

# Quadratic integrable system in magnetic field: beyond configuration space separability.

**Ondřej Kubů**

Faculty of Nuclear Sciences and Physical Engineering,  
Czech Technical University in Prague

Collaboration: **A. Marchesiello**, **L. Šnobl**, **P. Tempesta**

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# Superintegrability of systems with magnetic fields

**Integrable and superintegrable systems** are distinguished by their symmetry, which enables to solve their equations of motion **exactly**, in quadratures or even algebraically, respectively. Due to these properties they serve as a basis for perturbative models of more complex situations.

However, most integrable and superintegrable system were studied in the absence of **magnetic field**. However, there are many physical situations when magnetic field is crucial, e.g. **in plasma or accelerator physics**, hence the need for their study.

We (will) focus on **geometrical aspects** of these systems, often connected with **separation of variables**.

# Integrability and Superintegrability

## Integrability

A classical Hamiltonian system with  $n$  degrees of freedom is **integrable** if it admits  $n$  functionally independent integrals of motion in involution.

## Superintegrability

A classical Hamiltonian system with  $n$  degrees of freedom is **polynomially superintegrable** if it admits  $n + k$  functionally independent integrals of motion (where  $k \leq n - 1$ ), that are polynomial in the momenta and out of which  $n$  are in involution.

In 3D: 4 integrals = minimal superintegrability, 5 integrals = maximal superintegrability.

# Magnetic field and canonical momenta

The **vector potential** is represented by a 1-form

$$A(\vec{x}) = A_1(\vec{x})dx^1 + A_2(\vec{x})dx^2 + A_3(\vec{x})dx^3$$

and the **magnetic field** by a 2-form

$$B = dA = \left( \frac{\partial A_3}{\partial x^2} - \frac{\partial A_2}{\partial x^3} \right) dx^2 \wedge dx^3 + \text{cycl.} \equiv B^1(\vec{x})dx^2 \wedge dx^3 + \text{cycl.}$$

We use **momenta covariant** with respect to a time-independent gauge transformation  $A'(\vec{x}) = A(\vec{x}) + d\chi(\vec{x})$ ,  $V'(\vec{x}) = V(\vec{x})$  as

$$p_i^A = p_i + A_i = mv_i.$$

The **Hamiltonian** of the system on a Riemannian manifold reads

$$H = \frac{1}{2}g^{ij}(p_i + A_i)(p_j + A_j) + V(\vec{x}) = \frac{1}{2}g^{ij}p_i p_j + A^i p_j + U(\vec{x}).$$

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# Quadratic integrals

The condition for the integral of motion  $X$  is

$$\{H, X\} = 0.$$

For polynomial integrals it separates into PDEs for different coefficients of momenta.

For systems on 3D Euclidean space, the **highest order terms** of **quadratic** integrals are in the **enveloping algebra of the Euclidean algebra**  $\mathfrak{U}(\mathfrak{e}_3)$ .

$$\begin{aligned} X = & \sum_{i,j: i \leq j} \alpha_{ij} \ell_i^A \ell_j^A + \sum_{i,j} \beta_{ij} p_i^A \ell_j^A + \sum_{i,j: i \leq j} \gamma_{ij} p_i^A p_j^A + \\ & + s_1(\vec{x}) p_1^A + s_2(\vec{x}) p_2^A + s_3(\vec{x}) p_3^A + m(\vec{x}), \end{aligned}$$

where  $\ell_i^A = \sum_{j,k} \epsilon_{ijk} x^j p_k^A$ .

The **lower order ones** imply 10 linear PDEs on the functions  $\vec{s}$ ,  $m$ ,  $\vec{B}$ ,  $V$  which also depend on 20 constants  $\alpha_{ij}$ ,  $\beta_{ij}$  and  $\gamma_{ij}$ .

For integrability we need 2 such integrals in involution  $\rightsquigarrow$  naive brute-force approach is intractable.

# Quadratic integrability with magnetic field

When we consider **integrable systems on Euclidean 3D spaces**, i.e.

$$\{X_1, H\} = \{X_2, H\} = \{X_1, X_2\} = 0,$$

the restriction on the form of integrals comes from the involutivity. We can moreover choose the reference frame, i.e. perform Euclidean transformations, take linear combinations and use the geometric relation  $\vec{p} \cdot \vec{\ell} = 0$ .

The key to classification observed in [Marchesiello and Šnobl 2022] is to first classify pairs of quadratic terms (an algebraic problem) and only then consider lower order equations.

Somewhat surprisingly, **the highest order terms** of the integrals **may not be constrained** to the standard separable cases, i.e. **nonstandard integrals exist**. First such systems was found in [Marchesiello and Šnobl 2017]. The core of my PhD thesis [Kubů 2024] was classification of such systems.

# Possible forms of quadratic terms for integrable systems

As observed in [Marchesiello and Šnobl 2022], pairs of homogeneous quadratic integrals form **3D Abelian subalgebras**  $\text{span}\{h = \vec{p}^2, X_1, X_2\}$  of **quadratic commuting elements in  $\mathfrak{U}(\mathfrak{e}_3)$**  modulo equation  $\vec{p} \cdot \vec{\ell} = 0$  and transformations from the Euclidean group.

Their classification leads to **11 classes**, 9 of which may *a priori* admit **nonstandard systems** (i.e. not connected to separation of variables). For the latter we have to study the lower order equations.

In the following list we review the state of the art: the **blue** classes have been showed to possess nonstandard generalized systems, the **red** ones do not, **green** do not allow them due to their algebraic structure, on **bold** we do not have (complete) results.

# Review of investigated classes

- a**  $X_1 = \ell_1^2 + \ell_2^2 + \ell_3^2 + a\ell_3 p_3 + b p_3^2$ ,  $X_2 = \ell_3^2$ ,
- b**  $X_1 = \ell_1^2 + \ell_2^2 + \ell_3^2 + b(ap_2^2 + p_3^2)$ ,  $X_2 = a\ell_2^2 + \ell_3^2 - abp_1^2$ ,
- c**  $X_1 = \ell_1^2 + \ell_2^2 + \ell_3^2 + 2b(\ell_1 p_1 - (3a-1)\ell_2 p_2 - 2\ell_3 p_3) + 3b^2((1-4a)p_1^2 - (3a^2-2a-1)p_2^2 + 2(a-1)p_3^2)$ ,  $X_2 = a\ell_2^2 + \ell_3^2 + 6ab\ell_1 p_1 + 9ab^2(ap_3^2 + p_2^2)$ ,
- d**  $X_1 = \ell_3^2$ ,  $X_2 = \frac{1}{2}(\ell_1 p_2 + p_2 \ell_1 - \ell_2 p_1 - p_1 \ell_2) + a\ell_3 p_3$ ,
- e**  $X_1 = \ell_3^2 + 2a(\ell_1 p_1 - \ell_2 p_2) + a^2 p_3^2$ ,  $X_2 = \frac{1}{2}(\ell_1 p_2 + p_2 \ell_1 - \ell_2 p_1 - p_1 \ell_2) - ap_1 p_2$ ,
- f**  $X_1 = \ell_3^2 + a\ell_3 p_3 + bp_1^2 + cp_1 p_3 + dp_2 p_3$ ,  $X_2 = p_3^2$ ,
- g**  $X_1 = \ell_3^2 + ap_3^2$ ,  $X_2 = \ell_3 p_3 + bp_3^2$ ,
- h**  $X_1 = \ell_1 p_1 + a\ell_2 p_2 - (a+1)\ell_3 p_3 + bp_2^2$ ,  $X_2 = p_1^2 + \frac{2a+1}{a+2} p_2^2$ ,
- i**  $X_1 = \ell_1 p_1 + ap_2^2 + bp_2 p_3$ ,  $X_2 = p_1^2$ ,
- j**  $X_1 = \ell_1 p_1 + a\ell_2 p_2 - (a+1)\ell_3 p_3 + \frac{\omega}{2}(\ell_1 p_3 + p_3 \ell_1 - \ell_3 p_1 - p_1 \ell_3) + 2bp_1 p_2 + c(p_2^2 - p_3^2)$ ,  $X_2 = p_1^2 + \frac{6\omega}{4a-1} p_1 p_3 + \frac{a+2}{4a-1} p_2^2 - \frac{5a+1}{4a-1} p_3^2$ ,
- k**  $X_1 = p_1^2 + ap_2^2$ ,  $X_2 = p_2^2 + bp_1 p_2 + cp_1 p_3 + dp_2 p_3$ .

# Nonstandard systems

For example, **the extended cylindrical class** has the integrals

$$\begin{aligned}X_1 &= \ell_3^2 + a\ell_3 p_3 + b p_1^2 + c p_1 p_3 + d p_2 p_3 + \dots, \\X_2 &= p_3^2 + \dots,\end{aligned}$$

where the parameters  $a, b, c, d$  satisfy

$$a, b \in \mathbb{R}, c \geq 0, d \geq 0.$$

One of corresponding systems is ( $a \neq 0, c \neq 0, b = d = 0$ )

$$\begin{aligned}B^x &= \beta_2(ay + c), & B^y &= -\beta_2 ax, & B^z &= -(3\beta_2 r^2 + 2\beta_1), \\V &= \beta_2 \left[ -\frac{\beta_2}{4} r^6 - \frac{\beta_2 a^2 + 4\beta_1}{8} r^4 - \frac{ac\beta_2}{2} r^2 y + \frac{\beta_2 c^2}{2} x^2 \right. \\&\quad \left. + \omega_1 r^2 + \omega_2 x + \omega_3 y \right], & r &= \sqrt{x^2 + y^2}.\end{aligned}$$

The fields depend only on free parameters, no undetermined functions.

# Helical undulator in solenoid – a superintegrable system

Not much freedom for further restriction  $\rightsquigarrow$  superintegrable nonstandard systems will (probably) be rare.

In [Kubů et al. 2023] we have found one new superintegrable system,

$$\vec{B}(\vec{x}) = \left( -\beta_3 \cos\left(\frac{2z}{a}\right), \beta_3 \sin\left(\frac{2z}{a}\right), \beta_1 \right), \quad V(\vec{x}) = 0.$$

We need **both**  $\beta_1 \neq 0$  and  $\beta_3 \neq 0$ , otherwise the system was known from [Marchesiello, Šnobl, and Winternitz 2015].

It models motion of an electron in a nonrelativistic limit of a **helical undulator in an infinite solenoid**, neglecting the produced radiation. Helical undulator is a key component of **free-electron lasers**. The field can be produced by redistributing the constant magnetic field via a **ferromagnetic helix** [Balal et al. 2017] or by an **array of magnets** [Varfolomeev et al. 1993].

# Helical undulator in solenoid – Integrals

It admits 3 first-order integrals,

$$Y_1 = p_x^A + \beta_1 y + \frac{\beta_3 a}{2} \cos\left(\frac{2z}{a}\right),$$

$$Y_2 = p_y^A - \beta_1 x - \frac{\beta_3 a}{2} \sin\left(\frac{2z}{a}\right),$$

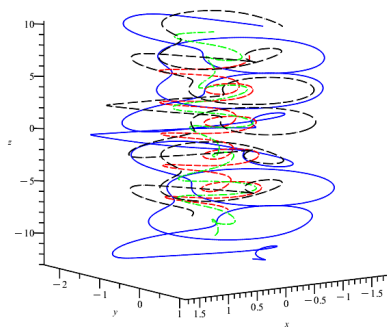
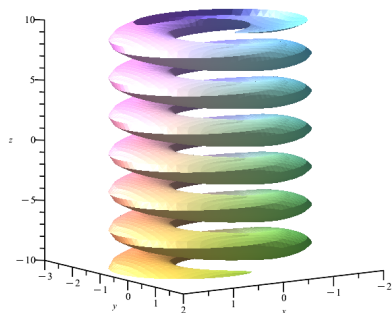
$$Y_3 = \ell_z^A - \frac{a}{2} p_z^A - \frac{1}{2} \left[ \beta_1 r^2 + \beta_3 a \sin\left(\frac{2z}{a}\right) x + \beta_3 a \cos\left(\frac{2z}{a}\right) y \right].$$

which **do not Poisson commute** pairwise if  $\beta_1 \neq 0$ ,

$$\{Y_1, Y_2\} = \beta_1 \cdot 1, \quad \{Y_1, Y_3\} = -Y_2, \quad \{Y_2, Y_3\} = Y_1.$$

If  $\beta_1^2 + \beta_3^2 \neq 0$ , it does **not** separate in any **orthogonal coordinate system** on configuration space. Work in progress on **separation on the phase space** using the Haantjes framework.

# Sample trajectories



**Figure:** Sample trajectories and the surface in configuration space with the integrals  $E = \frac{11}{8} + \frac{\pi^2}{8} + \frac{\sqrt{3}}{2}$ ,  $Y_1 = 1$ ,  $Y_2 = 0$ ,  $Y_3 = -\frac{1}{2} - \frac{\pi}{4}$  and the parameters  $a = 1$ ,  $b_1 = -\frac{\pi}{2}$ ,  $b_3 = -\sqrt{3}$ .

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# Nijenhuis and Haantjes operators

Let  $M$  be a differentiable manifold and  $K : TM \rightarrow TM$  be a (1,1) tensor field.

## Definition (Nijenhuis 1951)

The **Nijenhuis torsion** of  $K$  is the vector-valued 2-form defined by

$$\mathcal{T}_K(X, Y) := K^2[X, Y] + [KX, KY] - K([X, KY] + [KX, Y]),$$

where  $X, Y \in TM$  and  $[ , ]$  denotes the commutator of two vector fields.

## Definition (Haantjes, 1955)

The **Haantjes torsion** of  $K$  is the vector-valued 2-form defined by

$$\mathcal{H}_K(X, Y) := K^2\mathcal{T}_K(X, Y) + \mathcal{T}_K(KX, KY) - K(\mathcal{T}_K(X, KY) + \mathcal{T}_K(KX, Y)).$$

## Definition

A **Nijenhuis/Haantjes operator** is an operator whose Nijenhuis/Haantjes torsion identically vanishes.

# Examples of Nijenhuis and Haantjes operators

- **Every operator** on a 2-dimensional manifold is a **Haantjes operator**. In general, it is not a Nijenhuis one.
- Let  $M$  be an  $n$ -dimensional manifold and  $(x_1, \dots, x_n)$  a local chart on  $M$ . Let us consider the **simple** operators

$$\mathbf{K}_1 := \begin{pmatrix} \lambda_1(x_1) & 0 & \cdots & 0 \\ 0 & \lambda_2(x_2) & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n(x_n) \end{pmatrix},$$

$$\mathbf{K}_2 := \begin{pmatrix} \lambda_1(x_1, \dots, x_n) & 0 & \cdots & 0 \\ 0 & \lambda_2(x_1, \dots, x_n) & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n(x_1, \dots, x_n) \end{pmatrix}.$$

$\mathbf{K}_1$  is a **Nijenhuis** operator,  $\mathbf{K}_2$  is a **Haantjes** operator.

# Haantjes algebras: definition

An associative algebra of Haantjes operators of rank  $m$ , hereafter **Haantjes algebra**, is a pair  $(M, \mathcal{H})$  where

- $M$  is a differentiable manifold of dimension  $n$ ;
- $\mathcal{H}$  is a set of Haantjes operators  $\mathbf{K} : TM \rightarrow TM$ , that generate i) a **free module** of rank  $m$  over the ring of smooth functions on  $M$

$$\mathcal{H}_{(f(\mathbf{x})\mathbf{K}_1+g(\mathbf{x})\mathbf{K}_2)}(X, Y) = \mathbf{0}, \quad \forall \mathbf{K}_1, \mathbf{K}_2 \in \mathcal{H}, \quad \forall X, Y \in TM,$$

where  $f(\mathbf{x}), g(\mathbf{x})$  are arbitrary smooth functions on  $M$ . **Rank  $m$**  means that the module is generated by a “basis” with  $m$  elements.

ii) a **ring** w.r.t. the composition operation

$$\mathcal{H}_{(\mathbf{K}_1\mathbf{K}_2)}(X, Y) = \mathcal{H}_{(\mathbf{K}_2\mathbf{K}_1)}(X, Y) = \mathbf{0}, \quad \forall \mathbf{K}_1, \mathbf{K}_2 \in \mathcal{H}, \quad \forall X, Y \in TM.$$

- In addition, if  $\mathbf{K}_1\mathbf{K}_2 = \mathbf{K}_2\mathbf{K}_1, \forall \mathbf{K}_1, \mathbf{K}_2 \in \mathcal{H}$  the algebra  $\mathcal{H}$  will be called an **Abelian Haantjes algebra**.

# Basic examples

A natural Haantjes algebra is realized, in a local chart  $\{U, \mathbf{x} = (x^1, \dots, x^n)\}$ , by any set of **diagonal operators** of the form

$$\mathbf{K} = \sum_{k=1}^n \lambda_k(\mathbf{x}) \frac{\partial}{\partial x^k} \otimes dx^k, \quad (1)$$

where the smooth functions  $\lambda_k(\mathbf{x})$  play the role of **eigenvalue fields** of  $\mathbf{K}$ . Such operators generate an algebraic structure that will be said to be a *diagonal* Haantjes algebra.

More generally, we shall say that a Haantjes algebra is **semisimple** if each operator  $\mathbf{K} \in \mathcal{H}$  is semisimple, i.e. its (proper) eigenvectors form a frame in all open neighborhoods  $U \subset M$ .

P. Tempesta & G. Tondo, *Ann. Mat. Pura Appl.* (2021)

## Definition

A  $\omega\mathcal{H}$  (or **symplectic–Haantjes**) manifold of class  $m$  is a triple  $(M, \omega, \mathcal{H})$  which satisfies the following properties:

- i)  $(M, \omega)$  is a symplectic manifold of dimension  $2n$ ;
- ii)  $\omega$  is a symplectic form in  $M$ ;
- iii)  $\mathcal{H}$  is an **Abelian Haantjes algebra** of rank  $m$ ;
- iv)  $(\omega, \mathcal{H})$  are algebraically compatible, that is

$$\omega(X, \mathbf{K}Y) = \omega(\mathbf{K}X, Y) \quad \forall \mathbf{K} \in \mathcal{H}.$$

For our purposes, the manifold  $M$  will be the phase space  $M = T^*Q$ .

Proposition 2 in D. Reyes et al. (2024) implies that compatibility with  $\omega$  can only work with Abelian Haantjes algebras.

# Diagonalization of Haantjes algebras

P. Tempesta & G. Tondo. J. Geom. Phys. (2021).

## Theorem (Simultaneous diagonalization)

Let  $(M, \mathcal{H})$  be an **Abelian Haantjes algebra**.

i) If  $\mathcal{H}$  is a **semisimple Haantjes algebra**, then there exist local coordinate charts  $\{(x_1, \dots, x_n)\}$  where all  $\mathbf{K} \in \mathcal{H}$  can be simultaneously diagonalized.

Conversely, let  $\{\mathbf{K}_1, \dots, \mathbf{K}_m\}$  be a commuting set of  $m$   $C^\infty(M)$ -linearly independent operators. If they share a set of local coordinate charts in which they take a **diagonal form**, then they generate a semisimple Abelian Haantjes algebra (of rank not smaller than  $m$ ).

ii) More generally, if the Abelian Haantjes algebra  $(M, \mathcal{H})$  is **non-semisimple**, then there exist local coordinate charts  $\{(x_1, \dots, x_n)\}$  where all  $\mathbf{K} \in \mathcal{H}$  take a **block-diagonal form**.

# Haantjes chains, Darboux-Haantjes coordinates

## Definition

Let  $(M, \mathcal{H})$  be a Haantjes algebra of rank  $m$ . A smooth function  $H$  generates a **Haantjes chain** of 1-forms of length  $m$  if there exist a distinguished basis  $\{\tilde{K}_1, \dots, \tilde{K}_m\}$  of  $\mathcal{H}$  such that  $\tilde{K}_\alpha^T dH =: dH_\alpha$ ,  $\alpha = 1, \dots, m$  are locally exact.

## Theorem (D. Reyes, P. Tempesta & G. Tondo, 2022)

There is a 1:1 correspondence between complete **Louville integrability**, **separation of variables** for the **Hamilton–Jacobi equations** associated with the integrals  $\{H_1, H_2, \dots, H_n\}$ , and an  $\omega\mathcal{H}$  **manifold** of class  $n$  with a Haantjes chain generated by  $H_1$  with operators

$$K_\alpha = \sum_{i=1}^n \frac{\frac{\partial H_\alpha}{\partial p_i}}{\frac{\partial H_1}{\partial p_i}} \left( \frac{\partial}{\partial q^i} \otimes dq^i + \frac{\partial}{\partial p_i} \otimes dp_i \right) \quad \alpha = 1, \dots, n.$$

The **canonical coordinates**  $(\mathbf{q}, \mathbf{p})$  where  $K_\alpha$  take this diagonal form are called **Darboux-Haantjes** (DH) coordinates.

# Finding Darboux-Haantjes coordinates

P. Tempesta & G. Tondo. J. Geom. Phys. (2021).

- 1 Given an algebra  $\mathcal{H}$ , determine the nontrivial **joint eigen-distributions**  $\mathcal{V}_a(\mathbf{x})$  as the intersection of (generalized) eigen-distributions  $\mathcal{D}_{i_m}^{(j)}(\mathbf{x}) := \ker(\mathbf{K}^{(j)} - \lambda_{i_m} \mathbf{I})^{\rho_i}(\mathbf{x})$  corresponding to different operators  $\mathbf{K}^{(j)}$

$$\mathcal{V}_a(\mathbf{x}) := \bigoplus_{i_1, \dots, i_m}^{s_1, \dots, s_m} \mathcal{D}_{i_1}^{(1)}(\mathbf{x}) \cap \dots \cap \mathcal{D}_{i_m}^{(m)}(\mathbf{x}), \quad a = 1, \dots, v \leq n.$$

- 2 Construct a basis of closed 1-forms corresponding to each **annihilator**  $(\bigoplus_{a=1, \dots, \hat{i}, \dots, n} \mathcal{V}_a)^\circ$ , where we omit one  $\mathcal{V}_a(\mathbf{x})$  each time.
- 3 Find the local **characteristic and canonical coordinates** by integrating (suitable combinations of) the locally exact 1-forms.

# Application to 3D constant magnetic field

Let us consider an electron in the **constant magnetic field**

$$\vec{B}(\vec{x}) = b \vec{e}_z$$

with no electric potential on 3D Euclidean space. The corresponding Hamiltonian reads

$$H = \frac{1}{2} \left( \vec{p} + \vec{A}(\vec{x}) \right)^2 \equiv \frac{1}{2} \vec{\Pi}(\vec{x})^2.$$

$\vec{A}(\vec{x})$  is the vector potential generating the magnetic field  $\vec{B} = \nabla \times \vec{A}$ . The superintegrable system admits 4 independent **first-degree integrals**

$$H_1 = \Pi_x + by, \quad H_2 = \Pi_y - bx, \quad H_3 = \Pi_z, \quad H_4 = (x\Pi_y - y\Pi_x) - \frac{b}{2}(x^2 + y^2)$$

and the fifth, nonpolynomial one (A. Marchesiello, L. Šnobl & P. Winternitz, 2015).

$$H_5 = -\Pi_x \cos\left(\frac{bz}{\Pi_z}\right) - \Pi_y \sin\left(\frac{bz}{\Pi_z}\right).$$

# Cartesian $\omega\mathcal{H}$ structure I

In the following, we shall provide a set of **semisimple** Haantjes operators that solve the chain equations  $\mathbf{K}_i^T dH = dH_i$ , jointly with their spectral properties. We focus on the Cartesian  $\omega\mathcal{H}$  structure  $\mathcal{H}_1 = \{\mathbf{I}_{6 \times 6}, \mathbf{K}_1, \mathbf{K}_3\}$ .

$$\mathbf{K}_1 = \frac{1}{\Pi_x} \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & \partial_x A_y & \partial_x A_z & 1 & 0 & 0 \\ -\partial_x A_y & 0 & 0 & 0 & 0 & 0 \\ -\partial_x A_z & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$\lambda_1^{(1)} = 0, \mathcal{D}_1 = \left\langle \partial_x A_z \frac{\partial}{\partial y} - \partial_x A_y \frac{\partial}{\partial z}, \frac{\partial}{\partial y} - \partial_x A_y \frac{\partial}{\partial p_x}, \frac{\partial}{\partial p_y}, \frac{\partial}{\partial p_z} \right\rangle,$$
$$\lambda_2^{(1)} = \frac{1}{\Pi_x}, \mathcal{D}_2 = \left\langle \frac{\partial}{\partial x} - \partial_x A_y \frac{\partial}{\partial p_y} - \partial_x A_z \frac{\partial}{\partial p_z}, \frac{\partial}{\partial p_x} \right\rangle.$$

## Cartesian $\omega\mathcal{H}$ structure II

$$\mathbf{K}_3 = \frac{1}{\Pi_z} \left[ \begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & -\partial_z A_x & 0 & 0 & 0 \\ 0 & 0 & -\partial_z A_y & 0 & 0 & 0 \\ \partial_z A_x & \partial_z A_y & 0 & 0 & 0 & 1 \end{array} \right],$$

$$\lambda_1^{(3)} = 0, \mathcal{D}_3 = \left\langle \partial_z A_y \frac{\partial}{\partial x} - \partial_z A_x \frac{\partial}{\partial y}, \frac{\partial}{\partial x} - \partial_z A_x \frac{\partial}{\partial p_z}, \frac{\partial}{\partial p_x}, \frac{\partial}{\partial p_y} \right\rangle,$$

$$\lambda_2^{(3)} = \frac{1}{\Pi_z}, \mathcal{D}_4 = \left\langle \frac{\partial}{\partial z} - \partial_z A_x \frac{\partial}{\partial p_x} - \partial_z A_y \frac{\partial}{\partial p_y}, \frac{\partial}{\partial p_z} \right\rangle.$$

# DH coordinates for the Cartesian $\omega\mathcal{H}$ structure I

We now determine the **Darboux-Haantjes coordinates** for  $\mathcal{H}_1$ .

Defining the distributions

$$\mathcal{V}_1 = \mathcal{D}_1 \cap \mathcal{D}_3, \quad \mathcal{V}_2 = \mathcal{D}_1 \cap \mathcal{D}_4, \quad \mathcal{V}_3 = \mathcal{D}_2 \cap \mathcal{D}_3, \quad \mathcal{V}_4 = \mathcal{D}_2 \cap \mathcal{D}_4 = \emptyset,$$

we can work out the annihilators

$$(\mathcal{V}_1 \oplus \mathcal{V}_2)^\circ = \langle dx, \partial_x A_y dy + \partial_z A_x dz + dp_x \rangle,$$

$$(\mathcal{V}_1 \oplus \mathcal{V}_3)^\circ = \langle dz, \partial_x A_z dx + \partial_y A_z dy + dp_z \rangle,$$

$$(\mathcal{V}_2 \oplus \mathcal{V}_3)^\circ = \langle dy, \partial_x A_y dx + \partial_z A_y dz + dp_y \rangle.$$

Integrating them, we find a set of Darboux-Haantjes coordinates for each choice of gauge

$$\begin{aligned} q^1 &= x, & p_1 &= \Pi_x + by, \\ q^2 &= y, & p_2 &= \Pi_y, \\ q^3 &= z, & p_3 &= \Pi_z. \end{aligned} \tag{2}$$

# DH coordinates for the Cartesian $\omega\mathcal{H}$ structure II

In this chart, the operators of the algebra  $\mathcal{H}_1$  are **indeed diagonal**

$$\mathbf{K}_1 = \frac{1}{p_1 - bq^2} \text{diag}(1, 0, 0, 1, 0, 0),$$
$$\mathbf{K}_3 = \frac{1}{p_3} \text{diag}(0, 0, 1, 0, 0, 1);$$

the corresponding Hamiltonians  $H, H_1, H_3$  are **indeed separable**.

$$H = \frac{1}{2} [(p_1 - bq^2)^2 + p_2^2 + p_3^2],$$
$$H_1 = p_1, \quad H_2 = p_2 - bq^1, \quad H_3 = p_3,$$
$$H_4 = q^1 p_2 - q^2 p_1 + \frac{b}{2} ((q^2)^2 - (q^1)^2).$$

The form of  $H$  is the one we would obtain by choosing the gauge  $\vec{A} = (-bq^2, 0, 0)$  in the original Hamiltonian (23).

# Stäckel geometry

In a suitable choice of coordinates, a large class of separable systems satisfy the **Stäckel equation**

$$\begin{bmatrix} \tilde{H}_1 \\ \tilde{H}_2 \\ \tilde{H}_3 \end{bmatrix} = S^{-1} \begin{bmatrix} f_1(q^1, p_1) \\ f_2(q^2, p_2) \\ f_3(q^3, p_3) \end{bmatrix}, \quad S = \begin{bmatrix} S_{11}(q^1) & S_{12}(q^1) & S_{13}(q^1) \\ S_{21}(q^2) & S_{22}(q^2) & S_{23}(q^2) \\ S_{31}(q^3) & S_{31}(q^3) & S_{31}(q^3) \end{bmatrix},$$

where  $\tilde{H}_i$  are the commuting Hamiltonians. Note the dependence for the rows of the **Stäckel matrix**  $S$  and of the generalized **Stäckel functions**  $f_i$  (Arnold, Kozlov and Neishtadt 1997).

The classical Stäckel functions for systems without magnetic field are always quadratic in momenta  $f_k := \frac{1}{2}p_k^2 + W_k(q^k)$  and the associated Haantjes operator can be projected to a **Killing tensor**,

$$\mathbf{K}_{j-1} := \sum_{r=1}^n \frac{\tilde{S}_{jr}}{\tilde{S}_{1r}} \left( \frac{\partial}{\partial q^r} \otimes dq^r + \frac{\partial}{\partial p_r} \otimes dp_r \right) \xrightarrow{\pi} \tilde{\mathbf{K}}_{j-1} := \sum_{r=1}^n \frac{\tilde{S}_{jr}}{\tilde{S}_{1r}} \frac{\partial}{\partial q^r} \otimes dq^r.$$

Here  $\tilde{S}_{jk}$  denotes the cofactor of the element  $S_{kj}$ .

# Stäckel analysis for the system in a constant magnetic field

To show that for our constant magnetic field, we shall work in the coordinates (2) that diagonalize the Haantjes algebra  $\mathcal{H}_1$  and use a suitable functional combination of the Hamiltonians:

$$\begin{aligned}\tilde{H}_1 &= 2H - H_1^2 = -2bq^2 p_1 + b^2(q^2)^2 + p_2^2 + p_3^2, \\ \tilde{H}_2 &= H_1 = p_1, \quad \tilde{H}_3 = H_3 = p_3^2.\end{aligned}\tag{3}$$

(This does not change the underlying foliations by invariant tori.)  
Then it follows that the Stäckel equation is satisfied by choosing

$$S = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2bq^2 & -1 \\ 0 & 0 & 1 \end{bmatrix},\tag{4}$$

and therefore  $f_1(q^1, p_1) = p_1$ ,  $f_2(q^2, p_2) = p_2^2 + b^2(q^2)^2$  and  $f_3(q^3, p_3) = p_3^2$ . Note that  $f_1$  is **linear, i.e. generalized**.

By generalizing Stäckel matrix and functions we obtained new magnetic integrable systems on pseudo-Riemannian space.

# Further research directions

Our focus for the upcoming ICMAT postdoc is to use tools from differential geometry (Eisenhart lift,  $\omega\mathcal{H}$  algebras, twisted product, Riemannian coverings ect.) to study integrable systems, especially those with magnetic fields.

- (Non)separable helical undulator
- Nijenhuis/Haantjes approach for phase space separation. Haantjes condition is LPDE quartic in components
- Eisenhart lift  $\rightsquigarrow$  pp-waves or gyratons (special types of Kundt spacetimes) - algebraic classification in higher dimensions
- Generalize the algebraic geometry for (nonmagnetic) superintegrable systems a la [Kress, Schöbel, Vollmer 2023 and 2024] to Lorentzian spacetimes and combine with Eisenhart lift.

Conjecture: All (super)integrable systems can be interpreted as a reduction of a simple system from a higher dimensional space.

# Eisenhart lift

The key technical idea allowing the work with fields is the **Eisenhart-(Duval) lift** [Eisenhart 1929, Duval et al. 1985]: The added fields  $(\vec{A}, U)$  are included in a metric defined on an extended configuration space  $\hat{M} = M \times \mathbb{R}$

$$\hat{G}^{AB} = \left[ \begin{array}{c|c} \hat{G}^{ij} & \hat{G}^{i0} \\ \hline \hat{G}^{0j} & \hat{G}^{00} \end{array} \right] = \left[ \begin{array}{c|c} g^{ij} & A^i \\ \hline A^j & -2U \end{array} \right]. \quad (5)$$

Remark: In order to have a globally defined metric, it is more common to add two real axes,  $\tilde{M} = M \times \mathbb{R} \times \mathbb{R}$ , and work with a Lorentzian metric that admits a null Killing vector  $\partial_s$ :

$$\hat{G}^{AB} = \left[ \begin{array}{c|c|c} g^{ij} & A^i & 0 \\ \hline A^j & -2U & 1 \\ \hline 0 & 1 & 0 \end{array} \right]. \quad (6)$$






Integrals of motion become homogeneous in momenta, e.g. Hamiltonian

$$\hat{H} = H^{(2)} + p_s H^{(1)} + p_s^2 H^{(0)} + p_t p_s. \quad (7)$$






# Key take-aways

- We have seen the progress on nonstandard integrable classes, nonseparable helical undulator
  - $\omega\mathcal{H}$  algebras characterize integrability and separability on phase space, applied to constant magnetic field.
- ↪ Differential geometry provides powerful tools.







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