Symmetries and invariant solutions of PDEs on superspace

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Outline

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1. How do we find point symmetries of PDEs?
2. What are the symmetries good for – invariant solutions
3. Generalization of the method to equations on superspace
4. Conclusions

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Transformation of functions and prolongations

P. J. Olver, *Applications of Lie Groups to Differential Equations* (Springer–Verlag, New York, 1986).

Assume that an open neighborhood $U \subset R^n$ with coordinates x^i is given. Consider the graph of a given smooth function $f : U \to \mathbb{R}$ as a section of the (trivial) fiber bundle $\mathcal{J}^{(0)} = U \times \mathbb{R}$, $\sigma_f(\vec{x}) = (\vec{x}, f(\vec{x}))$. It naturally induces a section of the jet bundle, e.g. for the 2nd order jet bundle $\mathcal{J}^{(2)} = U \times \mathbb{R} \oplus \mathbb{R}^n \oplus \mathbb{R}^{n(n+1)/2}$

$$\sigma_f^{(2)}(\vec{x}) = (\,\vec{x}, f(\vec{x}), \partial_i f(\vec{x}), \partial_{ij} f(\vec{x})\,)$$

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Let u be the coordinate on $\mathbb{R}$, together with u_i and u_{ij} defining the coordinates on the fiber of $\mathcal{J}^{(2)}$.

Let[1] $\mathbf{v} = \xi^i(\vec{x}, u)\partial_i + \mathcal{U}(\vec{x}, u)\partial_u$ be the generator of a one–parametric group of transformations of $\mathcal{J}^{(0)}$. Assume that the graph of f and consequently the section σ_f is transformed by the flow of $\mathbf{v}$, defining a new function f_τ for each value of the flow parameter τ provided $|\tau|$ is small enough. Consider its prolongation $\sigma^{(2)}_{f_\tau}$. Is it generated from $\sigma^{(2)}_f$ by the flow of some vector field on $\mathcal{J}^{(2)}$?

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$$\mathrm{pr}^{(2)}(\mathbf{v}) = \xi^i \partial_i + \mathcal{U}\partial_u + \mathcal{U}_i \partial_{u_i} + \mathcal{U}_{ij}\partial_{u_{ij}},$$

where

$$\mathcal{U}_i = \mathcal{D}_i \mathcal{U} - \sum_j \mathcal{D}_i \xi^j u_j, \qquad \mathcal{U}_{ij} = \mathcal{D}_j \mathcal{U}_i - \sum_k \mathcal{D}_j \xi^k u_{ik}, \quad (1)$$

and $\mathcal{D}_i$ is the operator of the total derivative on the jet space

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When does the vector field $\mathbf{v} = \xi^i \partial_i + \mathcal{U}\partial_u$ generate a one–parametric group of symmetries of a given K–th order PDE

$$F(\vec{x}, f(\vec{x}), \partial_i f(\vec{x}), \partial_{i_1 i_2} f(\vec{x}), \ldots, \partial_{i_1 i_2 \ldots i_K} f(\vec{x})) = 0 \text{ ?} \qquad (2)$$

In other words start with an arbitrary solution f of PDE (2). When do the functions f_τ solve the same PDE (2), for any choice of f?

Provided that $\operatorname{grad} F|_{F=0} \neq 0$ on $\mathcal{J}^{(K)}$ there is an equivalence

$\mathbf{v} = \xi^i \partial_i + \mathcal{U}\partial_u$ is a symmetry generator of PDE (2) if and only if

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Computation of symmetry generators of PDEs in practice

1. For the given K–order PDE $F = 0$ find the prolongation of order K of an arbitrary vector field $\mathbf{v}$ on $\mathcal{J}^{(0)}$.
2. Solve $F(\vec{x}, u, u_i, u_{ij}, \ldots) = 0$ for a suitable "derivative" $u_{AB\ldots}$ and substitute for it and all its differential consequences, e.g. $\mathcal{D}_i u_{AB\ldots}$, into

$$\left(\mathrm{pr}^{(K)}(\mathbf{v})F\right)\left(\vec{x}, u, u_i, u_{ij}, \ldots\right) = 0.$$

3. The resulting expression is an equation for unknown functions $\xi^i(x^j, u), \mathcal{U}(x^j, u)$ which must hold for any value of the remaining jet coordinates $u_i, u_{ij}, \ldots$. This gives an overdetermined system of linear PDEs for $\xi^i, \mathcal{U}$. If it can be solved we find all symmetry generators of the given PDE $F = 0$.

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Construction of invariant solutions of PDEs

The knowledge of a 1–parametric group of symmetries of given PDE (such that its orbits have dimension one in the space of independent variables) generated by **v** allows reduction of the number of independent variables.

It works as follows: one finds the invariants I_k: $\mathbf{v}(I_k) = 0$, $k = 1, .., n$, of the action of the group on $\mathcal{J}^{(0)}$, and constructs the coordinates on $\mathcal{J}^{(0)}$ out of them and one of the original variables, say ω, functionally independent of I_k's. One of the invariants is chosen as the new dependent variable $\tilde{u} \equiv I_n$.

Once the PDE is expressed in these new dependent and independent variables, one assumes that its solution is invariant with respect to the action of the group, i.e. $\tilde{u}$ depends on $I_1, \ldots, I_{n-1}$ but not on ω.

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The symmetry of the equation guarantees that such reduced equation is consistent, i.e. ω drops out of it, and we obtain a PDE with one less independent variables. Repeating this procedure one is able to reduce PDE to ODE provided a suitable symmetry group is present at each step.

Of course, this procedure allows to find only special solutions of the original PDE, namely those invariant with respect to some 1–parametric symmetry group. But for nonlinear PDEs it is one of the few known methods giving at least some nontrivial solutions.

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Can the same method be applied to supersymmetric equations?

Consider some supersymmetric model formulated in terms of a superfield on superspace, e.g the supersymmetric sine–Gordon equation (SSG)

$$D_1 D_2 \Phi = \sin \Phi \qquad (3)$$

for a real bosonic superfield

$$\Phi(x_1, x_2, \theta_1, \theta_2) = \frac{u(x_1, x_2)}{2} + \theta_1 \phi(x_1, x_2) + \theta_2 \psi(x_1, x_2) + \theta_1 \theta_2 F(x_1, x_2).$$

The covariant derivative operators in Eq. (3) are

$$D_1 = \partial_{\theta_1} + \theta_1 \partial_{x_1} \qquad \text{and} \qquad D_2 = \partial_{\theta_2} + \theta_2 \partial_{x_2}.$$

The quantities x_1, x_2, ϕ, F have Grassmann–even, commuting character, $\theta_1, \theta_2, \phi, \psi$ are Grassmann–odd, anticommuting.

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Supersymmetry transformations

SSG (3) is invariant under the supersymmetry transformations

$$x \to x - \underline{\eta}_1\theta_1,\ \theta_1 \to \theta_1 + \underline{\eta}_1,\ t \to t - \underline{\eta}_2\theta_2,\ \theta_2 \to \theta_2 + \underline{\eta}_2,$$

where $\underline{\eta_1}$ and $\underline{\eta_2}$ are arbitrary constant fermionic parameters.

These transformations are generated by the infinitesimal supersymmetry generators

$$Q_1 = \partial_{\theta_1} - \theta_1\partial_{x_1} \qquad \text{and} \qquad Q_2 = \partial_{\theta_2} - \theta_2\partial_{x_2}, \tag{4}$$

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The form of the generator in the superfield approach

Explicitly, the SSG (3) reads

$$\theta_1\theta_2\Phi_{x_1x_2} - \theta_2\Phi_{x_2\theta_1} + \theta_1\Phi_{x_1\theta_2} - \Phi_{\theta_1\theta_2} = \sin\Phi, \tag{5}$$

where each successive subscript (from left to right) indicates a successive left partial derivative.

We use the generalized method of prolongations so as to include also the fermionic variables (introduced in M. A. Ayari and V. Hussin, Comput. Phys. Commun. **100** (1997) 157). We write

$$\mathbf{v} = \xi\partial_{x_1} + \tau\partial_{x_2} + \rho\partial_{\theta_1} + \sigma\partial_{\theta_2} + \Lambda\partial_{\Phi}, \tag{6}$$

where ξ, τ and Λ are supposed to be even–valued functions of $(x_1, x_2, \theta_1, \theta_2, \Phi)$, while ρ and σ are odd–valued functions.

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We need the fermionic analogues $\mathcal{D}_{\theta_1}, \mathcal{D}_{\theta_2}$ of the bosonic total derivatives $\mathcal{D}_{x_1}, \mathcal{D}_{x_2}$, e.g.

$$\begin{aligned}\mathcal{D}_{\theta_1} =& \partial_{\theta_1} + \Phi_{\theta_1}\partial_{\Phi} + \Phi_{x_1\theta_1}\partial_{\Phi_{x_1}} + \Phi_{x_2\theta_1}\partial_{\Phi_{x_2}} + \Phi_{\theta_2\theta_1}\partial_{\Phi_{\theta_2}} + \\ &+ \Phi_{x_1x_1\theta_1}\partial_{\Phi_{x_1x_1}} + \Phi_{x_1x_2\theta_1}\partial_{\Phi_{x_1x_2}} + \Phi_{x_1\theta_2\theta_1}\partial_{\Phi_{x_1\theta_2}} + \\ &+ \Phi_{x_2x_2\theta_1}\partial_{\Phi_{x_2x_2}} + \Phi_{x_2\theta_2\theta_1}\partial_{\Phi_{x_2\theta_2}},\end{aligned} \tag{7}$$

We note that due to the use of left derivatives the chain rule for a Grassmann–valued function $f(g(x))$ is

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial x} \cdot \frac{\partial f}{\partial g}$$

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The 2nd prolongation

Similarly as in the classical case, we derive the prolongation formulae. With proper respect for ordering they read

$$\begin{aligned} \mathrm{pr}^{(2)}\mathbf{v} =& \xi\partial_{x_1} + \tau\partial_{x_2} + \rho\partial_{\theta_1} + \sigma\partial_{\theta_2} + \Lambda\partial_{\Phi} + \Lambda_{x_1}\partial_{\Phi_{x_1}} + \\ & + \Lambda_{x_2}\partial_{\Phi_{x_2}} + \Lambda_{\theta_1}\partial_{\Phi_{\theta_1}} + \Lambda_{\theta_2}\partial_{\Phi_{\theta_2}} + \Lambda_{x_1x_1}\partial_{\Phi_{x_1x_1}} + \\ & + \Lambda_{x_1x_2}\partial_{\Phi_{x_1x_2}} + \Lambda_{x_1\theta_1}\partial_{\Phi_{x_1\theta_1}} + \Lambda_{x_1\theta_2}\partial_{\Phi_{x_1\theta_2}} + \Lambda_{x_2x_2}\partial_{\Phi_{x_2x_2}} + \\ & + \Lambda_{x_2\theta_1}\partial_{\Phi_{x_2\theta_1}} + \Lambda_{x_2\theta_2}\partial_{\Phi_{x_2\theta_2}} + \Lambda_{\theta_1\theta_2}\partial_{\Phi_{\theta_1\theta_2}} \end{aligned} \tag{8}$$

where the coefficients are defined by

$$\Lambda_A = \mathcal{D}_A\Lambda - \sum_B \mathcal{D}_A\zeta^B\Phi_B, \quad \Lambda_{AB} = \mathcal{D}_B\Lambda_A - \sum_C \mathcal{D}_B\zeta^C\Phi_{AC}, \tag{9}$$

and $A, B, C \in \{x_1, x_2, \theta_1, \theta_2\}$, $\zeta^A = (\xi, \tau, \rho, \sigma)$.

Applying the second prolongation (8) to the SSG equation (5), we obtain the following condition

$$\rho\left(\theta_2\Phi_{x_1x_2}+\Phi_{x_1\theta_2}\right)-\sigma\left(\theta_1\Phi_{x_1x_2}+\Phi_{x_2\theta_1}\right)-\Lambda\cos\Phi \\ +\Lambda_{x_1x_2}\,\theta_1\theta_2+\Lambda_{x_2\theta_1}\,\theta_2-\Lambda_{x_1\theta_2}\,\theta_1-\Lambda_{\theta_1\theta_2}=0. \tag{10}$$

Next, we substitute the SSG equation into (10), i.e. eliminate $\Phi_{\theta_1\theta_2}$, expand components of $\mathbf{v}$ into polynomials in θ_1, θ_2, and proceed as before, carefully keeping track of the ordering.

We find the full super–Poincaré algebra in $(1+1)$ dimensions, spanned by the generators

$$L=-2x\partial_{x_1}+2t\partial_{x_2}-\theta_1\partial_{\theta_1}+\theta_2\partial_{\theta_2},\quad P_1=\partial_{x_1},\quad P_2=\partial_{x_2}, \\ Q_1=-\theta_1\partial_{x_1}+\partial_{\theta_1},\qquad Q_2=-\theta_2\partial_{x_2}+\partial_{\theta_2}. \tag{11}$$

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Next, we substitute the SSG equation into (10), i.e. eliminate $\Phi_{\theta_1\theta_2}$, expand components of $\mathbf{v}$ into polynomials in θ_1, θ_2, and proceed as before, carefully keeping track of the ordering.

We find the full super–Poincaré algebra in $(1+1)$ dimensions, spanned by the generators

$$L=-2x\partial_{x_1}+2t\partial_{x_2}-\theta_1\partial_{\theta_1}+\theta_2\partial_{\theta_2}, \quad P_1=\partial_{x_1}, \quad P_2=\partial_{x_2}, \\ Q_1=-\theta_1\partial_{x_1}+\partial_{\theta_1}, \qquad Q_2=-\theta_2\partial_{x_2}+\partial_{\theta_2}. \tag{11}$$

Invariant solutions of SSG

- It is possible to reduce SSG without any difficulty to a system of ODEs when the 1–parametric subgroup is constructed out of bosonic generators L, P_1, P_2. Whether or not at least particular nontrivial solutions of these ODEs and the corresponding invariant solutions of SSG can be found explicitly depends on the chosen subgroup.
- When fermionic generators are included, problems may (not always) arise.

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Example

Consider the transformations generated by $\underline{\mu}Q_1$. The invariants are t, θ_2, Φ and any quantity of the form

$$\tau = \underline{\mu}f\left(x_1, x_2, \theta_1, \theta_2, \Phi\right).$$

Obviously, we cannot find adapted coordinates on the superspace in which $\underline{\mu}Q_1$ would become $\partial_{\tilde{x}}$ and consequently we do not obtain a reduced equation expressible in terms of the invariants only.

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Practical implementation of the method

- Determination of the symmetry algebra by hand is rather tedious even in the commuting case. In that case, computer programs implementing the algorithm exist. E.g. standard PDEtools package in Maple.
- In the superspace the standard procedures usually break down but the prolongations and the resulting determining equations can be constructed using e.g. the standard Physics package in Maple, which allows to perform algebraic manipulations and calculus with anticommuting variables.
- The resulting overdetermined system of linear PDEs for the symmetry generators we must solve by hand, using computer for algebraic manipulations. I don't know any solver of PDEs involving anticommuting functions.

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Conclusions

- With proper care it is possible to extend the conventional method of determination of symmetry algebra of PDEs to superspace.
- The supersymmetry then demonstrates itself as a point symmetry.
- In the case of the super–sine–Gordon equation no hidden, unexpected symmetries were found.
- In the symmetry reduction on superspace problems may arise due to nilpotency of some invariants.

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Thank you for your attention