Remembering Willard Miller: Beyond separation of variables, superintegrability

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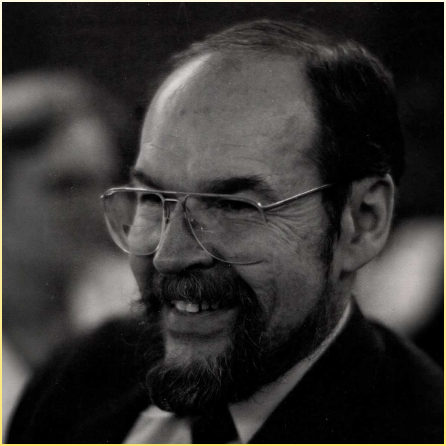

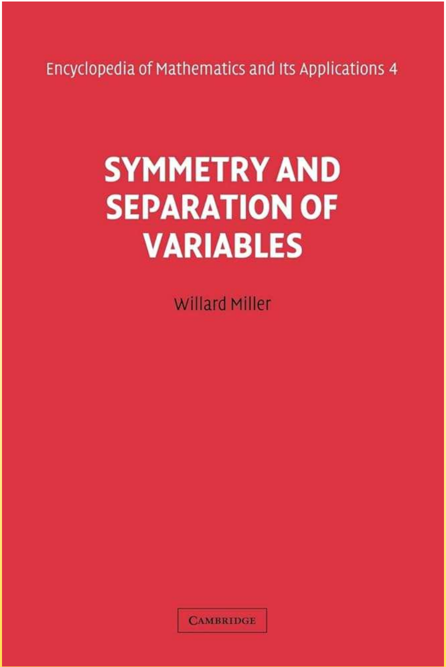

Encyclopedia of Mathematics and Its Applications 4

SYMMETRY AND
SEPARATION OF
VARIABLES

Willard Miller

CAMBRIDGE

Legacy of Willard Miller Jr

Three fundamental results:

- *Beyond of separation of variables*
- *Superintegrability: TTW model at rational index k*
- *Superintegrability: polynomial algebras of integrals*

(I)

Beyond of separation of variables

Take formal spectral problem in R^2

$$L(x, y; \partial_x, \partial_y)\, \phi \;=\; \lambda\, \phi$$

with linear differential operator L

If

$$L(x, y; \partial_x, \partial_y) \;=\; L_1(x; \partial_x) \;+\; L_2(y; \partial_y)$$

separation of variables occurs, all solutions are of the form

$$\phi(x, y) \;=\; f(x)\, g(y)$$

If

$$L(x, y; \partial_x, \partial_y) \;=\; L_1(x; \partial_x) \;+\; L_2(x, y, \partial_x; \partial_y)$$

(L_2 is “proportional” to ∂_y)
it is **beyond** of separation of variables, but some solutions are of the form

$$\phi(x, y) \;=\; f(x)$$

There exists a subfamily of solutions which depends on smaller number of variables!

$$L_1(x; \partial_x)\ \phi\ =\ \lambda\ \phi$$

This scheme can be extended to any number of x-variables and y-variables.

n-body problem, classical and quantum, as the example

of **beyond** of separation of variables

Hamiltonian

$$H \;=\; \sum_{i}^{n} \frac{P_i^2}{2m_i} \;+\; V(\{r_{ij}\})\ , \quad i \neq j$$

r_{ij} are relative distances

We name:

The Lagrange Problem (classical):

> *To find trajectories of n-body problem with d degrees of freedom which depend on relative distances r_{ij} ALONE*

(3-body case: J.-L. Lagrange (1772) - formulated and failed, F.D. Murnaghan (1936) - succeeded, G. Moore (1993) - concrete example: the remarkable Figure Eight trajectory (choreography))

The Lagrange Problem (quantum):

> *To find eigenfunctions of n-body Schrödinger equation which depend on relative distances $\{r_{ij}\}$ ALONE*

IDEA I. (after J.-L. Lagrange, 1772)

Let us separate relative distances $\{r_{ij}\}$ from other variables defining them as *relative* angles

$$(\vec{r_1}, \vec{r_2}, \ldots, \vec{r_n}) \to \left(\vec{R}_{CMS}\ ,\ \{r_{ij}\}\ ,\ \Omega\right)$$

thus, by introducing **generalized Euler coordinates!**

Meaning:

• If $d > (n-2)$, distances $\{r_{ij}\}$ are edges of n-vertex regular (non-degenerate) polytope, otherwise → there are constraints

Property:

• unlike celebrated Jacobi coordinates, $\{r_{ij}\}$ do **not** depend on masses

IDEA II. (W Miller et al, 2018)

Make change of variables in the **quantum** kinetic energy: it decomposes *naturally* into the sum of three operators

$$\sum_{i=1}^{n} \frac{1}{2m_i}\Delta_i^{(d)} \ = \ \Delta_{\mathbf{R}_{CMS}} + \Delta_{\rm rad} + \Delta_{\Omega}$$

$\Rightarrow$ $\Delta_{\mathbf{R}_{CMS}}$ is the center-of-mass Laplacian,

$\Rightarrow$ the "radial" operator $\Delta_{\rm rad}$ depends on the derivatives on relative distances, and written in coordinates

$\rho_{ij} = r_{ij}^2$, alone **(!)**

$\Rightarrow$ Δ_{Ω} annihilates any function of the relative distances alone, (Δ_{Ω} is "proportional" to derivatives in angles)

$\Delta_{\Omega} f(r_{ij}) = 0$ and then →

- **de-quantize** to **classical**: from quantum momentum to classical

$$-i\,\partial \to p$$

Key result (for $d > (n-2)$):

$$\Delta_{\mathrm{rad}}(\rho) \; = \; 2 \sum_{i\neq j, i\neq k, j<k}^{n} \frac{1}{m_i}(\rho_{ij}+\rho_{ik}-\rho_{jk})\partial_{\rho_{ij}}\partial_{\rho_{ik}} \; +$$

$$2\sum_{i<j}^{n}\left(\frac{m_i+m_j}{m_i m_j}\right)\rho_{ij}\partial^2_{\rho_{ij}} + d\sum_{i<j}^{n}\left(\frac{m_i+m_j}{m_i m_j}\right)\partial_{\rho_{ij}}$$

- it is **algebraic** operator (!) and also **positive-definite** (!)

No angular dependence in coeffs!

- *if $d \le (n-2)$ there are some constraints on ρ_{ij} ... -*

for arbitrary n no **explicit** formulae are known, except for $d=1$ or for $n=3,4$ at any d ! → *Open direction in distant geometry.*

This remarkable formula should be called by

Willard's name!

since he was the **first** who saw it

⇒ Proposed *decomposition* of kinetic energy leads to

Reduction:

- from Original *n*-body problem in $\mathbf{R^{nd}}$

$$\Big(-\sum_{i=1}^{n} \frac{1}{2m_i}\Delta_i^{(d)} + V(r_{ij}) \Big)\Psi(r_{ij},\Omega) \;=\; E\Psi(r_{ij},\Omega) \qquad (\star)$$

- to Reduced *n*-body problem in $\mathbf{R^{n(n-1)/2}}$

$$\Big(-\Delta_{\mathrm{rad}}(\rho) + V(\rho_{ij}) \Big)\Psi(r_{ij}) \;=\; E\Psi(r_{ij}) \qquad (\star\star)$$

$\Psi(r_{ij})$ does **not** depend on angles

- Ground state of $(\star\star)$ is the ground state of $(\star)$ (no nodes!) it is Theorem!

- We can always write it like

$$\Delta_{rad}(\rho) \ = \ g^{\mu\nu}\partial_\mu\partial_\nu \ + \ b^\mu\partial_\mu$$

where $g^{\mu\nu}$ is called the *n*-**body matrix**, it made out of coefficients in front of the second derivatives and b^μ is a column vector.

- There exists the **gauge factor** Γ such that

$$\Gamma^{-1}\Big(- \Delta_{\mathrm{rad}}(\rho) + V(\rho)\Big)\Gamma \ = \ -\Delta_g + V_{\mathrm{eff}}(\rho) + V(\rho) \ \equiv \ \mathcal{H}_{\mathrm{g}}$$

here Δ_g is the Laplace-Beltrami operator

$$\Delta_g \ = \ \sqrt{g}\partial_\mu g^{\mu\nu}\frac{1}{\sqrt{g}}\partial_\nu \ , \ g = \det g^{\mu\nu}$$

Hence, $g^{\mu\nu}$ is the contravariant metric tensor!

- Conjecture (Miller et al, '18)

Radial reduced Hamiltonian $(-\Delta_{\rm rad} + V)$ of the n-body problem is essentially self-adjoint with respect to the normalized radial measure,

$$dv_{\rm rad} \ = \ (V_n^2)^{\frac{d-n}{2}} \prod_1^{\frac{n(n-1)}{2}} d\rho_{ij}$$

It was checked for $n = 2, 3, 4$

(interval, triangle, tetrahedron: the cases important for applications)

V_n is the volume of n-vertex polytope

(II)

Superintegrability: TTW model at rational index k

F.Tremblay, A.T., P.Winternitz '09, **TTW** system/model:

$$\mathcal{H}_k(x,y;\alpha,\beta;k) \;=\; -\partial_r^2-\frac{1}{r}\partial_r-\frac{1}{r^2}\partial_\varphi^2+\omega^2 r^2+\frac{k^2\alpha}{r^2\cos^2 k\varphi}+\frac{k^2\beta}{r^2\sin^2 k\varphi}$$

- $k=1$ – Smorodinsky-Winternitz model
- $k=2$ – BC_2-rational model
- $k=3$ – Calogero ($\alpha=0$) and Wolfes ($\alpha\neq 0$) models
- integer k – I_{2k} rational model

It is superintegrable model for integer k: Second degree Integral I_1 and $2k$ degree Integral I_2 - in the Hamiltonian reduction by Harish Chandra

THEOREM (Miller et al, '11-12): It is superintegrable model for rational $k=p/q$ with deg $I_2=2(p+q-1)$

(III)

Superintegrability: polynomial algebras of integrals

Take a quantum system with Hamiltonian $\mathcal{H}$ in n-dimensional coordinate space.

- If there exist $(n-1)$ integrals of motion I_i,

$$[\mathcal{H}, I_i] \; = \; 0 \quad , \quad i = 1, 2, \ldots (n-1)$$

with property

$$[I_i, I_k] \; = \; 0$$

hence, span n-dimensional commutative algebra, such a system is called

integrable

- If there exist the p additional integrals $\mathcal{I}_m, m = 1, 2, \ldots p$, the system is called

super-integrable

- If number of these integrals is maximal, $p = (n - 1)$, the system is called

maximally-super-integrable

Usually, these additional integrals do not commute with each other and with integrals I_i.

What is the algebra of integrals if any?

Take the simplest non-trivial case $d = 2$ – two-dimensional quantum dynamics.

Polynomial algebra of integrals

(definition)

Any $2D$ superintegrable system is characterized by the Hamiltonian H and two integrals I_1, I_2:

$$[H, I_1] \ = \ [H, I_2] \ = \ 0$$

Commutator:

$$I_{12} \ = \ [I_1, I_2] \ \neq \ 0$$

evidently,

$$[H, I_{12}] \ = \ 0$$

Double-commutators:

$$[I_1, I_{12}] \ = \ P(H, I_1, I_2, I_{12}) \quad , \quad [I_2, I_{12}] \ = \ Q(H, I_1, I_2, I_{12})$$

P, Q are polynomials of finite degree

- Mathematically, the polynomial algebra is infinite-dimensional algebra of ordered monomials

$$H^n I_1^m I_2^p I_{12}^q$$

where n, m, p, q are non-negative integers

- In general, in 2008 Ernest G Kalnins, Willard Miller Jr and Sarah Post, and in 2011 S Post showed that for the bulk the second order superintegrable systems on the plane

  (with integrals given by the second order differential operators, thus, variables are separated in, at least, two coordinate systems)

  the polynomial algebra of integrals exists and is always

  Quadratic

  Polynomials P, Q are at most of the second degree
  (still no theorem proved)

• We suspect that all these systems are exactly-solvable with hidden algebra $gl(3)$

(i) Recently, JC Lopez Vieyra, MA Escobar Ruiz and AVT identified seven additional $2D$ superintegrable systems (in flat and curve spaces) with cubic, quartic, quintic polynomial algebras of integrals,

(ii) For all so far known $2D$ superintegrable systems in flat space the Hamiltonian and both integrals can be written as linear differential operators with polynomial coefficients - it is calculus of the differential operators in two variables,

(iii) For all so far known $2D$ superintegrable systems in flat space the Hamiltonian and both integrals can be rewritten in terms of generators of the so-called $g^{(k)}$ algebra with integer index k, $g^{(1)} = gl(3)$, discovered by S Lie circa 1880 as the algebra of vector fields of non-semisimple algebra $gl(2) \ltimes R^{k+1}$ and extended by Gonzales-Lopez - Kamran - Olver ($\sim$ 1990) to the algebra of the first order differential operators with reducible fin-dim reps.

Recently, it was further extended by adding non-the-first-order, higher order differential operators
(in relation with G_2 rational, trigonometric, elliptic Calogero-Sutherland-Moser integrable models with planar Newton diagram with ratio 1:2, it corresponds to $g^{(2)}$)
in order to have the irreducible reps (with finite-dimensional representation space),

(iv) **All** so far known 2D superintegrable systems are **exactly-solvable** with polynomial eigenfunctions and eigenvalues given by a second degree polynomial in two quantum numbers.
In general, eigenfunctions are related to non-orthogonal polynomials (a weight function depends on quantum numbers)

(v) Rational models of the Hamiltonian reduction $A_n, B_n, C_n, D_n, BC_n, G_2, F_4, E_{6,7,8}$ are maximally superintegrable and exactly solvable with polynomial eigenfunctions. Does exist a polynomial algebra of integrals? - It is open question.

The case $n = 2$ - A_2, BC_2, G_2 - is already studied.

Willard Miller Jr (1937 - 2023)

R.I.P.