PLASMONIC TIME CRYSTALS

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Superintegrability, Exact Solvability and Representation Theory

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Plasmonic Time Crystals:
an EM parametric oscillator utilizing plasmon modes

Context: intensive recent activity on

Photonic Time Crystals

time varying host dielectric with permittivity $\varepsilon(t)$

periodic time modulation $\varepsilon(t+T) = \varepsilon(t)$ $T = \frac{2\pi}{\Omega}$

modulation frequency

would typically require strong and abrupt modulation to observe at optical frequencies

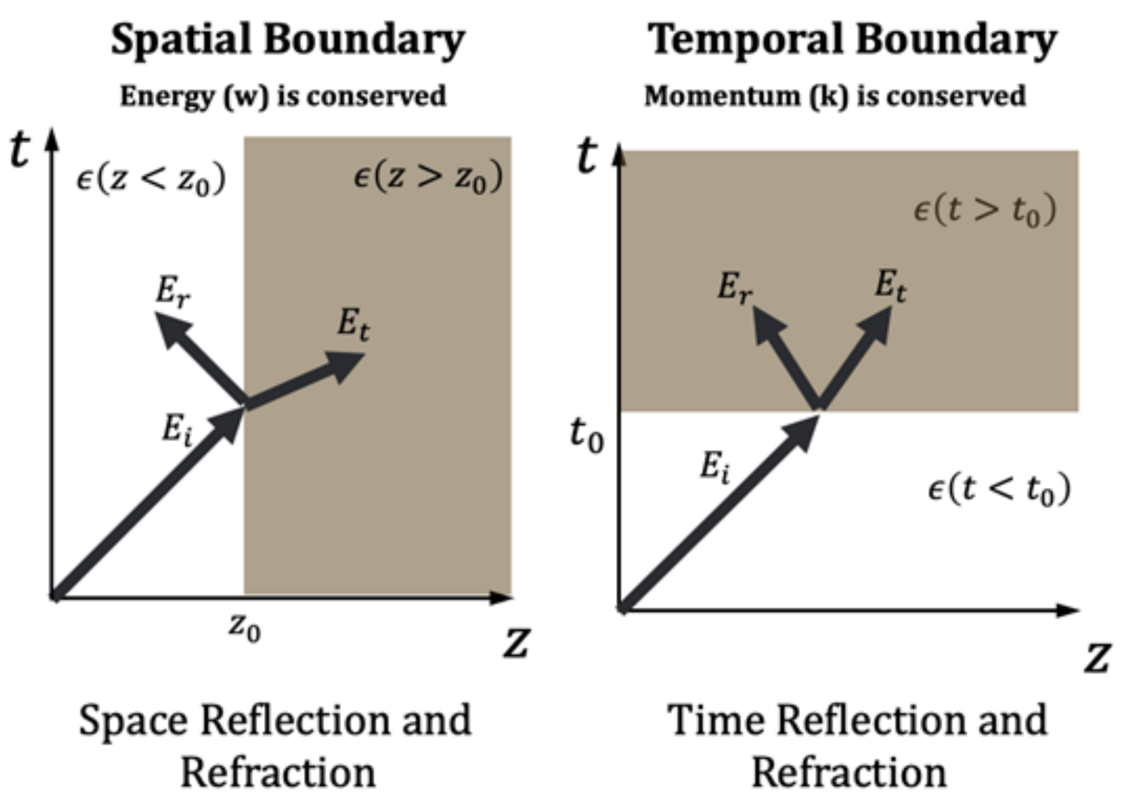


Fig. 1. Reflection by a spatial interface vs reflection by a temporal interface. A monochromatic plane wave incident on a spatial boundary vs a plane wave experiencing an abrupt change in permittivity (a temporal boundary), respectively.

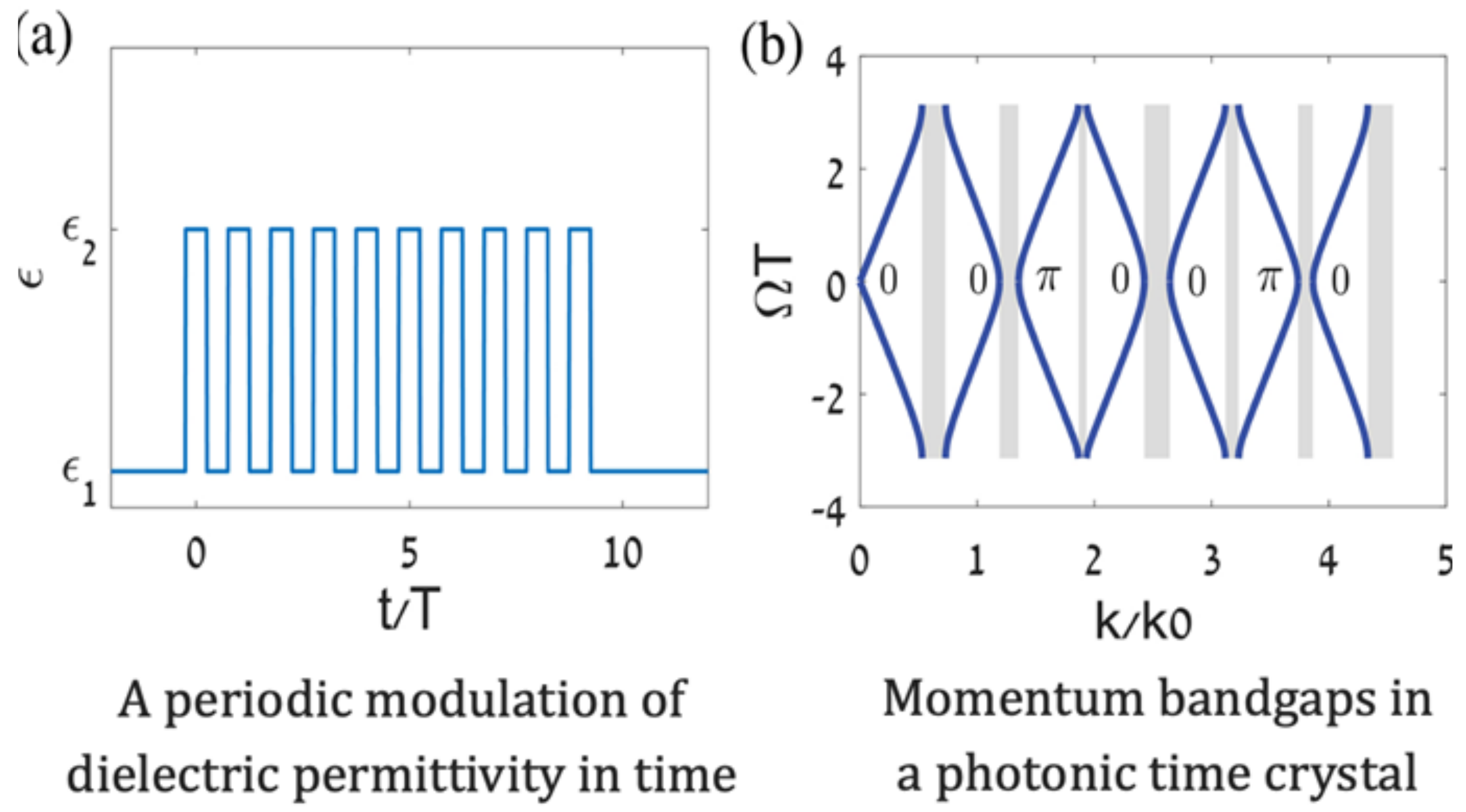


Fig. 2. Schematic of a photonic time-crystal concept where the dielectric permittivity of the material is modulated periodically in time (a), giving rise to dispersion relation characterized by bands separated by significant gaps in the momentum **k** (b).

from Boltasseva, Shalaev & Segev, *Optical Materials Express* **14**(3)(2024)592

see also Lustig, Segal, Shoam, Fruhling, Shalaev, Boltasseva & Segev, *Optics Express* **31**(6)(2023)9165

Plasmonic Time Crystals

periodically time modulated electron plasma in medium

modulate periodically in time *both* the permittivity $\varepsilon(t)$ of the hosting medium

and the electron concentration $N_0(t)$ in the medium

$$\varepsilon(t+T) = \varepsilon(t)$$

$$N_0(t+T) = N_0(t)$$

common modulation frequency $\Omega = \frac{2\pi}{T}$

time variation of ε may be attributed to the effect of the external driving on the bound electrons of the ambient crystal

modulation of both quantities could be taken to be weak

$$\varepsilon(t) = \varepsilon_0(1 + \delta\varepsilon(t))$$

constant permittivity of the medium in equilibrium

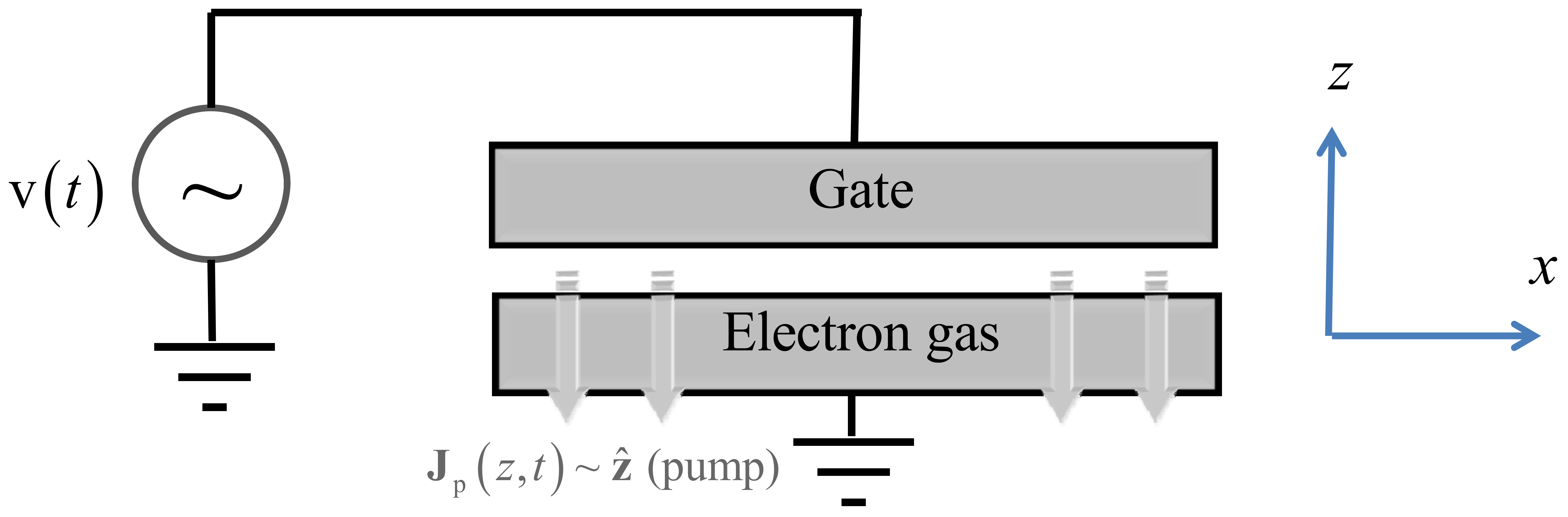
small (weak) fractional periodic dimensionless modulation profile

$$N_0(t) = n_0(1 + \delta n(t))$$

unmodulated electron density in the medium in steady state

small (weak) fractional dimensionless modulation profile

Experimental Realization



a grounded semiconductor slab of subwavelength thickness positioned beneath a gate plate which together form a capacitor

apply time-varying voltage across the capcitor to modulate the amount of charge $\propto N_0(t)$ in the semiconductor

charge may be injected or extracted by grounding the semiconductor

General: wave propagation in the time-modulated plasma

Maxwell's equations

$$\frac{1}{c}\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t} = -\nabla \times \mathbf{E}(\mathbf{r},t)$$

$$\frac{1}{c}\frac{\partial\left(\varepsilon(t)\mathbf{E}(\mathbf{r},t)\right)}{\partial t} = \nabla \times \mathbf{B}(\mathbf{r},t) - \frac{4\pi}{c}\mathbf{J}(\mathbf{r},t)$$

coupled with the *linearized* continuity and transport equations

$$\frac{\partial n(\mathbf{r},t)}{\partial t} + N_0(t)\nabla\cdot\mathbf{v}(\mathbf{r},t) = 0$$

$$\frac{\partial \mathbf{v}(\mathbf{r},t)}{\partial t} = \frac{e}{m}\mathbf{E}(\mathbf{r},t) - \nu\mathbf{v}(\mathbf{r},t)$$

ν collision frequency which accounts to losses in the system

linearization w.r.t. the small quantities

$n(\mathbf{r},t) \ll N_0(t)$ local fluctuating part of the electron density (riding atop the spatially uniform background density $N_0(t)$)

$\mathbf{v}(\mathbf{r},t)$ electron velocity field (much smaller than the velocity induced by the pump)

$\mathbf{J} = eN_0(t)\,\mathbf{v}$ the (linearized) fluctuating part of the electron current

Goal:

to characterize how the external driving (strong signal) affects propagation of another wave (weak signal) propagating in the plasma channel in the xy-plane.

the plasma is *spatially homogeneous*: $N_0(t)$ & $\varepsilon(t)$ are space independent

thus, it is enough to consider plane wave excitations with spatial dependence $\sim e^{i\mathbf{k}\cdot\mathbf{r}}$

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the plasma is *isotropic* $\Longrightarrow$ excitations are decoupled into longitudinal and transverse modes

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for lack of time we shall concentrate in the rest of this talk on the effect of time modulation of ***longitudinal plasmons***, which is enough to explain the concept of ***Plasmonic Time Crystal (PLTC)***

Longitudinal Plasmons

for these waves $\mathbf{B} = 0$ and $\mathbf{E} = -\nabla\phi$ ϕ =electric potential

Gauss law (dynamic Poisson equation) $\nabla \cdot \mathbf{E}(\mathbf{r}, t) \equiv -\nabla^2\phi(\mathbf{r}, t) = \dfrac{4\pi e}{\varepsilon(t)} n(\mathbf{r}, t)$

introduce the scalar function $f(\mathbf{r}, t) = \nabla \cdot \mathbf{v}(\mathbf{r}, t)$

for the sake of simplicity, we shall consider below lossless system and set $\nu = 0$

in addition, if $\varepsilon(t)$ never vanishes, there is no essential loss of generality by assuming time independent $\varepsilon(t) = \varepsilon_0$

the plasma frequency (squared) of the unmodulated medium in equilibrium $\omega_p^2 = \frac{4\pi e^2 n_0}{m\varepsilon_0}$

then it follows from Maxwell's equations and from the linearized continuity and transport equations that

$$\frac{\partial^2 f}{\partial t^2} + \omega_p^2 (1 + \delta n(t)) f = 0$$

namely, the equation of a parametric oscillator!

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a unique feature of this equation: it **does not** depend on the wave number k

this is specific to longitudinal plasmons, due to their k-independent dispersion relation (in the absence of modulation) $\omega = \pm\omega_p$

this implies that after time modulation is switched on, exponential growth of f (and all other relevant quantities) becomes possible, simultaneously for *all k-modes*

this collective instability facilitates interaction the plasma with the external driving circuitry, culminating in strong parametric amplification

Floquet Theory

periodic driving $\delta n(t+T) = \delta n(t)$

modulation period T

modulation frequency $\Omega = \frac{2\pi}{T}$

the two solutions of $\frac{\partial^2 f}{\partial t^2} + \omega_p^2(1+\delta n(t))f = 0$

are of the form $f_{1,2}(t) = e^{-i\omega_{1,2}t}u_{1,2}(t)$, where $\omega_{1,2}$ are the two fundamental frequencies, and $u_{1,2}(t)$ are periodic functions with period $T = 2\pi/\Omega$. Thus, $f_{1,2}(t+T) = e^{-i\omega_{1,2}T}f_{1,2}(t) = \Lambda_{1,2}f_{1,2}(t)$, where $\Lambda_{1,2} = e^{-i\omega_{1,2}T}$ are the eigenvalues of the temporal transfer matrix $\boldsymbol{T}$, which acts on the two dimensional space of solutions of the ODE for f and propagates them through one period of time. The fundamental frequencies $\omega_{1,2} = \omega'_{1,2} + i\omega''_{1,2}$ are typically complex. Thus, if at least one of the imaginary parts $\omega''_{1,2}$ is negative, the corresponding eigenvalues $\Lambda_{1,2}$ will be larger than 1 in absolute values. Therefore, the corresponding Floquet solution will grow as function of time, rendering the system unstable.

The eigenvalues $\Lambda_{1,2}$ depend on the modulation frequency Ω and on the parameters controlling the (dimensionless) modulation profile $\delta n(t)$. This parameter space is split into regions of stability and instability, separated by borderlines. In regions of stability, oscillations of $f(t)$ are bounded, while in regions of instability $f(t)$ exhibits resonant behavior, with unbounded oscillations.

Examples: various modulation profiles

(i) *harmonic modulation*: (Mathieu's equation)

$$\delta n(t) = \delta n_{\mathrm{M}} \cos \Omega t$$

it exhibits rich structure of stability and instability regions in the space of parameters Ω/ω_p and δn_{M}. In particular, for $\Omega = 2\omega_p$, it is well known that the amplitude of oscillations grows exponentially approximately as $\exp(\delta n_{\mathrm{M}} \omega_p t/4)$. It is instructive to interpret this instability as a collective resonance (i.e., for all values of

discussed clearly in L&L Mechanics

instructive interpretation: a collective resonance (i.e., for all values of the wavenumber k) between the two branches $\pm\omega_p$ of the static (unmodulated) crystal this instability persists also when Ω is slightly detuned away from $2\omega_p$, albeit with a slightly smaller growth exponent

$$\omega'' \simeq \frac{\omega_p}{4}\sqrt{\delta n_M^2 - 16\Delta^2}$$

where $\Delta = (\Omega/2\omega_p) - 1$ is the detuning parameter. The width of the instability region around $\Omega = 2\omega_p$ is determined by reality of ω''. In particular, this means that there is a minimal threshold value of $\delta n_M^{\mathrm{Th}} = 4|\Delta|$ required to induce instability. Similar but weaker instabilities occur also at modulation frequencies around

$\Omega_n = 2\omega_p/n$ with integer n

(ii) *periodic piecewise constant modulation*: In this case, in its simplest form, $\delta n(t)$ assumes one constant value δn_1 during the first part $0 \leq t < \tau$ of the modulation period and another constant value δn_2 during its remaining part $\tau \leq t < T$ [44,45]

the spatial analog of this temporal modulation is the celebrated Kronig-Penney model for electronic energy bands in one dimensional crystals focus on the modulation $\delta n(t) = \delta n_{\mathrm{M}} \operatorname{sgn}(\sin(\Omega t))$

which flips sign at the middle of the modulation period. Here, as in the case of Mathieu's equation, one obtains a rich chart of stability and instability regions in the plane of parameters Ω/ω_p and δn_M

in particular, for weak modulation amplitude δn_M one finds an infinite family of resonances analogous to the aforementioned resonance of the Mathieu equation, centered at modulation frequencies $\Omega_n^{\mathrm{odd}} = 2\pi/T = 2\omega_p/(2n+1)$ with integer n, and with growth exponents $\omega'' = \omega_p \sqrt{\frac{\delta n_M^2}{((2n+1)\pi)^2} - \Delta^2}$

where $\Delta = \Omega/\Omega_n^{\mathrm{odd}} - 1$ is the detuning parameter. Thus, at the center of the resonance ($\Delta = 0$), $\omega''_{\mathrm{Res}} = \omega_p \delta n_M/(2n+1)\pi$ is linear in δn_M, and is therefore of leading order in perturbation theory. Furthermore, similarly to the corresponding result (9) for the Mathieu case (8), detuning the modulation frequency away from the resonance requires a threshold value $\delta n_M^{\mathrm{Th}} = (2n+1)\pi|\Delta|$ of the modulation amplitude to induce instability with diminished ω''.

in addition to this family of 'linear' resonances, we also found yet another family of weaker resonances, centered in the vicinity of modulation frequencies $\Omega_n^{\text{even}} = 2\omega_p/2n = \omega_p/n$ with integer n, with growth exponents

$$\omega'' = \omega_p \sqrt{\left(\frac{\delta n_M^2}{4}\right)^2 - \left(\Delta + \frac{\delta n_M^2}{8}\right)^2}$$

where $\Delta = \Omega/\Omega_n^{\text{even}} - 1$ is the detuning parameter away from Ω_n^{even}. In contrast with (11), the centers of these resonances (i.e., maximal instability) occur at modulation frequencies $\Omega_n^{\text{Res}} = \Omega_n^{\text{even}}(1 - \delta n_M^2/8)$, which depend quadratically on δn_M. Furthermore, at $\Omega = \Omega_n^{\text{Res}}$, all these resonances share an *n-independent* common growth exponent $\omega''_{\text{Res}} = \omega_p \delta n_M^2/4$, quadratic in δn_M, and therefore of higher order in perturbation theory.

Comparison with the numerical solution

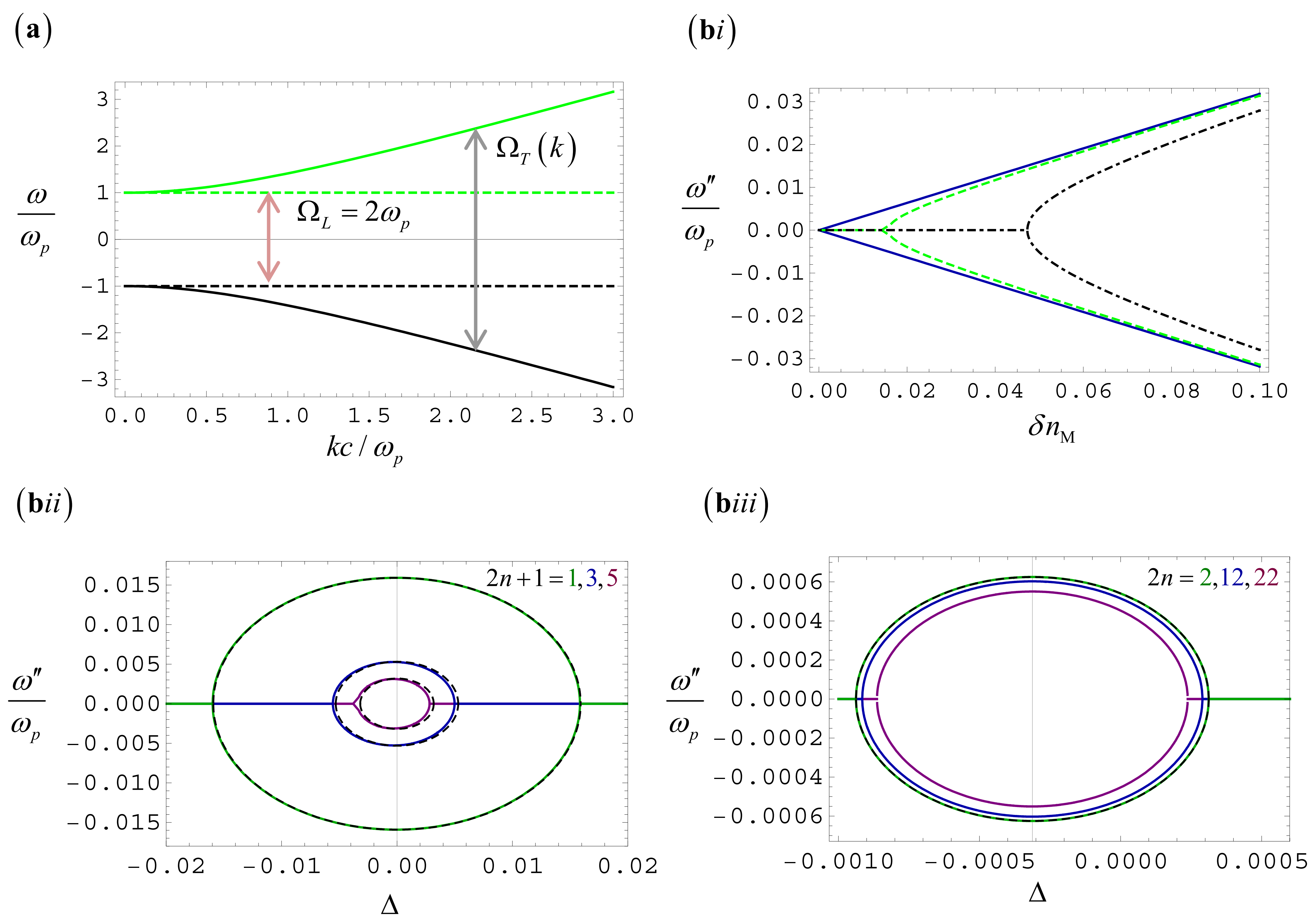

FIG. 1: **a)** Band structure of the static dispersive crystal, showing both positive and negative frequency bands. The arrows indicate possible interband transitions. **bi)** ω'' as a function of the (weak) modulation strength for i) $\Omega = 2\omega_p$ (blue solid lines), ii) $\Omega = (2 \pm 0.01)\omega_p$ (green dashed lines) and iii) $\Omega = (2 \pm 0.03)\omega_p$ (black dotted lines). **bii)** ω'' as a function of the detuning parameter Δ for fixed weak modulation amplitude $\delta n_{\mathrm{M}} = 0.05$ for the first three odd resonances $(2n+1 = 1, 3, 5)$ in (11). The dashed curves are the analytic predictions. The three resonances are centered at $\Delta = 0$ and the growth rate becomes smaller as n increases. **biii)** The same as in **bii)** with $\delta n_{\mathrm{M}} = 0.05$ for the even resonances $2n = 2, 12, 22$ in (12). Note the smaller vertical scale compared to the previous case. The three displayed resonances are are in good agreement with the n-independent prediction of (12), despite the large disparity of chosen values for n. Evidently, these three resonances are centered around a negative value of Δ consistent with $-\delta n_M^2/8$ as predicted in (12).

a simple perturbative result for the growth exponent of weak 'linear' resonances for arbitrary periodic weak modulation profile $\delta n(t)$ (at zero detuning $\Delta = 0$)

consider the Fourier decomposition $\delta n(t) = \sum_{l=1}^{\infty} (c_l \exp(il\Omega t) + c_l^* \exp(-il\Omega t))$

of the modulation amplitude. Resonant behavior of $f(t)$ should be expected when the modulation frequency Ω is commensurate with the gap $2\omega_p$ in the dispersion relation of the unmodulated longitudinal plasmon, such that $2\omega_p/\Omega = n$, with integer n. In this case, the nth Fourier mode of $\delta n(t)$ will become resonant, and will induce instability with growth rate $\omega'' = \frac{\omega_p |c_n|}{2}$

This result is linear in c_n, and therefore in δn, being the result of first order perturbation theory.

For example, in the Mathieu case $c_l = (\delta n_M/2)\delta_{l,1}$, so there is only one linear resonance which occurs for $n = 1$ at $\Omega = 2\omega_p$ and with the known growth exponent $\omega'' = \omega_p \delta n_M/4$.

For the piecewise constant modulation all even Fourier modes $c_{2n} = 0$ vanish, while the odd ones are $c_{2n+1} = 2\delta n_M/(2n+1)\pi i$. Thus, there is an *infinite set of linear resonances* at modulation frequencies $\Omega = 2\omega_p/(2n+1)$, with corresponding growth exponents $\omega'' = \omega_p \delta n_M/(2n+1)\pi$, which agrees with the aforementioned result

In this case, there are no *linear* resonances corresponding to even index $2n$ simply because there are no Fourier modes of even order in the modulation

Appendix

III. PIECEWISE CONSTANT MODULATION

A. Modulation Period with Two Different Amplitude Values

Here we shall derive equations (11) and (12) of the main text - the expressions for the growth exponents associated with resonances of odd and even indices, respectively. In the next subsection we shall also solve this model in the limit in which the piecewise modulation becomes a time-periodic Dirac comb of δ-impulses.

In the simplest form of piecewise constant periodic modulation, $\delta n(t)$ assumes one constant value δn_1 during the first part $0 \le t < \tau$ of the modulation period and another constant value δn_2 during its remaining part $\tau \le t < T$[1,2].

The common practice to solving (2) with periodic modulation is to analyze the initial value problem, and compute the two independent fundamental solutions $c(t)$ and $s(t)$ with initial conditions $c(0) = 1$; $\dot{c}(0) = 0$ and $s(0) = 0$; , $\dot{s}(0) = 1$ (namely, the cosine- and sine-like solutions). These solutions can be easily computed explicitly in our case of piecewise constant modulation. The transfer matrix $\boldsymbol{T}$ is then constructed from the values of these fundamental solutions and their derivatives at $t = T$[3]. Here we shall follow an alternative method[4–6] (familiar from the theory of Lyapunov stability) to computing $\boldsymbol{T}$, but in a basis different from that of $\{c(t), s(t)\}$. (The eigenvalues $\Lambda_{1,2}$ of $\boldsymbol{T}$ are of course basis independent.) To this end, we adopt a "hamiltonian" approach and rewrite (2) as a system of two coupled first order equations

$$\frac{d}{dt}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix} 0 & \omega_p \\ -\omega_p\left(1 - \frac{\nu^2}{4\omega_p^2} + \delta n(t)\right) & 0\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \tag{3}$$

with $\psi_1 = \omega_p g(t)$ and $\psi_2 = \dot{g}(t)$ (thus rendering ψ_1 and ψ_2 the same physical dimension). Equation (3) can be integrated formally by applying the time ordered exponent $\boldsymbol{U}(t, t_0) = \mathcal{T}\exp\left(\int_{t_0}^{t}\boldsymbol{L}(t')dt'\right)$ of the matrix $\boldsymbol{L}(t)$ (the "Liouvillian") on the right hand side of (3) to the vector of initial conditions at $t = t_0$. For constant δn, as in each part of the modulation period in the present example, one can exponentiate $\boldsymbol{L}$ explicitly and find

$$\boldsymbol{M}(\Theta; t) = e^{\boldsymbol{L}t} = \begin{pmatrix}\cos\Theta t & \frac{\omega_p}{\Theta}\sin\Theta t \\ -\frac{\Theta}{\omega_p}\sin\Theta t & \cos\Theta t\end{pmatrix} \tag{4}$$

with $\Theta = \omega_p\sqrt{1 - \frac{\nu^2}{4\omega_p^2} + \delta n}$. In the main text we focus on the modulation

$$\delta n\,(t) = \delta n_{\rm M}\,\mathrm{sgn}\,(\sin\,(\Omega t)) \tag{5}$$

which flips sign at the middle of the modulation period. In this case the transfer matrix for Eq.(2) is readily found as $\boldsymbol{T}_g = \boldsymbol{U}(T, 0) = \boldsymbol{M}(\Theta_-; T/2)\boldsymbol{M}(\Theta_+; T/2)$, where $\Theta_\pm = \omega_p\sqrt{1 - \frac{\nu^2}{4\omega_p^2} \pm \delta n_M}$.

From the relation $f = e^{-\nu t/2}g$ we have $(\omega_p f(t), \dot{f}(t))^T = e^{-\nu t/2}\boldsymbol{K}(\omega_p g(t), \dot{g}(t))^T$ with $\boldsymbol{K} = \begin{pmatrix}1 & 0\\ -\nu/2\omega_p & 1\end{pmatrix}$, and therefore the transfer matrix for Eq. (1) is $\boldsymbol{T}_f = e^{-\nu T/2}\boldsymbol{K}\boldsymbol{T}_g\boldsymbol{K}^{-1}$. Thus,

$$\begin{aligned}\frac{1}{2}\mathrm{Tr}\,\boldsymbol{T}_f &= \frac{1}{2}(\Lambda_1 + \Lambda_2) = \frac{1}{2}e^{-\nu T/2}\mathrm{Tr}\,\boldsymbol{T}_g\\ &= e^{-\nu T/2}\Bigg[\cos\left(\frac{\pi\Theta_+}{\Omega}\right)\cos\left(\frac{\pi\Theta_-}{\Omega}\right)\\ &-\frac{1}{2}\left(\frac{\Theta_+}{\Theta_-} + \frac{\Theta_-}{\Theta_+}\right)\sin\left(\frac{\pi\Theta_+}{\Omega}\right)\sin\left(\frac{\pi\Theta_-}{\Omega}\right)\Bigg]\end{aligned} \tag{6}$$

which coincides with the known result[1,2] in the absence of dissipation $\nu = 0$. (See also chapter 8 of[3] and note the obvious typo in Eq.(8.5) therein.)

For simplicity, from now on we shall focus on the non-dissipative case $\nu = 0$, and rewrite (6) more neatly as

$$\begin{aligned}\tfrac{1}{2}\ \mathrm{Tr}\,\boldsymbol{T}_f &= \cos\left(\frac{\pi(\Theta_+ + \Theta_-)}{\Omega}\right)\\ &- \frac{(\Theta_+ - \Theta_-)^2}{4\Theta_+\Theta_-}\cdot\Bigg[\cos\left(\frac{\pi(\Theta_+ - \Theta_-)}{\Omega}\right)\\ &\quad - \cos\left(\frac{\pi(\Theta_+ + \Theta_-)}{\Omega}\right)\Bigg]\,.\end{aligned} \tag{7}$$

In regions of stability, where $\Lambda_2^* = \Lambda_1 = e^{-i\omega T}$ (with real ω), we thus have $\frac{1}{2}\mathrm{Tr}\,\boldsymbol{T}_f = \cos\omega T$, so that $|\mathrm{Tr}\,\boldsymbol{T}_f/2| < 1$. In contrast, in the unstable regime we can always choose $\Lambda_1 = 1/\Lambda_2 = \pm e^{\omega'' T}$ with $\omega'' > 0$. Thus, $\omega = i\omega''$ for positive $\Lambda_{1,2}$, or $\omega = \pi/T + i\omega''$ for negative $\Lambda_{1,2}$ (where $\omega' = \pi/T$ is restricted to the first Brillouin zone). Therefore, in the unstable regime we have $\frac{1}{2}\mathrm{Tr}\,\boldsymbol{T}_f = \pm\cosh(\omega'' T)$ so that $|\mathrm{Tr}\,\boldsymbol{T}_f/2| > 1$. The boundaries separating stable and unstable regions are therefore given by the curves where $\mathrm{Tr}\,\boldsymbol{T}_f/2 = \pm 1$. This leads, like in the case of Mathieu's equation, to a rich chart of stability and instability regions in the plane of parameters Ω/ω_p and δn_M[1,2].

Let us now investigate the stability of this system perturbatively for weak modulation amplitude δn_M. In this case, the factor $(\Theta_+ - \Theta_-)^2/\Theta_+\Theta_-$ in the second term in (7) is clearly of order δn_M^2. Thus, the modulation frequencies which induce instabilities at infinitesimal $\delta n_M \to 0^+$ are determined in this limit by the leading term $\cos(\pi(\Theta_+ + \Theta_-)/\Omega) \simeq \cos(2\pi\omega_p/\Omega)$ in (7) tending to ± 1. The first few terms of the expansion of (7) around

$\delta n_M = 0$ are

$$\frac{1}{2}\mathrm{Tr}\boldsymbol{T}_f = \cos\left(\frac{2\pi\omega_p}{\Omega}\right) + \frac{1}{2}\left[\frac{\pi\omega_p}{2\Omega}\sin\left(\frac{2\pi\omega_p}{\Omega}\right) - \sin^2\left(\frac{\pi\omega_p}{\Omega}\right)\right]\delta n_M^2 + \frac{1}{32}\left[4\left(\frac{\pi\omega_p}{\Omega}\right)^2 - 6 + \left(6 - \left(\frac{\pi\omega_p}{\Omega}\right)^2\right)\cos\left(\frac{2\pi\omega_p}{\Omega}\right) + \frac{9}{2}\frac{\pi\omega_p}{\Omega}\sin\left(\frac{2\pi\omega_p}{\Omega}\right)\right]\delta n_M^4 + \mathcal{O}(\delta n_M^6)\,. \tag{8}$$

The instability near $\cos(2\pi\omega_p/\Omega) = -1$ occurs around modulation frequencies $\Omega = 2\pi/T = 2\omega_p/(2n+1)$ with integer n. (In the expansion of (7) around $\delta n_M = 0$ leading to (8), we tacitly assumed T was bounded, which means that the integer n, indexing the corresponding parametric resonance, is assumed to be bounded as well.) Thus, let us substitute $\Omega = 2\omega_p(1+\Delta)/(2n+1)$ in (8), with Δ a small detuning parameter. We find $1/2\mathrm{Tr}\boldsymbol{T}_f \equiv \cos((\omega' + i\omega'')T) = -1 - \left[\delta n_M^2 - (2n+1)^2\pi^2\Delta^2\right]/2 + \mathcal{O}(\delta n_M^2\Delta, \Delta^3) \simeq \cos\left(i\sqrt{\delta n_M^2 - (2n+1)^2\pi^2\Delta^2} \pm \pi\right)$. Therefore, up to the indicated accuracy, $\omega' = \pm\pi/T = \pm\omega_p(1+\Delta)/(2n+1)$ and

$$\omega'' = \omega_p\sqrt{\frac{\delta n_M^2}{((2n+1)\pi)^2} - \Delta^2}\,. \tag{9}$$

Thus, maximal instability (the center of the resonance) occurs at $\Delta = 0$, where $\omega''_{\mathrm{Res}} = \omega_p\delta n_M/(2n+1)\pi$ is linear in δn_M, and is therefore of leading order in perturbation theory. Furthermore, similarly to the corresponding result for the Mathieu case, given in Eq. (9) in the main text, detuning the modulation frequency away from the resonance requires a threshold value $\delta n_M^{\mathrm{Th}} = (2n+1)\pi|\Delta|$ of the modulation amplitude to induce instability with diminished ω''.

This is not the case for the parametric resonances around the other instability borderline at $\cos(2\pi\omega_p/\Omega) = +1$, which occur in the vicinity of modulation frequencies $\Omega = 2\omega_p/2n = \omega_p/n$ with integer n. By substituting $\Omega = \omega_p(1+\Delta)/n$ in (8) and expanding the resulting expression in powers of Δ, we find that the associated resonance lies in the parametric regime where Δ scales like δn_M^2 so that $1/2\mathrm{Tr}\boldsymbol{T}_f \equiv \cos((\omega' + i\omega'')T) = 1 + (1/2)(2\pi n)^2[\delta n_M^4/16 - (\Delta + \delta n_M^2/8)^2] + \mathcal{O}(\Delta^3, \delta n_M^2\Delta^2, \delta n_M^4\Delta, \delta n_M^6) \simeq \cosh\left[(2\pi n)\sqrt{\delta n_M^4/16 - (\Delta + \delta n_M^2/8)^2}\right]$. Thus, up to the indicated accuracy

$$\omega'' = \omega_p\sqrt{\left(\frac{\delta n_M^2}{4}\right)^2 - \left(\Delta + \frac{\delta n_M^2}{8}\right)^2} \tag{10}$$

(and of course $\omega' = 0$). Thus, in contrast with (9), these resonances (that is, maximal instability) are not centered at modulation frequency $\Omega = \omega_p/n$ where $\cos(\omega T) = 1$ reaches the upper border of the stability region, but rather at a slightly smaller and δn_M-dependent modulation frequency $\Omega_{\mathrm{Res}} = (1 - \delta n_M^2/8)\omega_p/n$. Moreover, notwithstanding the n-dependence of the location resonances of this type, (10) is *completely independent of* n, in contrast with (9), with a *common* maximal growth exponent $\omega''_{\mathrm{Res}} = \omega_p\delta n_M^2/4$, which is quadratic in δn_M, and is therefore of higher order in perturbation theory.

B. The Effect of Dissipation - Numerical Results

The discussion in the main text focused on lossless systems, $\nu = 0$. For completeness, in Fig.S3 we demonstrate schematically the effect of turning the damping coefficient ν on.

This figure shows the locus of the eigenfrequencies $\omega' + i\omega''$ in the complex plane, for values of the loss parameter ν in the range $0 < \nu < 5\omega_p$. The direction of increasing loss is indicated by arrows. The modulation frequency is $\Omega = 2\omega_p$ and $\delta n_{\mathrm{M}} = 0.5$. At $\nu = 0$, one of the eigenfrequencies resides in the upper half of the frequency plane, represented by the upper endpoints of the curves (note that due to the periodicity of the band diagram in ω' the two endpoints represent the same eigenfrequency). Its counterpart (not shown in the figure), is located symmetrically across the $\omega'' = 0$ horizontal axis. As the dissipation parameter increases, both modes descend towards the lower-half frequency plane, eventually converging in a bifurcation. The eigenfrequency of the mode experiencing gain intersects the $\omega'' = 0$ line when ν is approximately $0.32\omega_p$, highlighting a critical transition point influenced by dissipation. For comparison, the green points represent a similar study for the case when the modulation strength is vanishingly small ($\delta n_M \approx 0$). As expected, in this case the spectrum lies completely in the lower-half frequency plane.

C. Dirac Comb of δ-Impulses

Going back to the more general piecewise constant profile (with $\tau \neq T/2$, that is, uneven durations of the two constant values of the modulation amplitude δn), an interesting limit is obtained when, for example, we let $\tau \to 0^+, n_1 \propto 1/\tau$ and $n_2 = 0$. In this limit we obtain a Dirac comb of δ-impulses, namely,

$$\delta n(t) = \alpha\sum_{j=1}^{\infty}\delta(t - jT) \tag{11}$$

with time-independent parameters α and T. With this type of modulation, Eq. (1) is conveniently solved by computing explicitly the transfer matrix $\boldsymbol{T}$ (in the 'scattering basis'), as we now explain. Between impulses, say for $jT < t < (j+1)T$, $f(t)$ evolves as a linear combination $A_j e^{-i\omega_+(t-jT)} + B_j e^{-i\omega_-(t-jT)}$ of the two frequencies

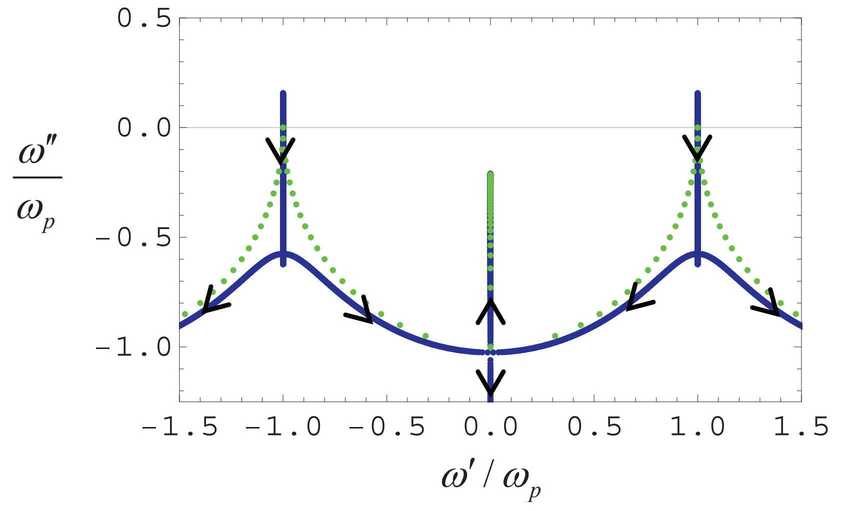

FIG. S2: Locus of $\omega' + i\omega''$ in the complex plane as a function of the damping strength ν at modulation frequency $\Omega = 2\omega_p$. The arrows indicate the direction of increasing ν. i) $\delta n_M = 0.5$ (solid blue lines), ii) $\delta n_M = 0^+$ (green dots). The horizontal grid line (in gray) separates the stable and unstable regions.

$\omega_\pm = -i\nu/2 \pm \omega_p'$ of the time-independent problem, with $\omega_p' = \sqrt{\omega_p^2 - \nu^2/4}$, under the further assumption that ω_p' is real-valued, namely, that the system is not overdamped. The solution $f(t)$ is continuous throughout the impulse at $t = (j+1)T$, while $\partial_t f$ suffers a jump discontinuity: $\partial_t f_+ - \partial_t f_- = -\alpha\omega_p^2 f$ (with obvious notations).

These properties uniquely determine the amplitudes A_{j+1} and B_{j+1} right after a given impulse as $(A_{j+1}, B_{j+1})^T = \boldsymbol{T}(A_j, B_j)^T$ where[7]

$$\boldsymbol{T} = e^{-\frac{\nu T}{2}} \begin{pmatrix} 1 - iu & -iu \\ iu & 1 + iu \end{pmatrix} \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \qquad (12)$$

with $u = \frac{\alpha\omega_p^2}{2\omega_p'}$ and $\theta = \omega_p' T$. The eigenvalues of $\boldsymbol{T}$ control stability of the system, namely, whether the amplitudes grow (regions of instability in parameter space) or remain bounded (regions of stability). For simplicity, let us analyze the stability properties of the PLTC in the absence of damping ($\nu = 0$), where $u = \frac{\alpha\omega_p}{2}$.

In this case $\det \boldsymbol{T} = 1$ and the two mutually reciprocal eigenvalues of $\boldsymbol{T}$ are $\Lambda_\pm = \mathrm{Tr}\,\boldsymbol{T}/2 \pm \sqrt{(\mathrm{Tr}\,\boldsymbol{T}/2)^2 - 1}$, where $\mathrm{Tr}\,\boldsymbol{T}/2 = \mathrm{Re}\,T_{11} = \cos\theta - u\sin\theta$.

Thus, if $|\mathrm{Re}T_{11}| > 1$, both eigenvalues are real, with either $|\Lambda_+|$ or $|\Lambda_-| > 1$, resulting in the growth of $f(t)$ after each kick. This is the region of instability. If, on the other hand, $|\mathrm{Re}T_{11}| < 1$, then $\Lambda_- = \Lambda_+^*$ so that $|\Lambda_\pm| = 1$ and $f(t)$ remains bounded as function of time. This is the region of stability. Clearly, the boundaries separating regions of stability and instability are determined by $\cos\theta - u\sin\theta = \pm 1$. Such points in the parameter space of $\boldsymbol{T}$ are exceptional points, where the matrix is non-diagonalizable, possessing only a single eigenvector. Straightforward calculation shows that at the instability threshold u and θ are related either by $u = \cot\theta/2$ (when $\mathrm{Tr}\,\boldsymbol{T}/2 = -1$) or $u = -\tan\theta/2$ (when $\mathrm{Tr}\,\boldsymbol{T}/2 = 1$). Let us denote the border stability lines of the first type by $\theta_-(u)$, and the border stability lines of the second type by $\theta_+(u)$. As illustrated in Fig.S4, the stability regions are the areas lying between these curves and the horizontal axis in the $u-\theta$ plane. The curves $\theta_-(u)$ terminate on the horizontal axis of Fig.S4 at points where $\theta/\pi =$ *an odd integer*, whereas the curves $\theta_+(u)$ terminate there at points where $\theta/\pi =$ *an even integer*. Let us consider now narrow strips in the unstable region immediately to the right of the border lines. That is, for a given u, we set $\theta = \theta_\pm(u) + \Delta$ with $0 < u\Delta \ll 1$. By expressing $\sin\theta_\pm(u)$ and $\cos\theta_\pm(u)$ in terms of u, it is straightforward to show in these strips that $\cos(\omega T) = \mathrm{Tr}\,\boldsymbol{T}/2 = \pm(\cos\Delta + u\sin\Delta) \simeq \pm\cosh(\sqrt{2u\Delta})$. Thus, immediately on the right of both lines $\theta_\pm(u)$ the instability growth rate is $\omega'' = \sqrt{2u\Delta}/T$, but such generic instability is non-resonant (in the sense that it does not have a local maximum as function of Δ). Resonances naturally appear near the horizontal axis in Fig.S4, where the 'tongues' of instability terminate, that is where also $\theta/\pi = n$ an integer. (We remind the reader that odd integers correspond to the $\theta_-(u)$ lines, and even integers to the $\theta_+(u)$ lines.) Thus, for $u \sim \Delta \ll 1$ we obtain $\cos(\omega T) = \pm(\cos\Delta + u\sin\Delta) \simeq (-1)^n \cosh(\sqrt{u^2 - (\Delta - u)^2}) + \mathcal{O}(\Delta^4, u\Delta^3)$. Therefore, in the *vicinity* of a given $\theta \simeq n\pi$ in the u–θ plane, the resonance is centered at $\Delta = u$ (that is, at $\theta_{\mathrm{Res}} = \theta_\pm(u) + u$), with maximal growth rate

$$\omega_n'' = \frac{u}{T} = \frac{\Omega u}{2\pi}\,, \qquad (13)$$

independently of n. The real part of the Floquet frequency depends on the the parity of n. Thus, $\omega' = 0$ for resonances corresponding to even n, while $\omega' = \pm\pi/T = \pm\Omega/2$ (in the first Brillouin zone) for resonances corre-

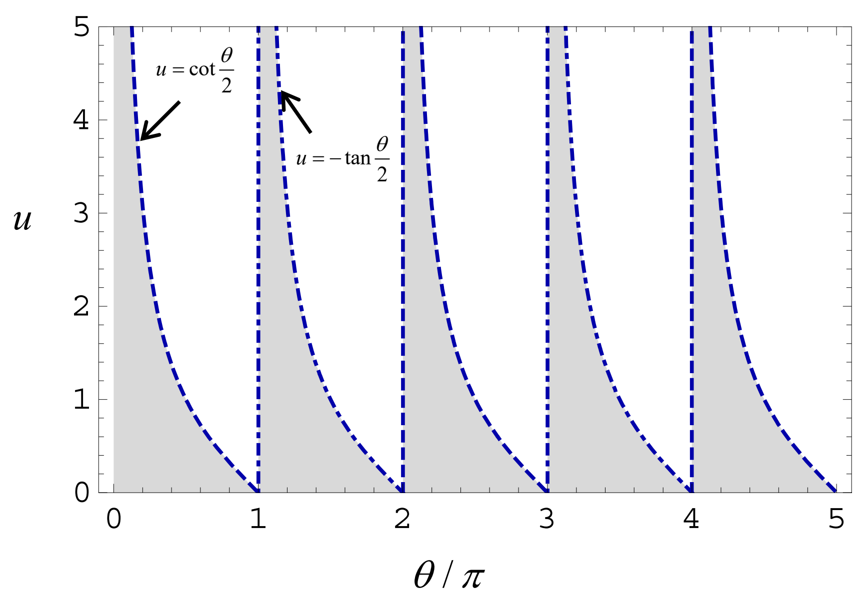

FIG. S3: Stability region of the $u - \theta$ plane for the Dirac impulse model (11). The zone shaded in gray is the stability region in parameter space. The boundary curves are determined by $u = \cot\theta/2$ (dashed curves) and $u = -\tan\theta/2$ (dot dashed curves).

sponding to odd n.

Finally, as can be seen in Fig.S4, the system is always unstable in the vicinity of $\theta_n = n\pi + 0^-$ (with integer n), which corresponds to modulation frequency $\Omega_n = 2\omega_p/n$. Furthermore, as $u \to 0+$ (no impulses) the gaps disappear, rendering the system always stable, while in the opposite limit $u \to +\infty$ the stability regions shrink to the points $\theta_n = n\pi + 0^+$.

IV. PARAMETRIC RESONANCES AT WEAK MODULATION - THE GENERAL CASE

We shall now offer a very simple demonstration of the existence of parametric resonances for generic periodic modulation at frequencies $\Omega = 2\omega_p/n$ with integer n, and derive the corresponding growth exponents at these resonances (i.e., at zero detuning Δ) given by Eq.(13) in the main text. For simplicity, we limit the discussion to the non-dissipative case $\nu = 0$. To this end, we consider the system of equations (3), where now $\psi_1 = \omega_p f(t)$ and $\psi_2 = \dot{f}(t)$. We further assume that the weak modulation $\delta n(t)$ oscillates equally between positive and negative values, so that its zeroth Fourier component $c_0 = \int_0^T \delta n(t)dt/T = 0$ (i.e., its mean value) vanishes.

In the absence of modulation ($\delta n = 0$), Eq.(3) is identical in form to the Schrödinger equation for a spin-1/2 (with magnetic moment normalized to 2) precessing in a constant magnetic field $\boldsymbol{B} = -\omega_p \hat{\mathbf{y}}$, whose eigen-solutions are

$$\Psi_\pm(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix} e^{\pm i\omega_p t} \tag{14}$$

and with corresponding eigenvalues $\mp\omega_p$. Now we turn the modulation on, and in the spirit of the discussion in[8] of parametric resonances in the Mathieu equation, seek a solution of (3) in the form $\Psi(t) = (\psi_1, \psi_2)^T = a(t)\Psi_+(t) + b(t)\Psi_-(t)$, with *slowly varying* amplitudes $a(t)$ and $b(t)$, which serve as amplitude envelopes to the harmonically oscillating factors. (This is reminiscent of transforming to the interaction picture in Quantum Mechanics.) Thus, if $f(t) = \psi_1(t)/\omega_p$ is the Floquet eigensolution of the equation with eigenvalue $\Lambda = e^{-i(\omega' + i\omega'')T}$, then the growth (or decay) coefficient ω'' should be encoded in the exponential envelope of $a(t)$ and $b(t)$, while the oscillatory parts of the latter should combine with the phase factors $e^{\pm i\omega_p t}$ in (14) to produce the real part ω' of the Floquet frequency. By substituting this form of $\Psi(t)$ in (3) and utilizing orthonormality of the eigenspinors (14) we obtain the equation for $a(t)$ and $b(t)$ as

$$\frac{d}{dt}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = i\frac{\omega_p \delta n(t)}{2} \begin{pmatrix} 1 & e^{-2i\omega_p t} \\ -e^{2i\omega_p t} & -1 \end{pmatrix} \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}. \tag{15}$$

This equation is exact. In what follows we shall assume the modulation $\delta n(t)$ is very weak, and contend ourselves with solving (15) perturbatively to leading order in $\delta n(t)$.

By assumption, $\delta n(t)$ oscillates evenly between positive and negative values, and its oscillations are in general incommensurate with those of the phases $e^{\pm 2i\omega_p t}$. In our perturbative solution of (15), we have to integrate its right-hand over a period of time t (starting at some initial time t_0). Let us assume t contains many modulation cycles. For generic modulation $\delta n(t)$ it is plausible to expect significant cancellations (or 'destructive interference') in this integral. Therefore, this integral should be dominated by the least oscillating terms in the quantities $(\delta n(t) \exp \pm 2i\omega_p t)$ in the integrand, which must be a combination of an appropriate Fourier mode of $\delta n(t)$ and the phases $e^{\pm 2i\omega_p t}$. Thus, we substitute the Fourier decomposition $\delta n(t) = \sum_{l=1}^\infty (c_l \exp(il\Omega t) + c_l^* \exp(-il\Omega t))$ in (15) and average it over one modulation period. In this procedure we encounter integrals of the form $(1/T)\int_0^T dt \exp[\pm i(l\Omega - 2\omega_p)t]$, whose phase is minimized for the pair of Fourier modes corresponding to $l = [2\omega_p/\Omega]$ (where $[x]$ is the integral part of the real number x). In particular, if Ω and $2\omega_p$ are commensurate such that $\Omega = 2\omega_p/n$ with integer n, this minimal phase, occurring for $l = n$ will be exactly null, and the dominant part in (15) will be

$$\frac{d}{dt}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \frac{\omega_p}{2} \begin{pmatrix} 0 & ic_n \\ -ic_n^* & 0 \end{pmatrix} \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}, \tag{16}$$

with eigenvalues $\pm\omega_p|c_n|/2$, leading to growth exponent

$$\omega'' = \frac{\omega_p |c_n|}{2} \tag{17}$$

for the instability at modulation frequency $\Omega = 2\omega_p/n$. This result is linear in c_n, and therefore in δn by construction. Thus, any such parametric resonance is one of the set of *dominant* resonances for the modulation profile $\delta n(t)$, analogous to the one associated with (9) in the case of piecewise constant modulation.

We comment that in the averaging procedure leading to (16) we have obviously lost all information about the oscillatory behavior of the amplitudes $a(t)$ and $b(t)$, and in particular, of the real part ω' of the Floquet frequency corresponding to this resonance. Indeed, the (approximate) amplitudes $a(t)$ and $b(t)$ resulting from (16) are not oscillatory and have purely real exponential behavior $e^{\pm\omega'' t}$. The only oscillatory part of the full solution for $\Psi(t) = (\psi_1(t), \psi_2(t))^T$ comes from the phases $e^{\pm i\omega_p t}$ in (14), which are independent of $\delta n(t)$ and therefore of the modulation frequency Ω. Indeed, the approximate growing (unstable) solution arising from (16) and (17) is $(a(t), b(t))^T \simeq (1, -i)^T e^{\omega'' t}$, leading to $(\psi_1, \psi_2)^T = (\omega_p f(t), \dot{f}(t)) \simeq (\cos\omega_p t, -\sin\omega_p t)e^{\omega'' t}$ (where a term of order $\omega''/\omega_p \propto |c_n|$ was neglected in the component $\psi_2 = \dot{f}$). Thus, the cosine factor in $f(t)$

oscillates periodically at frequency $\omega_p = n\Omega/2 \neq \Omega$, and is therefore not even a proper Floquet solution (unless $n = 2$ accidentally). Therefore, the averaged 'hamiltonian' (16) should only be used to determine the exponential envelope of the Floquet eigensolutions.

For example, in the Mathieu case (Eq.(8) in the main text), $c_l = (\delta n_M/2)\delta_{l,1}$, so there is only one dominant resonance which occurs for $n = 1$ at $\Omega = 2\omega_p$ and with the known growth exponent $\omega'' = \omega_p \delta n_M/4$.

For the piecewise constant modulation (5), the Fourier modes are $c_n = 0$ for even n and $c_n = 2\delta n_M/\pi i n$ for odd values of n. Thus, there is an *infinite set of dominant resonances* at modulation frequencies $\Omega = 2\omega_p/n$ with n an odd number, with corresponding growth exponents $\omega'' = \omega_p \delta n_M/\pi n$, which agrees with the result (9) obtained directly from the dispersion relation (7). There are no *dominant* resonances corresponding to n even simply because there are no Fourier modes of even order for the modulation (5), as we discovered by direct analysis of (8).

Finally, for the Dirac-comb impulse modulation (11), the Fourier modes are $c_n = \frac{\alpha}{T} = \frac{2u}{\omega_p T}$, independently of n, and indeed, we see from (13) and (17) that $\omega'' = \frac{u}{T} = \frac{\omega_p |c_n|}{2}$.

In order to determine higher order resonances of the modulated system, that is, resonances with growth exponents ω'' which depend on higher powers of the small amplitude δn, as those mentioned in[8] for the Mathieu equation, or those corresponding to (10) for the piecewise constant case, one should analyze higher orders in the time-dependent perturbative expansion of the solution of (15).

Thanks for your attention!