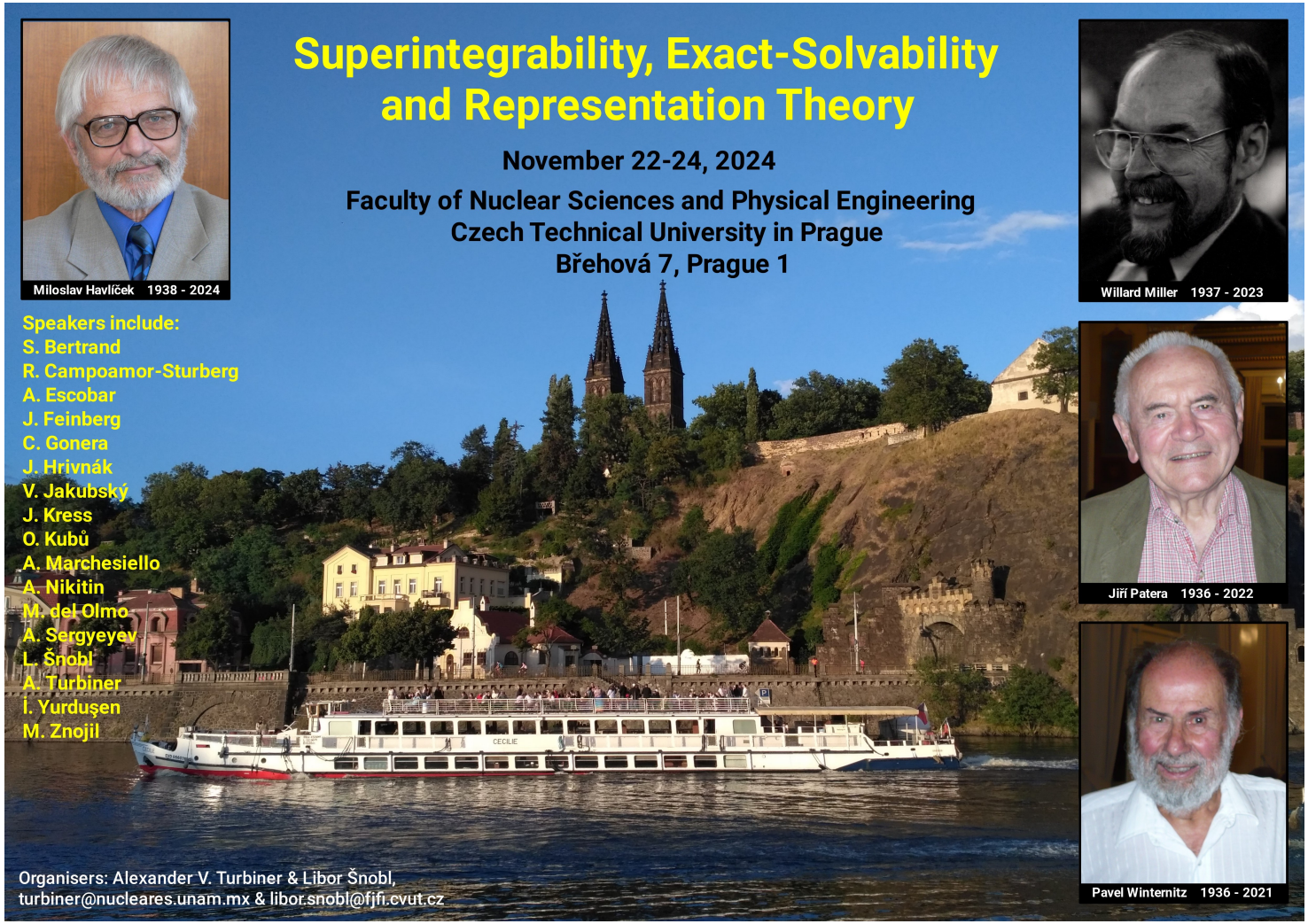
# Superintegrability, Exact-Solvability, and Representation Theory

Prague, November 22 – 24, 2024

## Programme and abstracts


Superintegrability, Exact-Solvability and Representation Theory
November 22-24, 2024
Faculty of Nuclear Sciences and Physical Engineering
Czech Technical University in Prague
Břehová 7, Prague 1
Miloslav Havlíček 1938 - 2024
Willard Miller 1937 - 2023
Jiří Patera 1936 - 2022
Pavel Winternitz 1936 - 2021
Speakers include:
S. Bertrand
R. Campoamor-Sturberg
A. Escobar
J. Feinberg
C. Gonera
J. Hrivnák
V. Jakubský
J. Kress
O. Kubů
A. Marchesiello
A. Nikitin
M. del Olmo
A. Sergyeyev
L. Šnobl
A. Turbiner
I. Yurduşen
M. Znojil
Organisers: Alexander V. Turbiner & Libor Šnobl,
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# Conference program

**Friday, November** $22^{nd}$

**Afternoon session**

Chairman: L. Šnobl

- 14:00 – 14:10 Conference opening
- 14:10 – 14:55 Alexander Turbiner: *Remembering Willard Miller Jr: Beyond separation of variables, superintegrability*
- 15:00 – 15:45 Jiří Hrivnák : *Jiří Patera*

break

Chairman: A. Turbiner

- 16:15 – 17:00 Libor Šnobl: *Memories of Pavel Winternitz*
- 17:05 – 17:50 Rutwig Campoamor–Stursberg: *The scientific legacy of Miloslav Havliček*

**Saturday, November** $23^{rd}$

**Morning session**

Chairman: J. Feinberg

- 09:30 – 10:00 Miloslav Znojil: *Exactly solvable Calogero–like models*
- 10:05 – 10:35 Lenka Motlochová: *Triangular graphene quantum dots with zigzag edges*

break

Chairman: A. Sergyeyev

- 11:05 – 11:35 Mariano del Olmo: *Pavel Winternitz and superintegrability: Superintegrable systems on the sphere*
- 11:40 – 12:10 Sébastien Bertrand: *Indecomposable representations of* $\mathfrak{sl}(2)$ *and the Laguerre polynomials*

**Afternoon session**

Chairman: M. del Olmo

- 14:00 – 14:30 Jonathan Kress: *The algebraic variety of abundant superintegrable systems*
- 14:35 – 15:05 Antonella Marchesiello: *Superintegrable families of magnetic monopoles in curved background*
- 15:10 – 15:40 Anatoly Nikitin: *Integrable and superintegrable systems admitting Lie symmetries*

break

Chairman: R. Campoamor–Stursberg

- 16:15 – 16:45 İsmet Yurduşen: *Superintegrability in the interaction of two particles with spin*
- 16:50 – 17:20 Ondřej Kubů: *Quadratic integrable system in magnetic field: beyond configuration space separability*

**Sunday, November** $24^{th}$

**Morning session**

Chairman: M. Znojil

- 09:30 – 10:00 Vít Jakubský: *Flat-band engineering of atomic chains via supersymmetric transformations*
- 10:05 – 10:35 Joshua Feinberg: *Plasmonic time crystals*

break

Chairman: J. Kress

- 11:05 – 11:35 Artur Sergyeyev: *Superintegrable Benenti systems through the Hamilton–Jacobi lens*
- 11:40 – 11:50 Conference closing

# Abstracts

**Sébastien Bertrand**

## Indecomposable representations of $\mathfrak{sl}(2)$ and the Laguerre polynomials

In this presentation, we will discuss the connection between representations of $\mathfrak{sl}(2)$ and special functions. More precisely, we will study two representations of $\mathfrak{sl}(2)$ and its association with the (generalized) Laguerre polynomials. The outcome is infinite-dimensional chains of polynomials linked by the $\mathfrak{sl}(2)$ generators, and satisfy non-homogeneous second-order ordinary differential equations. Several properties of the chains and polynomials are studied. This research was in collaboration with Sarah Post, Ian Marquette, and Willard Miller, and it was inspired by Willard's book on "Lie Theory and Special Functions".

**Rutwig Campoamor-Stursberg**
Instituto de Matemática Interdisciplinar, Universidad Complutense de Madrid

## The scientific legacy of Miloslav Havliček

In this talk, I will focus on the main research achievements of Miloslav Havliček in algebraic methods in Quantum Theory, representation theory of Lie algebras and superalgebras, as well as his extensive work on realizations of Lie algebras.

**Joshua Feinberg**
University of Haifa

## Plasmonic time crystals

We study plasmonic time crystals, which constitute an extension of the dielectric-based photonic time crystals to plasmonic media. A salient feature of plasmonic time crystals is their ability to support amplification of both longitudinal and transverse modes. In particular, we show that such systems support collective resonances of longitudinal modes, independently of the wave vector $k$. These resonances originate from the interaction between

the positive and negative frequency branches of the plasmonic dispersion relation of the unmodulated system, and from the divergence of the density of states near the plasma frequency $\omega_p$. The strongest resonance occurs at a modulation frequency of $\Omega = 2\omega_p$, associated with a direct inter-band transition. Higher-order resonances are associated with related mechanisms but at lower modulation frequencies. The mathematical formalism governing these resonances is Hill's equation. We demonstrate these resonances for various periodic modulation profiles, and also provide a generic perturbative formula (interesting on its own from the point of view of the theory of Hill's equation) for resonance widths in the limit of weak modulation amplitude, in the absence of losses. Our findings offer insights into using time-modulated plasmonic media to enhance optical gain.

**Jiří Hrivnák**
Czech Technical University in Prague

## Jiří Patera

The exceptional life and illustrious academic career of professor Jiří Patera, a Czech-Canadian mathematician and physicist, is summarized. Accomplished during the last sixty years with various international collaborators, numerous significant results concerning subalgebras of simple Lie algebras, gradings and contractions of simple Lie algebras and their representations, elements of finite order in simple Lie groups, discrete Fourier transforms associated with simple Lie groups, polytopes of 3D Coxeter groups and quasicrystals are outlined. The lasting legacy and wide-ranging influence of these achievements is discussed.

**Vít Jakubský**
Nuclear Physics Institute of CAS, Řež

## Flat-band engineering of atomic chains via supersymmetric transformations

Quasi-one-dimensional chains of atoms can be effectively described by one-dimensional Dirac-type equations. Crystal structure of the chain is reflected by the pseudo-spin of the quasi-particles. In the talk, there will be presented a simple framework where supersymmetric transformation is utilized to generate an interaction between two, initially non-interacting systems

described by pseudo-spin-one Dirac-type equations. In the presented example, the transformation converts two asymptotically non-interacting atomic chains into stub- or saw-chain locally. The model possesses a flat band whose energy can be fine-tuned deliberately.

**Jonathan Kress**
University of New South Wales

## The algebraic variety of abundant superintegrable systems

Non-degenerate second-order superintegrable systems in two- and three-dimensional conformally flat spaces were shown, by Willard Miller and others, to be to be parameterised by an algebraic variety. A maximally second-order superintegrable system has $2n-1$ functionally independent integrals quadratic in the momenta, and all known examples in fact have $n(n+1)/2$ linearly independent second-order integrals. Such systems are said to be abundant. In this talk I will describe some recent work with Konrad Schöbel and Andreas Vollmer describing the algebraic variety of abundant superintegrable systems in $n$ dimensions and has led to interesting connections between superintegrable systems and other seemly unrelated fields.

**Ondřej Kubů**
Czech Technical University in Prague

## Quadratic integrable system in magnetic field: beyond configuration space separability

We will review results contained in our PhD thesis and our recent research program. We first focus on systems not connected to separation in orthogonal coordinates, which we call nonstandard. Then we turn to Haantjes algebras that allow searching for separable coordinates on the whole phase space. In the last part of the talk we will present additional geometrical approaches that might bring better insight into integrable systems beyond configuration space separability.

**Mariano A. del Olmo**
Departamento de Física Teórica, Atómica y Óptica and IMUVA
Universidad de Valladolid, Spain

## Pavel Winternitz and superintegrability : *Superintegrable systems on the sphere*

In this talk we revisit a family of superintegrable systems on $O(p,q)$ hyperboloids developed by Pavel Winternitz, Miguel Angel Rodríguez and myself around 1993 ("Integrable Systems based on $SU(p,q)$ Homogeneous Manifolds". *J. Math. Phys.* **34**, 5118–5139 (1993)). We present several developments of this family from its beginnings to the present.

**Antonella Marchesiello**
Czech Technical University in Prague

## Superintegrable families of magnetic monopoles in curved background

We review some known results on the superintegrability of monopole systems in the three-dimensional (3D) Euclidean space and in the 3D generalized Taub-NUT spaces. We show that these results can be extended to certain curved backgrounds that, for suitable choice of the domain of the coordinates, can be related via conformal transformations to systems in Taub-NUT spaces. These include the multifold Kepler systems as special cases. The curvature of the space is not constant and depends on a rational parameter that is also related to the order of the integrals. New results on minimal superintegrability when the electrostatic potential depends on both radial and angular variables are also presented.

**Lenka Motlochová**
Czech Technical University in Prague

## Triangular graphene quantum dots with zigzag edges

The tight-binding zigzag Hamiltonians of the electron propagation in the triangular zigzag graphene quantum dots surrounded by the Neumann and Dirichlet walls up to the next-to-nearest neighbour coupling are expressed as the sums of the $E_0$–multiples of the identity operator and two hopping

operators corresponding to the nearest and next-to-nearest neighbour hopping of the electron including the boundary wall interactions. For the zigzag honeycomb dots are closely connected to the irreducible crystallographic root system $A_2$, the essential terminology related to the Weyl group of $A_2$ and pertinent properties of the zigzag orbit functions are presented to provide the explicit solutions of the corresponding time-independent zigzag Schrödinger equations.

**Artur Sergyeyev**

Silesian University in Opava, Czech Republic

## Superintegrable Benenti systems through the Hamilton–Jacobi lens

For two large classes of maximally superintegrable Benenti systems in $n$ dimensions with arbitrarily large $n$ we solve the Hamilton–Jacobi equation and comment upon solving the associated Schrödinger equation.

**Libor Šnobl**

Czech Technical University in Prague

## Memories of Pavel Winternitz

After briefly introducing Pavel's illustrious and peregrinating career I shall focus on his research achievements, spanning more than half a century of fruitful scientific life. To conclude, I also introduce his personality, recalling a few of his more adventurous nonscientific accomplishments.

**Alexander Turbiner**
ICN Universidad Nacional Autónoma de México, Stony Brook University and CRM, Montréal

## Remembering Willard Miller Jr: Beyond separation of variables, superintegrability

Three concepts associated with name of Willard Miller Jr (1937 - 2023) - towering figure in mathematics (analysis, PDE, special functions, orthogonal polynomials, theory of operators) and theoretical physics (planar and many-body quantum mechanics, (super)-integrable systems) - will be presented:

1. Beyond of separation of variables in many-body physics,
2. Superintegrability: Tremblay–Turbiner–Winternitz (TTW) model at rational index $k$,
3. Superintegrability: polynomial algebras of integrals.

**İsmet Yurduşen**
Hacettepe University, Ankara

## Superintegrability in the interaction of two particles with spin

I will focus on the systematic investigation of the interaction of two particles with spin. First the case where one of the particles spin is constrained to zero will be fully discussed and for one nontrivial example it will be shown how superintegrability leads to exact solvability. Then the similar systematic approach is applied to two non-relativistic particles, both with spin 1/2. Several new superintegrable systems will be presented. The presentation consists of joint works with Pavel Winternitz, Jean-Francois Désilets and Orhan Oğulcan Tuncer. Certain parts of this work was financially supported by TÜBİTAK (The Scientific and Technological Research Council of Türkiye) 1001 program with project number 123F161.

**Miloslav Znojil**
Nuclear Physics Institute of CAS, Řež and Institute of Systems Science,
Durban University of Technology

## Exactly solvable Calogero–like models

Two alternative generalizations of the popular Calogero's rational ($A_N$) $A-$body quantum model will be presented. The idea of the first innovation lies in an asymmetrization of the repulsive two-body barriers. The exact solvability appears to be preserved even when more freely variable couplings are admitted. The degeneracy of spectrum can partially or completely be removed as a consequence. The second, alternative innovation relies on making the two-body barriers soft and penetrable. What then emerges is a tunneling between Weyl chambers (WCh). Their "natural" numbering becomes necessary, with the task discussed and illustrated in more detail at $A = 4$, with WChs projected on tetrakishexahedron.

**References:**

[**1** ] Znojil, M. Calogero-like Model without Rearrangement Symmetry. Symmetry 2024, 16, 27.

[**2** ] Znojil, M.; Tater, M. Exactly solvable three-body Calogero-type model with translucent two-body barriers. Phys. Lett. A 2001, 284, 225–230.

[**3** ] Fring, A.; Znojil, M. PT-symmetric deformations of Calogero models. J. Phys. A Math. Gen. 2008, 41, 194010.

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