde{h})_a = (\partial_+ \tilde{h}\,\tilde{h}^{-1})_a &= -\rho_+(g)^b F(g)_{cb}(a(g)^{-1})_a^c, \\ \tilde{\rho}_+(\tilde{h})_a = (\partial_- \tilde{h}\,\tilde{h}^{-1})_a &= \rho_-(g)^b F(g)_{bc}(a(g)^{-1})_a^c. \end{aligned} \quad (12)$$

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Poisson–Lie T–plurality: the lift to D

When one defines $l = g\tilde{h} : \Sigma \to D$, it satisfies

$$\langle \partial_{\pm} l l^{-1}, \mathcal{E}^{\pm} \rangle = 0, \tag{13}$$

where $\mathcal{E}^{\pm}$ are two orthogonal subspaces in $\mathfrak{d}$, spanned by $T + E_0 \cdot \tilde{T}$, $T - E_0^t \cdot \tilde{T}$, respectively. (13) is invariant with respect to the choice of the decomposition $\mathfrak{g} \dot{+} \tilde{\mathfrak{g}}$ of $\mathfrak{d}$.

Vice versa, starting from a solution $l : \Sigma \to D$ of Eq. (13), one can locally decompose it into $l = g\tilde{h}$, where $g : \Sigma \to G$, $\tilde{h} : \Sigma \to \tilde{G}$ (the existence of such local decomposition is a property of Drinfel'd doubles). One can show that g satisfies the equations of motion of the Poison–Lie sigma model (10).

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Poisson–Lie T–plurality: dual sigma models

Decomposing $\mathfrak{d}$ into a different Manin triple $\hat{\mathfrak{g}} \dot{+} \bar{\mathfrak{g}}$, $l = \hat{g}\bar{h}$ we get solutions $\hat{g}$ of sigma models defined on $\hat{G}$ with the tensor $\hat{F}(\hat{g}) = (\hat{E}_0^{-1} + \hat{\Pi})^{-1}$. This procedure was called Poisson–Lie T–plurality by R. von Unge.

Explicitly, assuming that the pairs of dual bases of $\mathfrak{g}, \tilde{\mathfrak{g}}$ and $\hat{\mathfrak{g}}, \bar{\mathfrak{g}}$ are related by the linear transformation

$$\begin{pmatrix} T \\ \tilde{T} \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} \widehat{T} \\ \bar{T} \end{pmatrix}, \qquad (14)$$

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the σ–model obtained by the plurality transformation is constructed by substitution

$$\widehat{F}(\hat{g}) = (\widehat{E}_0^{-1} + \widehat{\Pi}(\hat{g}))^{-1}, \quad \widehat{\Pi}(\hat{g}) = \widehat{b}(\hat{g}) \cdot \widehat{a}(\hat{g})^{-1} = -\widehat{\Pi}(\hat{g})^t, \tag{15}$$

$$\widehat{E}_0 = (p + E_0 \cdot r)^{-1} \cdot (q + E_0 \cdot s) = (s^t \cdot E_0 - q^t) \cdot (p^t - r^t \cdot E_0)^{-1} \tag{16}$$

into the action (1).

Transformation of D-branes under Poisson–Lie T–plurality

Now that we know how the bulk equations of motion transform between dual or plural Poisson–Lie sigma models, we would like to investigate under what restrictions can the boundary conditions expressed in terms of the gluing operator $\mathcal{R}$ be transformed and whether the transformation preserves the minimal set of consistency conditions on $\mathcal{R}$ formulated before.

Transformation of $\mathcal{R}$

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If the original boundary conditions are prescribed in the form

$$R = e(g) \cdot F^t(g) \cdot C \cdot F^{-1}(g) \cdot e^{-1}(g), \qquad (17)$$

where C is a constant matrix orthogonal w.r.t $E_0^{-1} + E_0^{-t}$ due to (4), then the transformed solutions $\hat{g}$ satisfy boundary conditions in a form similar to (17) (i.e. with proper replacements by $\hat{F}(\hat{g}), \hat{e}(\hat{g})$) with

$$\hat{C} = M_-^{-1} \cdot C \cdot M_+, \qquad (18)$$

where the constant matrices M_+, M_- are given in terms of E_0 and the transformation matrix between the pairs of dual bases of $\mathfrak{g}, \tilde{\mathfrak{g}}$ and $\hat{\mathfrak{g}}, \bar{\mathfrak{g}}$ only.

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M_+, M_- explicitly

Let $\hat{\mathfrak{g}} \dotplus \bar{\mathfrak{g}}$ be another decomposition of the Lie algebra $\mathfrak{d}$. The pairs of dual bases of $\mathfrak{g}, \tilde{\mathfrak{g}}$ and $\hat{\mathfrak{g}}, \bar{\mathfrak{g}}$ are related by the linear transformation

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Lifted D–branes

D–branes defined by the gluing matrix (17) can be lifted to the Drinfel'd double. The lifted boundary conditions take the form

$$\partial_\tau l\, l^{-1}|_{\sigma=0,\pi} \in V_{\mathcal{D}} = \mathrm{span}(A \cdot T + B \cdot \tilde{T}), \qquad (21)$$

$$\text{where} \qquad A = E_0^{-t} + C \cdot E_0^{-1}, \quad B = C - \mathbf{1}.$$

It can be checked explicitly that $V_{\mathcal{D}}$ defined in this way is invariant with respect to Poisson–Lie T–plurality transformations, i.e. $V_{\mathcal{D}}$ provides an invariant description of boundary conditions.

Lifted D–branes

D–branes defined by the gluing matrix (17) can be lifted to the Drinfel'd double. The lifted boundary conditions take the form

$$\partial_\tau l\, l^{-1}|_{\sigma=0,\pi} \in V_{\mathcal{D}} = \mathrm{span}(A \cdot T + B \cdot \tilde{T}), \tag{21}$$

$$\text{where} \qquad A = E_0^{-t} + C \cdot E_0^{-1}, \quad B = C - \mathbf{1}.$$

It can be checked explicitly that $V_{\mathcal{D}}$ defined in this way is invariant with respect to Poisson–Lie T–plurality transformations, i.e. $V_{\mathcal{D}}$ provides an invariant description of boundary conditions.

Interpretation of the requirements on gluing operator $\mathcal{R}$

- The conformal condition (4) means that $V_{\mathcal{D}}$ is maximally isotropic.
- The reformulated boundary equation of motion (8) is equivalent to $V_{\mathcal{D}}$ being a subalgebra (after some rather tedious calculation).

Therefore, the lifted D–branes are cosets $\mathfrak{D}/I$ where $Lie(\mathfrak{D}) = V_{\mathcal{D}}$ as was predicted by C. Klimčík, P. Ševera[3].

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The integrability (9) of D–branes in G follows by coset projection and needs not to be imposed as an additional constraint, i.e. it is a consequence of the assumed form of the gluing operator (17) and the conditions (4) and (8).

Conversely, every maximally isotropic subalgebra can be written in the form $V_{\mathcal{D}}$ for some matrix C, provided the regularity condition

$$V_{\mathcal{D}} \cap \mathcal{E}^{+} = 0$$

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Conclusions

We have shown that the set of physical constraints on the gluing operator $\mathcal{R}$ as postulated in Eq. (4-7) is preserved under Poisson Lie T–duality (plurality) transformation provided the transformed electromagnetic field strength Δ is found from (6). On the other hand, the more restrictive set of constraints postulated by C. Albertsson, U. Lindström and M. Zabzine[4] is not invariant under Poisson Lie T–plurality.

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We have also shown that our formulation is equivalent to the description originally discovered by C. Klimčík and P. Ševera. Both approaches can be considered complementary; their being more geometric; in our it is easier to write explicitly both the original and transformed boundary conditions.

Thank you for your attention