Scientific Legacy of Miloslav Havlíček

November 2, 1938 –September 4, 2024

Superintegrability, Exact-Solvability, and Representation Theory
FJFI ČVUT, November 22-24, 2024

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Academic trajectory

- Ing. - Technical & Theoretical Physics, FJFI ČVUT 1961.

- 1969 "About representations of a certain infinite-dimensional Lie algebra", CSc dissertation, Fac. of Math. and Phys. of the Charles University, Prague.

- 1979 "Canonical realizations (of Lie algebras)", Dr.Sc. thesis in Lab. Theor. Phys. at JINR, Dubna

- 1990 Doc. Habilitation in Mathematical Physics.

- 1993 Professor of Mathematical Physics.

Professional trajectory

- 1961 Katedra matematiky, FJFI ČVUT
- 1970 Katedra teoretické fyziky, Faculty of Mathematics and Physics, Charles University
- 1973 Theoretical Physics Laboratory, JINR, Dubna
- 1977 Nuclear Centre, Faculty of Mathematics and Physics, Charles University
- 1988 Katedra matematiky, FJFI ČVUT
- 1990 Dean of FJFI ČVUT
- 1994 Head of the Department of Mathematics, FJFI
- 2000 Dean of FJFI ČVUT

Prizes and Memberships

- Prizes
  - First Prize for Theoretical Physics (1985), JINR Dubna
  - Rector's Prize ČVUT (1995)
  - Prize of the Minister of Education 1st degree (1998)
- Memberships
  - Doppler Institute FJFI [Foundation, 1993]
  - IAMP (Int. Ass. Math. Phys.)
  - JČMF (Jednota Českých Matematiků a Fyziků)
  - Scientific Council ČVUT (1990-1994, 2000-2024)
  - Scientific Council, Institute of Informatics of the AV ČR
  - Vice-Chair of the Academic Assessment Board, AV ČR (1993-99)
  - Scientific Council MFF, Charles University (1990-2024)

• Once asked about his experiences in leading the faculty, Havlíček answered "The faculty actually runs itself."

• “We don’t pretend to be an easy school, but for the effort we offer individual attention, quick involvement in faculty research teams and an emphasis on our students’ own creative work. These students are then able to establish themselves at the top workplaces in the Czech Republic and abroad"
[M. Havlíček on the occasion of the 50th anniversary of FJFI]

• “You may be surprised, but applied natural sciences penetrate into virtually all components of human activity, from theoretical physics and research into the construction of matter through medicine, energy, environmental protection, monument research, food science and many others. We have a good historical basis, excellent scientific and technological level and capable people, our students are among the elite of the Czech intelligentsia - what other field can say this?"
[M. Havlíček on the occasion of the 50th anniversary of FJFI]

Main scientific collaborators

- J. Votruba [son of Václav Votruba?]
- M. Bacovský, M. Bednář, J. Blank, Č. Burdík, G. Chadzitaskos, P. Exner, J. Hořejší, J. Kotrbatý, O. Navrátil, E. Pelantová, S. Pošta, J. Tolar, I. Úlehla, A. Vančura, [Prague]
- J. Patera, P. Winternitz [Montréal]
- A. U. Klimyk [Kiev]
- R. M. Asherova, A. A. Sakharuk, Yu. F. Smirnov, V. N. Tolstoy, [Moscow]
- W. Lassner [Leipzig]
- P. J. Moylan [Abington, Penn.]

The infinite dimensional Lie algebra $A(P, S)$

- Work related to the CSc thesis' topics

M. Havlíček, M.: About representations of a certain infinite-dimensional Lie algebra.

Cand. Diss., Fac. Math. Phys. of the Charles University, Prague, 1968.

- M. Havlíček and J. Votruba, On the physical representations of an infinite-dimensional Lie algebra, Czechoslovak J. Phys. B **16** (1966), 631–642
- M. Havlíček and J. Votruba, The tensor product of one-particle representations of an infinite-dimensional Lie algebra, Czechoslovak J. Phys. B **17** (1967), 809–821
- J. Votruba and M. Havlíček, On the representation of infinite-dimensional Lie algebra $A(P, S)$, in *High Energy Phys. Theory Elem. Particles (Internat. School Theoret. Phys., Yalta, 1966)*, pp. 330–335, "Naukova Dumka", Kiev
- M. Havlíček, Representations of an algebra of the Gell-Mann-Dashen type, Comm. Math. Phys. **13** (1969), 73–80
- M. Havlíček, About one class of representations of the Lie algebra, Comm. Math. Phys. **20** (1971), 130–142

- Brackets of $A(P, SU(2))$

$$\text{(A 1)} \qquad [\mathbf{L}_{\mu\nu}, \mathbf{L}_{\varrho\lambda}] = i(g_{\mu\varrho}\mathbf{L}_{\nu\lambda} - g_{\nu\varrho}\mathbf{L}_{\mu\lambda} + g_{\mu\lambda}\mathbf{L}_{\varrho\nu} - g_{\nu\lambda}\mathbf{L}_{\varrho\mu}),$$

$$\text{(A 2)} \qquad [\mathbf{P}_{\mu}, \mathbf{L}_{\varrho\nu}] = i(g_{\varrho\mu}\mathbf{P}_{\nu} - g_{\nu\mu}\mathbf{P}_{\varrho}),$$

$$\text{(A 3)} \qquad [\mathbf{P}_{\mu}, \mathbf{P}_{\nu}] = 0,$$

$$\text{(B 1)} \qquad [\mathbf{P}_{\mu}, \mathbf{T}^{(1)}_{\varrho_1 \cdots \varrho_m}] = -i\alpha \mathbf{T}^{(2)}_{\mu\varrho_1 \cdots \varrho_m},$$

$$\text{(B 2)} \qquad [\mathbf{P}_{\mu}, \mathbf{T}^{(2)}_{\varrho_1 \cdots \varrho_m}] = i\alpha \mathbf{T}^{(1)}_{\mu\varrho_1 \cdots \varrho_m},$$

$$(B\ 3) \qquad [\mathbf{P}_{\mu}, \mathbf{T}^{(3)}_{\varrho_1 \cdots \varrho_m}] = 0,$$

$$\text{(B 4)} \qquad [\mathbf{L}_{\mu\nu}, \mathbf{T}^{(i)}_{\varrho_1 \cdots \varrho_m}] = i \sum_{j=1}^{m} (g_{\nu\varrho_j}\mathbf{T}^{(i)}_{\varrho_1 \cdots \varrho_{j-1}\mu \cdots \varrho_m} - g_{\mu\varrho_j}\mathbf{T}^{(i)}_{\varrho_1 \cdots \varrho_{j-1}\nu\varrho_m}),$$

$$\text{(C 1)} \qquad [\mathbf{T}^{(i)}_{\mu_1 \cdots \mu_n}, \mathbf{T}^{(k)}_{\varrho_1 \cdots \varrho_m}] = i\varepsilon_{ikl}\mathbf{T}^{(e)}_{\mu_1 \cdots \mu_n \varrho_1 \cdots \varrho_m}.$$

- $P_\mu, L_{\nu\rho}, \mu, \nu, \rho = 0 \ldots 3; i, k, \ell = 1, 2, 3$

- Later appeared in the context of Kač–Moody algebras.

• Extends Formánek's work on nontrivial coupling $A(P, S)$ of internal and space-time symmetries.[1], combining the Poincaré algebra with $SU(2)$ in new representations.

• Determination of representations (IH, special case: one isomultiplet) and mass formulae (depending on the third component of the isotopic spin)

• Tensor products of one-particle representations studied. Reduction problem. New series possessing non-linear mass formulae found.

• Operator representations of pairs $(\mathfrak{g}', \mathfrak{g})$ in Hilbert spaces. Obtainment of analogue of Schur lemmata.

[1] J. Formánek 1966 *Czech. J. Phys.* **B 16** 1–7; 631–642.

Realizations of Lie algebras

- Subject of Dr.Sc. thesis in Dubna (1979):
  - M. Havlíček and P. Exner, On the minimal canonical realizations of the Lie algebra $\mathrm{O}_C(n)$, Ann. Inst. H. Poincaré Sect. A (N.S.) **23** (1975), no. 4, 313–333
  - M. Havlíček and P. Exner, Matrix canonical realizations of the Lie algebra $o(m, n)$. I. Basic formulae and classification, Ann. Inst. H. Poincaré Sect. A (N.S.) **23** (1975), no. 4, 335–347
  - M. Havlíček and W. Lassner, Canonical realizations of the Lie algebras $\mathrm{gl}(n, R)$ and $\mathrm{sl}(n, R)$. I. Formulae and classification, Rep. Mathematical Phys. **8** (1975), no. 3, 391–399
  - M. Havlíček and W. Lassner, Canonical realizations of the Lie algebras $\mathrm{gl}(n, R)$ and $\mathrm{sl}(n, R)$. II. Casimir operators, Rep. Mathematical Phys. **9** (1976), no. 2, 177–185
  - P. Exner, M. Havlíček and W. Lassner, Canonical realizations of classical Lie algebras, Czechoslovak J. Phys. **26B** (1976), no. 11, 1213–1228
  - M. Havlíček and W. Lassner, Canonical realizations of the Lie algebra $\mathrm{sp}(2n, R)$, Internat. J. Theoret. Phys. **15** (1976), no. 11, 867–876

- M. Havlíček and W. Lassner, On the "near to minimal" canonical realizations of the Lie algebra C_n, Internat. J. Theoret. Phys. **15** (1976), no. 11, 877–884
- M. Havlíček and W. Lassner, Matrix canonical realizations of the Lie algebra $\mathfrak{u}(p,q)$, Rep. Mathematical Phys. **12** (1977), no. 1, 1–8
- M. Havlíček and P. Exner, Matrix canonical realizations of the Lie algebra $\mathfrak{o}(m,n)$. II. Casimir operators, Czechoslovak J. Phys. B **28** (1978), no. 9, 949–962
- P. Exner, M. Havlíček and W. Lassner, Boson representations of classical Lie algebras, in *Proceedings of the International Conference on Operator Algebras, Ideals, and their Applications in Theoretical Physics (Leipzig, 1977)*, pp. 277–278, Teubner, Leipzig
- Č. Burdík and M. Havlíček, Boson realizations of semi-simple Lie algebras, in *Symmetry in physics*, 87–98, CRM Proc. Lecture Notes, 34, Amer. Math. Soc., Providence, RI, 2004.

- Some of the main results:

- Existence criteria for Schur realizations.

- Integrability properties of representations,

- Notion of matrix canonical realization.

$$\tau : \mathfrak{g} \to \mathcal{W}_{2N} \otimes \mathrm{Mat}_M \implies \hat{\tau} : U(\mathfrak{g}) \to \mathcal{W}_{2N} \otimes \mathrm{Mat}_M$$

- Canonical realizations of classical Lie algebras. Notion of minimality.

THEOREM 1. — If $N < n - 3$ then any canonical realization of $O_C(n)$ in W_{2N} does not exist.

THEOREM 2. — The minimal canonical realization of
i) $O_C(3)$ is in W_2,
ii) $O_C(4)$ is in W_4,
iii) $O_C(5)$ is in W_4,
iv) $O_C(6)$ is in W_6.

- Matrix canonical realizations of the Lie algebra $\mathfrak{o}(p,q)$, $\mathfrak{u}(p,q)$

i) the following formulæ define the skew-hermitean Schur-realization of $o(m,n)$ in $W_{2(m+n-2+N),M}$:

$$\begin{aligned} &\tau(P_i) = p_i, \\ &\tau(L_{ij}) = q_i p_j - q_j p_i + M_{ij}, \\ &\tau(R) = -(qp) - \left[\frac{1}{2}(m+n-2) - i\alpha\right] \mathbf{1} \quad , \quad \alpha \in \mathbf{R}, \\ &\tau(Q_i) = -q^2 . p_i - 2 q_i \tau(R) - 2 q^j M_{ji} \quad , \quad i, j = 1, 2, \dots, m+n-2, \end{aligned} \tag{3}$$

where

$$(qp) = q^i p_i \quad , \quad q^2 = q^i q_i \quad , \quad p_i = g_{ij} p^j \quad , \quad q^i = g^{ij} q_j,$$

- Extensive analysis of Casimir operators and their eigenvalues.

$$I_r^{(m,n)} = \beta^r + \bar{\beta}^r - \sum_{s=0}^{r-2} \left(\beta^{r-s-1} + \bar{\beta}^{r-s-1} + \frac{\bar{\beta}^{r-s-1} - \beta^{r-s-1}}{\beta - \bar{\beta}} \right) \times \times J_s^{(m-1,n-1)} - 2 J_{r-1}^{(m-1,n-1)} + J_r^{(m-1,n-1)}, \quad r = 0, 1, \ldots \tag{5}$$

where $\beta = \mathrm{i}\alpha + \frac{1}{2}(m + n - 2)$ and in the case when $m + n$ is even

$$\tilde{I}^{(m,n)} = \mathrm{i}\alpha(m + n)\, \tilde{I}^{(m-1,n-1)}\,. \tag{6}$$

- Canonical realizations of the Lie algebras $\mathrm{gl}(n, R)$. Classification.

THEOREM 1. *Let $F_{\mu\nu}$, $\mu, \nu = 1, 2, \ldots, n-1$, be a canonical realization of generators of $\mathrm{gl}(n-1, R)$ fulfilling (6) in W_{2m}. The following formulae define a realization $E_{ij} = E_{ij}(F_{\mu\nu}, \alpha)$ of $\mathrm{gl}(n, R)$ in $W_{2(n-1+m)}$.*

$$\begin{aligned} E_{\mu\nu} &= q_\mu p_\nu + F_{\mu\nu} + \tfrac{1}{2}\delta_{\mu\nu} 1, \\ E_{n\mu} &= -p_\mu, \\ E_{\mu n} &= q_\mu\left(q_\nu p_\nu + \frac{n}{2} - i\alpha\right) + q_\nu F_{\mu\nu}, \\ E_{nn} &= -q_\nu p_\nu - \left(\frac{n-1}{2} - i\alpha\right) 1, \quad \alpha \in C \end{aligned} \tag{11}$$

(summation over ν).

This realization has the following properties.

(i) *The realization is skew-Hermitian if α is real and if $F_{\mu\nu}$ are skew-Hermitian.*

(ii) *The realization is a Schur-realization if the realization of $\mathrm{gl}(n-1, R)$ is a Schur-realization.*

Realizations of Lie superalgebras

- Analogue analysis of graded Lie algebras (Lie superalgebras)[2]

  - J. Blank, M. Havlíček, P. Exner and W. Lassner,, Canonical representations of the Lie superalgebra $\mathrm{osp}(1, 4)$, Czechoslovak J. Phys. B **31** (1981), no. 11, 1286–1301
  - J. Blank, M. Havlíček, P. Exner and W. Lassner, Boson-fermion representations of Lie superalgebras: the example of $\mathrm{osp}(1, 2)$, J. Math. Phys. **23** (1982), no. 3, 350–353
  - J. Blank and M. Havlíček, *Irreducible $*$-representations of Lie superalgebras $B(0, n)$ with finite-degenerated vacuum*, JINR Dubna, E2-85-112, 1985
  - J. Blank and M. Havlíček, *Irreducible $*$-representations of Lie superalgebras $B(0, n)$ with finite-degenerated vacuum. Results for $B(0, 1)$*, JINR Dubna, E2-85-160, 1985
  - Yu. F. Smirnov, V. N. Tolstoy, M. Havlíček, Č Burdík and A. A. Sakharuk, The Dyson-type boson realizations for representations of the semisimple Lie algebras and superalgebras, in *Group theoretical methods in physics, Vol. 1–3 (Zvenigorod, 1982)*, 67–76, Harwood Academic Publ., Chur.

[2] In the terminology of L. Corwin, Y. Ne'eman and S. Sternberg 1975 *Graded Lie algebras in mathematics and physics (Bose–fermi symmetry)*, Rev. Mod. Phys. **47** 573-603; F. A. Berezin, G. I. Kač 1970, Lie groups with commuting and anticommuting parameters, Mat. Sbornik **124**, no. 3, 343-359.

- J. Blank and M. Havlíček, Irreducible $*$-representations of Lie superalgebras $B(0, n)$ with finite-degenerated vacuum, J. Math. Phys. **27** (1986), no. 12, 2823–2831
- J. Blank and M. Havlíček, Irreducible $*$-representations of the Lie superalgebras $B(0, n)$ with finite-degenerated vacuum II, J. Math. Phys. **29** (1988), no. 3, 546–559
- J. Blank and M. Havlíček, On the tensor product of supersingleton representations of $\mathrm{osp}(1, 2n)$, Czechoslovak J. Phys. B **39** (1989), no. 11, 1192–1207
- J. Blank and M. Havlíček, On the tensor product of supersingleton representations of Lie superalgebras $\mathrm{osp}(1, 2n)$, in *Selected Topics in QFT and Mathematical Physics (Liblice, 1989)*, 190–196, World Sci. Publ., Teaneck, NJ

- Important structural results for $B(0;n)$-type SAs:
  - New method for constructing infinite-dim. representations of superalgebras.
  - Description in terms of creation-annihilation operators of para-Bose systems with n degrees of freedom.
  - *-representations of Lie superalgebras of type $B(0;n)$ and real form $\mathfrak{osp}(1;2n)$.[3]
  - Tensor product decomposition of metaplectic representation σ_n.

Theorem 4.4: The *-representation $\tilde{\sigma}_n$ of $osp(1,2n)$ that is related to the tensor product of two supersingleton representations via the isomorphism (4.1) by $\tilde{\sigma}_n = \mathscr{U}\sigma_n^{\otimes}\mathscr{U}^{-1}$ equals the following direct sum:

$$\tilde{\sigma}_n = \sum_{J=0}^{\infty}{}^{\oplus} \pi_J^{(n)} .$$

  - Simplification of description of generators and relations in the odd sector.
  - Representation $P_M^{(n)}$ of $\mathfrak{sp}(n,\mathbb{R})$ restriction to even sector induces inequivalent extensions to $SU(n,n)$.

[3] $\Omega(z^*) = \Omega(z)^*$ adjoint operation adjoint of linear differential operator.

Quantum groups

- Structural study of $U_q(\mathrm{gl}(n, \mathrm{C}))$, $U_q(\mathrm{so}_m)$ etc.

- Č. Burdík, M. Havlíček and A. Vančura, Irreducible highest weight representations of quantum groups $U_q(\mathrm{gl}(n, \mathrm{C}))$, Comm. Math. Phys. **148** (1992), no. 2, 417–423
- M. Havlíček, E. Pelantová and A. U. Klimyk, Nonstandard $U_q(\mathrm{so}_3)$ and $U_q(\mathrm{so}_4)$: tensor products of representations, oscillator realizations and roots of unity, Czechoslovak J. Phys. **47** (1997), no. 1, 13–16
- M. Havlíček, E. Pelantová and A. U. Klimyk, Santilli-Fairlie algebra $U_q(\mathrm{so}_3)$: tensor products, oscillator realizations and root of unity, Hadronic J. **20** (1997), no. 6, 603–614
- M. Havlíček, S. Pošta and A. U. Klimyk, Representations of the cyclically symmetric q-deformed algebra $U_q(\mathrm{so}_3)$, Czechoslovak J. Phys. **48** (1998), no. 11, 1347–1353
- M. Havlíček, A. U. Klimyk and E. Pelantová, Representations of the q-deformed algebra $U_q(\mathrm{so}_4)$ for q a root of unity, Methods Funct. Anal. Topology **4** (1998), no. 3, 39–44
- M. Havlíček, A. U. Klimyk and S. Pošta, Representations of the cyclically symmetric q-deformed algebra $\mathrm{so}_q(3)$, J. Math. Phys. **40** (1999), no. 4, 2135–2161

- M. Havlíček, A. U. Klimyk and S. Pošta, Representations of the q-deformed algebra $U_q(\mathrm{iso}_2)$, J. Phys. A **32** (1999), no. 25, 4681–4690
- M. Havlíček, S. Pošta and A. U. Klimyk, Representations of the q-deformed algebra $\mathrm{so}_q(2,1)$, in *Symmetry in Nonlinear Mathematical Physics, Part 1, 2 (Kyiv, 1999)*, 280–287, Pr. Inst. Mat. Nats. Akad. Nauk Ukr. Mat. Zastos., 30, Part 1, 2, Natsīonal. Akad. Nauk Ukraïni, Īnst. Mat., Kiev
- M. Havlíček, A. U. Klimyk and S. Pošta, Central elements of the algebras $U_q'(\mathrm{so}_m)$ and $U_q(\mathrm{iso}_m)$, Czechoslovak J. Phys. **50** (2000), no. 1, 79–84
- M. Havlíček, A. U. Klimyk and S. Pošta, Classification of representations of the algebra $U_{q'}(\mathrm{so}_3)$ through examples, Czechoslovak J. Phys. **50** (2000), no. 11, 1235–1238
- M. Havlíček, S. Pošta and A. U. Klimyk, Some basic properties of nonstandard deformations $U_q'(\mathrm{so}_3)$, $U_q'(\mathrm{so}_4)$, in *Trends in Quantum Mechanics (Goslar, 1998)*, 10–17, World Sci. Publ., River Edge, NJ
- M. Havlíček and S. Pošta, On the classification of irreducible finite-dimensional representations of $U_q'(\mathrm{so}_3)$ algebra, J. Math. Phys. **42** (2001), no. 1, 472–500
- M. Havlíček, A. U. Klimyk and S. Pošta, Representations of the q-deformed algebra $U_q'(\mathrm{so}_4)$, J. Math. Phys. **42** (2001), no. 11, 5389–5416

- Č. Burdík, M. Havlíček, O. Navrátil and S. Pošta, Ideals of the enveloping algebra $U(\mathrm{osp}(1,2))$. J. Gen. Lie Theory Appl. 2 (2008), no. 3, 132–136
- R. M. Asherova, Č. Burdík, M. Havlíček, Yu. F. Smirnov and V. N. Tolstoy, q-analog of Gel'fand-Graev basis for the noncompact quantum algebra $U_q(\mathfrak{u}(n,1))$, SIGMA **6** (2010), Paper 010, 13 pp.
- M. Havlíček and S. Pošta, Central elements of quantum deformations, in *XXIX Workshop on Geometric Methods in Physics*, 125–130, AIP Conf. Proc., 1307, Amer. Inst. Phys., Melville, NY
- M. Havlíček and S. Pošta, Center of quantum algebra $U'_q(\mathrm{so}_3)$, J. Math. Phys. **52** (2011), no. 4, 943521, 15 pp.

- Exhaustive study of $U_q(\mathfrak{so}(n))$ for $n = 2, 3, 4$ and their representations.

  - Construction of tensor products in the Santilli-Fairlie algebra $U_q(\mathfrak{so}(3))$.[4]
  - General method for products of IRREPs of $U_q(\mathfrak{so}(3))$ for $q > 0$ given.
  - Study of explicit homomorphisms from $U'_q(\mathfrak{so}(n))$ to q-oscillator algebras $\mathcal{A}_q$ $[q^n - 1 = 0]$.
  - Homomorphism from $U_q(\mathfrak{so}(3))$ to $\widehat{U}_q(\mathfrak{sl}(2))$ and REPS

Theorem 2.2 *There exists a unique algebra homomorphism* $\psi : U_q(\mathrm{so}_3) \to \hat{U}_q(\mathrm{sl}_2)$ *such that*

$$\psi(I_1) = \mathrm{i}(q^H - q^{-H})/(q - q^{-1}), \qquad (2.11)$$
$$\psi(I_2) = (E - F)(q^H + q^{-H})^{-1}, \qquad (2.12)$$
$$\psi(I_3) = (\mathrm{i}q^{H-1/2}E + \mathrm{i}q^{-H-1/2}F)(q^H + q^{-H})^{-1}, \qquad (2.13)$$

where $q^{H+a} := q^H q^a$ *for* $a \in \mathbb{C}$.

$$R_l^{(1)} = T_l^{(1)} \circ \psi, \quad R_l^{(-1)} = T_l^{(-1)} \circ \psi, \quad R_l^{(\mathrm{i})} = T_l^{(\mathrm{i})} \circ \psi, \quad R_l^{(-\mathrm{i})} = T_l^{(-\mathrm{i})} \circ \psi$$

Theorem 3.1 *The representation* $R_l^{(1)}$ *of* $U_q(\mathrm{so}_3)$ *is irreducible. The representations* $R_l^{(\mathrm{i})}$ *and* $R_l^{(-\mathrm{i})}$ *are completely reducible.*

  - Study of infinite-dimensional IRREPs of $U_q(\mathfrak{so}(3))$ [obtainable or not from $\widehat{U}_q(\mathfrak{sl}(2))$].
  - Analogous study for the q-deformed algebra $U_q(\mathrm{iso}(2))$.[5]

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[4] Recall that no Hopf algebra structure is given.

[5] L. L. Vaksman and L. I. Korogodskij, Dokl. Akad. Nauk SSSR 304 (1989), no. 5, 1036–1040

- Analysis of center of quantum algebra $U'_q(\mathfrak{so}_3)$ [difficult!][6]

$$\begin{aligned} q^n - 1 \neq 0 &\implies \text{1 Casimir} \\ q^n - 1 = 0 &\implies \text{4 Casimirs (not algebraically independent)} \end{aligned}$$

- Central elements of $U'_q(\mathfrak{so}_m)$ and $U'_q(\mathfrak{iso}_m)$ and relation with $U'_q(\mathfrak{so}_3)$
- Quantum analogue of Gel'fand–Graev bases for $U_q(\mathfrak{u}(n,1))$
- Discrete series representations of $U_q(\mathfrak{u}(n,1))$

Theorem 3. 1) *Every Hermitian irreducible representation of the discrete series for the non-compact quantum algebra $U_q(u(n,1))$ with the extremal weight $\Lambda^{(\alpha)}_{n+1} = (\lambda_{1,n+1}, \ldots, \lambda_{n+1,n+1})$, where the integers $\lambda_{i,n+1}$ satisfy the inequalities $\lambda_{i,n+1} \geq \lambda_{i+1,n+1}$ $(i = 1, 2, \ldots, n)$, under the restriction $U_q(u(n,1)) \downarrow U_q(u(n))$ contains all multiplicity free irreducible representations of the compact subalgebra $U_q(u(n))$ with the highest weights $\Lambda_n = (\lambda_{1n}, \lambda_{2n}, \ldots, \lambda_{nn})$ satisfying the conditions:*

$$\begin{aligned} &\lambda_{1n} \geq \lambda_{1,n+1} \geq \lambda_{2,n} \geq \lambda_{2,n+1} \geq \cdots \geq \lambda_{\alpha n} \geq \lambda_{\alpha,n+1}, \\ &\lambda_{\alpha+2,n+1} \geq \lambda_{\alpha+1,n} \geq \lambda_{\alpha+3,n+1} \geq \cdots \geq \lambda_{n+1,n+1} \geq \lambda_{nn}. \end{aligned} \tag{5.2}$$

2) *The vectors*

$$|\Lambda^{(\alpha)}_{n+1}; \Lambda_n\rangle = F^{(\alpha)}_{-}(\Lambda_n; \Lambda^{(\alpha)}_{n+1}) \,|\, \Lambda^{(\alpha)}_{n+1}\rangle,$$

where the "lowering" operator $F^{(\alpha)}_{-}(\Lambda_n; \Lambda^{(\alpha)}_{n+1})$ is given by

$$\begin{aligned} F^{(\alpha)}_{-}(\Lambda_n; \Lambda^{(\alpha)}_{n+1}) = N^{(\alpha)}(\Lambda_n; \Lambda^{(\alpha)}_{n+1}) & z_\alpha^{\lambda_{\alpha n} - \lambda_{\alpha,n+1}} \cdots z_1^{\lambda_{1n} - \lambda_{1,n+1}} \\ & \times z_{-\alpha-1}^{\lambda_{\alpha+2,n+1} - \lambda_{\alpha+1,n}} \cdots z_{-n}^{\lambda_{n+1,n+1} - \lambda_{nn}}, \end{aligned} \tag{5.3}$$

for all highest wights $\Lambda_n = (\lambda_{1n}, \lambda_{2n}, \ldots, \lambda_{nn})$ constrained by the conditions (5.2) form an orthonormal basis in the space of the highest vectors with respect to the compact subalgebra $U_q(u(n))$. Here in (5.3) the normalized factor $N^{(\alpha)}(\Lambda_n; \Lambda^{(\alpha)}_{n+1})$ is given as follows:

$$N^{(\alpha)}(\Lambda_n; \Lambda^{(\alpha)}_{n+1}) = \left(\Lambda_n; \Lambda^{(\alpha)}_{n+1} | \Lambda^{(\alpha)}_{n+1}; \Lambda_n\right)^{-\frac{1}{2}}$$

[6] A. M. Gavrilik and A. U. Klimyk 1991, *Lett. Math. Phys.* 21, no. 3, 215–220.

Lie gradings

- Continuation and completion of the Patera–Zassenhaus classification

- M. Havlíček, J. Patera and E. Pelantová, On the fine gradings of simple classical Lie algebras, Internat. J. Modern Phys. A **12** (1997), no. 1, 189–194
- M. Havlíček, J. Patera and E. Pelantová, On Lie gradings. II, Linear Algebra Appl. **277** (1998), no. 1-3, 97–125
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- M. Havlíček, J. Patera, E. Pelantová and J. Tolar, Automorphisms of the fine grading of $\mathfrak{sl}(n, \mathbb{C})$ associated with the generalized Pauli matrices, J. Math. Phys. **43** (2002), no. 2, 1083-1094
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- M. Havlíček, J. Patera, E. Pelantová and J. Tolar, On Pauli graded contractions of $\mathrm{sl}(3, \mathbb{C})$, J. Nonlinear Math. Phys. **11** (2004), 37–42
- M. Havlíček, E. Pelantová and J. Tolar, On representations of Lie algebras compatible with a grading, Acta Polytechnica **50** (2010), no. 5, 30–39.

• Relevant results for the structural study of solvable (nilpotent) Lie algebras and deformations/contractions:

• Description of procedure for obtaining fine gradings in simple classical Lie algebras.[7]

• Analysis of $\mathfrak{gl}(n,\mathbb{C})$-fine gradings in connection with $\mathfrak{sl}(n,\mathbb{C})$-contractions.

• Detailed study of maximal Abelian diagonalizable groups of automorphisms.[8]

• Completion of the classification of fine gradings (Patera and Zassenhaus 1989, Linear Alg. Appl. **112**, 87).

• Systematic and exhaustive analysis of maximal Abelian diagonalizable groups of automorphisms in real forms of classical Lie algebras

• Discrete symmetry characterization in terms of $SL(2,\mathbb{F}_n)$ representations

• Applications to Pauli graded contractions of $\mathfrak{sl}(3,\mathbb{R})$.

• Role of $SL(2,\mathbb{Z}_n)$ analogue to the Weyl group in the case $\mathfrak{sl}(n,\mathbb{C})$ gradings.

• Compatibility criteria of representations with $\mathbb{Z}_2$-gradings.

[7] For $G \subset \mathrm{Aut}(\mathfrak{s})$ maximal Abelian of semisimple elements there is no grading s.t. $K_j \subset \mathfrak{s}_{i_j}$.

[8] Similar approach: D. A. Suprunenko, R. I. Tyshkevich, Commutative Matrices (Minsk: Nauka i Tehnika) 1966.

Differential equations

- I. Úlehla and M. Havlíček, New method for computation of discrete spectrum, Apl. Mat. **25** (1980), no. 5, 358–372
- I. Úlehla, M. Havlíček and J. Hořejší, Eigenvalues of the Schrödinger operator via the Prüfer transformation, Phys. Lett. A **82** (1981), no. 2, 64–66
- M. Havlíček, S. Pošta and P. Winternitz, Nonlinear superposition formulas based on imprimitive group action, J. Math. Phys. **40** (1999), no. 6, 3104–3122
- M. Havlíček, S. Pošta and P. Winternitz, Superposition formulas based on nonprimitive group action, in *Bäcklund and Darboux transformations. The Geometry of Solitons (Halifax, NS, 1999)*, 225–231, CRM Proc. Lecture Notes, 29, Amer. Math. Soc., Providence, RI

- Some relevant conclusions
  - New method for computing discrete spectra of quantum mechanical problems.
  - Transformation of radial Schrödinger equation into first-order ODE

$$\frac{\mathrm{d}z}{\mathrm{d}x} = (\ell+1)cos^2 z - \frac{1}{\ell+1}(v(x,\chi)+\chi^2)sin^2 z$$

  - Replacement of Ritz variational method by direct integration methods.

- Main theorem on nonlinear superposition formulae based on a nonlinear action of $SL(n,\mathbb{C})$ on homogeneous spaces

$$\tilde{x}_i=\frac{(\sum_{j=1}^{N-1} g_{ji}x_j)+g_{Ni}}{(\sum_{j=1}^{N-1} g_{jN}x_j)+g_{NN}},\quad i=1,\dots,N-1,$$

$$\tilde{x}_{i+N-1}=F(i)/F(N-1),\quad i=1,\dots,N-2, \tag{3.18}$$

$$\dot{x}_j=Z_{Nj}+\sum_{i=1}^{N-1} Z_{ij}x_i+x_j\sum_{i=1}^{N-1} Z_{iN}x_i,\quad 1\leqq j\leqq N-1, \tag{3.20}$$

$$\dot{x}_{N-1+j}=Z_{N-1j}+\sum_{i=1}^{N-2} Z_{ij}x_{N-1+i}-Z_{N-1N-1}x_{N-1+j}-x_{N-1+j}\sum_{i=1}^{N-2} Z_{iN-1}x_{N-1+i}$$
$$+(x_j-x_{N-1+j}x_{N-1})\sum_{i=1}^{N-2}(Z_{iN}x_{N-1+i}+Z_{N-1N}),\quad 1\leqq j\leqq N-2. \tag{3.21}$$

Theorem 5: (1) The nonlinear ODEs with superposition formulas, based on the action of SL(N,$\mathbb{C}$) on the space SL(N,$\mathbb{C}$)/$P(N)$ are given for all $N\geqq 3$ in Eqs. (3.20) and (3.21). The general form of the solution is given by Eq. (3.18).

(2) The number of equations for SL(N,$\mathbb{C}$) is $n=2N-3$. The group elements $g_{ij}(t)$ can be reconstructed from m particular generically chosen solutions with

$$m=\begin{cases} k+2 & \text{for } N=2k+1 \\ k+1 & \text{for } N=2k \end{cases}. \tag{4.32}$$

(3) Such a fundamental set of m solutions satisfies τ constraints with

$$\tau=mn-N^2+1=\begin{cases} 3k-2 & \text{for } N=2k+1 \\ k-2 & \text{for } N=2k \end{cases}. \tag{4.33}$$

(4) The reconstruction of the group action is linear in the sense that it requires the solution of $2N-3$ linear algebraic equations.

Representation theory

- Extensive work on the characterization of highest weights representations of SCLAs

- Č. Burdík, M. Havlíček and P. Exner, Highest-weight representations of the $\mathrm{sl}(n+1, \mathrm{C})$ algebras: maximal representations, J. Phys. A **14** (1981), no. 5, 1039–1054
- Č. Burdík, P. Exner and M. Havlíček, Highest-weight representations of $\mathrm{sl}(2, \mathrm{C})$ and $\mathrm{sl}(3, \mathrm{C})$ via canonical realizations, Czechoslovak J. Phys. B **31** (1981), no. 5, 459–469
- Č. Burdík, P. Exner and M. Havlíček, A complete set of irreducible highest-weight representations for $\mathrm{sl}(3, \mathrm{C})$, Czechoslovak J. Phys. B **31** (1981), no. 11, 1201–1206
- M. Havlíček and P. J. Moylan, An embedding of the Poincaré Lie algebra into an extension of the Lie field of $\mathrm{SO}_0(1,4)$, J. Math. Phys. **34** (1993), no. 11, 5320–5332
- S. Pošta and M. Havlíček, Note on Verma bases for representations of simple Lie algebras, Acta Polytechnica **53** (2013), no. 5, 450–456
- M. Havlíček, J. Kotrbatý, P. J. Moylan and S. Pošta, Construction of representations of Poincaré group using Lie fields, J. Math. Phys. **59** (2018), no. 2, 021702, 23 pp.

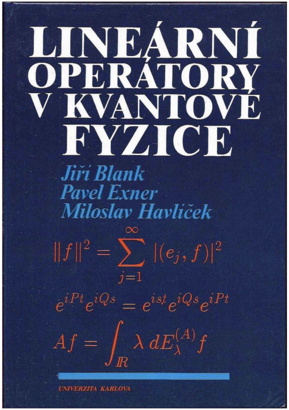
- Distinguished results
  - Construction of $\mathfrak{sl}(n+1,\mathbb{C})$-representations by means of boson realizations.
  - Determination and explicit expressions of maximal representation $D_M^{(n+1)}$ containing IRREP with highest weight M.
  - Irreducibility criteria for infinite-dimensional representations of the algebras $\mathfrak{sl}(2,\mathbb{C})$, $\mathfrak{sl}(3,\mathbb{C})$.
  - Determination of a complete set of irreducible highest-weight representations for $\mathfrak{sl}(3,\mathbb{C})$:

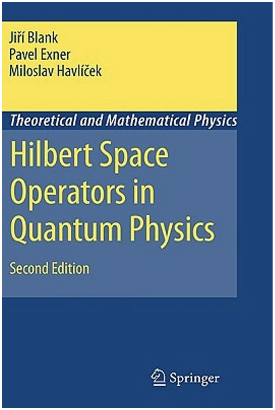
$$\Omega_F = \{\Lambda:\ \Lambda_i \in \mathbb{N}, i=1,2\}, \quad \Omega_1 = \{\Lambda:\ \Lambda_1 \in \mathbb{N}, \Lambda_2 \notin \mathbb{N}\},$$
$$\Omega_2 = \{\Lambda:\ \Lambda_1 \notin \mathbb{N}, \Lambda_2 \in \mathbb{N}\}, \quad \Omega_{12} = \{\Lambda:\ \Lambda_i \notin \mathbb{N}, i=1,2; 1+\Lambda_1+\Lambda_2 \in \mathbb{N}\},$$
$$\Omega_{\max} = \{\Lambda:\ \Lambda_i \notin \mathbb{N}, i=1,2; 1+\Lambda_1+\Lambda_2 \notin \mathbb{N}\}.$$

  - Alternative to the proof of Verma inequalities [P. Littelmann 1997 *Pure Appl Algebra* **117/118** 447–468].
  - Construction of $\tilde{P}_n$-representations by extension and localization of enveloping algebras.

A fundamental reference for QM

- A marvelous textbook with an adventurous background history[9]

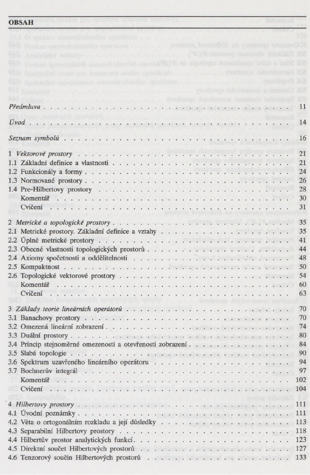




Karolinum, 1993; 1st English Edition, AIP 1994; 2nd Edition, Springer 2008

[9] Work began around 1973, for several reasons the first edition could not appear until 1993.

- Actually not one, but two books (both very valuable!)



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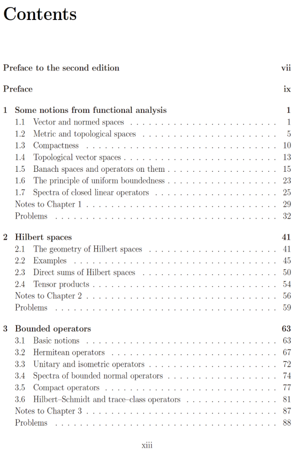

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- Some feedback on the book

  - "...The book can serve as an excellent text for graduate students."
    M. Gorzelańczyk (Wrocław) in Zentralblatt, 1994
  - "This is an excellent textbook for graduate students and young researchers in mathematics and theoretical physics..."
    Michael Demuth (Goslar) in Zentralblatt, 2008
  - "Each chapter contains valuable supplements... As the authors emphasize ... their aim is to describe the mathematical foundations of quantum theory in complete way. This aim is reached... "
    H. Baumgärtel (Berlin) in Mathematical Reviews, 1994
  - "I really enjoyed reading this work. It is very well written, by three real experts in the field. It stands quite alone...."
    John R. Taylor (Boulder, CO)
  - "Fantastic reference on mathematical quantum theory ... one of my favorite go-to references on the mathematically rigorous details of quantum theory."
    Extract from Amazon [100% of positive reviews]
  - "... A valuable feature of this book is the extensive background material and discussions collected into separate sections. ... The book does what it promises and does it well."
    Stig Stenholm in Contemporary Physics, 2010

- Detailed technical review: W. G. Faris, *Bull. Amer. Math. Soc. (N.S.)* **32** (1995), no. 3, 339–344.

AAMP meetings dedicated to Miloslav

• Analytic and Algebraic Methods in Physics IV, Villa Lanna, Prague, October 21st, 2008
Dedicated to subjects preferred by Prof. Havliček

• Analytic and Algebraic Methods in Physics XII, Villa Lanna, Prague, November 2nd, 2013
On the occasion of 75^{th} birthday of Prof. Havliček

Proceedings as Special Issue of Acta Polytechnica Vol. 53, No. 5, October 2013
[Guest Editor: Pavel Exner]

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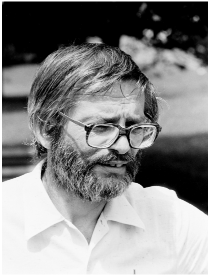
Miloslavův odkaz žije dál



Děkuji za pozornost

Thank you for your attention