Integrable 3D systems in a magnetic field corresponding to spherical separation of variables

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Contents

Introduction

We consider integrable and superintegrable systems in three spatial dimensions.

Integrability

A classical system in n degrees of freedom is integrable if it admits n functionally independent integrals of motion in involution.

Superintegrability

A classical system in n degrees of freedom is polynomially superintegrable if it admits $n + k$ functionally independent integrals of motion (where $k \leq n - 1$), that are polynomial in the momenta and out of which n are in involution.

Introduction, cont'd

Due to A.A. Makarov, J.A. Smorodinsky, K. Valiev, P. Winternitz, Il Nuovo Cimento LII A, 8881 (1967) when quadratic integrability is considered and the Hamiltonian involves only a kinetic term and a scalar potential, there are 11 classes of systems admitting pairs of commuting quadratic integrals, each uniquely determined by a pair of commuting quadratic elements in the enveloping algebra of the 3D Euclidean algebra.

These in turn correspond to a coordinate system in which the Hamilton-Jacobi equation separates.

Introduction, cont'd

When systems involving **vector potentials** are considered, quadratic integrability no longer implies separability.

In J. Bérubé, P. Winternitz. J. Math. Phys. 45 (2004), no. 5, 1959-1973 the structure of the gauge–invariant integrable and superintegrable systems involving vector potentials was considered in two spatial dimensions. It was shown there that under the assumption of integrals being of at most second order in momenta, no superintegrable system with nonconstant magnetic field exists in two dimensions.

Introduction, cont'd

Approaches to the problem in three spatial dimensions:

- A. Marchesiello, L. Šnobl, P. Winternitz, J. Phys. A: Math. Theor. 48, 395206 (2015): possibilities for integrability and superintegrability arising from first order integrals. 3D maximally superintegrable systems with nonconstant magnetic field were found. Among them magnetic monopole with Coulomb like potential is second order maximally superintegrable.
- A. Marchesiello, L. Šnobl, J. Phys. A: Math. Theor. 50, 245202 (2017): superintegrable systems which separate in Cartesian coordinates in the limit when the magnetic field vanishes, i.e. possess two second order integrals of motion of the so-called Cartesian type.

Outline

Here we study integrable systems involving vector potentials when two commuting quadratic spherical-type integrals

$$
\begin{aligned}
X_1 &= (l_3)^2 + \text{ lower order terms}, \\
X_2 &= L^2 + \text{ lower order terms}, \qquad L^2 = \sum_{i=1}^{3} (l_i)^2
\end{aligned}
$$

are present.

We find four classes of such integrable Hamiltonian systems. One of them is minimally quadratically superintegrable, the properties of the others are currently under investigation.

General structure of the integrals of motion

We consider the classical Hamiltonian describing the motion of a particle in three dimensions in a nonvanishing magnetic field

$$H = \frac{1}{2}(\vec{p} + \vec{A})^2 + W(\vec{x}), \qquad (1)$$

where $\vec{p}$ is the linear momentum, $\vec{A}$ is the vector potential and W is the electrostatic potential. The Newtonian equations of motion are gauge invariant – they are the same for the potentials

$$\vec{A}'(\vec{x}) = \vec{A}(\vec{x}) + \nabla\chi, \qquad W'(\vec{x}) = W(\vec{x})$$

for any choice of the function $\chi(\vec{x})$. Thus, the physically relevant quantity is the magnetic field $\vec{B} = \nabla \times \vec{A}$. $\vec{B}$ is assumed to be nonvanishing so that the system is not gauge equivalent to a system with only the scalar potential.

The general structure of the integrals of motion, cont'd

Let us consider integrals of motion which are at most second order in the momenta. Since our system is gauge invariant, we express the integrals in terms of gauge covariant expressions (velocities)

$$p_j^A = p_j + A_j(\vec{x}) \tag{2}$$

rather than the momenta themselves.

We write a general second order integral of motion as

$$X = \sum_{j=1}^{3} h^j(\vec{x}) p_j^A p_j^A + \sum_{j,k,l=1}^{3} \frac{1}{2} |\epsilon_{jkl}| n^j(\vec{x}) p_k^A p_l^A + \sum_{j=1}^{3} s^j(\vec{x}) p_j^A + m(\vec{x}), \tag{3}$$

where ϵ_{jkl} is the completely antisymmetric tensor with $\epsilon_{123} = 1$.

The general structure of the integrals of motion, cont'd

The condition that the Poisson bracket

$$\{a(\vec{x},\vec{p}), b(\vec{x},\vec{p})\}_{P.B.} = \sum_{j=1}^{3} \left(\frac{\partial a}{\partial x_j}\frac{\partial b}{\partial p_j} - \frac{\partial b}{\partial x_j}\frac{\partial a}{\partial p_j} \right) \qquad (4)$$

of the integral (3) with the Hamiltonian (1) vanishes

$$\{H, X\}_{P.B.} = 0 \qquad (5)$$

leads to terms of order $3, 2, 1$ and 0 in the momenta. The third order ones are the same as for the system with vanishing magnetic field and their explicit solution is known - they imply that the highest order terms in the integral (3) are linear combinations of products of the generators of the Euclidean group $p_1, p_2, p_3, l_1, l_2, l_3$ where $l_j = \sum_{l,k} \epsilon_{jkl} x_k p_l$, i.e. $\vec{h}$, $\vec{n}$ can be expressed in terms of 20 constants α_{ab}, $1 \leq a \leq b \leq 6$.

Spherical-type integrals

Let us now turn our attention to the situation when the Hamiltonian is integrable in the Liouville sense, with at most quadratic integrals. That means that in addition to the Hamiltonian itself there must be at least two independent integrals of motion of the form (3) which commute in the sense of Poisson bracket.

We assume such integrals to be of spherical type,

$$X_1 = (l_3^A)^2 + \text{ lower order terms},$$
$$X_2 = (L^A)^2 + \text{ lower order terms}, \tag{6}$$

where $l_j^A = \sum_{1\leq k,l\leq 3} \epsilon_{jkl} x_k p_l^A$, $(L^A)^2 = \sum_{i=1}^{3} (l_i^A)^2$. For vanishing magnetic field, these integrals would correspond to separation in spherical coordinates.

Spherical coordinates

Thus, we expect the problem to be most tractable in the spherical coordinates which we assume in the form

$$x = R\sin(\theta)\cos(\phi), \quad y = R\sin(\theta)\sin(\phi), \quad z = R\cos(\theta). \tag{7}$$

The momenta and the vector potential transform as the components of differential 1-forms, e.g.

$$\begin{aligned} p_x &= \sin(\theta)\cos(\phi)p_R + \frac{\cos(\theta)\cos(\phi)}{R}p_\theta - \frac{\sin(\phi)}{R\sin(\theta)}p_\phi, \\ p_y &= \sin(\theta)\sin(\phi)p_R + \frac{\cos(\theta)\sin(\phi)}{R}p_\theta + \frac{\cos(\phi)}{R\sin(\theta)}p_\phi, \\ p_z &= \cos(\theta)p_R - \frac{\sin(\theta)}{R}p_\theta. \end{aligned} \tag{8}$$

Magnetic field in spherical coordinates

The components of the magnetic field are components of the magnetic field 2-form $B = \mathrm{d}A$,

$$\begin{aligned} B &= B_x(\vec{x})\mathrm{d}y \wedge \mathrm{d}z + B_y(\vec{x})\mathrm{d}z \wedge \mathrm{d}x + B_z(\vec{x})\mathrm{d}x \wedge \mathrm{d}y \\ &= B_R(R,\theta,\phi)\mathrm{d}\theta \wedge \mathrm{d}\phi + B_\theta(R,\theta,\phi)\mathrm{d}\phi \wedge \mathrm{d}R + B_\phi(R,\theta,\phi)\mathrm{d}R \wedge \mathrm{d}\theta. \end{aligned} \tag{9}$$

Consequently, the relation between the components of the magnetic field in Cartesian and spherical coordinates is

$$\begin{aligned} B_x(\vec{x}) &= \frac{\cos(\phi)}{R^2} B_R + \frac{\cos(\theta)\cos(\phi)}{R\sin(\theta)} B_\theta - \frac{\sin(\phi)}{R} B_\phi, \\ B_y(\vec{x}) &= \frac{\sin(\phi)}{R^2} B_R + \frac{\cos(\theta)\sin(\phi)}{R\sin(\theta)} B_\theta + \frac{\cos(\phi)}{R} B_\phi, \\ B_z(\vec{x}) &= \frac{\cos(\theta)}{R^2\sin(\theta)} B_R - \frac{1}{R} B_\theta. \end{aligned} \tag{10}$$

Hamiltonian in spherical coordinates

The Hamiltonian (1) expressed in spherical coordinates reads

$$H = \frac{1}{2}\left((p_R + A_R(R,\theta,\phi))^2 + \left(\frac{p_\theta + A_\theta(R,\theta,\phi)}{R}\right)^2 + \right.$$
$$\left. + \left(\frac{p_\phi + A_\phi(R,\theta,\phi)}{R\sin(\theta)}\right)^2\right) + W(R,\theta,\phi). \quad (11)$$

The equations of motion as well as the determining equations for the integrals are derived using the Poisson brackets,

$$\{f,g\}_{P.B.} = \sum_{j=1}^{3}\left(\frac{\partial f}{\partial x_j}\frac{\partial g}{\partial p_j} - \frac{\partial g}{\partial x_j}\frac{\partial f}{\partial p_j}\right) = \sum_{\alpha\in\{R,\theta,\phi\}}\left(\frac{\partial f}{\partial \alpha}\frac{\partial g}{\partial p_\alpha} - \frac{\partial g}{\partial \alpha}\frac{\partial f}{\partial p_\alpha}\right). \quad (12)$$

Spherical-type integrals

In spherical coordinates we again introduce covariantized momenta

$$p_R^A = p_R + A_R(R,\theta,\phi),\ p_\theta^A = p_\theta + A_\theta(R,\theta,\phi),\ p_\phi^A = p_\phi + A_\phi(R,\theta,\phi) \tag{13}$$

and express the spherical-type integrals in the following form

$$\begin{aligned} X_1 &= (p_\phi^A)^2 + s_1^R(R,\theta,\phi)p_R^A + s_1^\theta(R,\theta,\phi)p_\theta^A + \\ &+ s_1^\phi(R,\theta,\phi)p_\phi^A + m_1(R,\theta,\phi), \end{aligned} \tag{14}$$

$$\begin{aligned} X_2 &= \left(p_\theta^A\right)^2 + \left(\frac{p_\phi^A}{\sin(\theta)}\right)^2 + s_2^R(R,\theta,\phi)p_R + s_2^\theta(R,\theta,\phi)p_\theta^A + \\ &+ s_2^\phi(R,\theta,\phi)p_\phi^A + m_2(R,\theta,\phi). \end{aligned} \tag{15}$$

The determining equations in spherical coordinates

The determining equations for the functions $s_k^R, s_k^\theta, s_k^\phi$ and m become:

- second order in momenta

$$
\begin{aligned}
\partial_R s_k^R &= n^\theta B_\theta - n_k^\phi B_\phi, \\
\partial_\theta s_k^R &= R^2 \left(n_k^R B_\theta - 2h_k^\theta B_\phi - \partial_R s_k^\theta\right) - n_k^\theta B_R + 2h_k^R B_\phi, \\
\partial_\theta s_k^\theta &= n_k^\phi B_\phi - n_k^R B_R - \frac{1}{R} s_k^R, \\
\partial_\phi s_k^\theta &= \sin^2(\theta) \left(n_k^\theta B_\phi - 2h_k^\phi B_R - \partial_\theta s_k^\phi\right) + 2h_k^\theta B_R - n_k^\phi B_\theta, \quad (16) \\
\partial_R s_k^\phi &= 2h_k^\phi B_\theta - n_k^R B_\phi + \frac{1}{R^2 \sin^2(\theta)} \left(n_k^\phi B_R - 2h_k^R B_\theta - \partial_\phi s_k^R\right), \\
\partial_\phi s_k^\phi &= n_k^R B_R - n_k^\theta B_\theta - \frac{\cos(\theta)}{\sin(\theta)} s_k^\theta - \frac{1}{R} s_k^R,
\end{aligned}
$$

The determining equations in spherical coordinates, cont'd

- first order in momenta

$$\begin{aligned} \partial_R m_k &= s_k^\phi B_\theta - s_k^\theta B_\phi + n_k^\theta \partial_\phi W + n_k^\phi \partial_\theta W + 2h_k^R \partial_R W, \\ \partial_\theta m_k &= R^2 \left(n_k^R \partial_\phi W + n_k^\phi \partial_R W + 2h_k^\theta \partial_\theta W \right) - s_k^\phi B_R + s_k^R B_\phi, \\ \partial_\phi m_k &= R^2 \sin^2(\theta) \left(2h_k^\phi \partial_\phi W + n_k^R \partial_\theta W + n_k^\theta \partial_R W \right) + s_k^\theta B_R - s_k^R B_\theta, \end{aligned} \tag{17}$$

- zero order in momenta

$$s_k^R \partial_R W + s_k^\theta \partial_\theta W + s_k^\phi \partial_\phi W = 0, \tag{18}$$

where

$$h_1^\phi = 1,\ h_1^R = 0,\ h_1^\theta = 0,\ n_\phi = 0,\ n_R = 0,\ n_\theta = 0,$$

$$h_2^\phi = \frac{1}{\sin^2(\theta)},\ h_2^R = 0,\ h_2^\theta = 1,\ n_2^\phi = 0,\ n_2^R = 0,\ n_2^\theta = 0.$$

X_1 and X_2 in involution

The involutivity of the integrals

$$\{X_1, X_2\}_{P.B.} = 0 \qquad (19)$$

order by order implies more conditions:

- second order in momenta

$$\begin{aligned} B_R &= \frac{1}{2}\left(\partial_\phi s_2^\theta - \partial_\theta s_1^\phi - \frac{1}{\sin^2(\theta)}\partial_\phi s_1^\theta\right), \\ \partial_\phi s_2^\phi &= \frac{1}{\sin^2(\theta)}\partial_\phi s_1^\phi + \frac{\cos(\theta)}{\sin^3(\theta)} s_1^\theta, \quad \partial_\phi s_1^R = \sin^2(\theta)\partial_\phi s_2^R, \\ \partial_\theta s_1^R &= 0, \quad \partial_\theta s_1^\theta = 0, \end{aligned} \qquad (20)$$

X_1 and X_2 in involution

- first order in momenta

$$
\begin{aligned}
s_1^\phi \partial_\phi s_2^R - s_2^\theta \partial_\theta s_1^R - s_2^\phi \partial_\phi s_1^R + s_1^\theta \partial_\theta s_2^R - s_2^R \partial_R s_1^R + s_1^R \partial_R s_2^R &= 0, \\
-2\partial_\theta m - 2s_1^\phi B_R + 2s_1^R B_\phi + s_1^R \partial_R s_2^\theta + s_1^\theta \partial_\theta s_2^\theta + s_1^\phi \partial_\phi s_2^\theta - & \\
-s_2^\theta \partial_\theta s_1^\theta - s_2^R \partial_R s_1^\theta - s_2^\phi \partial_\phi s_1^\theta &= 0, \\
2\partial_\phi m_2 - \frac{2}{\sin^2(\theta)} \partial_\phi m_1 - \frac{2}{\sin^2(\theta)} s_1^R B_\theta + 2s_2^R B_\theta + s_1^R \partial_R s_2^\phi + s_1^\theta \partial_\theta s_2^\phi + & \\
\frac{2}{\sin^2(\theta)} s_1^\theta B_R - 2s_2^\theta B_R - s_2^\theta \partial_\theta s_1^\phi + s_1^\phi \partial_\phi s_2^\phi - s_2^\phi \partial_\phi s_1^\phi - s_2^R \partial_R s_1^\phi &= 0,
\end{aligned} \tag{21}
$$

- zero order in momenta

$$
\begin{aligned}
(s_1^\theta s_2^\phi - s_1^\phi s_2^\theta) B_R + (s_1^\phi s_2^R - s_1^R s_2^\phi) B_\theta + (s_1^R s_2^\theta - s_1^\theta s_2^R) B_\phi - & \\
-s_2^R \partial_R m_1 - s_2^\theta \partial_\theta m_1 - s_2^\phi \partial_\phi m_1 + s_1^R \partial_R m_2 + s_1^\theta \partial_\theta m_2 + s_1^\phi \partial_\phi m_2 &= 0.
\end{aligned} \tag{22}
$$

Solution of conditions (16) and (20)

All second order conditions (16) and (20) can be easily solved, leading to the following structure of the functions $s_1^R, s_1^\theta, s_1^\phi, s_2^R, s_2^\theta, s_2^\phi$ and the magnetic field B_R, B_θ, B_ϕ

$$\begin{aligned}
s_1^R &= 0, \quad s_1^\theta = \partial_\phi \sigma(\phi), \quad s_1^\phi = \sin^2(\theta)\omega(R) + \tau(\theta) - \frac{\cos(\theta)}{\sin(\theta)}\sigma(\phi), \\
s_2^R &= \cos(\theta)\mathcal{S}, \quad s_2^\theta = -\sin(\theta)\frac{\mathcal{S}}{R}, \quad s_2^\phi = \omega(R), \\
B_R &= -\sin(\theta)\cos(\theta)\omega(R) - \frac{1}{2}\partial_\theta \tau(\theta) - \frac{\partial^2_{\phi\phi}\sigma(\phi) + \sigma(\phi)}{2\sin^2(\theta)}, \qquad (23) \\
B_\theta &= \frac{\sin^2(\theta)}{2}\partial_R \omega(R), \quad B_\phi = 0.
\end{aligned}$$

In the solution (23) the functions $\sigma(\phi), \tau(\theta), \omega(R)$ are arbitrary functions of the specified variables, $\mathcal{S}$ is an arbitrary constant.

Solution of lower order conditions

Once we know the general solution (23) of the second order conditions we need to consider the lower order condition (17) and (18) together with (21) and (22). We observe that the functions $m_1(R, \theta, \phi)$ and $m_2(R, \theta, \phi)$ show up in them only through their first derivatives. Their compatibility conditions

$$\partial_\mu (\partial_\nu m_k) \;=\; \partial_\nu (\partial_\mu m_k), \qquad \mu, \nu = R, \theta, \phi, \tag{24}$$

after substituting the relations (17) for the derivatives of m_1 and m_2, give conditions independent of m_1 and m_2. We also substitute (17) into (21) and (22), obtaining equations independent of the functions m_1 and m_2 as well. Once these equations together with the compatibility conditions (24) are solved, the (local) existence of the functions m_1 and m_2 follows.

Solution of lower order conditions, cont'd

This is the stage at which our calculation starts to split into various subcases. The major splitting is based on the equation

$$\sin(\theta)\mathcal{S}\partial_\phi\sigma(\phi) = 0 \tag{25}$$

which arises as one of the first order terms (21) in $\{X_1, X_2\}_{P.B.}$ after using equations (23), (17). Thus we have to consider separately

- **Case A** $\mathcal{S} = 0$ and $\sigma(\phi)$ is assumed to be a nonconstant function,
- **Case B** $\sigma(\phi) = s_0 = \text{const.}$

Case A

In this case we immediately find that the conditions (21), the compatibility conditions (24) and the equations (17) for the integral X_2 imply

$$\omega(R) = 0, \quad W(R,\theta,\phi) = U(R) + \frac{V(\theta,\phi)}{R^2}, \quad m_2(R,\theta,\phi) = 2V(\theta,\phi), \tag{26}$$

thus we have $s_2^R = s_2^\theta = s_2^\phi = 0$ and the condition (18) for X_2 is satisfied trivially. To be able to solve the remaining equations in a closed form we introduce $\tilde{\tau}(\theta)$ via

$$\tau(\theta) = \sin^2(\theta)\partial_\theta\tilde{\tau}(\theta). \tag{27}$$

Case A, cont'd

The magnetic field (23) becomes

$$B_R(R,\theta,\phi) = -\frac{1}{2}\left(\partial_\theta\left(\sin^2(\theta)\partial_\theta\tilde{\tau}(\theta)\right) + \frac{1}{\sin^2(\theta)}\left(\partial^2_{\phi\phi}\sigma(\phi)+\sigma(\phi)\right)\right),$$
$$B_\theta(R,\theta,\phi) = 0, \qquad B_\phi(R,\theta,\phi)=0. \tag{28}$$

Solving the compatibility conditions (24) we find

$$V(\theta,\phi) = \frac{\sigma(\phi)\partial_\theta\left(\sin(\theta)\partial_\theta\tilde{\tau}(\theta)\right)}{4\sin(\theta)} + \frac{\left(\tilde{\tau}(\theta)\partial^2_{\phi\phi}\sigma(\phi) - 4G(\phi)\right)}{4\sin^2(\theta)} + \frac{1}{4\sin^4(\theta)}\left(\sigma(\phi)^2 + \frac{1}{2}\partial_\phi\left(\sigma(\phi)\partial_\phi\sigma(\phi)\right)\right) + \frac{1}{4}F(\theta). \tag{29}$$

It must satisfy the only remaining condition (22) which becomes

$$\frac{\sin^3(\theta)\partial_\theta\tilde{\tau}(\theta) - \cos(\theta)\sigma(\phi)}{\sin(\theta)}\partial_\phi V(\theta,\phi) + \partial_\phi\sigma(\phi)\partial_\theta V(\theta,\phi) = 0. \tag{30}$$

Case A, cont'd

Thus we need to solve the sole remaining equation (30) into which the function $V(\theta, \phi)$ expressed in terms of four single-variable functions $\tilde{\tau}(\theta), F(\theta), G(\phi), \sigma(\phi)$, i.e. (29), is substituted.

We find two solutions, presented as Class I and Class II below.

Case B

In this case we have two subcases: $\mathcal{S} \neq 0$ and $\mathcal{S} = 0$.

- $\mathcal{S} \neq 0$. We find

$$\begin{aligned} B_R(R,\theta,\phi) &= \kappa_1 R^2 \cos(\theta)\sin(\theta),\ B_\theta = -\kappa_1 R \sin^2(\theta),\ B_\phi = 0, \\ W(R,\theta,\phi) &= -\frac{\kappa_1^2}{8}(R\sin(\theta))^2 - \frac{\lambda_1}{(R\sin(\theta))^2}. \end{aligned} \tag{31}$$

In Cartesian coordinates the system has constant magnetic field

$$\vec{B}(\vec{x}) = (0,0,\kappa_1) \tag{32}$$

and the potential reads

$$W(\vec{x}) = -\frac{\kappa_1^2}{8}(x^2+y^2) - \frac{\lambda_1}{x^2+y^2}. \tag{33}$$

Case B, cont'd

This system has two first order integrals

$$\tilde{X}_1 = p_\phi^A - \frac{\kappa_1}{2}\left(R\sin(\theta)\right)^2 = l_z^A - \frac{\kappa_1}{2}\left(x^2+y^2\right),$$
$$X_3 = \cos(\theta)p_R^A - \frac{\sin(\theta)}{R}p_\theta^A = p_z^A. \quad (34)$$

The integral X_1 can be expressed in terms of $\tilde{X}_1$ as quadratic polynomial thus we can replace X_1 by the first order integral $\tilde{X}_1$.

On the other hand, the integral X_3 is independent of $H, \tilde{X}_1$ and X_2, i.e. the system defined by (31) is (at least) minimally superintegrable.

Case B, cont'd

Looking for additional integrals at most quadratic in momenta, we find only one more integral, namely

$$X_4 = \frac{\cos(\theta)\left(p_\theta^A\right)^2}{R} + \frac{\cos(\theta)\left(p_\phi^A - \frac{\kappa_1 R^2 \sin^2(\theta)}{2}\right)^2}{R\sin^2(\theta)} + \sin(\theta) p_R^A p_\theta^A - \frac{2\lambda_1 \cos(\theta)}{R\sin^2(\theta)}$$
$$= p_x^A l_y^A - p_y^A l_x^A - \kappa_1 z l_z^A + \frac{\kappa_1^2}{4}(x^2+y^2)z - 2\frac{\lambda_1 z}{x^2+y^2}. \quad (35)$$

However the integral (35), similar to the z–component of the Laplace–Runge–Lenz vector, is not independent of $H, \tilde{X}_1, X_2, X_3$. It satisfies the relation

$$X_4^2 = 2H\left(X_2 - \tilde{X}_1^2 + 2\lambda_1\right) - X_2 X_3^2 + \kappa_1\left(\tilde{X}_1^3 - \tilde{X}_1 X_2\right) - 2\kappa_1\lambda_1\tilde{X}_1.$$

Thus we conclude that the system with the magnetic field (32) and the electrostatic potential (33) possesses four independent integrals at most quadratic in momenta, i.e. it is minimally but not maximally quadratically superintegrable.

Case B, cont'd

- $\mathcal{S} = 0$. We find

$$\begin{aligned} B_R &= -\sin(\theta)\chi(R)R^2\cos(\theta) - \frac{1}{2}\partial_\theta\tau(\theta), \\ B_\theta &= R\sin^2(\theta)\left(\frac{1}{2}R\partial_R\chi(R) + \chi(R)\right),\ B_\phi = 0, \\ W &= -\frac{1}{8}R^2\sin^2(\theta)\chi(R)^2 - \frac{1}{4}\tau(\theta)\chi(R) + F_1(R) + \frac{1}{R^2}F_2(\theta). \end{aligned} \tag{36}$$

This case for a particular choice of the arbitrary functions $\chi(R) = \frac{\kappa_1}{R^2}$ and $\tau(\theta) = \kappa_1\cos^2(\theta) + 2\kappa_2\cos(\theta) + \kappa_3$ contains the magnetic monopole field, i.e.

$$B_R = \kappa_2\sin(\theta), \quad B_\theta = 0, \quad B_\phi = 0, \ \text{i.e.}\ \vec{B}(\vec{x}) = \frac{\kappa_2}{R^3}\vec{x}. \tag{37}$$

Results – integrable system with spherical-type integrals

Four classes of second order spherical type integrable systems exist. For all of them the Hamiltonian H has the form (11) and the two further integrals in involution have the form X_1, X_2 as in (14) and (15). To specify them completely we must specify the functions $B_\alpha(R,\theta,\phi)$ and $W(R,\theta,\phi)$ describing the system, (and the functions $s_j^\alpha(R,\theta,\phi)$ and $m_j(R,\theta,\phi)$ for each integral where $\alpha = R,\theta,\phi$ and $j = 1,2$ - cf. our paper).

In order to make their structure easier to comprehend we present them in Cartesian coordinates.

We denote $R = \sqrt{x^2+y^2+z^2}$.

Results – integrable system with spherical-type integrals

- Class I.

$$\vec{B}(\vec{x}) = -\frac{1}{2}\left(\frac{\kappa_1}{R^3} - 3\kappa_2\frac{z^2}{R^5}\right)\vec{x}, \qquad (\kappa_1,\kappa_2) \neq (0,0), \quad (38)$$

$$W(\vec{x}) = U(R) + \frac{\kappa_2^2 z^4}{8R^6} + \frac{\kappa_2\kappa_1(x^2+y^2)}{4R^4} + \frac{\kappa_2(\lambda_2 y+\lambda_3 x-\lambda_1 z)}{2R^3} + \frac{\lambda_4}{4R^2}.$$

- Class II.

$$\vec{B}(\vec{x}) = \frac{(4\kappa_1\kappa_2-\kappa_3^2)}{16\sqrt{\kappa_1 x^2+\kappa_2 y^2+\kappa_3 xy}^3}\vec{x}, \qquad \kappa_1+\kappa_2>0,\ 4\kappa_1\kappa_2>\kappa_3^2,$$

$$W(\vec{x}) = U(R) - \frac{\lambda_1}{\kappa_1 x^2+\kappa_2 y^2+\kappa_3 xy}. \quad (39)$$

Results – integrable system with spherical-type integrals

- Class III.

$$\vec{B}(\vec{x}) = (0, 0, \kappa_1), \quad W(\vec{x}) = -\frac{\kappa_1^2}{8}(x^2 + y^2) - \frac{\lambda_1}{x^2 + y^2}, \ \kappa_1 \neq 0.$$

- Class IV.

$$\begin{aligned} B_x(\vec{x}) &= \frac{1}{2}\left(\frac{z}{R}\partial_R\chi(R) - \frac{1}{R^3\sin(\theta)}\partial_\theta\tau(\theta)\right)x, \\ B_y(\vec{x}) &= \frac{1}{2}\left(\frac{z}{R}\partial_R\chi(R) - \frac{1}{R^3\sin(\theta)}\partial_\theta\tau(\theta)\right)y, \\ B_z(\vec{x}) &= \frac{1}{2}\left(\frac{z}{R}\partial_R\chi(R) - \frac{1}{R^3\sin(\theta)}\partial_\theta\tau(\theta)\right)z - \frac{1}{2}R\partial_R\chi(R) - \chi(R), \\ W(\vec{x}) &= -\frac{1}{8}R^2\sin^2(\theta)\chi(R)^2 - \frac{1}{4}\tau(\theta)\chi(R) + F_1(R) + \frac{1}{R^2}F_2(\theta). \end{aligned} \tag{40}$$

Future outlook

Further work on spherical type integrable systems in a magnetic field is in progress in two directions:

- To determine which subclasses of the 4 classes found above can be extended to superintegrable systems, to find the additional integrals and particle trajectories.
- To find all quantum integrable and superintegrable systems of spherical type and analyze them. In particular to verify whether the conjecture, that all maximally superintegrable systems are exactly solvable, holds also in the presence of magnetic fields.

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