Integrable systems of the ellipsoidal, paraboloidal and conical type with magnetic field

Libor Šnobl in collaboration with A. Marchesiello and M. F. Hoque

Czech Technical University in Prague

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Outline

We construct integrable Hamiltonian systems with magnetic fields of the ellipsoidal, paraboloidal and conical type, i.e. systems that generalize natural Hamiltonians separating in the respective coordinate systems to include nonvanishing magnetic field. In the ellipsoidal and paraboloidal case each this classification results in three one-parameter families of systems, each involving one arbitrary function of a single variable and a parameter specifying the strength of the magnetic field of the given fully determined form. In the conical case the results are more involved, there are two one-parameter families like in the other cases and one class which is less restrictive and so far resists full classification.

Formulation of the problem

Let us consider the Hamiltonian system in the three–dimensional Euclidean space of the form

$$H = \frac{1}{2}\vec{p}^2 + \vec{A}(\vec{x})\cdot\vec{p} + V(\vec{x}) = \frac{1}{2}(\vec{p}^A)^2 + W(\vec{x}), \quad \vec{p}^A = \vec{p} + \vec{A}(\vec{x}) \tag{1}$$

and its integrals of motion polynomial in the momenta. The leading order terms of such an integral must belong to a representation of the universal enveloping algebra $\mathfrak{U}(\mathfrak{e}_3)$ of the Euclidean algebra $\mathfrak{e}_3 = \mathrm{span}\{p_1, p_2, p_3, l_1, l_2, l_3\}$ such that

$$\vec{p}\cdot\vec{l} = \sum_{j=1}^{3} p_j l_j = 0 \tag{2}$$

between the linear momenta $\vec{p} = (p_1, p_2, p_3)$ and the angular momenta $\vec{l} = (l_1, l_2, l_3)$, $l_j = \sum_{k,l} \epsilon_{jkl} x_k p_l$, holds. (I.e. the quadratic Casimir invariant $\vec{p}\cdot\vec{l}$ of $\mathfrak{e}_3$ vanishes in the representations relevant for our physical application.)

Formulation of the problem

Restricting ourself to the most tractable situation of quadratic integrals of motion we are looking for pairs of commuting quadratic elements in $\mathfrak{U}(\mathfrak{e}_3)$ which obviously also commute with the quadratic Casimir invariant $h = \vec{p}^2 = \sum_j p_j^2$ of $\mathfrak{e}_3$ and together with it may define leading order terms of a triple of commuting integrals of motion (including the Hamiltonian).

Formulation of the problem

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Classification: quadratic 3D Abelian subalgebras in $\mathfrak{U}(\mathfrak{e}_3)$ I

Let us briefly summarize the possible structures of those leading order terms and show for which classes the existence of systems with generalized form of their integrals was by now studied and with what conclusion (includes both published results and ones submitted for publication). In the following list the blue classes have been showed to possess generalized systems, the red ones do not, green do not allow any generalized structures already by their algebraic structure, black we didn't yet conclude.

Classification: quadratic 3D Abelian subalgebras in $\mathfrak{U}(\mathfrak{e}_3)$

(a) $X_1 = l_1^2 + l_2^2 + l_3^2 + a l_3 p_3 + b p_3^2, \quad X_2 = l_3^2,$

(b) $X_1 = l_1^2 + l_2^2 + l_3^2 + b(a p_2^2 + p_3^2), \quad X_2 = a l_2^2 + l_3^2 - ab p_1^2,$

(c) $X_1 = l_1^2 + l_2^2 + l_3^2 + 2b(l_1 p_1 - (3a-1) l_2 p_2 - 2 l_3 p_3) + 3b^2((1-4a)p_1^2 - (3a^2 - 2a - 1)p_2^2 + 2(a-1)p_3^2), \quad X_2 = a l_2^2 + l_3^2 + 6ab l_1 p_1 + 9ab^2(a p_3^2 + p_2^2),$

(d) $X_1 = l_3^2, \quad X_2 = \frac{1}{2}(l_1 p_2 + p_2 l_1 - l_2 p_1 - p_1 l_2) + a l_3 p_3,$

(e) $X_1 = l_3^2 + 2a(l_1 p_1 - l_2 p_2) + a^2 p_3^2, \quad X_2 = \frac{1}{2}(l_1 p_2 + p_2 l_1 - l_2 p_1 - p_1 l_2) - a p_1 p_2,$

(f) $X_1 = l_3^2 + a l_3 p_3 + b p_1^2 + c p_1 p_3 + d p_2 p_3, \quad X_2 = p_3^2,$

(g) $X_1 = l_3^2 + a p_3^2, \quad X_2 = l_3 p_3 + b p_3^2,$

(h) $X_1 = l_1 p_1 + a l_2 p_2 - (a+1) l_3 p_3 + b p_2^2, \quad X_2 = p_1^2 + \frac{2a+1}{a+2} p_2^2,$

(i) $X_1 = l_1 p_1 + a p_2^2 + b p_2 p_3, \quad X_2 = p_1^2,$

(j) $X_1 = l_1 p_1 + a l_2 p_2 - (a+1) l_3 p_3 + \frac{\omega}{2}(l_1 p_3 + p_3 l_1 - l_3 p_1 - p_1 l_3) + 2b p_1 p_2 + c\left(p_2^2 - p_3^2\right), X_2 = p_1^2 + \frac{6\omega}{4a-1} p_1 p_3 + \frac{a+2}{4a-1} p_2^2 - \frac{5a+1}{4a-1} p_3^2,$

(k) $X_1 = p_1^2 + a p_2^2, \quad X_2 = p_2^2 + b p_1 p_2 + c p_1 p_3 + d p_2 p_3.$

Ellipsoidal, paraboloidal and conical classes

In the rest of this talk let us focus on the classes (b) and (e), i.e. the integrals of ellipsoidal, conical (together forming class (b)) and paraboloidal type class (e) , cf. F. Hoque, AM and LŠ, J. Phys. A 57 (2024) 225201. We find it convenient rotate our reference frame to work with the integrals of the form **IX**, **X** and **XI** of Makarov et al.

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IX Conical $X_1 = l_1^2 + l_2^2 + l_3^2 + \ldots,$

$X_2 = b^2 l_2^2 + c^2 l_3^2 + \ldots, \quad c > b > 0.$

X Ellipsoidal

$X_1 = l_1^2 + l_2^2 + l_3^2 + (a^2 + b^2) p_1^2 + a^2 p_2^2 + b^2 p_3^2 + \ldots,$

$X_2 = b^2 l_2^2 + a^2 l_3^2 + a^2 b^2 p_1^2 + \ldots, b > a > 0.$

XI Paraboloidal

$X_1 = l_3^2 + a\,(l_1 p_2 + p_2 l_1) - b\,(l_2 p_1 + p_1 l_2) - abp_3^2 + \ldots,$

$X_2 = (l_1 p_2 + p_2 l_1 - l_2 p_1 - p_1 l_2) - a(p_2^2 + p_3^2) - b(p_1^2 + p_3^2) + \ldots, \quad b > a > 0.$

Ellipsoidal, paraboloidal and conical classes

Without magnetic fields these forms of the commuting integrals imply separation in the respective orthogonal coordinate system. We find it convenient to use these curvilinear coordinates also in the presence of magnetic field, which we interpret as a 2–form, i.e.

$$B = B^x(\vec{x})\mathrm{d}y \wedge \mathrm{d}z + B^y(\vec{x})\mathrm{d}z \wedge \mathrm{d}x + B^z(\vec{x})\mathrm{d}x \wedge \mathrm{d}y \quad (= \mathrm{d}A) \tag{3}$$

and similarly in any curvilinear coordinate system, e.g. in the ellipsoidal one

$$B = B^s(s,t,u)\mathrm{d}t \wedge \mathrm{d}u + B^t(s,t,u)\mathrm{d}u \wedge \mathrm{d}s + B^u(s,t,u)\mathrm{d}s \wedge \mathrm{d}t. \tag{4}$$

Ellipsoidal class

Let us start by considering the ellipsoidal case, i.e. **X**. The ellipsoidal coordinates are defined by

$$x^2 = \frac{(s-a^2)(t-a^2)(a^2-u)}{a^2(b^2-a^2)}, \quad y^2 = \frac{(s-b^2)(b^2-t)(b^2-u)}{b^2(b^2-a^2)}, \quad z^2 = \frac{stu}{a^2 b^2}, \tag{5}$$

where $s > b^2 > t > a^2 > u > 0$.

Ellipsoidal class

Let us start by considering the ellipsoidal case, i.e. **X**. The ellipsoidal coordinates are defined by

$$x^2 = \frac{(s-a^2)(t-a^2)(a^2-u)}{a^2(b^2-a^2)}, \quad y^2 = \frac{(s-b^2)(b^2-t)(b^2-u)}{b^2(b^2-a^2)}, \quad z^2 = \frac{stu}{a^2b^2}, \tag{5}$$

where $s > b^2 > t > a^2 > u > 0$.

In these coordinates, our Hamiltonian (1) reads

$$H = \frac{2s(s-a^2)(s-b^2)}{(s-t)(s-u)}\left(p_s^A\right)^2 + \frac{2t(t-a^2)(b^2-t)}{(s-t)(t-u)}\left(p_t^A\right)^2 + \frac{2u(a^2-u)(b^2-u)}{(s-u)(t-u)}\left(p_u^A\right)^2 + W(s,t,u) \tag{6}$$

Ellipsoidal class

and the integrals take the form

$$X_1 = \frac{4s(s-b^2)(s-a^2)(t+u)}{(s-t)(s-u)}(p_s^A)^2 + \frac{4t(b^2-t)(t-a^2)(s+u)}{(s-t)(t-u)}(p_t^A)^2 + \frac{4u(b^2-u)(a^2-u)(s+t)}{(t-u)(s-u)}(p_u^A)^2 + \sum_{\alpha=s,t,u} s_1^\alpha(s,t,u)p_\alpha^A + m_1(s,t,u), \tag{7}$$

$$X_2 = \frac{4stu(s-b^2)(s-a^2)}{(s-t)(s-u)}(p_s^A)^2 + \frac{4stu(b^2-t)(t-a^2)}{(s-t)(t-u)}(p_t^A)^2 + \frac{4stu(b^2-u)(a^2-u)}{(s-u)(t-u)}(p_u^A)^2 + \sum_{\alpha=s,t,u} s_2^\alpha(s,t,u)p_\alpha^A + m_2(s,t,u). \tag{8}$$

Ellipsoidal class

In order to classify all new integrable systems allowing integrals of motion of the form (7) and (8), we have to solve the involutivity conditions of the Hamiltonian and the integrals of motion, that is,

$$\{H, X_1\}_{P.B.} = \{H, X_2\}_{P.B.} = \{X_1, X_2\}_{P.B.} = 0, \tag{9}$$

expressed in the elliptic coordinates. Notice that the momenta p_x, p_y, p_z transform to p_s, p_t, p_u so that we have again Darboux coordinates, i.e.

$$\{F, G\}_{P.B.} = \sum_{\alpha=s,t,u} \frac{\partial F}{\partial \alpha}\frac{\partial G}{\partial p_\alpha} - \frac{\partial G}{\partial \alpha}\frac{\partial F}{\partial p_\alpha}. \tag{10}$$

We first consider the leading order, i.e. quadratic, determining equations coming from $\{H, X_1\}_{P.B.} = 0$:

Ellipsoidal class

$$
\begin{aligned}
\frac{\partial s_1^s}{\partial s} =& \frac{1}{2(s-b^2)(s-a^2)(s-u)(s-t)s}\Big((s^4-2(t+u)s^3+((t+u-b^2)a^2 \\
&+(t+u)b^2+3tu)s^2-2stu(a^2+b^2)+a^2b^2tu)s_1^s \\
&+(s-b^2)(s-a^2)s((s-u)s_1^t+(s-t)s_1^u)\Big)\,, \\
\frac{\partial s_1^s}{\partial t} =& \frac{(s-b^2)(s-a^2)s}{(s-u)(b^2-t)(t-a^2)t}\left((u-t)\frac{\partial s_1^t}{\partial s}-8t(b^2-t)(t-a^2)B^u\right)\,, \\
\frac{\partial s_1^s}{\partial u} =& \frac{(s-b^2)(s-a^2)s}{(t-s)(a^2-u)(b^2-u)u}\left((t-u)\frac{\partial s_1^u}{\partial s}-8u(a^2-u)(b^2-u)B^t\right)\,, \\
\frac{\partial s_1^t}{\partial t} =& -\frac{1}{2(b^2-t)(t-a^2)(t-u)(s-t)t}\Big((-t^4+2(s+u)t^3+((b^2-s-u)a^2 \\
&-(s+u)b^2-3us)t^2+2stu(a^2+b^2)-a^2b^2su)s_1^t-(t-a^2)(b^2-t)t((u-t)s_1^s \\
&+(s-t)s_1^u)\Big)\,, \\
\frac{\partial s_1^t}{\partial u} =& -\frac{(t-a^2)(b^2-t)t}{(s-t)(a^2-u)(b^2-u)u}\left((s-u)\frac{\partial s_1^u}{\partial t}+8u(a^2-u)(b^2-u)B^s\right)\,, \\
\frac{\partial s_1^u}{\partial u} =& \frac{1}{2(b^2-u)(a^2-u)(t-u)(s-u)u}\Big((u^4-2(s+t)u^3+((s+t-b^2)a^2 \\
&+(s+t)b^2+3st)u^2-2stu(a^2+b^2)+a^2b^2st)s_1^u-(u-a^2)(u-b^2)u((t-u)s_1^s \\
&+(s-u)s_1^t)\Big)
\end{aligned}
\tag{11}
$$

Ellipsoidal class

and similarly from the other two Poisson brackets. The general solution of all the leading order equations can be found, depends on three integration constants $s^t_{20}, s^u_{220}, s^u_{210}$ and reads

Ellipsoidal class

$$
\begin{aligned}
s_1^s(s,t,u) &= \frac{\sqrt{s(s-a^2)(s-b^2)}}{(s-t)(s-u)(t-u)\sqrt{t}} \left(s_{210}^u (s-t)\sqrt{tu(a^2-u)(b^2-u)} + s_{20}^t (s-u)t\sqrt{(t-a^2)(b^2-t)} \right), \\
s_1^t(s,t,u) &= \frac{\sqrt{t(t-a^2)(b^2-t)}}{(s-t)(s-u)(u-t)} \left(s_{220}^u (s-t)\sqrt{u(a^2-u)(b^2-u)} + s_{20}^t (t-u)\sqrt{s(s-a^2)(s-b^2)} \right), \\
s_1^u(s,t,u) &= \frac{\sqrt{u(a^2-u)(b^2-u)}}{(s-u)(t-u)(t-s)\sqrt{st}} \left(s_{210}^u (u-t)s\sqrt{t(s-a^2)(s-b^2)} + s_{220}^u (u-s)t\sqrt{s(t-a^2)(b^2-t)} \right), \\
s_2^s(s,t,u) &= \frac{\sqrt{ts(s-a^2)(s-b^2)}}{(s-t)(s-u)(t-u)} \left(s_{210}^u (s-t)\sqrt{tu(b^2-u)(a^2-u)} + s_{20}^t u(s-u)\sqrt{(t-a^2)(b^2-t)} \right), \\
s_2^t(s,t,u) &= \frac{\sqrt{t(t-a^2)(b^2-t)}}{(s-t)(s-u)(u-t)} \left(s_{220}^u s(s-t)\sqrt{u(b^2-u)(a^2-u)} + s_{20}^t u(t-u)\sqrt{s(s-a^2)(s-b^2)} \right), \\
s_2^u(s,t,u) &= \frac{\sqrt{stu(a^2-u)(b^2-u)}}{(s-t)(s-u)(u-t)} \left(s_{210}^u (u-t)\sqrt{t(s-a^2)(s-b^2)} + s_{220}^u (u-s)\sqrt{s(t-a^2)(b^2-t)} \right)
\end{aligned}
\tag{12}
$$

and similarly B^s, B^t, B^u determined as functions of the coordinates s, t, u and the parameters $s_{20}^t, s_{220}^u, s_{210}^u$.

Ellipsoidal class

Proceeding with lower order terms in the involutivity conditions one finds from the commutativity $\{X_1, X_2\}_{P.B.} = 0$ at the first order level the conditions

$$\begin{aligned}
-s_1^u \frac{\partial s_2^u}{\partial u} + s_2^t \frac{\partial s_1^u}{\partial t} + s_2^u \frac{\partial s_1^u}{\partial u} - s_1^t \frac{\partial s_2^u}{\partial t} - s_1^s \frac{\partial s_2^u}{\partial s} + s_2^s \frac{\partial s_1^u}{\partial s} &=0, \\
-s_1^s \frac{\partial s_2^s}{\partial s} + s_2^s \frac{\partial s_1^s}{\partial s} + s_2^t \frac{\partial s_1^s}{\partial t} + s_2^u \frac{\partial s_1^s}{\partial u} - s_1^t \frac{\partial s_2^s}{\partial t} - s_1^u \frac{\partial s_2^s}{\partial u} &=0, \\
-s_1^s \frac{\partial s_2^t}{\partial s} + s_2^s \frac{\partial s_1^t}{\partial s} + s_2^t \frac{\partial s_1^t}{\partial t} + s_2^u \frac{\partial s_1^t}{\partial u} - s_1^t \frac{\partial s_2^t}{\partial t} - s_1^u \frac{\partial s_2^t}{\partial u} &=0,
\end{aligned} \tag{13}$$

which depend only on the functions $s_1^\alpha(s,t,u)$ and $s_2^\alpha(s,t,u)$, $\alpha = s, t, u$. Plugging (12) into (13) we arrive at complicated expressions that must vanish for all values of s, t, u and do not involve any arbitrary functions.

Ellipsoidal class

Isolating functionally independent terms in these and solving for vanishing coefficients, we find a set of algebraic equations whose solution implies that two of the three integration constants in (12) must vanish and lead to three very similar, however due to different ranges on the coordinates s, t, u not equivalent, solutions:

(*i*) $s^u_{210} = 0, \quad s^u_{220} = 0;$

(*ii*) $s^t_{20} = 0, \quad s^u_{210} = 0;$

(*iii*) $s^t_{20} = 0, \quad s^u_{220} = 0.$

Proceeding with the remaining lower order determining equations in each case, we arrive at the general solution of our problem, i.e. at the list of all quadratically integrable Hamiltonian systems with vector potential with integrals of the ellipsoidal type.

Ellipsoidal class

In the absence of the magnetic field the assumption of the existence of the pair of commuting integrals (7) and (8) implies that potential must have the separable form

$$W(s,t,u) = \frac{(t-u)f(s) + (s-u)g(t) + (s-t)h(u)}{(s-t)(t-u)(s-u)}. \qquad (14)$$

In the presence of the magnetic field the assumption of the existence of the pair of commuting integrals (7) and (8) implies three possibilities, each having one of the arbitrary functions from (14) and one parameter (β below) determining the strength of the magnetic field.

Ellipsoidal class

$$
\begin{aligned}
B^s(s,t,u) &= \frac{(t-u)\left(s^4+(t+u-2a^2-2b^2)s^3+3(a^2b^2-tu)s^2+(2(a^2+b^2)tu-a^2b^2(t+u))s-a^2b^2tu\right)\beta}{16(s-u)^2(s-t)^2\sqrt{tu(t-a^2)(b^2-t)(b^2-u)(a^2-u)}},\\
B^t(s,t,u) &= -\frac{\sqrt{tu(t-a^2)(b^2-t)}}{8u\sqrt{(a^2-u)(b^2-u)}(s-t)^2}\beta,\\
B^u(s,t,u) &= -\frac{\sqrt{u(b^2-u)(a^2-u)}}{8(s-u)^2\sqrt{t(t-a^2)(b^2-t)}}\beta,\\
W(s,t,u) &= \frac{f(s)}{(s-u)(s-t)}+\frac{s(s-b^2)(s-a^2)}{32(s-u)^2(s-t)^2}\beta^2, \qquad (15)
\end{aligned}
$$

Ellipsoidal class

$$
\begin{aligned}
B^s(s,t,u) &= -\frac{\sqrt{su(s-b^2)(s-a^2)}}{8u\sqrt{(a^2-u)(b^2-u)}(s-t)^2}\beta, \\
B^t(s,t,u) &= \frac{(s-u)\left(-t^4+(2a^2+2b^2-s-u)t^3+3(su-a^2b^2)t^2+(a^2b^2(s+u)-2(a^2+b^2)su)t+a^2b^2su\right)\beta}{16(t-u)^2(s-t)^2\sqrt{su(s-a^2)(s-b^2)(b^2-u)(a^2-u)}}, \\
B^u(s,t,u) &= \frac{\sqrt{u(b^2-u)(a^2-u)}}{8(t-u)^2\sqrt{s(s-b^2)(s-a^2)}}\beta, \\
W(s,t,u) &= \frac{g(t)}{(s-t)(t-u)} - \frac{t(b^2-t)(a^2-t)}{32(s-t)^2(t-u)^2}\beta^2, \qquad (16)
\end{aligned}
$$

Ellipsoidal class

$$B^s(s,t,u) = \frac{\sqrt{s(s-a^2)(s-b^2)}}{8\sqrt{t(t-a^2)(b^2-t)}(s-u)^2}\beta,$$

$$B^t(s,t,u) = \frac{\sqrt{t(t-a^2)(b^2-t)}}{8\sqrt{s(s-a^2)(s-b^2)}(t-u)^2}\beta,$$

$$B^u(s,t,u) = \frac{(s-t)\left(u^4-(2a^2+2b^2-s-t)u^3-3(st-a^2b^2)u^2-(a^2b^2(s+t)-2(a^2+b^2)st)u-a^2b^2st\right)\beta}{16(t-u)^2(s-u)^2\sqrt{st(s-a^2)(s-b^2)(t-a^2)(b^2-t)}},$$

$$W(s,t,u) = \frac{h(u)}{(t-u)(s-u)} + \frac{u(b^2-u)(a^2-u)}{32(t-u)^2(s-u)^2}\beta^2. \quad (17)$$

Paraboloidal class

For the paraboloidal class **XI** the results are structurally very similar to the ellipsoidal case. Let us recall that the paraboloidal coordinates are defined by

$$x^2 = \frac{(\mu - a)(a - \nu)(\lambda - a)}{b - a}, \qquad y^2 = \frac{(\mu - b)(b - \nu)(b - \lambda)}{b - a},$$
$$z = \frac{1}{2}(\mu + \nu + \lambda - a - b), \tag{18}$$

where $\mu > b > \lambda > a > \nu > 0$.

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$$z = \frac{1}{2}(\mu + \nu + \lambda - a - b), \tag{18}$$

where $\mu > b > \lambda > a > \nu > 0$.

In these coordinates, the Hamiltonian (1) reads

$$H = \frac{2(\mu - a)(\mu - b)}{(\mu - \nu)(\mu - \lambda)} \left(p_\mu^A\right)^2 + \frac{2(\lambda - a)(b - \lambda)}{(\mu - \lambda)(\lambda - \nu)} \left(p_\lambda^A\right)^2 + \frac{2(a - \nu)(b - \nu)}{(\mu - \nu)(\lambda - \nu)} \left(p_\nu^A\right)^2 + W(\mu, \nu, \lambda), \tag{19}$$

Paraboloidal class

and the integrals take the form

$$X_1 = -\frac{4\nu\lambda(\mu-a)(\mu-b)}{(\mu-\nu)(\mu-\lambda)}\left(p_\mu^A\right)^2 - \frac{4\mu\nu(\lambda-a)(b-\lambda)}{(\mu-\lambda)(\lambda-\nu)}\left(p_\lambda^A\right)^2 - \tag{20}$$

$$-\frac{4\lambda\mu(a-\nu)(b-\nu)}{(\mu-\nu)(\lambda-\nu)}\left(p_\nu^A\right)^2 + \sum_{\alpha=\mu,\nu,\lambda} s_1^\alpha(\mu,\nu,\lambda)p_\alpha^A + m_1(\mu,\nu,\lambda),$$

$$X_2 = -\frac{4(\mu-a)(\mu-b)(\nu+\lambda)}{(\mu-\nu)(\mu-\lambda)}\left(p_\mu^A\right)^2 - \frac{4(\lambda-a)(b-\lambda)(\mu+\nu)}{(\mu-\lambda)(\lambda-\nu)}\left(p_\lambda^A\right)^2$$

$$-\frac{4(a-\nu)(b-\nu)(\mu+\lambda)}{(\mu-\nu)(\lambda-\nu)}\left(p_\nu^A\right)^2 + \sum_{\alpha=\mu,\nu,\lambda} s_2^\alpha(\mu,\nu,\lambda)p_\alpha^A + m_2(\mu,\nu,\lambda). \tag{21}$$

Paraboloidal class

In the absence of the magnetic field the assumption of the existence of the pair of commuting integrals (20) and (21) implies that potential must have the separable form

$$W(\mu,\nu,\lambda) = \frac{(\lambda-\nu)f(\mu) + (\mu-\lambda)g(\nu) + (\mu-\nu)h(\lambda)}{(\mu-\lambda)(\lambda-\nu)(\mu-\nu)}, \quad (22)$$

In the presence of the magnetic field the assumption of the existence of the pair of commuting integrals (20) and (21) again implies three possibilities, each having one of the arbitrary functions from (22) and one parameter (β below) determining the strength of the magnetic field.

Paraboloidal class

As they are all formally related by permutation of the coordinates (however not allowed by the parameter ranges), we shall present only one of them:

$$
\begin{aligned}
B^{\mu}(\mu,\nu,\lambda) = & -\frac{(\lambda-\nu)}{16(\mu-\lambda)^2(\mu-\nu)^2\sqrt{(\nu-a)(\nu-b)(b-\lambda)(\lambda-a)}}\left(2\mu^3-3(a+b)\mu^2\right. \\
& \left.+((4b+\nu+\lambda)a+(\nu+\lambda)b-2\lambda\nu)\mu+(-2(\nu+\lambda)b+\lambda\nu)a+b\nu\lambda\right)\beta, \\
B^{\nu}(\mu,\nu,\lambda) = & \frac{\sqrt{(a-\nu)(b-\nu)}}{8(\mu-\nu)^2\sqrt{(b-\lambda)(\lambda-a)}}\beta, \\
B^{\lambda}(\mu,\nu,\lambda) = & \frac{\sqrt{(b-\lambda)(\lambda-a)}}{8(\mu-\lambda)^2\sqrt{(b-\nu)(a-\nu)}}\beta, \\
W(\mu,\nu,\lambda) = & \frac{f(\mu)}{(\mu-\nu)(\mu-\lambda)}+\frac{(\mu-a)(\mu-b)}{32(\mu-\lambda)^2(\mu-\nu)^2}\beta^2.
\end{aligned}
\tag{23}
$$

Conical class

For the conical class **IX** the derivation is similar, however the results are structurally somewhat different. The conical coordinates are defined by

$$x = \frac{r}{b}\sqrt{\frac{(b^2-\theta^2)(b^2-\lambda^2)}{b^2-c^2}}, \quad y = \frac{r}{c}\sqrt{\frac{(\theta^2-c^2)(c^2-\lambda^2)}{b^2-c^2}}, \quad z = \frac{r\theta\lambda}{bc}, \qquad (24)$$

where $r > 0$, $b^2 > \theta^2 > c^2 > \lambda^2 > 0$, $b > c > 0$, and the Hamiltonian reads

$$H = \frac{1}{2}\left(p_r^A\right)^2 + \frac{(\theta^2-c^2)(b^2-\theta^2)}{2r^2(\theta^2-\lambda^2)}\left(p_\theta^A\right)^2 + \frac{(c^2-\lambda^2)(b^2-\lambda^2)}{2r^2(\theta^2-\lambda^2)}\left(p_\lambda^A\right)^2 + W(r,\theta,\lambda). \qquad (25)$$

Conical class

In the absence of the magnetic field the assumption of the existence of the conical pair of commuting integrals of the form IX implies that potential must have the separable form

$$W(r,\theta,\lambda) = f(r) + \frac{g(\theta) + h(\lambda)}{r^2(\theta^2 - \lambda^2)}, \tag{26}$$

In the presence of the magnetic field we find two possibilities structurally similar to the ellipsoidal and paraboloidal cases, with the function $g(\theta)$ and $h(\lambda)$, respectively, and one parameter determining the strength of the magnetic field, for example

Conical class

$$
\begin{aligned}
B^r(r,\theta,\lambda) &= \frac{\lambda}{r\sqrt{(b^2-\theta^2)(\theta^2-c^2)}}\beta, \\
B^\theta(r,\theta,\lambda) &= \frac{\theta\lambda\sqrt{(b^2-\theta^2)(\theta^2-c^2)}}{r^2(\theta^2-\lambda^2)^2}\beta, \\
B^\lambda(r,\theta,\lambda) &= -\frac{(3\lambda^4-2(b^2+c^2)\lambda^2)\theta^2+b^2c^2(\theta^2+\lambda^2)-\lambda^6)}{2r^2(\theta^2-\lambda^2)^2\sqrt{(b^2-\theta^2)(\theta^2-c^2)}}\beta, \\
W(r,\theta,\lambda) &= \frac{h(\lambda)}{2r^2(\theta^2-\lambda^2)} + \frac{\lambda^2(c^2-\lambda^2)(b^2-\lambda^2)}{8r^4(\theta^2-\lambda^2)^2}\beta^2,
\end{aligned}
\tag{27}
$$

Conical class

However, there is another possibility, for which only partial separation

$$W(r,\theta,\lambda) = w_1(r) + \frac{w_2(\theta,\lambda)}{r^2} \tag{28}$$

of the potential is accomplished, and the solution depends on the solution of one PDE coming from the leading order conditions, namely

$$\frac{\partial^2 S^{\theta}_{21}}{\partial\theta\partial\lambda} = \frac{1}{\lambda(\theta^2-\lambda^2)^2}\left((\theta^4-\lambda^4)\frac{\partial S^{\theta}_{21}}{\partial\theta} - \theta\lambda^2 S^{\theta}_{21}\right), \tag{29}$$

where $S^{\theta}_{21}(\theta,\lambda)$ is a yet to be determined part of the function $s^{\theta}_2(r,\theta,\lambda)$.

Conical class

Explicit examples of systems of this form were found by symmetry reduction of (29),

$$
\begin{aligned}
B^r(r,\theta,\lambda) =& \frac{\lambda^2-\theta^2}{2\sqrt{(b^2-\theta^2)(\theta^2-c^2)(c^2-\lambda^2)(b^2-\lambda^2)}}\kappa_4,\\
B^\theta(r,\theta,\lambda) =& 0, \quad B^\lambda(r,\theta,\lambda)=0, \quad W(r,\theta,\lambda)=w_1(r)
\end{aligned}
\tag{30}
$$

and

$$
\begin{aligned}
B^r(r,\theta,\lambda) =& -\frac{\left(b^2+c^2-\frac{3}{2}(\theta^2+\lambda^2)\right)(\theta^2-\lambda^2)}{\sqrt{(b^2-\theta^2)(\theta^2-c^2)(c^2-\lambda^2)(b^2-\lambda^2)}}\kappa_3,\\
B^\theta(r,\theta,\lambda) =& 0, \quad B^\lambda(r,\theta,\lambda)=0,\\
W(r,\theta,\lambda) =& w_1(r)+\frac{(\theta^2-\lambda^2)^2}{8r^2}(\kappa_3)^2-\frac{\omega}{r^2(\theta^2-\lambda^2)}.
\end{aligned}
\tag{31}
$$

Conclusions

We discussed integrable systems of ellipsoidal, paraboloidal and conical type. We have seen that for ellipsoidal and paraboloidal integrals, there are one–parameter families, each involving one of the arbitrary functions in the separated potential in the absence magnetic field.

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Conclusions

We discussed integrable systems of ellipsoidal, paraboloidal and conical type. We have seen that for ellipsoidal and paraboloidal integrals, there are one–parameter families, each involving one of the arbitrary functions in the separated potential in the absence magnetic field.

In the conical case the situation is more complicated and in one branch the computation did not lead to a full classification.

One of the open problems is how to rewrite the magnetic fields of these integrable systems in the Cartesian coordinates, as the inversion of the transformation between these curvilinear coordinates and the Cartesian ones is not known. We tried to guess the corresponding form of the magnetic field but without any success.

Conclusions

The above mentioned difficulty in expressing the resulting systems in the Cartesian coordinates also implies that search for hypothetical superintegrability of the constructed integrable systems is a very complicated task. Expressing the hypothetical additional integral X_3 of the general form in the respective curvilinear coordinates and computing the Poisson bracket $\{H, X_3\}_{P.B.}$, one arrives at very cumbersome conditions which seem to be rather impossible to analyze. At the same time, it is impossible to express these conditions in the Cartesian coordinates since the magnetic field and the potential are not known in those.

Conclusions

Another nontrivial question is whether the integrable systems introduced above can be interpreted also quantum mechanically. It is known that if the (totally symmetrized) integrals are interpreted as quantum mechanical operators on the configuration space, the determining equations may obtain quantum corrections.

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Another nontrivial question is whether the integrable systems introduced above can be interpreted also quantum mechanically. It is known that if the (totally symmetrized) integrals are interpreted as quantum mechanical operators on the configuration space, the determining equations may obtain quantum corrections. In the case of $[H, X_i] = 0$ these arise only at the 0th order level and modify the classical equations

$$\sum_{\alpha} s_i^{\alpha} \partial_{\alpha} W = 0 \tag{32}$$

by $\hbar^2$–proportional terms expressible in terms of the derivatives of the magnetic field and the leading order terms of the integrals. E.g., for the paraboloidal case we have quantum corrections:

$$\begin{aligned} X_1 : &\quad 2\hbar^2 \left(b\partial_x B^y - a\partial_y B^x - x\partial_y B^z + y\partial_x B^z \right), \\ X_2 : &\quad 2\hbar^2 \left(\partial_x B^y - \partial_y B^x \right). \end{aligned} \tag{33}$$

Conclusions

After transforming these corrections to the respective curvilinear coordinates and evaluating them for the classical solutions presented above we have found the following (preliminary) conclusions:

- in the paraboloidal case the quantum corrections to $\{H, X_i\} = 0$ vanish for all three solutions,
- in the conical case the quantum corrections to $\{H, X_i\} = 0$ vanish for all the explicit solutions, but we were not able to reach this conclusion by assuming the equation (29) only,
- in the ellipsoidal case the quantum corrections to $\{H, X_i\} = 0$ do not vanish for any of the three solutions.

Conclusions

However, there is still the involutivity condition $[X_1, X_2] = 0$ to consider. For it, the general form of its quantum corrections is not known. It seems that they may appear already at the level of first order conditions. Computing the determining equations, converting them to paraboloidal coordinates and inserting into them the solution (23) it seems (after a long computation in Maple, relying on its capabilities in manipulating huge expressions and derivatives in various coordinate systems) that the quantum integrals X_1 and X_2 corresponding to the system (23) do not commute thus the system is not quantum integrable in the Liouville sense. We find this result unexpected and we are working on its confirmation / refutation.

Thank you for your attention!