Superintegrability in the presence of magnetic fields

Libor Šnobl

Czech Technical University in Prague

In collaboration with A. Marchesiello and S. Bertrand, based on SIGMA 14 (2018) 092 and J. Phys. A: Math. Theor. 52 (2019) 195201.

XIth International Symposium Quantum Theory and Symmetries, Montréal, July 2, 2019

Abstract

In my talk I shall review our recent results concerning classical superintegrability with magnetic fields. In particular, I shall focus on two concrete examples in three spatial dimensions:

Abstract

In my talk I shall review our recent results concerning classical superintegrability with magnetic fields. In particular, I shall focus on two concrete examples in three spatial dimensions:

1. 3–parameter family of maximally superintegrable systems with constant magnetic field, which are quadratically minimally superintegrable and which for rational values of one of its parameters $\kappa = \frac{m}{n}$ (where $m, n \in \mathbb{N}$ are incommensurable) possess an additional integral of the order $m + n - 1$; and

Abstract

In my talk I shall review our recent results concerning classical superintegrability with magnetic fields. In particular, I shall focus on two concrete examples in three spatial dimensions:

1. 3–parameter family of maximally superintegrable systems with constant magnetic field, which are quadratically minimally superintegrable and which for rational values of one of its parameters $\kappa = \frac{m}{n}$ (where $m, n \in \mathbb{N}$ are incommensurable) possess an additional integral of the order $m + n - 1$; and
2. 6–parameter family of minimally superintegrable systems with the magnetic field of the form

$$\vec{B}(\vec{x}) = b_m \frac{\vec{x}}{|\vec{x}|^3} + \frac{b_n}{|\vec{x}|^3} \left(xz, yz, |\vec{x}|^2 + z^2 \right) + (0, 0, b_z)$$

which seems to possess closed bounded trajectories (based on numerical experiments), thus hinting at a hypothetical maximal superintegrability.

Contents

Contents

Contents

Contents

Contents

Introduction

We consider integrable and superintegrable systems in three spatial dimensions.

Introduction

We consider integrable and superintegrable systems in three spatial dimensions.

Integrability

A classical Hamiltonian system with n degrees of freedom is integrable if it admits n functionally independent integrals of motion in involution.

Introduction

We consider integrable and superintegrable systems in three spatial dimensions.

Integrability

A classical Hamiltonian system with n degrees of freedom is integrable if it admits n functionally independent integrals of motion in involution.

Superintegrability

A classical Hamiltonian system with n degrees of freedom is polynomially superintegrable if it admits $n+k$ functionally independent integrals of motion (where $k \leq n-1$), that are polynomial in the momenta and out of which n are in involution.

Introduction, cont'd

Due to A.A. Makarov, J.A. Smorodinsky, K. Valiev, P. Winternitz, Il Nuovo Cimento LII A, 8881 (1967) when quadratic integrability is considered and the Hamiltonian involves only a kinetic term and a scalar potential, there are 11 classes of systems admitting pairs of commuting quadratic integrals, each uniquely determined by a pair of commuting quadratic elements in the enveloping algebra of the 3D Euclidean algebra. These in turn correspond to a coordinate system in which the Hamilton-Jacobi equation separates.

Introduction, cont'd

Due to A.A. Makarov, J.A. Smorodinsky, K. Valiev, P. Winternitz, Il Nuovo Cimento LII A, 8881 (1967) when quadratic integrability is considered and the Hamiltonian involves only a kinetic term and a scalar potential, there are 11 classes of systems admitting pairs of commuting quadratic integrals, each uniquely determined by a pair of commuting quadratic elements in the enveloping algebra of the 3D Euclidean algebra. These in turn correspond to a coordinate system in which the Hamilton-Jacobi equation separates.

When systems involve vector potentials, quadratic integrability no longer implies separability, cf. e.g. J. Bérubé, P. Winternitz. J. Math. Phys. 45 (2004), no. 5, 1959-1973.

Introduction, cont'd

Approaches to the problem in three spatial dimensions:

Introduction, cont'd

Approaches to the problem in three spatial dimensions:

- A. Marchesiello, L. Šnobl, P. Winternitz, J. Phys. A: Math. Theor. 48, 395206 (2015): possibilities for integrability and superintegrability arising from first order integrals.

Introduction, cont'd

Approaches to the problem in three spatial dimensions:

- A. Marchesiello, L. Šnobl, P. Winternitz, J. Phys. A: Math. Theor. 48, 395206 (2015): possibilities for integrability and superintegrability arising from first order integrals.
- A. Marchesiello, L. Šnobl, J. Phys. A: Math. Theor. 50, 245202 (2017): superintegrable systems which separate in Cartesian coordinates in the limit when the magnetic field vanishes, i.e. possess two second order integrals of motion of the so-called Cartesian type.

Introduction, cont'd

Approaches to the problem in three spatial dimensions:

- A. Marchesiello, L. Šnobl, P. Winternitz, J. Phys. A: Math. Theor. 48, 395206 (2015): possibilities for integrability and superintegrability arising from first order integrals.
- A. Marchesiello, L. Šnobl, J. Phys. A: Math. Theor. 50, 245202 (2017): superintegrable systems which separate in Cartesian coordinates in the limit when the magnetic field vanishes, i.e. possess two second order integrals of motion of the so-called Cartesian type.
- A. Marchesiello, L. Šnobl, P. Winternitz, J. Phys. A: Math. Theor. 51, 135205 (2018): (super)integrability with spherical type integrals.

Introduction, cont'd

Approaches to the problem in three spatial dimensions:

- A. Marchesiello, L. Šnobl, P. Winternitz, J. Phys. A: Math. Theor. 48, 395206 (2015): possibilities for integrability and superintegrability arising from first order integrals.
- A. Marchesiello, L. Šnobl, J. Phys. A: Math. Theor. 50, 245202 (2017): superintegrable systems which separate in Cartesian coordinates in the limit when the magnetic field vanishes, i.e. possess two second order integrals of motion of the so-called Cartesian type.
- A. Marchesiello, L. Šnobl, P. Winternitz, J. Phys. A: Math. Theor. 51, 135205 (2018): (super)integrability with spherical type integrals.
- S. Bertrand and L. Šnobl, J. Phys. A: Math. Theor. 52, 195201 (2019): (super)integrability with nonsubgroup type integrals incl. at least one angular momentum component.

General structure of the integrals of motion

We consider the classical Hamiltonian describing the motion of a particle in three dimensions in a nonvanishing magnetic field

$$H = \frac{1}{2}(\vec{p} + \vec{A}(\vec{x}))^2 + W(\vec{x}), \tag{1}$$

where $\vec{p}$ is the linear momentum, $\vec{A}(\vec{x})$ is the vector potential and $W(\vec{x})$ is the electrostatic potential.

General structure of the integrals of motion

We consider the classical Hamiltonian describing the motion of a particle in three dimensions in a nonvanishing magnetic field

$$H = \frac{1}{2}(\vec{p} + \vec{A}(\vec{x}))^2 + W(\vec{x}), \tag{1}$$

where $\vec{p}$ is the linear momentum, $\vec{A}(\vec{x})$ is the vector potential and $W(\vec{x})$ is the electrostatic potential. The Newtonian equations of motion are gauge invariant – they are the same for the potentials

$$\vec{A}'(\vec{x}) = \vec{A}(\vec{x}) + \nabla\chi, \qquad W'(\vec{x}) = W(\vec{x})$$

for any choice of the function $\chi(\vec{x})$. Thus, the physically relevant quantity is the magnetic field $\vec{B}(\vec{x}) = \nabla \times \vec{A}$.

General structure of the integrals of motion

We consider the classical Hamiltonian describing the motion of a particle in three dimensions in a nonvanishing magnetic field

$$H = \frac{1}{2}(\vec{p} + \vec{A}(\vec{x}))^2 + W(\vec{x}), \qquad (1)$$

where $\vec{p}$ is the linear momentum, $\vec{A}(\vec{x})$ is the vector potential and $W(\vec{x})$ is the electrostatic potential. The Newtonian equations of motion are gauge invariant – they are the same for the potentials

$$\vec{A}'(\vec{x}) = \vec{A}(\vec{x}) + \nabla\chi, \qquad W'(\vec{x}) = W(\vec{x})$$

for any choice of the function $\chi(\vec{x})$. Thus, the physically relevant quantity is the magnetic field $\vec{B}(\vec{x}) = \nabla \times \vec{A}$. $\vec{B}(\vec{x})$ is assumed to be nonvanishing so that the system is not gauge equivalent to a system with only the scalar potential.

The general structure of the integrals of motion, cont'd

Let us consider integrals of motion which are at most second order in the momenta. Since our system is gauge invariant, we express the integrals in terms of gauge covariant expressions

$$p_j^A = p_j + A_j(\vec{x}), \qquad L_j^A = \sum_{l,k} \epsilon_{jkl} x_k p_l^A \tag{2}$$

rather than the linear and angular momenta themselves. (ϵ_{jkl} is the completely antisymmetric tensor with $\epsilon_{123} = 1$.)

The general structure of the integrals of motion, cont'd

Let us consider integrals of motion which are at most second order in the momenta. Since our system is gauge invariant, we express the integrals in terms of gauge covariant expressions

$$p_j^A = p_j + A_j(\vec{x}), \qquad L_j^A = \sum_{l,k} \epsilon_{jkl} x_k p_l^A \tag{2}$$

rather than the linear and angular momenta themselves. (ϵ_{jkl} is the completely antisymmetric tensor with $\epsilon_{123} = 1$.)

We write a general second order integral of motion as

$$X = \sum_{j=1}^{3} h^j(\vec{x}) p_j^A p_j^A + \sum_{j,k,l=1}^{3} \frac{1}{2} |\epsilon_{jkl}| n^j(\vec{x}) p_k^A p_l^A + \sum_{j=1}^{3} s^j(\vec{x}) p_j^A + m(\vec{x}). \tag{3}$$

The general structure of the integrals of motion, cont'd

The condition that the Poisson bracket

$$\{a(\vec{x},\vec{p}), b(\vec{x},\vec{p})\}_{P.B.} = \sum_{j=1}^{3} \left(\frac{\partial a}{\partial x_j}\frac{\partial b}{\partial p_j} - \frac{\partial b}{\partial x_j}\frac{\partial a}{\partial p_j} \right) \tag{4}$$

of the integral (3) with the Hamiltonian (1) vanishes

$$\{H, X\}_{P.B.} = 0 \tag{5}$$

leads to terms of order $3, 2, 1$ and 0 in the momenta.

The general structure of the integrals of motion, cont'd

The condition that the Poisson bracket

$$\{a(\vec{x},\vec{p}), b(\vec{x},\vec{p})\}_{P.B.} = \sum_{j=1}^{3} \left(\frac{\partial a}{\partial x_j} \frac{\partial b}{\partial p_j} - \frac{\partial b}{\partial x_j} \frac{\partial a}{\partial p_j} \right) \qquad (4)$$

of the integral (3) with the Hamiltonian (1) vanishes

$$\{H, X\}_{P.B.} = 0 \qquad (5)$$

leads to terms of order $3, 2, 1$ and 0 in the momenta. The third order ones are the same as for the system with vanishing magnetic field and their explicit solution is known - they imply that the quadratic terms in the integral (3) are linear combinations of products of the generators of the Euclidean group p_1, p_2, p_3, L_1, L_2, L_3, i.e. $\vec{h}$, $\vec{n}$ can be expressed in terms of 20 constants α_{ab}, $1 \leq a \leq b \leq 6$. The lower order ones imply conditions (PDEs) on the functions $\vec{s}, m, \vec{B}, W$ which also depend on the constants α_{ab}.

The general structure of the integrals of motion, cont'd

Let us now turn our attention to the situation when the Hamiltonian is integrable in the Liouville sense, with at most quadratic integrals. That means that in addition to the Hamiltonian itself there must be at least two independent integrals of motion of the form (3) which commute in the sense of Poisson bracket.

The general structure of the integrals of motion, cont'd

Let us now turn our attention to the situation when the Hamiltonian is integrable in the Liouville sense, with at most quadratic integrals. That means that in addition to the Hamiltonian itself there must be at least two independent integrals of motion of the form (3) which commute in the sense of Poisson bracket.

In the papers mentioned above we have studied such systems for various possible structures of the leading order terms in the integrals. In this talk I shall focus on two special cases which we find particularly interesting.

First example

Let us first consider the system with

$$\vec{B}(\vec{x}) = (-\Omega_1, \Omega_2, 0), \quad W(\vec{x}) = \frac{\Omega_1 \Omega_2}{2S}(Sx - y)^2 \qquad (6)$$

where Ω_1, Ω_2, S are real constants such that $S \neq 0$ and $\Omega_1^2 + \Omega_2^2 \neq 0$.

The system (6) is known to be minimally superintegrable. In addition to the Hamiltonian it possesses the following three independent integrals

$$\begin{aligned} X_1 &= (p_1^A)^2 - 2\Omega_2 x p_3^A - \Omega_2^2 x^2 + \Omega_1 \Omega_2 x (Sx - 2y), \\ X_2 &= (p_2^A)^2 - 2\Omega_1 y p_3^A - \Omega_1^2 y^2 + \frac{\Omega_1 \Omega_2}{S} y (y - 2Sx), \\ X_3 &= p_1^A + S p_2^A - (S\Omega_1 + \Omega_2) z. \end{aligned} \qquad (7)$$

First example - trajectories

The trajectories of the system (6) are known:

$$\begin{aligned} x(t) &= \frac{1}{\omega_1^2}\left(\left(\omega_1^2 x_0 - \Omega_2 p_{30}\right)\cos(\omega_1 t) + \omega_1 p_{10}\sin(\omega_1 t) + \Omega_2 p_{30}\right), \\ y(t) &= \frac{1}{\omega_2^2}\left(\left(\omega_2^2 y_0 - \Omega_1 p_{30}\right)\cos(\omega_2 t) + \omega_2 p_{20}\sin(\omega_2 t) + \Omega_1 p_{30}\right), \qquad (8) \\ z(t) &= \frac{1}{\Omega_1 S + \Omega_2}\Big(p_{10}\left(\cos(\omega_1 t) - 1\right) + S p_{20}\left(\cos(\omega_2 t) - 1\right) + \\ &\quad + \frac{\Omega_2 p_{30} - \omega_1^2 x_0}{\omega_1}\sin(\omega_1 t) + \frac{\Omega_1 p_{30} - \omega_2^2 y_0}{\omega_2}\sin(\omega_2 t)\Big) + z_0, \end{aligned}$$

where we introduced the constants

$$\omega_1 = \sqrt{\Omega_2(\Omega_1 S + \Omega_2)}, \quad \omega_2 = \sqrt{\frac{\Omega_1}{S}(\Omega_1 S + \Omega_2)} = \sqrt{\frac{\Omega_1}{S\Omega_2}}\omega_1. \qquad (9)$$

First example - closed trajectories

We observe that whenever

$$S = \frac{\Omega_1}{\Omega_2}\kappa^2, \quad \text{where} \;\; \kappa = \frac{m}{n}, \;\; m, n \in \mathbb{N} \;\; \text{are incommensurable,} \tag{10}$$

the trajectories (8) are periodic (or, equivalently, closed).

First example - closed trajectories

We observe that whenever

$$S = \frac{\Omega_1}{\Omega_2}\kappa^2, \quad \text{where } \kappa = \frac{m}{n}, \quad m, n \in \mathbb{N} \text{ are incommensurable,} \tag{10}$$

the trajectories (8) are periodic (or, equivalently, closed).

We shall see that for $\kappa = \frac{m}{n}$ the system (6) is actually maximally superintegrable, with the fifth integral of the order $m + n - 1$ in the momenta p_1, p_2, p_3.

Cf. A. Marchesiello and L. Šnobl, SIGMA 14 (2018) 092.

First example - canonical transformation

By the canonical transformation

$$\begin{aligned} x &= X + \frac{\Omega_2 P_3}{\Omega_2^2 + \Omega_1^2 \kappa^2}, \qquad y = Y + \frac{\Omega_1 P_3 \kappa^2}{\Omega_2^2 + \Omega_1^2 \kappa^2}, \qquad (11) \\ z &= \frac{\Omega_2 P_1}{\Omega_2^2 + \Omega_1^2 \kappa^2} + \frac{\Omega_1 P_2 \kappa^2}{(\Omega_2^2 + \Omega_1^2 \kappa^2)} + Z, \\ p_j &= P_j, \quad \text{j=1,2,3} \end{aligned}$$

the Hamiltonian for $\kappa = \frac{m}{n}$ reduces to

$$H_2 = \frac{1}{2}(P_1^2 + P_2^2) + \frac{1}{2}\omega^2(m^2 X^2 + n^2 Y^2), \;\; \omega^2 = \frac{\Omega_1^2}{n^2} + \frac{\Omega_2^2}{m^2}. \quad (12)$$

i.e. two dimensional anisotropic oscillator without magnetic field, with rational frequency ratio $\kappa = \frac{m}{n}$, plus a constant motion in Z.

First example - integrals of the system (12)

Thus, we immediately see two integrals of the system given by

$$P_3 = p_3, \quad Z = (\Omega_2 + \frac{\Omega_1^2}{\Omega_2}\kappa^2)\, X_3,$$

since both Z and P_3 are cyclic (notice: they are not in involution).

First example - integrals of the system (12)

Thus, we immediately see two integrals of the system given by

$$P_3 = p_3, \quad Z = (\Omega_2 + \frac{\Omega_1^2}{\Omega_2}\kappa^2)\, X_3,$$

since both Z and P_3 are cyclic (notice: they are not in involution).

And we have the other three independent integrals of the two dimensional anisotropic oscillator.

First example - integrals of the system (12)

Thus, we immediately see two integrals of the system given by

$$P_3 = p_3, \quad Z = (\Omega_2 + \frac{\Omega_1^2}{\Omega_2}\kappa^2)\, X_3,$$

since both Z and P_3 are cyclic (notice: they are not in involution).

And we have the other three independent integrals of the two dimensional anisotropic oscillator.

Thus, the original system (6) is maximally superintegrable.

First example - explicit construction of the integral

After introducing complex coordinates

$$z_1 = iP_1 + m\omega X, \;\; z_2 = iP_2 + n\omega Y$$

the generators of the ring of the integrals of the 2D oscillator can by easily written as

$$I_1 \;\; = \;\; z_1 \bar{z}_1, \quad I_2 = z_2 \bar{z}_2, \quad I_3 = \mathrm{Re}\,(z_1^n \bar{z}_2^m), \;\; I_4 = \mathrm{Im}\,(z_1^n \bar{z}_2^m).$$

They are clearly not independent; they satisfy the relation

$$I_3^2 + I_4^2 = I_1^n I_2^m. \tag{13}$$

By inverting the canonical transformations, we see that I_j, $j = 1, 2$ correspond to the Cartesian type integrals X_1, X_2 and I_3 (or I_4) provides a new independent integral X_4, of order at most $n + m$ in the momenta.

First example - explicit construction of the integral

The integrals I_j can also be expressed explicitly, in terms of Chebyshev polynomials. This provides a polynomial expression for X_4 in the original 3D phase space. In the gauge covariant form it reads

$$
\begin{aligned}
X_4 &= \sum_{k=0}^{[\frac{n-1}{2}]} (-1)^k \binom{n}{2k+1} (m\omega \tilde{X}^A)^{n-2k-1} (p_1^A)^{2k+1} \cdot \\
&\cdot \sum_{k=0}^{[\frac{m-1}{2}]} (-1)^k \binom{m}{2k+1} (n\omega \tilde{Y}^A)^{m-2k-1} (p_2^A)^{2k+1} + \\
&+ \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n}{2k} (m\omega \tilde{X}^A)^{n-2k} (p_1^A)^{2k} \sum_{k=0}^{[\frac{m}{2}]} (-1)^k \binom{m}{2k} (n\omega \tilde{Y}^A)^{m-2k} (p_2^A)^{2k}
\end{aligned}
$$

where

$$
\tilde{X}^A = x - \frac{n^2\Omega_2(p_3^A + \Omega_2 x + \Omega_1 y)}{n^2\Omega_2^2 + m^2\Omega_1^2}, \quad \tilde{Y}^A = y - \frac{m^2\Omega_1(p_3^A + \Omega_2 x + \Omega_1 y)}{n^2\Omega_2^2 + m^2\Omega_1^2}
$$

(and similarly for X_5).

First example - simplification of the integral

We notice that the terms of order $m+n$ in X_4 are only of the form

$$\alpha_k \gamma_j p_1^{2k} p_2^{2j} p_3^{n+m-2(k+j)}, \quad k = 0, \ldots, \left[\tfrac{n}{2}\right], j = 0, \ldots, \left[\tfrac{m}{2}\right]$$
$$\beta_k \delta_j p_1^{2k+1} p_2^{2j+1} p_3^{n+m-2(k+j+1)}, \quad k = 0, \ldots, \left[\tfrac{n-1}{2}\right], j = 0, \ldots, \left[\tfrac{m-1}{2}\right]$$

where α_j, β_j, γ_j, δ_j are some coefficients. Such terms can be eliminated by subtracting the integrals

$$\alpha_k \gamma_j p_3^{n+m-2(k+j)} X_1^k X_2^j, \quad k = 0, \ldots \left[\tfrac{n}{2}\right], j = 0, \ldots \left[\tfrac{m}{2}\right].$$

$$\frac{\beta_k \delta_j}{2} p_3^{n+m-2(k+j+1)} X_1^k X_2^j \left(\frac{\Omega_2}{\kappa^2 \Omega_1} (X_3^2 - X_1) - \kappa^2 \frac{\Omega_1}{\Omega_2} X_2 \right),$$

$k = 0, \ldots, \left[\frac{n-1}{2}\right]$, $j = 0, \ldots, \left[\frac{m-1}{2}\right]$. Therefore the order of the integral X_4 can be reduced to $m+n-1$.

First example - explicit form of the integral

Explicitly, e.g. for $n = 2$ and $m = 3$, the integral X_4 is of order $n + m - 1 = 4$. It has the leading order term in momenta of the following form

$$\begin{aligned} X_4^{(\mathrm{h.o.})} &= \frac{1}{\sqrt{9\Omega_1^2 + 4\Omega_2^2}} \Bigg(\left(\frac{16\Omega_2^3}{9\Omega_1} + 4\Omega_1\Omega_2 \right) L_2 p_2^2 p_3 \\ & -4\Omega_1\Omega_2 \left(3L_2 p_3 + 8L_3 p_2\right) p_3^2 - \qquad (14) \\ & - \left(4\Omega_2^2 + 9\Omega_1^2\right) \left(L_1 p_3 + L_3 p_1\right) p_2^2 + 27\Omega_1^2 \left(L_1 p_3 + L_3 p_1\right) p_3^2 \Bigg). \end{aligned}$$

First example - explicit form of the integral

and the remaining terms are

$$
\begin{aligned}
X_4 - X_4^{(\mathrm{h.o.})} \quad = \quad & 2\Omega_1 \tau y^2 p_1^2 p_3 - 2\tau \left(3\Omega_1 x + \frac{8}{9}\Omega_2 y \right) y p_1 p_2 p_3 - \frac{8\Omega_2 \tau}{9} y z p_1 p_3^2 \\
& + \tau \left(\frac{\Omega_1}{2}(9x^2 + y^2 - z^2) + 2\Omega_2 x y + \frac{2}{9}\frac{\Omega_2^2}{\Omega_1}(x^2 - z^2) \right) p_2^2 p_3 \\
& - \frac{1}{2\tau} \left(27 \left(x^2 - \frac{1}{3} y^2 - z^2 \right) \Omega_1^3 - 36\Omega_1^2 \Omega_2 x y \right. \\
& \left. + 4\Omega_2^2 \Omega_1 (3x^2 + 4y^2 - 3z^2) - \frac{64\Omega_2^3}{9} x y \right) p_3^3 \\
& - 2\Omega_1 \tau y z p_2 p_3^2 - \frac{\tau^3}{27} y^3 p_1^2 + \frac{\tau^3}{3} x y^2 p_1 p_2 + \frac{4\Omega_2 \tau^3}{81\Omega_1} y^2 z p_1 p_3 \\
& - \frac{\tau^3}{4} x^2 y p_2^2 + \frac{\tau^3}{9} y^2 z p_2 p_3 \\
& - \tau \left(\Omega_1^2 \left(9\frac{x^2}{4} + 2y^2 - z^2 \right) + \frac{4\Omega_2^2}{9} \left(x^2 - \frac{1}{3} y^2 - z^2 \right) + \frac{16\Omega_2^3}{81\Omega_1} x y \right) y p_3^2 \\
& + \frac{1}{18\Omega_1} \tau^3 \left(\left(\Omega_1 y - \frac{2}{3}\Omega_2 x \right)^2 - \left(\Omega_1^2 + \frac{4}{9}\Omega_2^2 \right) z^2 \right) y^2 p_3 + \frac{\tau^5}{108} y^3 x^2,
\end{aligned}
$$

where $\tau = \sqrt{9\Omega_1^2 + 4\Omega_2^2} = 6\omega$.

Second example

Next, let us consider a system which possesses two intersecting pairs of commuting quadratic integrals – one corresponding to the spherical case, i.e. of the form $L^2 + \dots$ and $L_z^2 + \dots$, the other corresponding to the circular parabolic case, $L_z^2 + \dots$ and $p_y L_x - p_x L_y + \dots$.

These assumptions imply the structure of the magnetic field

$$B(\vec{x}) = B_z(\vec{x}) + B_m(\vec{x}) + B_n(\vec{x}).$$

where $\vec{B}_z = (0, 0, b_z)$ is a constant magnetic field,

$$\vec{B}_m(\vec{x}) = b_m \frac{\vec{x}}{R^3}, \qquad R = \sqrt{x^2 + y^2 + z^2}.$$

is the field of the magnetic monopole and the last component takes the form

$$\vec{B}_n(\vec{x}) = \frac{b_n}{R^3}\big(xz, yz, (R^2 + z^2)\big).$$

Second example - (minimally) superintegrable Hamiltonian

The potential and thus also the Hamiltonian

$$
\begin{aligned}
H &= \frac{(p_x^A)^2+(p_y^A)^2+(p_z^A)^2}{2} + \frac{u_1}{x^2+y^2} + \frac{u_2}{R} + \frac{u_3 z}{(x^2+y^2)\,R} \\
&\quad + \frac{b_m^2}{2R^2} + \frac{b_z b_m z}{2R} - \frac{b_z b_n\left(x^2+y^2\right)}{2R} \\
&\quad + \frac{b_m b_n z}{R^2} - \frac{b_n^2(x^2+y^2)}{2R^2} - \frac{1}{8} b_z^2\left(x^2+y^2\right) \\
&= \frac{p_x^2+p_y^2+p_z^2}{2} + \left(-\frac{b_m z}{R(x^2+y^2)} + \frac{b_n}{R} + \frac{b_z}{2}\right) L_z \\
&\quad + \frac{b_m^2}{2(x^2+y^2)} + \frac{u_1}{x^2+y^2} + \frac{u_2}{R} + \frac{u_3 z}{(x^2+y^2)R}
\end{aligned}
$$

involve three additional arbitrary constants u_1, u_2, u_3. Notice that in the second form of the Hamiltonian we used the gauge choice $\vec{A}(\vec{x}) = \left(\frac{b_m yz}{(x^2+y^2)R} - \frac{b_n y}{R} - \frac{b_z y}{2}, -\frac{b_m xz}{(x^2+y^2)R} + \frac{b_n x}{R} + \frac{b_z x}{2}, 0\right)$.

Second example - known integrals of motion

$$
\begin{aligned}
X_1 = \;& p_y^A L_x^A - p_x^A L_y^A + \left(\frac{b_m}{R} + \frac{b_n z}{R} + b_z z \right) L_z^A \\
& - \frac{b_m b_z \left(x^2+y^2\right)}{2R} - \frac{b_n b_z z \left(x^2+y^2\right)}{2R} - \frac{b_z^2 z}{4} \left(x^2+y^2\right) \\
& - \frac{2u_1 z}{x^2+y^2} - \frac{u_2 z}{R} - \frac{u_3 \left(R^2+z^2\right)}{\left(x^2+y^2\right) R}, \\
X_2 = \;& L_z^A + \frac{b_m z}{R} - \frac{b_n \left(x^2+y^2\right)}{R} - \frac{b_z}{2} \left(x^2+y^2\right) = L_z, \\
Y_3 = \;& \left(L^A\right)^2 - \left(2b_n R + b_z R^2\right) L_z^A + \frac{2u_1 z^2}{x^2+y^2} + \frac{2u_3 z R}{x^2+y^2} \\
& + b_n b_z \left(x^2+y^2\right) R + b_n^2 \left(x^2+y^2\right) + \frac{1}{4} b_z^2 \left(x^2+y^2\right) R^2.
\end{aligned}
$$

The algebra of these integrals of motion closes polynomially and there exists no additional first or second order integral.

Second example - existence of closed trajectories

Nevertheless, we observe in numerical experiments that the trajectories for generic rational parameters are closed (when bounded).

Second example - existence of closed trajectories

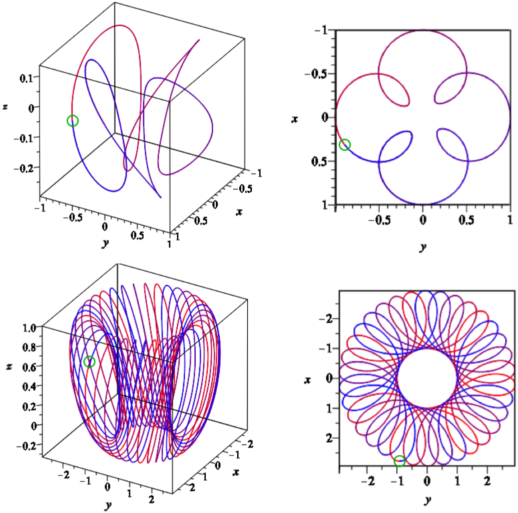
Nevertheless, we observe in numerical experiments that the trajectories for generic rational parameters are closed (when bounded).

On the next page are plots of two trajectories, for the following values of parameters and initial data

- $b_z = -2/7$, $b_m = -1/2$, $b_n = -5/2$, $u_1 = 1/6$, $u_2 = -3/2$, $u_3 = 0$ with the initial conditions $x(0) = 1$, $y(0) = 0$, $z(0) = 0$, $p_x(0) = 0$, $p_y(0) = 1$, $p_z(0) = 1/2$,
- $b_z = 0$, $b_m = 0$, $b_n = -2$, $u_1 = 1/2$, $u_2 = -1$, $u_3 = -1/4$ with the initial conditions $x(0) = 1$, $y(0) = 0$, $z(0) = 0$, $p_x(0) = 0$, $p_y(0) = 1$, $p_z(0) = 1/2$.

The point of closure is highlighted by a green circle, the flow of time is denoted by a gradual change of color from red to blue.

Closed trajectories for generic rational values of parameters



Second example - hypothetical maximal superintegrability

Based on this observation, we conjecture that also this system is maximally superintegrable for rational ratios of its parameters.

Second example - hypothetical maximal superintegrability

Based on this observation, we conjecture that also this system is maximally superintegrable for rational ratios of its parameters.

We also expect that the order of the hypothetical additional integral depends on the values of the the parameters $b_m, b_n, b_z, u_1, u_2, u_3$ of the system.

Second example - hypothetical maximal superintegrability

Based on this observation, we conjecture that also this system is maximally superintegrable for rational ratios of its parameters.

We also expect that the order of the hypothetical additional integral depends on the values of the the parameters $b_m, b_n, b_z, u_1, u_2, u_3$ of the system.

So far we have no clue about the structure of this conjectured integral. We know that it must be at least of third order in the momenta.

Future outlook

Further work on superintegrable systems in a magnetic field is in progress in several directions:

Future outlook

Further work on superintegrable systems in a magnetic field is in progress in several directions:

- Developing more efficient techniques to determine higher order integrals.
- Extending these results to relativistic mechanics.
- Studying properties of quantum analogues of the considered systems.

Acknowledgments

The research of A. M. and L. Š was supported by the Czech Science Foundation (Grant Agency of the Czech Republic), project 17-11805S. S.B. was supported by a postdoctoral fellowship provided by the Fonds de Recherche du Québec : Nature et Technologie (FRQNT).

Thank you for your attention!