Superintegrable 3D systems in a magnetic field corresponding to Cartesian separation of variables

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Contents

Contents

Contents

Contents

Contents

Contents

Introduction

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These in turn correspond to a coordinate system in which the Hamilton-Jacobi or Schrödinger equation separates.

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Outline

Here we extend the analysis in two directions:

- We study superintegrable 3D systems involving vector potentials when two quadratic commuting Cartesian integrals are present.
- We show that more general classes for quadratic integrals than the ones corresponding to separation in absence of magnetic field, should be considered.

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Classical Hamiltonian

We consider the Hamiltonian describing the motion of a spinless particle in three dimensions in a nonvanishing magnetic field, i.e. classically

$$H = \frac{1}{2}(\vec{p} + \vec{A})^2 + W(\vec{x}) \tag{1}$$

where $\vec{p}$ is the momentum, $\vec{A}$ is the vector potential and V is the electrostatic potential. The magnetic field $\vec{B} = \nabla \times \vec{A}$ is assumed to be nonvanishing so that the system is not gauge equivalent to a system with only the scalar potential. We chose the units in which the mass of the particle has the numerical value 1 and the charge of the particle is -1 (having an electron in mind as the prime example).

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Gauge invariance

We recall that the equations of motion of the Hamiltonian (1) are gauge invariant, i.e. that they are the same for the potentials

$$\vec{A}'(\vec{x}) = \vec{A}(\vec{x}) + \nabla\chi, \qquad V'(\vec{x}) = V(\vec{x})$$

for any choice of the function $\chi(\vec{x})$ (we are considering only the static situation here). Thus, the physically relevant quantity is the magnetic field

$$\vec{B} = \nabla \times \vec{A}, \qquad \text{i.e.} \qquad B_j = \epsilon_{jkl} \frac{\partial A_l}{\partial x_k} \tag{2}$$

rather than the vector potential $\vec{A}(\vec{x})$.

Quantum Hamiltonian

We shall also consider the quantum Hamiltonian defined as the (properly symmetrized) analogue of (1) in terms of the operators of the linear momenta $\hat{P}_j = -\mathrm{i}\hbar\frac{\partial}{\partial x_j}$ and coordinates $\hat{X}_j = x_j$:

$$\begin{aligned} \hat{H} &= \frac{1}{2}\sum_j \left(\hat{P}_j + \hat{A}_j(\vec{x})\right)^2 + \hat{W}(\vec{x}) \\ &= \frac{1}{2}\sum_j \left(\hat{P}_j\hat{P}_j + \hat{P}_j\hat{A}_j(\vec{x}) + \hat{A}_j(\vec{x})\hat{P}_j + \hat{A}_j(\vec{x})^2\right) + \hat{W}(\vec{x}). \end{aligned}$$

The operators $\hat{A}_j(\vec{x})$ and $\hat{W}(\vec{x})$ act on wavefunctions as multiplication by the functions $A_j(\vec{x})$ and $W(\vec{x})$, respectively.

Quantum gauge invariance

On the quantum level, the gauge transformation demonstrates itself as a unitary transformation of the Hilbert space. Namely, let us take

$$\hat{U}\psi(\vec{x}) = \exp\left(\frac{\mathrm{i}}{\hbar}\chi(\vec{x})\right) \cdot \psi(\vec{x}). \tag{3}$$

Applying (3) on the states and the observables we get an equivalent description of the same physical reality in terms of

$$\psi \to \psi' = \hat{U}\psi, \qquad \hat{O} \to \hat{O}' = \hat{U}\hat{O}\hat{U}^\dagger. \tag{4}$$

In particular, the following observables transform covariantly

$$(\hat{P}_j + \hat{A}_j) \to \hat{U}(\hat{P}_j + \hat{A}_j)\hat{U}^\dagger = P_j + \hat{A}'_j, \qquad \hat{V} \to \hat{U}\hat{V}\hat{U}^\dagger = \hat{V}.$$

The general structure of the integrals of motion

Let us consider integrals of motion which are at most second order in the momenta. Since our system is gauge invariant, we express the integrals in terms of gauge covariant expressions

$$p_j^A = p_j + A_j, \qquad \hat{P}_j^A = \hat{P}_j + \hat{A}_j \tag{5}$$

rather than the momenta themselves. The operators (5) no longer commute among each other.

They satisfy

$$[\hat{P}_j^A, \hat{P}_k^A] = -\mathrm{i}\hbar\epsilon_{jkl}\hat{B}_l, \qquad [\hat{P}_j^A, \hat{X}_k] = -\mathrm{i}\hbar\mathbf{1}, \tag{6}$$

where $\hat{B}_l$ is the operator of the magnetic field strength,

$$\hat{B}_j\psi(\vec{x}) = B_j(\vec{x})\psi(\vec{x}) = \epsilon_{jkl}\frac{\partial A_l}{\partial x_k}\psi(\vec{x})$$

and ϵ_{jkl} is the completely antisymmetric tensor with $\epsilon_{123} = 1$.

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and ϵ_{jkl} is the completely antisymmetric tensor with $\epsilon_{123} = 1$.

The general structure of the integrals of motion, cont'd

Classically, we write a general second order integral of motion as

$$X = \sum_{j=1}^{3} h_j(\vec{x}) p_j^A p_j^A + \sum_{j,k,l=1}^{3} \frac{1}{2} |\epsilon_{jkl}| n_j(\vec{x}) p_k^A p_l^A + \sum_{j=1}^{3} s_j(\vec{x}) p_j^A + m(\vec{x}). \tag{7}$$

The condition that the Poisson bracket

$$\{a(\vec{x},\vec{p}), b(\vec{x},\vec{p})\}_{P.B.} = \sum_{j=1}^{3} \left(\frac{\partial a}{\partial x_j} \frac{\partial b}{\partial p_j} - \frac{\partial b}{\partial x_j} \frac{\partial a}{\partial p_j} \right) \tag{8}$$

of the integral (7) with the Hamiltonian (1) vanishes

$$\{H, X\}_{P.B.} = 0 \tag{9}$$

leads to terms of order 3, 2, 1 and 0 in the momenta:

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The conditions for the integrals of motion

Third order

$$\begin{aligned}
&\partial_x h_1 = 0, && \partial_y h_1 = -\partial_x n_3, && \partial_z h_1 = -\partial_x n_2,\\
&\partial_x h_2 = -\partial_y n_3, && \partial_y h_2 = 0, && \partial_z h_2 = -\partial_y n_1,\\
&\partial_x h_3 = -\partial_z n_2, && \partial_y h_3 = -\partial_z n_1, && \partial_z h_3 = 0,\\
& && \nabla \cdot \vec{n} = 0. && \qquad (10)
\end{aligned}$$

Second order

$$\begin{aligned}
\partial_x s_1 &= n_2 B_2 - n_3 B_3,\\
\partial_y s_2 &= n_3 B_3 - n_1 B_1,\\
\partial_z s_3 &= n_1 B_1 - n_2 B_2, \qquad \text{i.e.} \qquad \nabla \cdot \vec{s} = 0,\\
\partial_y s_1 + \partial_x s_2 &= n_1 B_2 - n_2 B_1 + 2(h_1 - h_2) B_3, \qquad (11)\\
\partial_z s_1 + \partial_x s_3 &= n_3 B_1 - n_1 B_3 + 2(h_3 - h_1) B_2,\\
\partial_y s_3 + \partial_z s_2 &= n_2 B_3 - n_3 B_2 + 2(h_2 - h_3) B_1.
\end{aligned}$$

The conditions for the integrals of motion, cont'd

First order terms

$$\begin{aligned}
\partial_x m &= 2h_1 \partial_x W + n_3 \partial_y W + n_2 \partial_z W + s_3 B_2 - s_2 B_3, \\
\partial_y m &= n_3 \partial_x W + 2h_2 \partial_y W + n_1 \partial_z W + s_1 B_3 - s_3 B_1, \\
\partial_z m &= n_2 \partial_x W + n_1 \partial_y W + 2h_3 \partial_z W + s_2 B_1 - s_1 B_2.
\end{aligned} \quad (12)$$

Zeroth order

$$\vec{s} \cdot \nabla W = 0. \quad (13)$$

Equations (10) are the same as for the system with vanishing magnetic field and their explicit solution is known - they imply that the highest order terms in the integral (7) are linear combinations of products of the generators of the Euclidean group $p_1, p_2, p_3, l_1, l_2, l_3$ where $l_j = \sum_{l,k} \epsilon_{jkl} x_k p_l$, i.e. $\vec{h}$, $\vec{n}$ can be expressed in terms of 20 constants α_{ab}, $1 \leq a \leq b \leq 6$.

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First order terms

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The conditions for the integrals of motion, cont'd

In the quantum case we have to consider a properly symmetrized analogue of (7). We choose

$$\hat{X} = \sum_{j=1}^{3}\{h_j(\vec{x}), \hat{P}_j^A \hat{P}_j^A\} + \sum_{j,k,l=1}^{3} \frac{|\epsilon_{jkl}|}{2}\{n_j(\vec{x}), \hat{P}_k^A \hat{P}_l^A\} + + \sum_{j=1}^{3}\{s_j(\vec{x}), \hat{P}_j^A\} + m(\vec{x}), \tag{14}$$

where $\{\ ,\ \}$ denotes the symmetrization. Only (13) obtains an $\hbar^2$–proportional correction

$$\vec{s}\cdot\nabla W + \frac{\hbar^2}{4}\left(\partial_z n_1 \partial_z B_1 - \partial_y n_1 \partial_y B_1 + \partial_x n_2 \partial_x B_2 - \partial_z n_2 \partial_z B_2 + +\partial_y n_3 \partial_y B_3 - \partial_x n_3 \partial_x B_3 + \partial_x n_1 \partial_y B_2 - \partial_y n_2 \partial_x B_1\right) = 0. \tag{15}$$

Cartesian type second order integrals

Let us now turn our attention to the situation where the Hamiltonian is integrable in the Liouville sense, with at most quadratic integrals. That means that in addition to the Hamiltonian itself there must be at least two independent integrals of motion of the form (7) or (14) which commute in the sense of Poisson bracket or commutator, respectively.

We assume such integrals to be of Cartesian type,

$$X_j = (p_j^A)^2 + \sum_{\ell=1}^{3} \mathcal{S}_j^\ell(\vec{x}) p_\ell^A + m_j(\vec{x}), \quad j = 1, 2. \tag{16}$$

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Integrable systems in the Cartesian Case

For such pair of Cartesian-type integrals to exist, the magnetic field $\vec{B}$ must be of the form

$$\begin{aligned} B_1(\vec{x}) &= F_2'(z) + k_3'(y), \\ B_2(\vec{x}) &= -F_1'(z) - g_3'(x), \\ B_3(\vec{x}) &= g_2'(x) - k_1'(y) \end{aligned} \qquad (17)$$

where the functions F_1, F_2, g_ℓ and k_ℓ must satisfy the following compatibility constraints

$$\begin{aligned} F_1(z)g_2'(x) - g_3(x)F_2'(z) &= 0, \\ F_2(z)k_1'(y) - k_3(y)F_1'(z) &= 0, \\ g_2(x)k_3'(y) - k_1(y)g_3'(x) &= 0. \end{aligned} \qquad (18)$$

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Integrable systems in the Cartesian Case, cont'd

The functions g_ℓ and k_ℓ are related to the first order terms in (16) through

$$\begin{aligned} \mathcal{S}_1^1(\vec{x}) &= 2(F_1(z) - k_1(y)), \\ \mathcal{S}_1^\ell(\vec{x}) &= 2g_\ell(x), & \ell = 2, 3, \\ \mathcal{S}_2^2(\vec{x}) &= 2(F_2(z) - g_2(x)), \\ \mathcal{S}_2^\ell(\vec{x}) &= 2k_\ell(y), & \ell = 1, 3. \end{aligned} \tag{19}$$

Superintegrability in the Cartesian case

Conditions (17) and (18) prescribe the structure of the magnetic field that leads to 5 classes of integrable systems with nonvanishing magnetic field. [1]

Here we investigate which choices of the potentials render the system with integrals (16) not only integrable, but superintegrable. Namely, we look for conditions for a third independent integral to exist.

A "brute force" approach, which directly looks for an additional second order integral solving (11)-(13), presently appears intractable due to the computational complexity.

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First order additional integral

Instead, let us assume that superintegrability arises in the simplest possible way, requiring the third integral to be of first order in the momenta:

$$X_3 = \sum_{\ell=1}^{3} s_\ell(\vec{x}) p_\ell^A + m_3(\vec{x}). \tag{20}$$

Next, for each minimally superintegrable system found, we can investigate the possibilities for another integral, this time allowing second order terms, so to obtain a maximally superintegrable system.

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First order additional integral, cont'd

By setting $\vec{h} = \vec{n} = 0$, we see that in turn the first order term in X_3 must lie in the enveloping algebra of the Euclidean algebra:

$$\begin{aligned} s_1(x,y,z) &= \beta_{12}y + \beta_{13}z + \beta_{11}, \\ s_2(x,y,z) &= -\beta_{12}x + \beta_{23}z + \beta_{22}, \\ s_3(x,y,z) &= -\beta_{13}x - \beta_{23}y + \beta_{33}, \end{aligned} \tag{21}$$

where $\beta_{ij} \in \mathbb{R}$.

Also we see that the zero order equations in the classical and quantum case now coincide for all three integrals, i.e. we cannot discover any purely quantum integrable systems in this setting.

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Superintegrability in the Cartesian case

Going through the solution of the remaining determining equations, we have found that minimally superintegrable systems with first order additional integral can exist only in three of the five classes of integrable systems existing in the Cartesian case. (Cases A, B, C in the following).

Next, we look for maximally superintegrable systems among them. Thus, we must go through equations (11)-(13), looking for another independent integral of at most second order, i.e. of the form (7). The existence of three known integrals significantly restricts the structure of the system, thus we are able to find all its solutions, if any.

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Superintegrable systems found: Case A

Case A. Let us start with the integrable system given by

$$W(\vec{x}) = \frac{1}{2}(u_1(x)+u_2(y)-(g_3(x)+k_3(y))^2), \; \vec{B} = (k_3'(y), -g_3'(x), 0).$$

We choose the gauge so that

$$\vec{A}(\vec{x}) = (0, 0, g_3(x) + k_3(y))$$

and the two Cartesian integrals read

$$X_1 = p_1^2 + 2g_3(x)p_3 + u_1(x), \qquad X_2 = p_2^2 + 2k_3(y)p_3 + u_2(y).$$

It follows immediately that p_3 is an integral, however not independent on the others since

$$2H - X_1 - X_2 = p_3^2.$$

Superintegrable systems found: Case A.1

We have $g_3 = \frac{\Omega}{2}x^2$, $k_3 = \frac{\Omega}{2}\Omega y^2$, $\Omega \in \mathbb{R}\backslash\{0\}$, thus

$$\vec{A}(\vec{x}) = \left(0, 0, \frac{\Omega}{2}(x^2 + y^2)\right), \qquad \vec{B}(\vec{x}) = (\Omega y, -\Omega x, 0).$$

The remaining arbitrary functions in the effective potential are

$$u_1 = \frac{U}{2}x^2, \qquad u_2 = \frac{U}{2}y^2, \qquad U \in \mathbb{R}$$

so that

$$W(\vec{x}) = -\frac{\Omega^2}{4}(x^2 + y^2)^2 + \frac{U}{2}(x^2 + y^2).$$

The additional independent first order integral turns out to be

$$X_3 = l_3.$$

For no choice of the nonvanishing magnetic field an additional independent integral of at most second order exists, i.e. this system is never maximally quadratically superintegrable.

For $2\Omega p_{30} + U > 0$, the solution of the equations of motion takes the form of a deformed spiral.

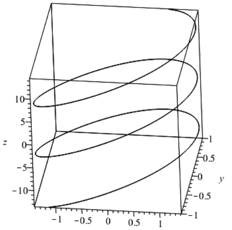

Figure: $\Omega = 1, U = 0, p_{10} = 1, p_{20} = 1, p_{30} = 1, x_0 = 1, y_0 = 0, z_0 = 0$

Superintegrable systems found: Case A.1, cont'd

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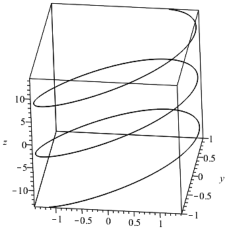

Figure: $\Omega = 1, U = 0, p_{10} = 1, p_{20} = 1, p_{30} = 1, x_0 = 1, y_0 = 0, z_0 = 0.$

Superintegrable systems found: Case A.1, cont'd

When the initial conditions are such that $2\Omega p_{30} + U = 0$ the solution becomes polynomial in time with all three momenta p_1, p_2, p_3 conserved.

For $2\Omega p_{30} + U < 0$ the solution is expressed in terms of hyperbolic functions and is not bounded in any spatial direction.

Superintegrable systems found: Case A.1, cont'd

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For $2\Omega p_{30} + U < 0$ the solution is expressed in terms of hyperbolic functions and is not bounded in any spatial direction.

Superintegrable systems found: Case A.2

We have $g_3 = -\Omega_2 x$, $k_3 = -\Omega_1 y$, $\Omega_1, \Omega_2 \in \mathbb{R}$, which implies

$$\vec{A} = (0, 0, -\Omega_2 x - \Omega_1 y), \qquad \vec{B}(\vec{x}) = (-\Omega_1, \Omega_2, 0).$$

The effective potential takes the form

$$W(\vec{x}) = \frac{\Omega_1 \Omega_2}{2S}(Sx - y)^2 + \frac{U}{2}(Sx - y),$$

since

$$\begin{aligned} u_1 &= \Omega_2^2 x^2 + S(U + \Omega_1 \Omega_2 x)x \\ u_2 &= \Omega_1^2 y^2 - (U - \frac{\Omega_1 \Omega_2}{S} y)y, \quad S \in \mathbb{R}. \end{aligned}$$

The third integral is given by

$$X_3 = p_1 + S p_2 - (S\Omega_1 + \Omega_2) z. \tag{22}$$

Superintegrable systems found: Case A.2, cont'd

Under the assumption $\Omega_1\Omega_2 \neq 0$, a shift of the coordinates accompanied by a gauge transformation allows us to set $U = 0$. When either of the Ω_i vanishes, the potential W becomes a linear function of the coordinates.

When $\Omega_1 S + \Omega_2 = 0$, the Hamiltonian becomes

$$H = \frac{p_1^2 + p_2^2 + p_3^2}{2} - \Omega_1 y p_3 - \Omega_2 x p_3, \quad W(\vec{x}) = -\frac{(\Omega_1 y + \Omega_2 x)^2}{2}.$$

We can rotate our coordinates around the z-axis to set $\Omega_2 = 0$ and the integrals reduce to

$$X_1 = p_1^2, \quad X_2 = p_2^2 - 2\Omega_1 y p_3, \quad X_3 = p_1, \qquad (23)$$

i.e. X_3 becomes equal to X_1.

Superintegrable systems found: Case A.2, cont'd

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Superintegrable systems found: **Case A.2**, cont'd

However, there are two additional independent second order integrals

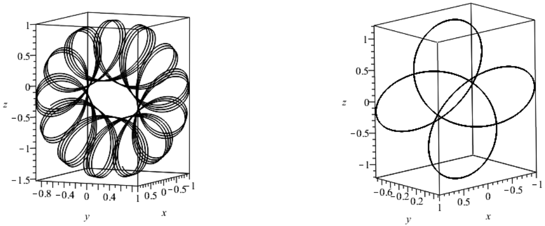
$$X_4 = p_1 l_3 - \frac{\Omega_1}{2} x^2 p_3, \qquad X_5 = \frac{1}{3} p_2 l_3 - l_2 p_3 - \frac{2}{3} \Omega_1 xy p_3$$

which classically make the system maximally quadratically superintegrable. The classical trajectories are unbounded for almost all initial conditions. Its quantum properties are not clear since the potential $W(\vec{x})$ is not bounded from below.

Superintegrable systems found: Case A.2, cont'd

Coming back to $\Omega_1 S + \Omega_2 \neq 0$: there are bounded trajectories, when the frequency ratio satisfies the rationality condition

$$S\frac{\Omega_2}{\Omega_1} = k^2, \qquad k \in \mathbb{Q}. \tag{24}$$



Left: $\Omega_1 = 1, \Omega_2 = 3, S = 10$, irrational frequency ratio (24).

Right: $\Omega_1 = 1, \Omega_2 = 3, S = 3$, i.e. rational frequency ratio (24) $k = 3$.

Superintegrable systems found: Case A.2, cont'd

For $1 \neq k = \sqrt{S\frac{\Omega_2}{\Omega_1}} \in \mathbb{Q}$ there is no additional first order integral. An independent second order integral exists for particular values of S. Namely, for $S = \frac{\Omega_1}{4\Omega_2}$, i.e. $k = \frac{1}{2}$,

$$\begin{aligned} X_4 &= -p_1 l_3 - p_3 l_1 + 4\frac{\Omega_2}{\Omega_1} p_3 l_2 \\ &+ \left(\left(2\frac{\Omega_2^2}{\Omega_1} + \frac{\Omega_1}{2}\right)(x^2 - z^2) + 2\Omega_2 xy + \frac{\Omega_1}{2} y^2\right) p_3 \\ &- \frac{\Omega_1^2}{4} x^2 y - \Omega_2^2 x^2 y. \end{aligned}$$

For $S = 4\frac{\Omega_1}{\Omega_2}$, i.e. $k = 2$, one finds a similar integral which should not come as a surprise - the two cases can be brought one into the other by the following exchange of coordinates and parameters $x \leftrightarrow y, \quad p_1 \leftrightarrow p_2, \quad \Omega_1 \leftrightarrow \Omega_2$.

Superintegrable systems found: Case A.2, cont'd

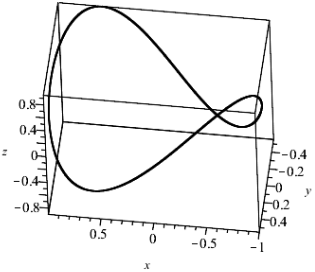


Figure: $\Omega_1 = 1, \Omega_2 = \frac{5}{6}, S = \frac{\Omega_1}{4\Omega_2} = \frac{3}{10}, p_{20} = 1, p_{10} = 0, p_{30} = 0, x_0 = 1, y_0 = \frac{1}{5}, z_0 = \frac{1}{2}$, i.e. rational frequency ratio $k = \frac{1}{2}$, closed trajectory, maximally superintegrable.

Superintegrable systems found: Case B

Let us consider the integrable system with the magnetic field

$$\vec{B}(\vec{x}) = (0, 0, f''(x) - g''(y)), \quad g_2 = f', \quad k_1 = g' \qquad (25)$$

where f and g satisfy the elliptic equations:

$$\begin{aligned} f''(x) &= \alpha f(x)^2 + \beta f(x) + \gamma, \\ g''(y) &= \alpha g(y)^2 + \delta g(y) + \xi. \end{aligned}$$

and the effective potential reads

$$\begin{aligned} W(\vec{x}) &= V(z) - \frac{1}{6}(f(x) + g(y))\,(6(\eta + \gamma - \xi) + \\ &\quad +(f(x) + g(y))(3(\beta + \delta) + 2\alpha(f(x) + g(y)))\,, \end{aligned} \qquad (26)$$

where $\alpha, \beta, \gamma, \delta, \xi, \eta \in \mathbb{R}$ and $V(z)$ is an arbitrary function of z. The gauge is chosen so that $\vec{A}(\vec{x}) = (k_1'(y), g_2'(x), 0)$.

Superintegrable systems found: Case B, cont'd

The system is minimally superintegrable if $\alpha = \beta = \delta = \xi = 0$ and $\eta = -\gamma$, so that

$$\vec{B}(\vec{x}) = (0, 0, \gamma), \qquad W(\vec{x}) = V(z).$$

The Hamiltonian reads

$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \gamma x p_2 + \frac{\gamma^2}{2} x^2 + V(z).$$

In order to have nonvanishing magnetic field we must assume $\gamma \neq 0$. The integrals read

$$X_1 = p_1 + \gamma y, \qquad X_2 = p_2, \qquad X_3 = 2l_3 + \gamma\left(x^2 - y^2\right).$$

Although these three first order integrals don't commute among themselves, the system is Liouville integrable because H, X_2 and $X_1^2 + \gamma X_3$ form a commuting triple of integrals.

Superintegrable systems found: Case B, cont'd

The system turns to be maximally superintegrable, if

$$V(z) = \frac{c}{z^2} + \frac{\gamma^2 z^2}{8}, \qquad (27)$$

$$\text{or } V(z) = \frac{\gamma^2}{2} z^2. \qquad (28)$$

In the potential (27) we shall assume that $c \geq 0$; otherwise, the energy is not bounded from below and the system allows fall on the singular plane $z = 0$ where the dynamical equations are ill-defined. Nevertheless, at the algebraic level the structure of the integrals described below is the same also for $c < 0$.

Superintegrable systems found: Case B, cont'd

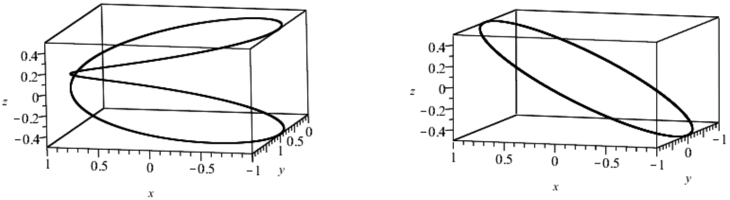


Figure: Left: Sample trajectory for the effective potential (27) with $\gamma = 2, c = 0, p_{10} = 0, p_{20} = 0, p_{30} = 0, x_0 = -1, y_0 = 1, z_0 = 1/2$.
Right: Sample trajectory for the effective potential (28) with $\gamma = 2, p_{10} = 0, p_{20} = 0, p_{30} = 0, x_0 = -1, y_0 = 0, z_0 = -1/2$.

Superintegrable systems found: Case B, algebra of integrals

We construct also the algebras of integrals of motion. E.g. for the potential (27) we have the integral

$$\begin{aligned} X_4 &= l_1^2 + l_2^2 + \frac{\gamma}{4}\left(\gamma(x^2+y^2) + 4yp_1 + 4xp_2\right) z^2 - \\ &\quad -2\gamma xyzp_3 + 2\frac{c}{z^2}\left(x^2+y^2\right). \end{aligned} \tag{29}$$

and two more second order functionally dependent integrals

$$\begin{aligned} X_5 &= \frac{1}{2}\{X_4, X_2\}_{P.B.} = p_3 l_1 + \frac{\gamma}{2} z^2 p_1 - z\gamma x p_3 + \frac{\gamma^2}{4} y z^2 + 2c\frac{y}{z^2}, \\ X_6 &= \frac{1}{2}\{X_4, X_1\}_{P.B.} = p_3 l_2 - \frac{\gamma}{2} z^2 p_2 - \frac{\gamma^2}{4} z^2 x + 2c\frac{x}{z^2}. \end{aligned} \tag{30}$$

We notice that X_3, X_4 commute, i.e. provide another choice of integrals demonstrating Liouville integrability of (27).

Superintegrable systems found: Case B, algebra of integrals

$$
\begin{aligned}
\{X_1, X_2\}_{P.B.} &= \gamma, \qquad \{X_1, X_3\}_{P.B.} = -2X_2, \qquad \{X_2, X_3\}_{P.B.} = 2X_1, \\
\{X_1, X_5\}_{P.B.} &= 0, \qquad \{X_1, X_6\}_{P.B.} = X_1^2 + X_2^2 - 2H + \gamma X_3, \\
\{X_2, X_5\}_{P.B.} &= X_1^2 + X_2^2 - 2H + \gamma X_3, \qquad \{X_2, X_6\}_{P.B.} = 0, \\
\{X_3, X_5\}_{P.B.} &= -2X_6, \qquad \{X_3, X_6\}_{P.B.} = 2X_5, \\
\{X_4, X_5\}_{P.B.} &= 2\left(-X_2X_4 + X_3X_6 - 2cX_2\right), \\
[X_4, X_6\}_{P.B.} &= -2\left(X_1X_4 + X_3X_5 + 2cX_1\right), \\
\{X_5, X_6\}_{P.B.} &= 2\left(\left(X_1^2 + X_2^2 - 2H + \gamma X_3\right) X_3 - X_1X_5 + X_2X_6 - c\gamma\right).
\end{aligned}
\tag{31}
$$

Superintegrable systems found: **Case C**

Let us consider the last relevant integrable system given by

$$W(\vec{x}) = W(z), \qquad \vec{B}(\vec{x}) = (B_1(z), B_2(z), 0)\,,\ X_j = p_j,\ j = 1, 2.$$

We already obtained this case via a different approach in A. Marchesiello, L. Šnobl, P. Winternitz J. Phys. A: Math. Theor. 48, 395206 (2015) .

Two superintegrable systems have been found and they are described there. They are both maximally superintegrable but one of the integrals is not a polynomial function in the momenta.

Need of a “generalised” Cartesian case

Let us consider the system ($a, \beta, c, \Omega_1, \Omega_2 \in \mathbb{R}$, $a, \beta, \Omega_2 \neq 0$).

$$\begin{aligned} B_1 &= \Omega_1,\ B_2 = \Omega_2,\ B_3 = 0, \\ W &= acx + \left(\frac{1}{2}\Omega_2 \left(\beta^2\Omega_2 - \Omega_1\right) - a \right) y^2 - az^2 \end{aligned}$$

This system admits two quadratic commuting integrals. One of these integrals still has the Cartesian form (16) with

$$\begin{aligned} \vec{\mathcal{S}}_1 &= (-2\beta\Omega_2 z, 0, -c\beta\Omega_2), \\ m_1(\vec{x}) &= \beta^2(z^2 - cx)\Omega_2^2 + \beta\Omega_1\Omega_1 cy + 2acx. \end{aligned}$$

Need of a “generalised” Cartesian case, cont'd

However, the second integral reads

$$X_2 = (p_2^A) + 2\beta p_1^A p_2^A + \sum_{\ell+1}^{\ell} \mathcal{K}_\ell(\vec{x}) p_\ell^A + m_2(\vec{x}) \qquad (32)$$

where

$$\begin{aligned}
\mathcal{K}_1(\vec{x}) &= 2\beta\Omega_1 z, \quad \mathcal{K}_2(\vec{x}) = -\frac{4az}{\Omega_2}, \\
\mathcal{K}_3(\vec{x}) &= 2\left(\Omega_1 - \beta^2\Omega_2 + \frac{2a}{\Omega_2}\right) y + c\beta\Omega_1, \\
m_2(\vec{x}) &= c\beta^2\Omega_2\Omega_1 x + \beta c(2a - \Omega_1^2) y + \\
&\frac{\Omega_1 + \Omega_2}{\Omega_2}\left(\beta^2\Omega_2^2 - \Omega_1\Omega_2 - 2a\right) y^2 - \\
&-\frac{\Omega_1}{\Omega_2}\left(\beta^2\Omega 2^2 + 2a\right) z^2.
\end{aligned}$$

Need of a "Generalised" Cartesian case, cont'd

Equivalently, putting $\tilde{X}_2 = X_2 + \beta^2 X_1$ the second integral can be written with the leading order term of the form $(p_2 + \beta p_1)^2$.

For $\beta \neq 0$ it can be easily seen that no Euclidean transformation or linear combination can reduce the integrals X_1 and X_2 to the form of a Cartesian-type integral.

And we cannot reduce the system to some of the classes corresponding to separation in some other coordinate systems for vanishing magnetic field.

This means that, for nonvanishing magnetic field, other pairs of integrals also need to be considered !

Need of a "Generalised" Cartesian case, cont'd

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Conclusions

- We investigated possibilities for superintegrability to arise in the Cartesian case. We found three classes of minimally superintegrable systems which lead to four maximally superintegrable subclasses.
- It was already established that quadratic integrability does not imply separability anymore. Now we also have an example showing that more general structure of the pairs of integrals needs to be considered.
- How to find a systematic way to classify all commuting quadratic integrals still remains an open problem under investigation.
- There can exist purely quantum systems, with no non-trivial classical counterpart. However, the conditions imposed here were too restrictive to allow such behavior.

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Thank you for your attention!