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On the Poisson-Lie *T*-plurality of boundary conditions

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Conditions for the gluing matrix defining consistent boundary conditions of two-dimensional nonlinear σ-models are analyzed and reformulated. Transformation properties of the right-invariant fields under the Poisson-Lie T-plurality are used to derive a formula for the transformation of the boundary conditions. Examples of transformation of D-branes in two and three dimensions are presented. We investigate obstacles arising in this procedure and propose possible solutions. © *2008 American Institute of Physics.* [DOI: 10.1063/1.2832622]

I. INTRODUCTION

T-duality of strings may be realized as a canonical transformation acting on the fields in a two-dimensional nonlinear σ-model. This model describes the worldsheet theory of a string propagating on some target manifold equipped with a background tensor field $\mathcal{F}_{\mu\nu}$ which is a convenient rearrangement of the metric and the Kalb-Ramond B-field. For open strings, the worldsheet has boundaries, by definition confined to D-branes; hence, the action is subject to boundary conditions. Imposing extra symmetries, e.g., conformal invariance, restricts these conditions. They determine the dynamics of the ends of the string and hence the embedding of D-branes in the target space. Applying duality transformations yields the dual boundary conditions and hence the geometry of D-branes in the dual target.

Traditional T-duality requires the presence of an isometry group leaving the σ-model invariant, a rather severe restriction. In the Poisson-Lie T-duality,[1] isometries are not necessary, provided that the two dual target spaces are both Poisson-Lie group manifolds (or at least Poisson-Lie groups act freely on them) whose Lie algebras constitute a *Drinfel'd double*. That is, they are maximally isotropic Lie subalgebras in the decomposition of a Lie bialgebra $\mathfrak{d}=\mathfrak{g}+\tilde{\mathfrak{g}}$, where $\tilde{\mathfrak{g}} \equiv \mathfrak{g}^*$. The background $\mathcal{F}_{\mu\nu}$ is related to the Poisson structure on the target manifold and satisfies the *Poisson-Lie condition*, a restriction that replaces the traditional isometry condition.

Recently, the transformation of worldsheet boundary conditions under the Poisson-Lie T-duality was derived in Ref. 2. The key formulas were transformations of left-invariant fields[1]

$$\widetilde{L}^t_{\pm}(\tilde{g}) = \widetilde{E}^{-t}(\tilde{g}) \cdot E_0^{-t} \cdot E^t(g) \cdot L^t_{\pm}(g), \tag{1}$$

[1]The dot denotes matrix multiplication, t denotes transposition, $E^{-t} \equiv (E^t)^{-1}$, where E is a general background field in the Lie algebra frame, and E_0 is a constant matrix.

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$$\tilde{L}^t_{=}(\tilde{g}) = -(\tilde{E}(\tilde{g}))^{-1}\cdot E_0^{-1}\cdot E(g)\cdot L^t_{=}(g), \tag{2}$$

obtained from the canonical transformations derived in Refs. 3 and 4. Here, g and $\tilde{g}$ are elements of the groups corresponding to $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$, respectively, and the subscripts $+\!\!\!+$ and $=$ refer to the worldsheet lightcone coordinates.

Poisson-Lie T-plurality[5] is a further generalization of T-duality, where the mutually dual target spaces do not necessarily belong to the same Lie algebra decomposition of the Drinfel'd double (i.e., they belong to different *Manin triples*).

In Refs. 6 and 7 we found classical solutions of σ-models in curved backgrounds by applying Poisson-Lie T-plurality transformations to flat σ-models. Unfortunately, we were not able to control the boundary conditions necessary for string solutions in the curved background or, more precisely, to identify the conditions for the flat solution that transform to suitable conditions in the curved background.

Our goal here is to derive a transformation of boundary conditions under the Poisson-Lie T-plurality that could enable us to control the boundary conditions in the transformed σ-model. Analogs of the formulas (1) and (2) for the Poisson-Lie T-plurality were derived in Ref. 8 so that we can easily write down the transformation of the boundary conditions. As the σ-models investigated in Refs. 6 and 7 and other papers of ours are formulated in terms of right-invariant fields $\partial_\pm g g^{-1}$, we shall use this formulation here.[2]

In Sec. II, we review the Poisson-Lie T-plurality and introduce the framework necessary for the subsequent analysis. In Sec. III, we list and discuss the set of boundary conditions required to define consistent σ-models, describing them in terms of a gluing matrix. In Sec. IV, we derive the T-plurality transformation of the gluing matrix and show that it does not automatically yield well-defined boundary conditions on the T-plural side. In Secs. V and VI, we analyze two explicit examples, one three dimensional and one two dimensional, demonstrating how different D-branes transform under the Poisson-Lie T-plurality. In the process, we discuss the conditions necessary to eliminate any interdependence of the gluing matrices on coordinates of the different target spaces involved. Finally, Sec. VII contains our conclusions.

II. ELEMENTS OF POISSON-LIE *T*-PLURALITY

The classical action of the σ-model under consideration is

$$S_{\mathcal{F}}[\phi] = \int_\Sigma d^2x\,\partial_-\phi^\mu \mathcal{F}_{\mu\nu}(\phi)\partial_+\phi^\nu, \tag{3}$$

where $\mathcal{F}$ is a tensor on a Lie group G and the functions $\phi^\mu:\Sigma\subset\mathbb{R}^2\to\mathbb{R}$, $\mu=1,2,\ldots,\dim G$ are obtained by the composition $\phi^\mu=y^\mu\circ g$ of a map $g:\Sigma\to G$ and components of a coordinate map y of a neighborhood U_g of an element $g(x_+,x_-)\in G$. For the purpose of this paper, we shall assume that the worldsheet Σ has the topology of a strip infinite in timelike direction, $\Sigma=\langle 0,\pi\rangle\times\mathbb{R}$.

On a Lie group G, the tensor $\mathcal{F}$ can be written as

$$\mathcal{F}_{\mu\nu} = e_\mu{}^a(g)F_{ab}(g)e_\nu{}^b(g), \tag{4}$$

where $e_\mu{}^a(g)$ are components of the right-invariant Maurer-Cartan forms dgg^{-1} and $F_{ab}(g)$ are matrix elements of bilinear nondegenerate form $F(g)$ on $\mathfrak{g}$, the Lie algebra of G. The action of the σ-model then reads

[2]Left-invariant fields were used in Ref. 8.

$$S_F[g] = \int_\Sigma d^2x \rho_-(g) \cdot F(g) \cdot \rho_+(g)^t, \tag{5}$$

where the right-invariant vector fields $\rho_\pm(g)$ are given by[3]

$$\rho_\pm(g)^a \equiv (\partial_\pm g g^{-1})^a = \partial_\pm \phi^\mu e_\mu{}^a(g), \quad (\partial_\pm g g^{-1}) = \rho_\pm(g) \cdot T, \tag{6}$$

and T_a are basis elements of $\mathfrak{g}$. [Note that $\rho_\pm(g)$ is written in a condensed notation; in full detail, it would read $\rho_\pm(g(x_+,x_-),\partial_\pm g(x_+,x_-))$ since it is a map $\Sigma \to \mathfrak{g}$.]

The σ-models that are transformable under the Poisson-Lie T-duality can be formulated (see Refs. 1 and 9) on a Drinfel'd double $D \equiv (G|\tilde{G})$, a Lie group whose Lie algebra $\mathfrak{d}$ admits a decomposition $\mathfrak{d} = \mathfrak{g} \dotplus \tilde{\mathfrak{g}}$ into a pair of subalgebras maximally isotropic with respect to a symmetric ad-invariant nondegenerate bilinear form $\langle .,. \rangle$. The matrices $F_{ab}(g)$ for the dualizable σ-models are of the form[1]

$$F(g) = (E_0^{-1} + \Pi(g))^{-1}, \quad \Pi(g) = b(g) \cdot a(g)^{-1} = -\Pi(g)^t, \tag{7}$$

where E_0 is a constant matrix, Π defines the Poisson structure on the group G, and $a(g), b(g)$ are submatrices of the adjoint representation of G on $\mathfrak{d}$. They satisfy

$$gTg^{-1} \equiv \mathrm{Ad}(g) \triangleright T = a^{-1}(g) \cdot T, \quad g\tilde{T}g^{-1} \equiv \mathrm{Ad}(g) \triangleright \tilde{T} = b^t(g) \cdot T + a^t(g) \cdot \tilde{T}, \tag{8}$$

where $\tilde{T}^a$ are elements of dual basis in the dual algebra $\tilde{\mathfrak{g}}$, i.e., $\langle T_a, \tilde{T}^b \rangle = \delta_a^b$. The matrix $a(g)$ relates the left- and right-invariant fields on G via

$$(g^{-1}\partial_\pm g) = L_\pm(g) \cdot T, \quad L_\pm(g) = \rho_\pm(g) \cdot a(g). \tag{9}$$

The equations of motion of the dualizable σ-models can be written as Bianchi identities for the right-invariant fields $\tilde{\rho}_\pm(\tilde{h})$ on the dual algebra $\tilde{\mathfrak{g}}$ satisfying[9]

$$\tilde{\rho}_+(\tilde{h}) \cdot \tilde{T} \equiv (\partial_+ \tilde{h}\tilde{h}^{-1}) = -\rho_+(g) \cdot F(g)^t \cdot a^{-t}(g) \cdot \tilde{T}, \tag{10}$$

$$\tilde{\rho}_-(\tilde{h}) \cdot \tilde{T} \equiv (\partial_- \tilde{h}\tilde{h}^{-1}) = +\rho_-(g) \cdot F(g) \cdot a^{-t}(g) \cdot \tilde{T}. \tag{11}$$

This is a consequence of the fact that the equations of motion of the dualizable σ-model can be written as the following equations on the Drinfel'd double:[1]

$$\langle \partial_\pm l l^{-1}, \mathcal{E}^\pm \rangle = 0, \tag{12}$$

where $l = g\tilde{h}$ and $\mathcal{E}^\pm$ are two orthogonal subspaces in $\mathfrak{d}$. On the other hand, the solution $g(x_+,x_-)$ of the equations of motion of the action (5) gives us a flat connection (10) and (11), which is therefore locally pure gauge, and the gauge potential $\tilde{h}(x_+,x_-)$ is determined up to right multiplication by a constant element $\tilde{h}_0$. Therefore, we find $l(x_+,x_-) = g(x_+,x_-) \cdot \tilde{h}(x_+,x_-)$, the so-called lift of the solution $g(x_+,x_-)$ to the Drinfel'd double, determined up to the constant shift

$$l \to l\tilde{h}_0, \quad \tilde{h}_0 \in \tilde{G}. \tag{13}$$

In general, as was realized already in Ref. 1 and then further developed in Ref. 5, there are several decompositions (Manin triples) of a Drinfel'd double. Let $\hat{\mathfrak{g}} \dotplus \bar{\mathfrak{g}}$ be another decomposition of the Lie algebra $\mathfrak{d}$. The pairs of dual bases of $\mathfrak{g}, \tilde{\mathfrak{g}}$ and $\hat{\mathfrak{g}}, \bar{\mathfrak{g}}$ are related by the linear transformation

[3]Note that while matrix multiplication is denoted by dot, for group multiplication we use concatenation.

$$\begin{pmatrix} T \\ \tilde{T} \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} \hat{T} \\ \bar{T} \end{pmatrix}, \tag{14}$$

where the duality of both bases requires

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix}^{-1} = \begin{pmatrix} s^t & q^t \\ r^t & p^t \end{pmatrix}, \tag{15}$$

i.e.,

$$\begin{aligned} p \cdot s^t + q \cdot r^t &= \mathbf{1}, \\ p \cdot q^t + q \cdot p^t &= 0, \\ r \cdot s^t + s \cdot r^t &= 0. \end{aligned} \tag{16}$$

The σ-model obtained by the plurality transformation is then defined analogously to the original one, namely, by substituting

$$\hat{F}(\hat{g}) = (\hat{E}_0^{-1} + \hat{\Pi}(\hat{g}))^{-1}, \quad \hat{\Pi}(\hat{g}) = \hat{b}(\hat{g}) \cdot \hat{a}(\hat{g})^{-1} = - \hat{\Pi}(\hat{g})^t, \tag{17}$$

$$\hat{E}_0 = (p + E_0 \cdot r)^{-1} \cdot (q + E_0 \cdot s) = (s^t \cdot E_0 - q^t) \cdot (p^t - r^t \cdot E_0)^{-1} \tag{18}$$

into (4) and (5). Solutions of the two σ-models are related by two possible decompositions of $l \in D$, namely,

$$l = g\tilde{h} = \hat{g}\bar{h}. \tag{19}$$

The transformed solution $\hat{g}$ is determined by the original solution $g(x_+, x_-)$ up to a choice of constant shift (13).

III. BOUNDARY CONDITIONS AND *D*-BRANES

The properties of D-branes in the groups G and $\hat{G}$ can be derived from the so-called gluing operators $\mathcal{R}$ and $\hat{\mathcal{R}}$, respectively; the number of their -1 eigenvalues determines the number of Dirichlet directions and hence the dimension of the D-branes. Moreover, the explicit form of the operator in principle yields the embedding of a brane in the target space.

We impose the boundary conditions for open strings in the form of the gluing operator $\mathcal{R}$ relating the left and right derivatives of field $g:\Sigma \to G$ on the boundary of Σ,

$$\partial_- g|_{\sigma=0,\pi} = \mathcal{R} \partial_+ g|_{\sigma=0,\pi}, \quad \sigma \equiv x_+ - x_-. \tag{20}$$

As we have to work with several choices of coordinates, we denote the matrices corresponding to the operator $\mathcal{R}$ in the bases of coordinate derivatives as R_ϕ, R_λ, etc., e.g.,

$$\partial_- \phi|_{\sigma=0,\pi} = \partial_+ \phi \cdot R_\phi|_{\sigma=0,\pi} \tag{21}$$

or

$$\partial_- \lambda|_{\sigma=0,\pi} = \partial_+ \lambda \cdot R_\lambda|_{\sigma=0,\pi}, \tag{22}$$

where $\partial_- \phi, \partial_- \lambda$ are row vectors of the derivatives of the respective coordinates (therefore, matrices of operators in our notation may differ by a transposition from expressions in other papers). Nevertheless, we suppress the indices ϕ, λ in expressions valid in any choice of coordinates, R having the obvious meaning of the matrix of the gluing operator, the tensor $\mathcal{F}$ is assumed to be expressed in the same coordinates, etc.

We define the Dirichlet projector $\mathcal{Q}$ that projects vectors onto the space normal to the D-brane, which is identified with the eigenspace of $\mathcal{R}$ with the eigenvalue -1, and the Neumann projector $\mathcal{N}$, which projects onto the tangent space of the brane. The corresponding matrices Q, N are given by the axioms

$$Q^2 = Q, \quad Q \cdot R = -Q, \quad N = \mathbf{1} - Q. \tag{23}$$

In so-called adapted coordinates λ^α (where $\alpha=1,\ldots,\dim G$), the gluing matrix can be written as[10]

$$R_\lambda = \begin{pmatrix} R_m{}^n & 0 \\ 0 & -\delta_i^j \end{pmatrix}, \quad m,n = 1,\ldots,p+1, \quad i,j, = p+2,\ldots,\dim G. \tag{24}$$

If the B-field of the model vanishes, one can choose $R_m{}^n = \delta_m{}^n$. In such coordinates, the terminology becomes clearer as λ^i become coordinates in the Dirichlet directions,

$$\partial_\tau \lambda^i = \tfrac{1}{2}(\partial_+ + \partial_-)\lambda^i = 0,$$

whereas λ^m are Neumann directions. This is a traditional misnomer; it is actually a generalization of the Neumann boundary conditions

$$\partial_\sigma \lambda^m = \tfrac{1}{2}(\partial_+ - \partial_-)\lambda^m = 0$$

to the cases with nonvanishing B-field (a better notation might be free boundary conditions, but we shall stick to the traditional "Neumann").

To obtain the corresponding boundary conditions written in terms of right-invariant fields $\rho_\pm(g)$, we must first express the gluing operator in the group coordinates y as

$$R_\phi = T(y) \cdot R_\lambda \cdot T(y)^{-1},$$

where

$$T(y)_\mu{}^\alpha = \frac{\partial \lambda^\alpha}{\partial y^\mu}(y),$$

and then transform it into the basis of the Lie algebra of right-invariant fields,

$$R_\rho = e^{-1}(g) \cdot R_\phi \cdot e(g) = e^{-1}(g) \cdot T(y) \cdot R_\lambda \cdot T(y)^{-1} \cdot e(g), \tag{25}$$

where $e(g)$ are the right-invariant vielbeins on G introduced in Eq. (6). The boundary conditions may then be expressed in terms of the right-invariant fields as

$$\rho_-(g)\big|_{\sigma=0,\pi} = \rho_+(g) \cdot R_\rho\big|_{\sigma=0,\pi}. \tag{26}$$

Of course, not every operator-valued function on the target space, in our case the group G, can be interpreted as a gluing operator, giving consistent boundary conditions for the σ-model in question. There are several restrictions on $\mathcal{R}$ derived, e.g., in Ref. 10. We shall briefly recall how these conditions arise and rewrite them in a slightly more compact but equivalent form.

First, in order that the adapted coordinates exist in a particular point, we must impose

$$R \cdot Q = Q \cdot R. \tag{27}$$

This is essentially a part of the definition of Q; otherwise, Q is not fully determined because to define a projector we need to specify its image and its kernel. Equation (23) defines the image of Q to be an eigenspace of R, while Eq. (27) implies that the kernel is the sum of all the remaining (generalized) eigenspaces of R. On the other hand, condition (27) is a restriction on R since it tells us that the geometrical[4] and algebraic[5] multiplicities of the eigenvalue -1 are equal. If this condition does not hold, one cannot find adapted coordinates (24), and the boundary conditions cannot be split into Dirichlet and (generalized) Neumann directions.

The distribution defined by the image of the Neumann projector must be integrable in order to be a tangent space to a submanifold, i.e., the brane. We find using the Frobenius theorem on integrability of distributions that the distribution must be in involution. When expressed in terms of the matrix N of the Neumann projector, this condition reads in any coordinates,

$$N_\kappa{}^\mu N_\lambda{}^\nu \partial_{[\mu} N_{\nu]}{}^\rho = 0. \tag{28}$$

In an arbitrary, noncoordinate frame, e.g., when expressed in terms of the right-invariant fields, the condition (28) appears more complicated. It may in general be expressed using covariant derivatives but for simplicity we shall use only the coordinate expression (28).

Since our σ-models are studied with applications to string theory in mind, they are often viewed as gauge fixed Polyakov actions. This imposes a further constraint on the solutions, in the form of a vanishing stress tensor

$$\mathcal{T}_{++} = \mathcal{T}_{--} = 0$$

(the trace $\mathcal{T}_{+-}$ vanishes automatically). Enforcing this condition not only in the bulk but also on the boundary leads to the consistency condition that the gluing operator preserves the metric on the target space; in other words, it is orthogonal with respect to the metric. If this condition were not satisfied, the σ-model would not allow generic string solutions. Explicitly, we have

$$R \cdot \mathcal{G} \cdot R^t = \mathcal{G}, \tag{29}$$

where the metric is written as $\mathcal{G}=(\mathcal{F}+\mathcal{F}^t)/2$. Equivalently, in the Lie algebra frame $\{T_a\}$, we express the metric as $(F+F^t)/2$ and consequently we have

$$R_\rho \cdot (F + F^t) \cdot R^t_\rho = (F + F^t). \tag{30}$$

We moreover require that what we identified as Dirichlet and Neumann directions are indeed orthogonal with respect to the metric on the target space,

$$N \cdot \mathcal{G} \cdot Q^t = 0. \tag{31}$$

When the metric on the target space is positive (or negative) definite, this is an automatic consequence of (29). In the pseudo-Riemannian signature, it is an additional constraint weeding out pathological configurations.

Finally, a crucial condition follows from the field variation of the action. Since the boundary conditions should be such that the variation of the action vanishes not only in the bulk but also on the boundary (that is why we impose the boundary conditions in the first place), we find by inspection of the boundary term arising in the variation that under the assumption of locality[6] we must impose

$$\delta\phi \cdot N_\phi \cdot \; (\mathcal{F} \cdot \partial_+ \phi^t - \mathcal{F}^t \cdot \partial_- \phi^t)\big|_{\sigma=0,\pi} = 0,$$

which after the use of Eq. (21) becomes

$$\delta\phi \cdot N_\phi \cdot (\mathcal{F} - \mathcal{F}^t \cdot R^t_\phi) \cdot \; \partial_+ \phi^t\big|_{\sigma=0,\pi} = 0. \tag{32}$$

Because $\delta\phi = \delta\phi \cdot N_\phi$ (i.e., $\delta\phi$ is tangent to the brane) and since $\partial_+\phi^t$ are not further restricted, we find

[4] i.e., the dimension of the eigenspace

[5] i.e., the multiplicity of the root of the characteristic polynomial

[6] That is, the integrand itself, not only the integral $\int_{\partial\Sigma}(\cdots)$, vanishes.

$$N \cdot (\mathcal{F} - \mathcal{F}^t \cdot R^t) = 0, \tag{33}$$

which, using Eqs. (27) and (31) as well as the following consequences of the definition of the projectors (23):

$$N \cdot (\mathbf{1} + R) = \mathbf{1} + R, \quad N \cdot (\mathbf{1} - R) = \mathbf{1} - R - 2Q, \tag{34}$$

can be rewritten in an equivalent form originally deduced and used in Ref. 10,

$$N \cdot \mathcal{F} \cdot N^t - N \cdot \mathcal{F}^t \cdot N^t \cdot R^t = 0. \tag{35}$$

In fact, once we impose condition (27), the pair of conditions (31) and (35) is equivalent to condition (33). For example, assuming (33), we can establish (31) as follows:

$$2N \cdot \mathcal{G} \cdot Q^t = N \cdot (\mathcal{F} + \mathcal{F}^t) \cdot Q^t = N \cdot (\mathcal{F} \cdot Q^t - \mathcal{F}^t \cdot R^t \cdot Q^t) = N \cdot (\mathcal{F} - \mathcal{F}^t \cdot R^t) \cdot Q^t = 0,$$

where we have used first Eq. (23) and then Eq. (33). Moreover, once we have established that the condition (31) holds, we know that Eqs. (33) and (35) are equivalent.

To summarize, we are lead to the following conditions on a consistent gluing operator $\mathcal{R}$:

$$\begin{gathered} Q^2 = Q, \quad N = \mathbf{1} - Q, \quad R \cdot Q = Q \cdot R = -Q, \\ N_\kappa{}^\mu N_\lambda{}^\nu \partial_{[\mu} N_{\nu]}{}^\rho = 0, \\ R \cdot \mathcal{G} \cdot R^t = \mathcal{G}, \\ N \cdot (\mathcal{F} - \mathcal{F}^t \cdot R^t) = 0. \end{gathered} \tag{36}$$

Next, we investigate whether or not these conditions are preserved under the Poisson-Lie *T*-plurality. As we shall see by investigation of explicit examples, they are not preserved in general.

IV. POISSON-LIE *T*-PLURALITY TRANSFORMATIONS OF RIGHT-INVARIANT FIELDS AND BOUNDARY CONDITIONS

The derivation of Poisson-Lie *T*-plurality transformations of left-invariant fields was presented in Ref. 8 but we find it instructive to repeat it here for the right-invariant fields. In particular, we derive the formulas generalizing Eqs. (1) and (2).

Let us write the right-invariant field $(\partial_+ l l^{-1})$ on the Drinfel'd double in terms of $\rho_+(g)$ and $\tilde{\rho}_+(\tilde{h})$,

$$(\partial_+ l l^{-1}) = (\partial_+(g\tilde{h})(g\tilde{h})^{-1}) = \rho_+(g) \cdot T + \tilde{\rho}_+(\tilde{h}) \cdot g\tilde{T}g^{-1} = \rho_+(g) \cdot T + \tilde{\rho}_+(\tilde{h}) \cdot [b^t(g) \cdot T + a^t(g) \cdot \tilde{T}]. \tag{37}$$

Using the equations of motion (10) and the formula (7) for $F(g)$, we get

$$(\partial_+ l l^{-1}) = \rho_+(g) \cdot T - \rho_+(g) \cdot F(g)^t \cdot [a^{-t}(g) \cdot b^t(g) \cdot T + \tilde{T}] = \rho_+(g) \cdot F(g)^t \cdot [E_0^{-t} \cdot T - \tilde{T}]. \tag{38}$$

Similarly, from the decomposition $l = \hat{g}\bar{h}$, we get

$$(\partial_+ l l^{-1}) = \hat{\rho}_+(\hat{g}) \cdot \hat{F}(\hat{g})^t \cdot [\hat{E}_0^{-t} \cdot \hat{T} - \bar{T}]. \tag{39}$$

Substituting the relation (14) into Eq. (38) and comparing coefficients of $\hat{T}$ and $\bar{T}$ with those in (39), we find the transformation of right-invariant fields under the Poisson-Lie *T*-plurality,

$$\hat{\rho}_+(\hat{g}) = -\rho_+(g)\cdot F^t(g)\cdot[(E_0^t)^{-1}\cdot q - s]\cdot \hat{F}^{-t}(\hat{g}). \tag{40}$$

In the same way, we can derive

$$\hat{\rho}_-(\hat{g}) = \rho_-(g)\cdot F(g)\cdot[E_0^{-1}\cdot q + s]\cdot \hat{F}^{-1}(\hat{g}). \tag{41}$$

Formulas (1) and (2) for T-duality are obtained if $q=\mathbf{1}$, $s=0$, $F(g)=E(g^{-1})$, $\rho_+(g)=-L_=(g^{-1})$, and $\rho_-(g)=-L_{\#}(g^{-1})$, in agreement with the alternative version for the σ-model action used in Ref. 2,

$$S_E[g] = \int_\Sigma d^2x L_{\#}(g)\cdot E(g)\cdot L_=^t(g). \tag{42}$$

Substituting Eqs. (40) and (41) into the gluing condition (26), we find the T-plural boundary condition

$$\hat{\rho}_-(\hat{g})\big|_{\sigma=0,\pi} = \hat{\rho}_+(\hat{g})\cdot \hat{R}_\rho\big|_{\sigma=0,\pi}, \tag{43}$$

where the T-plural gluing matrix is given by

$$\hat{R}_\rho = \hat{F}^t(\hat{g})\cdot M_-^{-1}\cdot F^{-t}(g)\cdot R_\rho(g)\cdot F(g)\cdot M_+\cdot \hat{F}^{-1}(\hat{g}) \tag{44}$$

and

$$M_+ \equiv s + E_0^{-1}\cdot q, \quad M_- \equiv s - E_0^{-t}\cdot q. \tag{45}$$

Equation (44) defines the transformation of the gluing matrix R_ρ under the Poisson-Lie T-plurality. For the Poisson-Lie T-duality, i.e., for $q=r=\mathbf{1}$, $p=s=0$, the map (44) reduces (up to transpositions due to the different notations for matrices) to the duality map found in Ref. 2,

$$\tilde{R} = -\tilde{E}^{-1}\cdot E_0^{-1}\cdot E\cdot R\cdot E^{-t}\cdot E_0^t\cdot \tilde{E}^t. \tag{46}$$

An obvious problem is that the transformed gluing matrix $\hat{R}_\rho$ may depend not only on $\hat{g}$ but also on g, i.e., after performing the lift into the double $g\tilde{h}=\hat{g}\bar{h}$, it may depend on the new dual group elements $\bar{h}\in\bar{G}$, which contradicts any reasonable geometric interpretation of the dual boundary conditions. Nevertheless, as we shall see in Sec. V, if g and $\hat{g}$ represent the maps $\Sigma \to G$ and $\Sigma\to\hat{G}$ related by the plurality transformation, the boundary conditions (26) and (43) are equivalent in the sense that they result in the same conditions on arbitrary functions [see e.g., (85)] occurring in solutions of the Euler-Lagrange equation of the action (5).

The T-plural counterparts of the Dirichlet and Neumann projectors may be consistently introduced in the same manner as for the T-dual case,[2] letting the relations $\hat{R}\cdot\hat{Q}=\hat{Q}\cdot\hat{R}=-\hat{Q}$ and $\hat{N}=\mathbf{1}-\hat{Q}$ define $\hat{Q}$ and $\hat{N}$ on $\hat{G}$. When the conditions (36) are satisfied also for $\hat{R},\hat{Q},\hat{N}$, then given a nonlinear σ-model on G with well-defined boundary conditions, we find a σ-model on $\hat{G}$ with well-defined boundary conditions.

The conformal condition (29) is preserved under the Poisson-Lie T-plurality, i.e., Eq. (30) implies

$$\hat{R}_\phi\cdot\hat{\mathcal{G}}\cdot\hat{R}_\phi{}^t = \hat{\mathcal{G}}, \quad \hat{R}_\rho\cdot\hat{G}(g)\cdot\hat{R}_\rho{}^t = \hat{G}(g). \tag{47}$$

This is seen by using Eqs. (30) and (44), as well as the identities

$$F(g)^{-t}\cdot G(g)\cdot F(g)^{-1} = E_0^{-1} + E_0^{-t} = M_\pm\cdot(\hat{E}_0^{-1}+\hat{E}_0^{-t})\cdot M_\pm^t, \tag{48}$$

which follow from Eqs. (16)–(18).

Imposing the condition (33) on the T-plural model and working in the basis of right-invariant fields, we may substitute Eq. (44) in the left-hand side of Eq. (33) to obtain

$$\hat{N}\cdot(\hat{F}-\hat{F}^t\cdot\hat{R}_\rho{}^t)=\hat{N}\cdot(\hat{F}-\hat{F}^t\cdot\hat{F}^{-t}\cdot(s+E_0^{-1}\cdot q)^t\cdot C^t\cdot(s-E_0^{-t}\cdot q)^{-t}\hat{F}), \tag{49}$$

where we have defined $C\equiv F^{-t}(g)\cdot R_\rho(g)\cdot F(g)$. This simplifies to

$$\hat{N}\cdot((s-E_0^{-t}\cdot q)^t-(s+E_0^{-1}\cdot q)^t\cdot C^t)\cdot(s-E_0^{-t}\cdot q)^{-t}\cdot\hat{F}.$$

The last two terms are by construction regular matrices and can be omitted while investigating when expression (49) vanishes. Consequently, the T-plural version of condition (33) has the form

$$\hat{N}\cdot((s-E_0^{-t}\cdot q)^t-(s+E_0^{-1}\cdot q)^t\cdot C^t)=0. \tag{50}$$

To gain a better understanding of Eq. (50), consider the particular case of originally purely Neumann boundary conditions, i.e., free endpoints. In this case $R_\rho(g)=F^t(g)\cdot F^{-1}(g)$, i.e., $C=\mathbf{1}$, and the transformation (44) is well defined (i.e., $\hat{R}_\rho$ is a function of $\hat{g}$ only) on any of the groups in any decomposition of the Drinfel'd double. This means that any T-plural $\hat{R}$ depends on the coordinates on the respective group $\hat{G}$ only. In this case, condition (50) further simplifies to

$$\hat{N}\cdot q^t=0, \tag{51}$$

where again regular matrices have been omitted in the product. We conclude that in the case of Poisson-Lie T-duality, where $q=\mathbf{1}$, the dual gluing operator satisfies condition (33) only if it is completely Dirichlet, in which case the dual version of (33) is trivially satisfied.

A possible solution to this problem, considered already in Ref. 11, comes from the fact that condition (33) is modified if the endpoints of the string are electrically charged. Let us modify the action by boundary terms

$$S_{\mathcal{F}}[\phi]\to S_{\mathcal{F}}[\phi]+S_{\text{boundary}}[\phi], \tag{52}$$

where

$$S_{\text{boundary}}[\phi]=q_0\int_{\sigma=0}A_\mu\frac{\partial\phi^\mu}{\partial\tau}d\tau+q_\pi\int_{\sigma=\pi}A_\mu\frac{\partial\phi^\mu}{\partial\tau}d\tau \tag{53}$$

corresponds to electrical charges q_0,q_π associated with the two endpoints of the string interacting with electric field(s) present on the respective D-branes. In order to make the following derivation easily comprehensible, let us assume that the potential A_μ can be in an arbitrary but smooth way extended to the neighborhood of the respective brane[7] and denote the field strength of the potential A_μ by[8]

$$\Delta_{\mu\nu}=\frac{1}{2}\left(\frac{\partial A_\nu}{\partial y^\mu}-\frac{\partial A_\mu}{\partial y^\nu}\right),\quad \text{i.e.,}\quad \Delta=dA. \tag{54}$$

Consequently, the equations of motion in the bulk obtained by the variation of the action are left unchanged but we find on the boundary

$$\delta\phi\cdot N_\phi\cdot(\mathcal{F}-\mathcal{F}^t\cdot R^t_\phi+q_0\Delta\cdot(1+R^t_\phi))\cdot\partial_+\phi^t|_{\sigma=0}=0 \tag{55}$$

together with

[7]Generalization to the case when this is not possible will be explained below.

[8]Recall that y^μ are coordinates on G and $\phi^\mu=y^\mu\circ g$.

$$\delta\phi \cdot N_\phi \cdot (\mathcal{F} - \mathcal{F}^t \cdot R^t_\phi - q_\pi \Delta \cdot (1 + R^t_\phi)) \cdot \partial_+ \phi^t|_{\sigma=\pi} = 0, \tag{56}$$

instead of (32). Therefore, by similar arguments as before, we find the following conditions instead of (33):

$$N \cdot \ (\mathcal{F} - \mathcal{F}^t \cdot R^t + q_0 \Delta \cdot (1 + R^t))|_{\sigma=0} = 0,$$

$$N \cdot \ (\mathcal{F} - \mathcal{F}^t \cdot R^t - q_\pi \Delta \cdot (1 + R^t))|_{\sigma=\pi} = 0.$$

Because these conditions should hold irrespective of which of the two endpoints lies on the considered brane (i.e., on any given brane a string may begin, end, or both,) we see that the endpoints are oppositely charged (and by proper choice of convention for A_μ, we set the charge to unity),

$$q_0 = - q_\pi = 1. \tag{57}$$

This means that condition (33) modified by the presence of electric charge at the endpoints reads

$$N \cdot ((\mathcal{F} + \Delta) - (\mathcal{F} + \Delta)^t \cdot R^t) = 0. \tag{58}$$

In fact, recalling Eq. (34) and writing

$$N \cdot (\Delta - \Delta^t \cdot R^t) = N \cdot \Delta \cdot (\mathbf{1} + R^t) = N \cdot \Delta \cdot N^t \cdot (\mathbf{1} + R^t), \tag{59}$$

we see that only derivatives of A_μ along the brane are relevant in the variation of the action $S_\mathcal{F}[\phi]+S_{\text{boundary}}[\phi]$, i.e., the resulting condition (58) does not depend on the way we extend A_μ outside the brane. If such an extension is impossible, the definition (54) of Δ is obviously meaningless and must be corrected in the following way. We introduce the embedding ι of the brane $\mathcal{B}$,

$$\iota:\mathcal{B} \to G, \quad \mathcal{B} \simeq \iota(\mathcal{B}) \subset G$$

and construct the electric field on the brane as

$$\Delta_\mathcal{B} = d_\mathcal{B} A \in \Omega^2(\mathcal{B}). \tag{60}$$

Then, we may pointwise extend $\Delta_\mathcal{B}|_p$ to a two-form $\Delta|_{\iota(p)}$ with values in $\Omega^2_{\iota(p)}(G)$ [i.e., a two-form on G in the point $\iota(p)$],

$$\Delta\ (V,W)|_{\iota(p)} = \Delta_\mathcal{B}\ (\mathcal{N}(V),\mathcal{N}(W))|_p, \quad p \in \mathcal{B}, V,W \in T_{\iota(p)}G \tag{61}$$

[where the natural identification $T_p\mathcal{B} \simeq \iota_*(T_p\mathcal{B}) = \mathrm{Im}(\mathcal{N})|_{\iota(p)}$ is assumed]. With this understanding in mind, condition (58) remains the same as before but supplemented by a consequence of (61)

$$\Delta = N \cdot \Delta \cdot N^t. \tag{62}$$

Consequently, even if the target group G is foliated by D-branes and Δ constructed as a collection of Δ's on different branes may be well defined and smooth on G (or its open subset), Δ may nonetheless not be closed—only its restrictions $\Delta|_\mathcal{B}$ to the respective branes need to be closed in order to allow the potential A_μ along the brane.

In the following, we shall use condition (58) to look for a suitable background electric field strength Δ such that the boundary equations of motion are satisfied in the transformed models. Taking into account (59), we see that (58) determines $\Delta = N \cdot \Delta \cdot N^t$ uniquely and generically smoothly (except when N changes rank). The self-consistency of such a procedure of course requires that Δ found in this way is closed along the branes, i.e.,

$$N_\kappa{}^\nu N_\lambda{}^\rho N_\mu{}^\sigma \partial_{[\nu} \Delta_{\rho\sigma]} = 0 \tag{63}$$

and hence[9] gives rise to the potential A_μ.

We should note that the case of free endpoints, i.e., purely Neumann boundary condition $R_\rho(g) = F^t(g) \cdot F^{-1}(g)$, was investigated in Ref. 11. The approach used there was based on symplectic geometry and it was shown that the Poisson-Lie T-dual configuration corresponds to D-branes as symplectic leaves of the Poisson structure on the dual group $\tilde{G}$ [once one fixes one end of the dual string at the origin of $\tilde{G}$ using the freedom of a constant shift (13)] and that the correction Δ in this case exists and is obtained from the Semenov-Tian-Shansky symplectic form on the Drinfel'd double as a symplectic form on the symplectic leaves and is therefore closed along the branes. These results are in accord with the analysis here. Also, it is clear from the conclusions of Ref. 11 that in this particular case, the integrability condition (28) is automatically satisfied on the dual since the symplectic leaves are submanifolds.

V. THREE-DIMENSIONAL EXAMPLE

As mentioned in Sec. I, there are several explicitly solvable σ-models whose solutions are related by the Poisson-Lie T-plurality. We can construct their gluing matrices corresponding to D-branes and check the equivalence of Eqs. (26) and (43). Here, we present a three-dimensional example, where one of the solutions is flat with vanishing B-field, while the T-plural one is curved and torsionless. They are given by a six-dimensional Drinfel'd double with decompositions into, on the one hand, the Bianchi 5 and Bianchi 1 algebras and, on the other hand, the Bianchi 6_0 and Bianchi 1 algebras. On Bianchi 5, the background is given by

$$E_0 = F(g) = \begin{pmatrix} 0 & 0 & \kappa \\ 0 & \kappa & 0 \\ \kappa & 0 & 0 \end{pmatrix}, \quad \kappa \in \mathbb{R}. \tag{64}$$

The right-invariant vielbein in a convenient parametrization $g = g(y^\mu)$ of the solvable group corresponding to Bianchi 5 is

$$e(g) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-y^1} & 0 \\ 0 & 0 & e^{-y^1} \end{pmatrix}, \tag{65}$$

so that the tensor field of the conformal σ-model that lives on this group reads

$$\mathcal{F}_{\mu\nu}(y) = \begin{pmatrix} 0 & 0 & \kappa e^{-y^1} \\ 0 & \kappa e^{-2y^1} & 0 \\ \kappa e^{-y^1} & 0 & 0 \end{pmatrix}. \tag{66}$$

The metric of this model is indefinite and flat. The general solution of the equations of motion is[7]

[9] up to possible topological obstructions which we shall neglect here

$$\phi^1(x_+,x_-) = -\ln(-W_1 - Y_1),$$

$$\phi^2(x_+,x_-) = -\frac{W_2+Y_2}{W_1+Y_1}, \tag{67}$$

$$\phi^3(x_+,x_-) = W_3 + Y_3 + \frac{(W_2+Y_2)^2}{2(W_1+Y_1)},$$

where $W_j = W_j(x_+)$ and $Y_j = Y_j(x_-)$ are arbitrary functions.

The σ-model related to that on Bianchi 5 by the Poisson-Lie T-plurality lives on the solvable group corresponding to Bianchi 6_0 and its tensor field obtained from

$$\hat{E}_0 = \hat{F}(\hat{g}) = \begin{pmatrix} \frac{1}{\kappa} & \frac{1}{\kappa} & \frac{\kappa}{2} \\ \frac{1}{\kappa} & \frac{1}{\kappa} & -\frac{\kappa}{2} \\ \frac{\kappa}{2} & -\frac{\kappa}{2} & 0 \end{pmatrix} \tag{68}$$

and

$$\hat{e}(\hat{g}) = \begin{pmatrix} \cosh \hat{y}^3 & -\sinh \hat{y}^3 & 0 \\ -\sinh \hat{y}^3 & \cosh \hat{y}^3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{69}$$

reads

$$\hat{\mathcal{F}}_{\mu\nu}(\hat{y}) = \begin{pmatrix} \frac{1}{\kappa}e^{-2\hat{y}^3} & \frac{1}{\kappa}e^{-2\hat{y}^3} & \frac{\kappa}{2}e^{\hat{y}^3} \\ \frac{1}{\kappa}e^{-2\hat{y}^3} & \frac{1}{\kappa}e^{-2\hat{y}^3} & -\frac{\kappa}{2}e^{\hat{y}^3} \\ \frac{\kappa}{2}e^{\hat{y}^3} & -\frac{\kappa}{2}e^{\hat{y}^3} & 0 \end{pmatrix}. \tag{70}$$

The Ricci tensor of this metric is nontrivial so that the background is curved but has a zero Gauss curvature.

The transformation (14) between the bases of decompositions of the Lie algebra of the Drinfel'd double into Bianchi $5\dot{+}$Bianchi 1 and Bianchi $6_0\dot{+}$Bianchi 1 is given by the matrix

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}, \tag{71}$$

and the coordinate transformation on the Drinfel'd double that follows from this reads (see Ref. 7)

$$\hat{y}^1 = -y^3 + \tfrac{1}{2}\tilde{h}_2,$$
$$\hat{y}^2 = y^3 + \tfrac{1}{2}\tilde{h}_2, \qquad (72)$$
$$\hat{y}^3 = -y^1,$$

$$\bar{h}_1 = -\tfrac{1}{2}\tilde{h}_3 + y^2,$$
$$\bar{h}_2 = \tfrac{1}{2}\tilde{h}_3 + y^2, \qquad (73)$$
$$\bar{h}_3 = -\tilde{h}_1 + \tilde{h}_2 y^2,$$

where y, $\tilde{h}$, $\hat{y}$, $\bar{h}$ are coordinates on the respective subgroups G, $\tilde{G}$, $\hat{G}$, $\bar{G}$ that correspond to the different decompositions of the Drinfel'd double. Inserting Eq. (67) and the solution of Eqs. (10) and (11), into Eq. (72), we obtain the solution[7] of the equations of motion for the σ-model in the curved background given by $\hat{F}$,

$$\hat{\phi}^1(x_+,x_-) = \frac{1}{2}\kappa[Y_1(x_-)W_2(x_+) - Y_2(x_-)W_1(x_+)] - [W_3(x_+) + Y_3(x_-)] - \frac{1}{2}\frac{[W_2(x_+) + Y_2(x_+)]^2}{(W_1(x_+) + Y_1(x_-)]}$$
$$+ \frac{1}{2}\kappa(\alpha(x_+) + \beta(x_-)),$$

$$\hat{\phi}^2(x_+,x_-) = \frac{1}{2}\kappa[Y_1(x_-)W_2(x_+) - Y_2(x_-)W_1(x_+)] + [W_3(x_+) + Y_3(x_-)] + \frac{1}{2}\frac{(W_2(x_+) + Y_2(x_-))^2}{W_1(x_+) + Y_1(x_-)}$$
$$+ \frac{1}{2}\kappa(\alpha(x_+) + \beta(x_-)), \qquad (74)$$

$$\hat{\phi}^3(x_+,x_-) = \ln(-W_1(x_+) - Y_1(x_-)),$$

where α, β satisfy (primes denote differentiation)

$$\alpha' = W_1 W_2' - W_2 W_1',$$
$$\beta' = Y_2 Y_1' - Y_1 Y_2'. \qquad (75)$$

A. *D*-branes

In the following, we analyze examples of D-branes for which the adapted coordinates λ^α of the flat model are equal to those that bring the metric of the flat model to the diagonal form

$$F_{kl}(\lambda) = \begin{pmatrix} -\kappa & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & \kappa \end{pmatrix},$$

namely,

$$\lambda^1(y) = \lambda_0^1 - \tfrac{1}{\sqrt{2}}\Big[y^3 + \tfrac{1}{2}(y^2)^2 e^{-y^1} + e^{-y^1}\Big],$$

$$\lambda^2(y) = \lambda_0^2 + y^2 e^{-y^1}, \tag{76}$$

$$\lambda^3(y) = \lambda_0^3 + \tfrac{1}{\sqrt{2}}\Big[y^3 + \tfrac{1}{2}(y^2)^2 e^{-y^1} - e^{-y^1}\Big].$$

In these coordinates, the gluing matrices R_λ by assumption become diagonal.[10]

- $D2$-branes. The Dirichlet projector is zero (and the Neumann projector is the identity) in this case and as the tensor $\mathcal{F}$ is symmetric, it follows from Eq. (33) that the gluing matrices are

$$R_\lambda = R_\phi = R_\rho = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{77}$$

The conditions (36) are trivially satisfied. The condition (26), or equivalently (22), then gives the boundary conditions for the solution (67),

$$W_j'(x_+)|_{\sigma=0,\pi} = Y_j'(x_-)|_{\sigma=0,\pi}, \quad j = 1,2,3. \tag{78}$$

From Eq. (44), we get

$$\hat{R}_\rho = \hat{R}_\phi = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{79}$$

This matrix has eigenvalues $(-1,1,1)$ and the eigenvector corresponding to the eigenvalue -1 is spacelike in the (curved) metric (70) so that the $D2$-brane is transformed to a $D1$-brane. The Dirichlet projector obtained from Eqs. (23) and (27) is

$$\hat{Q} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{80}$$

and the conditions (36) are satisfied for the matrix (79). Using Eqs. (74) and (78), one can verify that

$$\partial_-\hat{\phi}|_{\sigma=0,\pi} = \partial_+\hat{\phi}\cdot \hat{R}_\phi|_{\sigma=0,\pi}, \tag{81}$$

which is equivalent to Eq. (43). Note that unlike the $D1$-branes and $D0$-branes discussed below, in this case neither the matrix R_ρ nor $\hat{R}_\rho$ depends on elements of the groups G and $\hat{G}$.

- $D1$-branes. We have chosen the branes as coordinate planes of the flat coordinates, i.e.,

$$R_\lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{82}$$

which in y coordinates gives the y-dependent gluing matrix

$$R_\phi = \begin{pmatrix} \frac{(y^2)^2}{2} & \frac{1}{2}y^2[(y^2)^2-2] & -\frac{1}{4}e^{-y^1}[(y^2)^2-2]^2 \\ -y^2 & 1-(y^2)^2 & \frac{1}{2}e^{-y^1}y^2[(y^2)^2-2] \\ -e^{y^1} & -e^{y^1}y^2 & \frac{(y^2)^2}{2} \end{pmatrix}. \quad (83)$$

The Dirichlet projector obtained from Eqs. (23) and (27) is

$$Q = \begin{pmatrix} \frac{1}{4}[2-(y^2)^2] & \frac{1}{4}[2(y^2)-(y^2)^3] & \frac{1}{8}e^{-y^1}[(y^2)^2-2]^2 \\ \frac{1}{2}(y^2)^2 & \frac{1}{2}(y^2)^2 & -\frac{1}{4}e^{-y^1}(y^2)[(y^2)^2-2] \\ \frac{1}{2}e^{y^1} & \frac{1}{2}e^{y^1}(y^2) & \frac{1}{4}[2-(y^2)^2] \end{pmatrix}, \quad (84)$$

and the conditions (36) are satisfied. The condition (26) then gives

$$\begin{aligned} W_1'(x_+)|_{\sigma=0,\pi} &= Y_3'(x_-)|_{\sigma=0,\pi}, \\ W_2'(x_+)|_{\sigma=0,\pi} &= Y_2'(x_-)|_{\sigma=0,\pi}, \\ W_3'(x_+)|_{\sigma=0,\pi} &= Y_1'(x_-)|_{\sigma=0,\pi}. \end{aligned} \quad (85)$$

From Eq. (44), we obtain $\hat{R}_\rho$ and $\hat{R}_\phi$, which, however, are too complicated to be displayed here. The matrix $\hat{R}_\phi$ depends on the coordinates on both $\hat{G}$ and G and consequently on $\bar{G}$; nevertheless, we have checked that the boundary condition (43) for the solution (74) implies again the relations (85). In this sense, the conditions (26) and (43) are equivalent.

The eigenvalues of $\hat{R}_\phi$ are 1, $-1+(y^2)^2 \pm \sqrt{(y^2)^4-2(y^2)^2}$ so that for $y^2 \neq 0$, the projectors are $\hat{Q}=0, \hat{N}=\mathbf{1}$, and the condition (33) is not satisfied.

On the other hand, the hypersurface $y^2=0$ does not coincide with a $D1$-brane in the original model since the tangent vector $\partial_{y^2}|_{y^2=0}$ is Neumann. Consequently, if at a given time the endpoint of a string is located at $y^2=0$, it might not stay there at later times. We conclude that in this case, the transformed D-brane configuration is not well defined due to the dependence of $\hat{R}_\rho$ on the coordinates on $\bar{G}$.

- *D*0-branes. We choose

$$R_\lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad (86)$$

so that

$$R_\phi = \begin{pmatrix} \frac{(y^2)^2}{2} & \frac{(y^2)^3}{2}+y^2 & -\frac{1}{4}e^{-y^1}[(y^2)^2+2]^2 \\ -y^2 & -(y^2)^2-1 & \frac{1}{2}e^{-y^1}y^2[(y^2)^2+2] \\ -e^{y^1} & -e^{y^1}y^2 & \frac{(y^2)^2}{2} \end{pmatrix}. \quad (87)$$

The Dirichlet projector is

$$Q=\begin{pmatrix} \frac{1}{4}[2-(y^2)^2] & \frac{1}{4}[-(y^2)^3-2(y^2)] & \frac{1}{8}e^{-y^1}[(y^2)^2+2]^2 \\ \frac{(y^2)}{2} & \frac{1}{2}[(y^2)^2+2] & -\frac{1}{4}e^{-y^1}(y^2)[(y^2)^2+2] \\ \frac{1}{2}e^{y^1} & \frac{1}{2}e^{y^1}(y^2) & \frac{1}{4}[2-(y^2)^2] \end{pmatrix}, \tag{88}$$

and the conditions (36) are satisfied. The condition (26) yields

$$\begin{aligned} W_1'(x_+)|_{\sigma=0,\pi} &= -Y_3'(x_-)|_{\sigma=0,\pi}, \\ W_2'(x_+)|_{\sigma=0,\pi} &= -Y_2'(x_-)|_{\sigma=0,\pi}, \\ W_3'(x_+)|_{\sigma=0,\pi} &= -Y_1'(x_-)|_{\sigma=0,\pi}. \end{aligned} \tag{89}$$

The matrix $\hat{R}_\phi$ is again rather complicated and depends on the coordinates of both G and $\hat{G}$, but once again using Eqs. (74) and (89), one can verify that conditions (26) and (43) are equivalent in the sense explained above. The eigenvalues of $\hat{R}_\phi$ are $-1,1+(y^2)^2\pm\sqrt{(y^2)^4+2(y^2)^2}$ and the Dirichlet projector $\hat{Q}$ obtained from Eqs. (23) and (27) reads

$$\hat{Q}=\begin{pmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{e^{2y^1+\hat{y}^3}}{2(y^2)^2+4} \\ -\frac{1}{4} & \frac{1}{4} & -\frac{e^{2y^1+\hat{y}^3}}{2(y^2)^2+4} \\ \frac{1}{4}e^{-2y^1-\hat{y}^3}((y^2)^2+2) & -\frac{1}{4}e^{-2y^1-\hat{y}^3}((y^2)^2+2) & \frac{1}{2} \end{pmatrix}. \tag{90}$$

Due to (72) and (73), namely, $y^2=\frac{1}{2}(\bar{h}_1+\bar{h}_2)$, the projector $\hat{Q}$ depends both on $\hat{G}$ and $\bar{G}$. The conditions (36) are again satisfied only for $y^2=0$ but now the tangent vector $\partial_{y^2}|_{y^2=0}$ is Dirichlet. We can therefore consistently restrict ourselves in the original model to $D0$-branes inside the hypersurface $y^2=0$. Their plural counterparts are given by a gluing matrix of the form

$$\hat{R}_\phi=\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2}e^{-\hat{y}^3} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2}e^{-\hat{y}^3} \\ -e^{\hat{y}^3} & e^{\hat{y}^3} & 0 \end{pmatrix}, \tag{91}$$

where we have used the coordinate (72). Its eigenvalues are $(-1,1,1)$ and the eigenvector corresponding to the eigenvalue -1 is spacelike so that the matrix (91) defines a $D1$-brane in the dual model.

- $D(-1)$-branes. We have

$$R_\lambda=R_\phi=R_\rho=\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{92}$$

The Dirichlet projector is the identity in this case so that the conditions (36) are trivially satisfied. The condition (26) then gives the boundary conditions for the solution (67),

$$W_j'(x_+)|_{\sigma=0,\pi} = -Y_j'(x_-)|_{\sigma=0,\pi}, \quad j=1,2,3. \tag{93}$$

From Eq. (44), we find

$$\hat{R}_\rho = \hat{R}_\phi = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{94}$$

and the *T*-plural Dirichlet projector is

$$\hat{Q} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{95}$$

The conditions (36) are satisfied and the condition (93) implies both Eqs. (26) and (43). The matrix (94) has eigenvalues $(-1,-1,1)$, where $+1$ corresponds to a spacelike direction. Hence, we get a Euclidean $D0$-brane. Similarly to the $D2$-brane case, also here neither R_ρ nor $\hat{R}_\rho$ depends on elements of the groups G and $\hat{G}$.

B. Gluing matrices that produce $\hat{R}$ dependent only on $\hat{G}$

The lesson we have learned from the previous subsection is that in some cases the transformation of coordinates (72) may cure the problem of dependence of the gluing matrix $\hat{R}$ on elements of the group $\bar{G}$. In particular, in our three-dimensional example, it turned out that if $D0$-branes in Bianchi 5 are contained in the hypersurface of constant y^2 located at $y^2=0$, then due to Eq. (72) the plural gluing matrices are well defined.

In the present section, we address the problem of coordinate cross dependence from another point of view. We shall assume that the plural gluing matrix depends on elements of $\hat{G}$ only, i.e., it is independent of the dual coordinates on $\bar{G}$, and we derive the gluing matrices on both sides of the plurality that make this assumption possible. Inspecting the transformation formula (44) for the gluing operator, we find that the *T*-plural gluing matrix $\hat{R}_\rho$ is a function on $\hat{G}$ if and only if the matrix-valued function

$$C(g) = F^{-t}(g) \cdot R_\rho(g) \cdot F(g)$$

extended to a function on the whole Drinfel'd double as $C_D(l)=C(g)$, where $l=g\tilde{h}$, satisfies

$$C_D(\hat{g}\bar{h}) = C_D(\hat{g}). \tag{96}$$

In our particular setting, where the relations between original and new coordinates on the Drinfel'd double D are given by Eqs. (72) and (73), we find that (only) the following combinations of $\hat{y}$'s can be written in terms of the original y's:

$$\hat{y}^2 - \hat{y}^1 = 2y^3, \quad \hat{y}^3 = -y^1.$$

Consequently, if the original gluing matrix has the form

$$R_\rho(g) = F^t(g) \cdot C \cdot F^{-1}(g), \tag{97}$$

where $C=C(y^1,y^3)$, then the gluing matrices $\hat{R}_\rho$ given by Eq. (44) and $\hat{R}_\phi$, $\hat{R}_\lambda$ given by Eq. (25) can be expressed as functions on $\hat{G}$ only, i.e., they are well defined. The condition (30) that R_ρ of the form (97) preserve the metric yields

$$C \cdot (E_0^{-1} + E_0^{-t}) \cdot C^t = (E_0^{-1} + E_0^{-t}). \tag{98}$$

In other words, the matrices C belong to the representation of the group $O(n, \dim G - n)$ given by the constant symmetric matrix $(E_0^{-1} + E_0^{-t})$ with signature n.

For E_0 of the form (64), we get the following possibilities:

$$C = \begin{pmatrix} -\frac{\alpha^2}{2\beta} & \alpha & \beta \\ -\epsilon\frac{\alpha}{\beta} & \epsilon & 0 \\ \frac{1}{\beta} & 0 & 0 \end{pmatrix}, \tag{99}$$

$$C = \begin{pmatrix} \frac{(\alpha+\epsilon)^2}{4\beta} & \frac{1-\alpha^2}{2\gamma} & -\frac{(\alpha-\epsilon)^2\beta}{2\gamma^2} \\ -\frac{(\alpha+\epsilon)\gamma}{2\beta} & \alpha & \frac{(\alpha-\epsilon)\beta}{\gamma} \\ -\frac{\gamma^2}{2\beta} & \gamma & \beta \end{pmatrix}, \tag{100}$$

$$C = \begin{pmatrix} \frac{1}{\beta} & \alpha & -\frac{\alpha^2\beta}{2} \\ 0 & \epsilon & -\epsilon\alpha\beta \\ 0 & 0 & \beta \end{pmatrix}, \tag{101}$$

where $\epsilon = \pm 1$ and α, β, γ are arbitrary functions of y^1 and y^3. In addition, the matrices R_ϕ and $\hat{R}_\phi$ calculated from Eqs. (25), (44), and (97) must satisfy the conditions (36) so that further restrictions on the matrices C are imposed.

- Case (99): The conditions (36) for R are satisfied only if $\alpha = 0$. The gluing matrices then read

$$R_\rho = \begin{pmatrix} 0 & 0 & \frac{1}{\beta} \\ 0 & \epsilon & 0 \\ \beta & 0 & 0 \end{pmatrix}, \quad \epsilon = \pm 1, \quad \beta = \beta(y^1, y^3), \tag{102}$$

$$\hat{R}_\rho = \frac{1}{2}\begin{pmatrix} -\epsilon & -\epsilon & \beta \\ -\epsilon & -\epsilon & -\beta \\ \frac{2}{\beta} & -\frac{2}{\beta} & 0 \end{pmatrix}, \quad \epsilon = \pm 1, \quad \beta = \beta\left(-\hat{y}^3, \frac{\hat{y}^2 - \hat{y}^1}{2}\right). \tag{103}$$

  The conditions (36) are satisfied for $\hat{R}$ as well. This corresponds to the transformation of $D1$-branes to $D0$-branes for $\epsilon = +1$ and $D0$-branes to $D1$-branes for $\epsilon = -1$.
- Case (100): The conditions (36) are satisfied for R only if $\alpha = -\epsilon - 2\beta$. The gluing matrices then read

$$R_\rho = \begin{pmatrix} \beta & \gamma & -\frac{\gamma^2}{2\beta} \\ -\frac{2(\beta+\epsilon)\beta}{\gamma} & -\epsilon-2\beta & \gamma \\ -\frac{2(\beta+\epsilon)^2\beta}{\gamma^2} & -\frac{2(\beta+\epsilon)\beta}{\gamma} & \beta \end{pmatrix}, \quad \beta=\beta(y^1,y^3), \quad \gamma=\gamma(y^1,y^3), \tag{104}$$

$$\hat{R}_\rho = \begin{pmatrix} \frac{(\beta+\epsilon)\beta\kappa^2+\gamma(3\beta+\epsilon)\kappa+2\gamma^2}{2\kappa\gamma} & \frac{(\beta+\epsilon)\beta\kappa^2+\gamma(\beta+\epsilon)\kappa-2\gamma^2}{2\kappa\gamma} & -\frac{(2\gamma+\kappa(\beta+\epsilon))(\beta+\epsilon)\beta}{\kappa\gamma^2} \\ \frac{-(\beta+\epsilon)\beta\kappa^2+\gamma(\beta+\epsilon)\kappa+2\gamma^2}{2\kappa\gamma} & \frac{-(\beta+\epsilon)\beta\kappa^2+\gamma(3\beta+\epsilon)\kappa-2\gamma^2}{2\kappa\gamma} & \frac{(\kappa(\beta+\epsilon)-2\gamma)(\beta+\epsilon)\beta}{\kappa\gamma^2} \\ -\frac{\gamma(\gamma+\kappa\beta)}{2\beta} & \frac{\gamma(\gamma-\kappa\beta)}{2\beta} & \beta \end{pmatrix}, \tag{105}$$

where $\beta=\beta(-\hat{y}^3,(\hat{y}^2-\hat{y}^1)/2)$, $\gamma=\gamma(-\hat{y}^3,(\hat{y}^2-\hat{y}^1)/2)$. For $\epsilon=-1$, the dependence of β and γ on y^1, y^3 is constrained by the condition (28) that yields

$$e^{y^1}\gamma^2\left(\gamma\frac{\partial\beta}{\partial y^3}-\beta\frac{\partial\gamma}{\partial y^3}\right)=2\beta^2\left(\gamma\frac{\partial\beta}{\partial y^1}+\frac{\partial\gamma}{\partial y^1}-\beta\frac{\partial\gamma}{\partial y^1}\right). \tag{106}$$

For $\epsilon=1$, we do not get any constraint on the functions β, γ.
The condition (33) is not satisfied for the matrix $\hat{R}$ unless we replace $\hat{\mathcal{F}}$ by $\hat{\mathcal{F}}+\hat{\Delta}$, where $\hat{N}\cdot\hat{\Delta}\cdot\hat{N}^t=\hat{\Delta}$.
For $\epsilon=1$,

$$\hat{\Delta} = \begin{pmatrix} 0 & -\frac{\beta}{\gamma} & -\frac{\gamma e^{-\hat{y}^3}}{2+2\beta} \\ \frac{\beta}{\gamma} & 0 & -\frac{\gamma e^{-\hat{y}^3}}{2+2\beta} \\ \frac{\gamma e^{-\hat{y}^3}}{2+2\beta} & \frac{\gamma e^{-\hat{y}^3}}{2+2\beta} & 0 \end{pmatrix}, \tag{107}$$

and it is closed along the branes for arbitrary β, γ. This case corresponds to the transformation of $D0$-branes to $D1$-branes.
For $\epsilon=-1$,

$$\hat{\Delta} = \begin{pmatrix} 0 & -\frac{\beta-1}{\gamma} & -\frac{\gamma e^{-\hat{y}^3}}{2\beta} \\ \frac{\beta-1}{\gamma} & 0 & -\frac{\gamma e^{-\hat{y}^3}}{2\beta} \\ \frac{\gamma e^{-\hat{y}^3}}{2\beta} & \frac{\gamma e^{-\hat{y}^3}}{2\beta} & 0 \end{pmatrix}, \tag{108}$$

and it is closed along the branes due to (106). This case corresponds to the transformation of $D1$-branes to $D2$-branes.

- Case (101): The conditions (36) for both R and $\hat{R}$ are satisfied if $\beta=\pm 1$ and $\alpha=0$. This corresponds to the transformation of $D2$-branes to $D1$-branes and $D(-1)$-branes to $D0$-branes,

$$R_\rho = \pm\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad \hat{R}_\rho = \pm\begin{pmatrix}0&-1&0\\-1&0&0\\0&0&1\end{pmatrix}, \tag{109}$$

as presented in Sec. V A and

$$R_\rho = \pm\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix},\quad \hat{R}_\rho = \pm\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \tag{110}$$

which correspond to transformations of $D1$-branes to $D2$-branes and of $D0$-branes to $D(-1)$-branes.

Besides that, the conditions (36) for R are also satisfied for

$$\beta = -\epsilon,\quad \frac{\partial\alpha}{\partial y^1} = 0. \tag{111}$$

However, to satisfy the condition (27) for $\hat{R}$, i.e., $\hat{R}\cdot\hat{Q}=\hat{Q}\cdot\hat{R}=-\hat{Q}$ with $\beta=-\epsilon=-1$, we must set $\alpha=0$. Thus, for $\beta=-\epsilon=-1$, we see that in general (i.e., for $\alpha\neq 0$) the Poisson-Lie T-plurality does not preserve the condition (27).

If $\alpha\neq 0$ and $\beta=-\epsilon=1$, then we have $\hat{Q}=0$ and the condition (27) holds trivially. We can satisfy the condition (33) by replacing $\hat{\mathcal{F}}$ by $\hat{\mathcal{F}}+\hat{\Delta}$, where

$$\hat{\Delta} = \begin{pmatrix}0&\frac{1}{2}\alpha&0\\-\frac{1}{2}\alpha&0&0\\0&0&0\end{pmatrix}. \tag{112}$$

This form is closed due to (111). The gluing matrices in this case,

$$R_\rho = \begin{pmatrix}1&0&0\\\alpha&-1&0\\-\frac{\alpha^2}{2}&\alpha&1\end{pmatrix},\quad \hat{R}_\rho = \begin{pmatrix}1-\frac{\alpha\kappa}{4}&-\frac{\alpha\kappa}{4}&\frac{\alpha}{\kappa}-\frac{\alpha^2}{4}\\\frac{\alpha\kappa}{4}&\frac{\alpha\kappa}{4}+1&\frac{\alpha(\alpha\kappa+4)}{4\kappa}\\0&0&1\end{pmatrix}, \tag{113}$$

correspond to the transformation of $D1$-branes to $D2$-branes.

We remark that in three dimensions, the integrability condition (28) is nontrivial only if the rank of the Neumann projector N is equal to 2; otherwise, the distribution $\Delta=\mathrm{Im}(N)$ is integrable on dimensional grounds. In two dimensions, investigated below, the condition (28) is always trivially satisfied.

VI. TWO-DIMENSIONAL EXAMPLE

The only σ-models with two-dimensional targets that can be transformed under T-plurality with nonisomorphic decompositions of a Drinfel'd double are generated by the semi-Abelian four-dimensional Drinfel'd double of Ref. 12. It has decompositions into two different pairs of maximally isotropic Lie subalgebras, namely, the semi-Abelian Manin triple with basis $T_1,T_2,\tilde{T}^1,\tilde{T}^2$ and Lie brackets (only nontrivial brackets are displayed)

$$[T_1,T_2]=T_2,\quad [\tilde{T}^2,T_1]=\tilde{T}^2,\quad [\tilde{T}^2,T_2]=-\tilde{T}^1, \tag{114}$$

and the so-called type B non-Abelian Manin triple with basis $\hat{T}_1,\hat{T}_2,\bar{T}^1,\bar{T}^2$ and Lie brackets

$$[\hat{T}_1,\hat{T}_2]=\hat{T}_2, \quad [\bar{T}^1,\bar{T}^2]=\bar{T}^1,$$
$$[\hat{T}_1,\bar{T}^1]=\hat{T}_2, \quad [\hat{T}_1,\bar{T}^2]=-\hat{T}_1-\bar{T}^2, \quad [\hat{T}_2,\bar{T}^2]=\bar{T}^1. \tag{115}$$

A simple transformation between the bases of these two decompositions is given by

$$\hat{T}_1=-T_1+T_2, \quad \hat{T}_2=\tilde{T}^1+\tilde{T}^2,$$
$$\bar{T}^1=\tilde{T}^2, \quad \bar{T}^2=T_1, \tag{116}$$

which corresponds to the transformation matrix

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{117}$$

The coordinate transformation on the Drinfel'd double that follows from this reads

$$\hat{x}^1=-\ln(-x^2+1), \quad \hat{x}^2=-\frac{\tilde{x}^1}{x^2-1},$$
$$\bar{x}^1=\frac{\tilde{x}^1\exp(x^1)+x^2\tilde{x}^2-\tilde{x}^2}{x^2-1}, \quad \bar{x}^2=-\ln(-x^2+1)+x^1. \tag{118}$$

We shall consider examples of two-dimensional σ-models given by the matrices

$$E_0=\begin{pmatrix} 1 & 0 \\ 0 & \kappa \end{pmatrix}, \quad \hat{E}_0=\begin{pmatrix} \kappa & 1 \\ -1 & 1 \end{pmatrix}, \tag{119}$$

where κ is a real constant. The corresponding tensors $\mathcal{F}$, $\hat{\mathcal{F}}$ are calculated from Eqs. (7) and (17), where

$$g=\exp(x^2T_2)\exp(x^1T_1), \quad \hat{g}=\exp(\hat{x}^2\hat{T}_2)\exp(\hat{x}^1\hat{T}_1). \tag{120}$$

They read

$$\mathcal{F}(x^\mu)=\begin{pmatrix} \kappa(x^2)^2+1 & -\kappa x^2 \\ -\kappa x^2 & \kappa \end{pmatrix}, \tag{121}$$

$$\hat{\mathcal{F}}(\hat{x}^\mu)=\frac{1}{-2e^{\hat{x}^1}\kappa+\kappa+e^{2\hat{x}^1}(\kappa+1)}\begin{pmatrix} (\hat{x}^2)^2+\kappa & -\kappa+e^{\hat{x}^1}(\kappa+1)-\hat{x}^2 \\ \kappa-e^{\hat{x}^1}(\kappa+1)-\hat{x}^2 & 1 \end{pmatrix}. \tag{122}$$

Unfortunately, the metrics of both models are curved and we are not able to solve the equations of motion. Nevertheless, we can at least find the gluing matrices that satisfy the conditions (36). Moreover, we require that the gluing matrices depend only on the coordinates where the σ-models live so that we have to take R_ρ of the form (97), with C depending only on x^2 in order to satisfy Eq. (96).

The condition (98) restricts C to the form

$$C = \begin{pmatrix} \epsilon_1\sqrt{1-\gamma^2\kappa} & \epsilon_2\gamma\kappa \\ \gamma & -\epsilon_1\epsilon_2\sqrt{1-\gamma^2\kappa} \end{pmatrix}, \tag{123}$$

where γ is an arbitrary function of x^2 and ϵ_1, $\epsilon_2 = \pm 1$. The conditions (27) and (31) are then satisfied for all corresponding matrices R. The condition (33) is satisfied only if $\epsilon_2 = 1$ or $\epsilon_2 = -1$, $\gamma = 0$.

If $\epsilon_2 = -1$, $\gamma = 0$, then the conditions (36) are satisfied for the transformed σ-model as well. The gluing matrices are

$$R_\rho = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_1 \end{pmatrix}, \quad \hat{R}_\rho = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & -\epsilon_1 \end{pmatrix}, \tag{124}$$

so that the boundary conditions for the σ-model on G are purely Dirichlet or purely Neumann. Interpretation of the boundary condition for the σ-model on $\hat{G}$ as either usual $D0$-branes or Euclidean (spacelike) $D1$-branes depends on the signature of the metric, i.e., on the sign of κ.

If $\epsilon_2 = 1$, then

$$R_\rho = \begin{pmatrix} -\epsilon_1\epsilon_2\sqrt{1-\gamma^2\kappa} & \gamma \\ \epsilon_2\gamma\kappa & \epsilon_1\sqrt{1-\gamma^2\kappa} \end{pmatrix}, \tag{125}$$

The transformed gluing matrix $\hat{R}_\rho$ is easily obtained from (44) but it is too complicated to display here. The conditions (27) and (31) are satisfied for all these matrices $\hat{R}_\rho$. The condition (33) can be always satisfied by replacing $\hat{\mathcal{F}}$ by $\hat{\mathcal{F}} + \hat{\Delta}$, where

$$\hat{\Delta} = \begin{pmatrix} 0 & \hat{\Delta}_{12} \\ -\hat{\Delta}_{12} & 0 \end{pmatrix}, \tag{126}$$

$$\hat{\Delta}_{12} = \frac{1 + \gamma\kappa - \epsilon_1\sqrt{1-\gamma^2\kappa} + e^{\hat{x}^1}(\gamma(1-\kappa) + 2\epsilon_1\sqrt{1-\gamma^2\kappa})}{\gamma\kappa + e^{2\hat{x}^1}(\gamma(-1+\kappa) - 2\epsilon_1\sqrt{1-\gamma^2\kappa}) + 2e^{\hat{x}^1}(-(\gamma\kappa) + \epsilon_1\sqrt{1-\gamma^2\kappa})}. \tag{127}$$

In the case when the denominator of $\hat{\Delta}_{12}$ vanishes, i.e., for $\hat{x}^1$ satisfying (recall that γ is a function of $\hat{x}^1$)

$$\frac{\epsilon_3 - \epsilon_1\gamma\kappa + \sqrt{1-\gamma^2\kappa}}{\epsilon_1\gamma(1-\kappa) + 2\sqrt{1-\gamma^2\kappa}} = e^{\hat{x}^1}, \tag{128}$$

where $\epsilon_3 = \pm 1$, we get

$$\hat{R}_\rho = -\epsilon_1\epsilon_3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{129}$$

The eigenvalues of R_ρ are $+1$, -1 corresponding to either usual $D0$-branes or Euclidean $D1$-branes, while the boundary conditions for the σ-model on $\hat{G}$ are purely Neumann except for $\hat{x}^1$ satisfying (128) with $\epsilon_3 = \epsilon_1$ in which case they are purely Dirichlet and $\hat{\Delta}_{12}$ becomes singular.[10]

VII. CONCLUSIONS

We have derived a formula (44) for the transformation of boundary conditions under the Poisson-Lie T-plurality. The examples in Sec. V A confirm that the formula works for solutions of

[10]whereas when (128) holds with $\epsilon_3 = -\epsilon_1$, we have $\hat{R}_\rho = \mathbf{1}$ and the singularity of $\hat{\Delta}_{12}$ is only apparent—it becomes an expression of the form $\frac{0}{0}$ with a finite and well-defined limit.

the equations of motion of the σ-models. This is not surprising since it was derived using these equations. The problem is that the transformed gluing matrix may depend on elements of the original group (and hence on elements of the dual group $\bar{G}$), so that only special forms of gluing matrices are transformable under the Poisson-Lie T-plurality.

To ensure that the gluing matrices transformed by the Poisson-Lie T-plurality depend only on the coordinates of the groups where the σ-models live, we can restrict them to the form (97)

$$R_\rho = F^t(g) \cdot C \cdot F^{-1}(g).$$

The matrix C must be constant or depend only on a particular subset of coordinates on G that transform into coordinates on $\hat{G}$.

Another problem is that not all conditions (36) for consistent D-branes are preserved under the Poisson-Lie T-plurality. We have proven that the condition (29), i.e., $R \cdot \mathcal{G} \cdot R^t = \mathcal{G}$, is always preserved. In Euclidean signature, this implies the preservation of conditions (27) and (31), i.e., $R \cdot Q = Q \cdot R$ and $N \cdot \mathcal{G} \cdot Q^t = 0$. As we have seen in the investigation of the matrix C of the form (101) in Sec. V B, it is not necessarily so in the case of indefinite signature. In that case, the transformed gluing matrix may become nondiagonalizable (in the sense of nondiagonal Jordan canonical form) and consequently the projector on (-1)-eigenspace cannot satisfy $R \cdot Q = Q \cdot R$. Nevertheless, when such an obstruction did not arise, the conditions (27) and (31) were satisfied in all cases investigated here also in the indefinite signature. Similarly, the integrability condition (28) was preserved in all examples.

On the other hand, we have seen explicitly that the condition (33), i.e., $N \cdot (\mathcal{F} - \mathcal{F}^t \cdot R^t) = 0$, is not preserved in general under the Poisson-Lie T-plurality and that in the transformed background it must be modified by the presence of an electric field constrained to the branes and interacting with oppositely charged endpoints of the string. We have moreover seen in several cases with nonconstant matrix C in (97) that the closedness (63) of this additional electric field is intimately related to the integrability of the Neumann distribution (28) in the original model. It is an open question whether and how this behavior can be proven in general or whether it happens just in the low dimensions investigated here.

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