Identification of Lie algebras

Def: Lie algebra $(\mathfrak{g}, [,])$ vector space with a bilinear map $[,]: \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ such that

$[x,y] = -[y,x] \quad \forall x,y\in\mathfrak{g}$ (antisymmetry)

$[[x,y],z]+[[y,z],x]+[[z,x],y]=0 \quad \forall x,y,z\in\mathfrak{g}$ (Jacobi identity)

Lie algebras arise in two natural ways

(1) from associative algebras $(A,\circ)\to(A,[,])$ $[X,Y]=X\circ Y - Y\circ X$ and their subalgebras

e.g. $A=\mathcal{L}(V)=\mathfrak{gl}(V)\supset \mathfrak{sl}(V)\supset \mathfrak{so}(V,(,))$ etc.

(2) Lie algebra of vector fields on a manifold ($\dim=+\infty$) and its subalgebras, finite or infinite dimensional

e.g. $L_1 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$, $L_2 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$, $L_3 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$

$\operatorname{span}\{L_1,L_2,L_3\}\subset \mathfrak{X}(\mathbb{R}^3)$

The subject of this minicourse: given a concrete Lie algebra, e.g. an algebra of infinitesimal symmetries of some ODE/PDE or Killing vectors of some spacetime, how can one identify its structure and in the best case scenario find an isomorphism bringing it to a form present in some known classification?

Some basic notions:

derivation D of $\mathfrak{g}$: $D\in\mathfrak{gl}(\mathfrak{g})$ $D[x,y]=[Dx,y]+[x,Dy]$

representation ϱ of $\mathfrak{g}$ on V: $\varrho:\mathfrak{g}\to\mathfrak{gl}(V)$, ϱ linear

$\varrho([x,y]) = \varrho(x)\varrho(y)-\varrho(y)\varrho(x)$

adjoint representation of $\mathfrak{g}$ $\operatorname{ad}:\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g})$ $\operatorname{ad}(x)y=[x,y]$

often $\operatorname{ad}(x)\equiv \operatorname{ad}_x$

$\forall x\in\mathfrak{g}$ ad_x is a derivation of $\mathfrak{g}$

D s.t. $\exists x\in\mathfrak{g}$ $D=\operatorname{ad}_x$ .. inner derivation, all other .. outer derivations

ideal I in $\mathfrak{g}$ — I subspace of $\mathfrak{g}$ s.t. $[I,\mathfrak{g}] \subset I$ (= span$\{[X,Y] \mid X\in I, Y\in\mathfrak{g}\}$)

$\mathfrak{g}^k = [\mathfrak{g},[\mathfrak{g},[\ldots,\mathfrak{g}]\ldots]]$ ($\mathfrak{g}$ k times) — similarly for any $V\subset\mathfrak{g}$

center $\mathcal{Z}(\mathfrak{g}) = \{X\in\mathfrak{g} \mid \forall Y\in\mathfrak{g}: [X,Y]=0\}$

Killing form $K(X,Y) = tr(ad_X ad_Y)$ (satisfies $K([X,Y],Z)+K(Y,[X,Z])$

Two basic types of Lie algebras

Def: $\mathfrak{g}$ is semisimple iff it does not have nonvanishing commutative ideals

Examples: $so(p,q)$, $p+q>2$, $sl(n)$, $n>1$, $su(n)$, $n>1$

Def: $\mathfrak{g}$ is solvable iff $(((\mathfrak{g}^2)^2)^2)\cdots \to 0$

Examples: • upper triangular matrices and any subalgebra of them

• $iso(2) = \text{span}\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\}$

What about algebras which are neither semisimple nor solvable?

Theorem (Levi) Any $\mathfrak{g}$, $\dim\mathfrak{g}<+\infty$ has unique maximal solvable ideal (radical) r. There exists a semisimple subalgebra s, unique up to isomorphism of $\mathfrak{g}$ such that

$\mathfrak{g} = s \oplus r$ (as vector spaces), i.e. $\mathfrak{g}$ is semidirect sum of s, r, $\mathfrak{g} = s \dotplus r$

i.e $[s,s]=s$, $[s,r]\in r$, $[r,r]\subsetneq r$

Ex: $iso(3,1)$ i.e. Poincaré algebra $iso(3,1) = so(3,1) \dotplus \mathbb{R}^4$

How to find out into which class a given Lie algebra belongs?

Theorem (Cartan) (1) $\mathfrak{g}$ is semisimple iff K is non-degenerate.

(2) $\mathfrak{g}$ is solvable iff $K(X,X)=0$ for all $X\in\mathfrak{g}^2$.

Next we shall study in some detail the structure of semisimple, Levi-type and solvable algebras.

Semisimple algebras are in fact direct sums of simple ones

$\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_N$, $[\mathfrak{g}_i,\mathfrak{g}_j] = \delta_{ij}\mathfrak{g}_i$

Def: $\mathfrak{g}$ is <u>simple</u> iff $\dim \mathfrak{g} > 1$ & only ideals of $\mathfrak{g}$ are $\{0\}$ and $\mathfrak{g}$.

Simple Lie algebras were fully classified. The classification is simpler over $\mathbb{C}$, over $\mathbb{R}$ several different real forms of a given complex simple algebra exist. The convenient mathematical object allowing to distinguish them is the signature of their Killing form.

Ex: $sl(2,\mathbb{C})$ has two real forms

$su(2) = so(3,\mathbb{R})$ $\quad sgn K = (0,3)$ i.e. negative definite

$sl(2,\mathbb{R})$ $\quad sgn K = (2,1)$

$so(4,\mathbb{C}) = so(3,\mathbb{C}) \oplus so(3,\mathbb{C})$ has real forms

$so(4,\mathbb{R}) = so(3,\mathbb{R}) \oplus so(3,\mathbb{R})$ $\quad sgn K = (0,6)$

$so(3,1)$ $\quad sgn K = (3,3)$

$so(2,2) = sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})$ $\quad sgn K = (4,2)$

<u>Review of Cartan's classification of complex simple Lie algebras</u>

Def: <u>rank</u> ℓ of simple Lie algebra $\mathfrak{g}$

$$\ell = \min_{X \in \mathfrak{g}} \dim \ker(ad_X)^K \quad \text{for } K \text{ big enough, e.g. } K = \dim \mathfrak{g}$$

In fact, for generic $X \in \mathfrak{g}$ $\ell = \dim \ker ad_X$. Such X is called regular. For regular $X \in \mathfrak{g}$, $\mathfrak{g}_0 = \ker ad_X$ is an Abelian subalgebra of $\mathfrak{g}$ and is called <u>Cartan subalgebra</u>. For a given choice of X, i.e. of $\mathfrak{g}_0$, we can always find a basis of $\mathfrak{g}$ $(H_\alpha, \alpha \in \Delta_p) \cup (E_\alpha, \alpha \in \Delta)$ where $\Delta_p \subseteq \Delta \subset (\mathfrak{g}_0)^*$ such that $\mathfrak{g}_0 = span\{H_\alpha\}$

$$\mathfrak{g} = \mathfrak{g}_0 \dot{+} \sum_{\alpha \in \Delta} span\{E_\alpha\}$$

We call $\alpha \in \Delta_p$ ... simple root

$\alpha \in \Delta$ ... root

H_α ... $\alpha(H) = K(H_\alpha, H)$, $\forall H \in \mathfrak{g}_0$

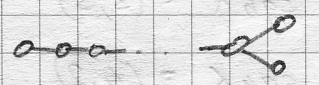
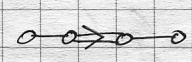
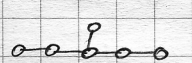
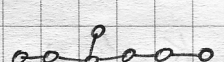
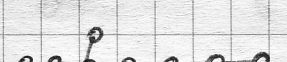
$[H_\alpha, H_\beta] = 0$, $\quad [H, E_\alpha] = \alpha(H) E_\alpha$, $\forall H \in \mathfrak{g}_0, \alpha \in \Delta$, $\quad [E_\alpha, E_{-\alpha}] \in \mathfrak{g}_0$

$[E_\alpha, E_\beta] \subset span\{E_{\alpha+\beta}\}$ when $\alpha + \beta \neq 0$ and $\alpha + \beta \in \Delta$. If $\alpha + \beta \notin \Delta \wedge \alpha + \beta \neq 0$ then $[E_\alpha, E_\beta] = 0$.

$\Delta = \Delta_+ \cup \Delta_-$ where $\alpha \in \Delta_+$ iff $-\alpha \in \Delta_-$. $\Delta_p \subseteq \Delta_+$ is a basis of $span\{\Delta\} = (\mathfrak{g}_0)^*$ such that any $\alpha \in \Delta_+$ is a linear combination of $\beta \in \Delta_p$ with integer nonnegative coefficients.

In fact, all information about root system and consequently about given simple Lie algebra, is encoded in angles between simple roots (where scalar product comes from Killing form restricted to $\mathbb{R}$-span$\{H_\alpha\}_{\alpha\in\Delta_p}$ and is positive definite). It turns out that these angles can be only $\frac{\pi}{2}, \frac{2}{3}\pi, \frac{3}{4}\pi, \frac{5}{6}\pi$.

If we associate to root systems the so-called Dynkin diagrams (i.e. graphs with vertices $\approx$ simple roots, # of lines joining any given pair of vertices given by angle: $0\sim\frac{\pi}{2}$, $1\sim\frac{2}{3}\pi$, $2\sim\frac{3}{4}\pi$, $3\sim\frac{5}{6}\pi$, arrow on 2 or 3 lines from longer to shorter vertex), we have the following celebrated classification

		rank	dim
	$A_\ell = sl(\ell+1,\mathbb{C})$	$\ell\geq 1$	$\ell(\ell+2)$
	$B_\ell = so(2\ell+1,\mathbb{C})$	$\ell\geq 2$	$\ell(2\ell+1)$
	$C_\ell = sp(2\ell,\mathbb{C})$	$\ell\geq 3$	$\ell(2\ell+1)$
	$D_\ell = so(2\ell,\mathbb{C})$	$\ell\geq 4$	$\ell(2\ell-1)$
	G_2	2	14
	F_4	4	52
	E_6	6	78 $= 6(2\cdot 6+1)$
	E_7	7	133
	E_8	8	248

where $sp(2\ell,\mathbb{C}) = \{A\in\mathbb{C}^{2\ell,2\ell} \mid A^TJ + JA^T = 0,\ J = \begin{pmatrix} 0 & I \\ -I & 0\end{pmatrix}\}$.

It is clear that determination of dimension and rank of a given simple Lie algebra takes us almost to its identification. The only problems are: B_ℓ and C_ℓ for the same ℓ have the same dimension, and for $\ell=6$ we have B_6, C_6 and E_6 with the same dim and rank.

I don't know about any simpler criterion for their identification other than finding the Cartan subalgebra, roots, simple roots and check the length of simple roots – for E_6 all have the same length, in B_ℓ one is shorter, in C_ℓ one is longer.

Examples:

Note: $sl(n) \Rightarrow K(X,Y) = 2n\,\mathrm{tr}\,X\cdot Y$

- $sl(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \middle| a,b,c \in \mathbb{C} \right\}$ i.e. $\mathrm{tr}\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = 0$

take $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\Rightarrow$ any $Y \in sl(2,\mathbb{C})$ such that $[H,Y]=0$ is multiple of H

$\Rightarrow g_0 = \mathrm{span}\{H\}$, eigen spaces of ad_H are $\mathrm{span}\{E_\pm\}$, $E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$ad_H X_+ = \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] = 2X_+ \Rightarrow \alpha(H) = 2$ $\qquad ad_H X_- = -\alpha(H) X_-$

H_α: $2 = \alpha(H) = K(H_\alpha, H) = 4\,\mathrm{tr}\,H_\alpha H \Rightarrow H_\alpha = \frac{1}{4}H$

$\Rightarrow sl(2) = \mathrm{span}\left\{\frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\} + \mathrm{span}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \mathrm{span}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\Delta = \{\alpha, -\alpha\}$, $\Delta_+ = \Delta_p = \{\alpha\}$

How can one obtain real forms of given complex simple algebra?

A general theorem claims that these can be found by looking for antilinear

$\sigma : g \to g$, $[\sigma(X), \sigma(Y)] = \sigma([X,Y])$, $\sigma^2 = \mathbb{1}$ and restricting the algebra to its real subspace on which $\sigma = id$.

In our example two possibilities exist

a) $\sigma(H) = H_\alpha$, $\sigma(E_+) = E_+$, $\sigma(E_-) = E_-$ $\Rightarrow sl(2,\mathbb{R})$

b) $\sigma(H) = -H_\alpha$, $\sigma(E_\pm) = -E_\mp$ $\Rightarrow$

$$su(2) = \mathbb{R}\text{-span}\left\{ iH_\alpha,\ E_+ - E_-,\ i(E_+ + E_-) \right\} \qquad (su(n) = \{A \in \mathbb{C}^{n,n} \mid A + A^\dagger = 0\})$$

In general, $\sigma = id$ on H_α, E_α gives us a real form of g with Killing form of signature $\left(\frac{\ell + \dim g}{2}, \frac{\dim g - \ell}{2}\right)$ whereas $\sigma = -id$ on $\mathbb{R}\text{-span}\{H_\alpha\}$, $\sigma(E_\alpha) = -E_{-\alpha}$ gives a real form

$$g_{comp} = \mathbb{R}\text{-span}_{\alpha\in\Delta_p}\{iH_\alpha\} + \mathbb{R}\text{-span}_{\alpha\in\Delta_+}\{E_\alpha - E_{-\alpha}\} + \mathbb{R}\text{-span}_{\alpha\in\Delta_+}\{i(E_\alpha + E_{-\alpha})\}$$

which has negative definite Killing form. Such real form is unique and is called compact because its simply connected Lie group is a compact manifold. Other real forms may or may not exist.

- Exercise: $sl(3,\mathbb{C})$ $\qquad g_0 = \mathrm{span}\left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}\right\}$, $E_\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = E_{12}$, $E_\beta = E_{23}$,

$E_{\alpha+\beta} = E_{13}$, $E_{-\alpha} = E_{21}$, $E_{-\beta} = E_{32}$, $E_{-\alpha-\beta} = E_{31}$

$\alpha\begin{pmatrix} a & & \\ & b & \\ & & -a-b \end{pmatrix} = a - b$, $\beta\begin{pmatrix} a & & \\ & b & \\ & & -a-b \end{pmatrix} = a + 2b$, $H_\alpha = \frac{1}{6}\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$, $H_\beta = \frac{1}{6}\begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}$

$\sphericalangle(\alpha,\beta) = \sphericalangle(H_\alpha, H_\beta)$: $\cos\sphericalangle = \dfrac{K(H_\alpha, H_\beta)}{\sqrt{K(H_\alpha,H_\alpha)K(H_\beta,H_\beta)}} = -\frac{1}{2} \Rightarrow \boxed{\sphericalangle = \frac{2\pi}{3}}$ $\Rightarrow$ Dynkin $\overset{\alpha}{\circ}\!\!-\!\!\!-\!\!\overset{\beta}{\circ}$

Also check that the compact form is $su(3)$.

How to distinguish simple and semisimple Lie algebras?

- dimension and rank (which are sums of those of simple ideals) is clearly not sufficient. Example: $so(32)$ vs. $E_8 \oplus E_8$ $\quad \dim = 496,\ \text{rank} = 16$
- one possibility over $\mathbb{C}$ is to find all ad-invariant symmetric bilinear forms, i.e. $\omega(ad_X Y, Z) + \omega(Y, ad_X Z) = 0$, $\omega(X,Y) = \omega(Y,X)$. The algebra is simple iff the only solutions are multiples of K. (On semisimple algebras any combination of Killing forms on simple ideals will satisfy the conditions.) This is a linear problem but doesn't immediately provide the decomposition if semisimple.
- A general procedure for decomposition into direct sum works as follows (for arbitrary algebra)

1) Find center of $\mathfrak{g}$ if any and write $\mathcal{Z}(\mathfrak{g}) = (\mathcal{Z}(\mathfrak{g}) \cap \mathfrak{g}^2) \dot{+} \mathfrak{a}$. ($\mathfrak{a}$: non-unique)
If $\mathfrak{a} \neq \{0\}$ write $\mathfrak{g} = \tilde{\mathfrak{g}} \dot{+} \mathfrak{a}$, where $\tilde{\mathfrak{g}}$ is any complement of $\mathfrak{a}$ in $\mathfrak{g}$ containing $\mathfrak{g}^2$. $\mathfrak{a}$ is called a central component and we have $\mathfrak{g} = \tilde{\mathfrak{g}} \oplus \mathfrak{a}$

2) Assume $\mathcal{Z}(\mathfrak{g}) = \mathcal{Z}(\mathfrak{g}) \cap \mathfrak{g}^2$. Construct $ad_{\mathfrak{g}} = \{ad_X \mid X \in \mathfrak{g}\} \subset \mathfrak{gl}(\mathfrak{g})$.
Find all $A \in \mathfrak{gl}(\mathfrak{g})$ such that $[A, ad_X] = 0$ for all $X \in \mathfrak{g}$.

3) Look for nontrivial idempotents among As, i.e. $A^2 = A \neq \mathbb{1}, 0$.
Iff such A exists, the algebra $\mathfrak{g}$ is a direct sum. Bring A to the form $A = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}$ by a change of basis in $\mathfrak{g}$. This realizes the decomposition into direct sum. Repeat on components of the sum.

Example: $\mathfrak{g} = \text{span}\{z_1, z_2, z_3, z_4\}$, $[z_1, z_2] = [z_3, z_4] = 0$, $[z_1, z_3] = [z_2, z_3] = z_3$,
$[z_1, z_4] = -[z_2, z_4] = z_4 \Rightarrow$

$$ad_{z_1} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix} \quad ad_{z_2} = \begin{pmatrix} 0 & 0 \\ 0 & \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \end{pmatrix} \quad ad_{z_3} = \begin{pmatrix} 0 & 0 \\ \begin{smallmatrix} -1 & -1 \\ 0 & 0 \end{smallmatrix} & 0 \end{pmatrix} \quad ad_{z_4} = \begin{pmatrix} 0 & 0 \\ \begin{smallmatrix} 0 & 0 \\ -1 & 1 \end{smallmatrix} & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} \begin{smallmatrix} a & b \\ b & a \end{smallmatrix} & \\ & \begin{smallmatrix} a+b & \\ & a-b \end{smallmatrix} \end{pmatrix}$$, nontrivial idempotent $a = b = \frac{1}{2}$ (from $\det = 0$ & $A^2 = A$) or $a = -b = \frac{1}{2}$

$a = b = \frac{1}{2} \Rightarrow$ eigenspaces $\text{span}\{z_1 - z_2, z_4\} \to 0$, $\text{span}\{z_1 + z_2, z_3\} \to 1 \Rightarrow$ change of basis

$\tilde{z}_1 = \frac{1}{2}(z_1 + z_2)$, $\tilde{z}_2 = z_3$, $\tilde{z}_3 = \frac{1}{2}(z_1 - z_2)$, $\tilde{z}_4 = z_4 \Rightarrow$ $\boxed{\mathfrak{g} = \text{span}\{\tilde{z}_1, \tilde{z}_2\} \oplus \text{span}\{\tilde{z}_3, \tilde{z}_4\}}$ $\quad [\tilde{z}_1, \tilde{z}_2] = \tilde{z}_2$, $[\tilde{z}_3, \tilde{z}_4] = \tilde{z}_4$

Algebras with nontrivial Levi decomposition

Example: $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \oplus S_{3,1}(1)$

$S_{3,1}(1) = \mathrm{span}\{e_1, e_2, e_3\}$ $\quad [e_1,e_2] = e_2 \quad [e_1,e_3] = e_3 \quad [e_2,e_3] = 0$

i.e. $[e_1, \begin{pmatrix} e_2 \\ e_3 \end{pmatrix}] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_2 \\ e_3 \end{pmatrix}$

$\mathfrak{sl}(2,\mathbb{R}) = \mathrm{span}\{H, X_+, X_-\}$ $\quad [H, X_+] = 2X_+, \quad [H, X_-] = -2X_-,$

$[X_+, X_-] = H$

$\mathfrak{sl}(2,\mathbb{R}) \triangleright S_{3,1}(1)$: $[S, e_1] = 0 \qquad [H, \begin{pmatrix} e_2 \\ e_3 \end{pmatrix}] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e_2 \\ e_3 \end{pmatrix}$

$[X_+, \begin{pmatrix} e_2 \\ e_3 \end{pmatrix}] = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e_2 \\ e_3 \end{pmatrix}, \quad [X_-, \begin{pmatrix} e_2 \\ e_3 \end{pmatrix}] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e_2 \\ e_3 \end{pmatrix}$

What do the Jacobi identities of semisimple sum tell us?

$\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}$

$[s,s,s]+\ldots = 0 \iff \mathfrak{s}$ is Lie algebra

$[r,r,r]+\ldots = 0 \iff \mathfrak{r}$ is Lie algebra

$[s,s,h]+\ldots = 0 \iff \mathrm{ad}_{\mathfrak{s}}$ defines a representation of $\mathfrak{s}$ on $\mathfrak{r}$

$[s,r,r]+\ldots = 0 \iff \forall X \in \mathfrak{s}: \mathrm{ad}_X$ is derivation of $\mathfrak{r}$

⇒ Two necessary conditions for nontrivial Levi decomposition of an algebra with ideal $\mathfrak{r}$ and semisimple subalgebra $\mathfrak{s}$ (i.e. other than $\mathfrak{s} \oplus \mathfrak{r}$)

- $\mathfrak{r}$ as an vector space must allow nontrivial representation of $\mathfrak{s}$

  e.g. ✗ nontrivial $\mathfrak{so}(3,\mathbb{R}) \oplus \mathbb{R}^2$

  but the representation may be reducible even for indecomposable $\mathfrak{g}$ (i.e. such that there is no decomposition into direct sum)

- the algebra of derivations of $\mathfrak{r}$ must contain $\mathfrak{s}$ as a subalgebra (otherwise at least some simple components of $\mathfrak{s}$ act trivially on $\mathfrak{r}$ and we have again direct sum). E.g. for $\dim \mathfrak{r} = 3$ only Abelian, Heisenberg and above introduced $S_{3,1}(1)$ have (semi) simple subalgebra of $\mathrm{Der}\,\mathfrak{r}$, all others (e.g. $S_{3,1}(a)$: $[e_1, \begin{pmatrix} e_2 \\ e_3 \end{pmatrix}] = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} e_2 \\ e_3 \end{pmatrix}$, $[e_2, e_3] = 0$, $a \neq 1$) have solvable algebra of derivations. Similarly, $\mathrm{Der}\, S_{3,1}(1)$ contains $\mathfrak{sl}(2,\mathbb{R})$ subalgebra but not $\mathfrak{so}(3)$ ⇒ no $\mathfrak{so}(3) \oplus S_{3,1}(1)$ exists.

This observation is starting point for classification in low dimensions.

Given $\mathfrak{g}$, e.g. $\mathfrak{iso}(3,1)$, how does one find its Levi decomposition?

1) Identify $\mathfrak{r}$ as orthogonal complement w.r.t. K of $\mathfrak{g}^2$.

2) Construct semisimple factor algebra $\mathfrak{g}/\mathfrak{r}$ and find its structure $\rightarrow$ the structure of $\mathfrak{s}$.

3) Find the concrete immersion of $\mathfrak{s}$ into $\mathfrak{g}$ (this is the only non-unique part of the structure). This task can be linearized by reduction to the computation with $\mathfrak{r}$ Abelian in the following way

a) make sure $\mathfrak{g}$ is indecomposable - otherwise go to its components,

b) check whether $\mathfrak{g}$ is perfect, i.e. $\mathfrak{g}^2=\mathfrak{g}$, if not go to $\mathfrak{g}^2 \subset \mathfrak{g}$ (necessarily $\mathfrak{s} \subset \mathfrak{g}^2$) and repeat,

c) check whether $\mathfrak{r}^2=0$. If not find $\overline{\mathfrak{g}}=\mathfrak{g}/\mathfrak{r}^2$ with Abelian radical and its decomposition $\overline{\mathfrak{g}}=\overline{\mathfrak{p}}\dotplus R(\overline{\mathfrak{g}})$. Find preimage $\mathfrak{g}_1$ of $\overline{\mathfrak{p}}$ in $\mathfrak{g}$, i.e $\mathfrak{g}_1 \subsetneq \mathfrak{g}$, $\mathfrak{g}_1/\mathfrak{r}^2=\overline{\mathfrak{p}}$ and repeat for $\mathfrak{g}_1$)

d) $\mathfrak{g} \supset \mathfrak{r}$, $\mathfrak{r}^2=0$. Consider basis $(R_1,\dots,R_m,S_1,\dots,S_e)$ of $\mathfrak{g}$, where $(R_1,\dots,R_m)$ is basis of $\mathfrak{r}$. We necessarily have $[S_i,S_j]=f_{ij}{}^k S_k + g_{ij}{}^n R_n$. Construct a transformation $S_i \rightarrow \tilde{S}_i = S_i + A_i{}^n R_n$. Then $[\tilde{S}_i,R_n]=[S_i,R_n]$ and we request $[\tilde{S}_i,\tilde{S}_j]=f_{ij}{}^k \tilde{S}_k$, i.e.

$$A_j{}^h \underbrace{[S_i,R_h]}_{h_{in}{}^p R_p} + A_i{}^n \underbrace{[R_n,S_j]}_{-h_{jn}{}^p R_p} = f_{ij}{}^k A_k{}^h R_h + g_{ij}{}^n R_n)$$

Note: without $\mathfrak{r}^2=0$ a quadratic term in A would appear.

a system of inhomogeneous linear equations for $A_k{}^n$, highly overdetermined but a solution exists by virtue of Levi's theorem. Pick one and $(R_1,\dots,R_m,\tilde{S}_1,\dots,\tilde{S}_e)$ is the desired decomposition.