Poisson–Lie T–plurality as Canonical Transformation

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Elements of Poisson–Lie T–duality of σ–models

The σ–models (without spectators) are given by the action

$$S[g] = \frac{1}{2}\int d^2x\, L_+(g)\cdot E(g)\cdot L_-^t(g) = \frac{1}{2}\int d^2x\, \partial_+\phi^\mu \mathcal{E}_{\mu\nu}(\phi)\partial_-\phi^\nu \tag{1}$$

where the map g maps $V \subset \mathbb{R}^2$ into the group G whose Lie algebra has basis $\{T_a\}$,

$$L_\pm(g)^a := (g^{-1}\partial_\pm g)^a, \quad g^{-1}\partial_\pm g = L_\pm(g)\cdot T$$

$\phi^\mu : V \subset \mathbb{R}^2 \to \mathbb{R}^{\dim G}$ is the same map as g but written in some group coordinates,

$$L_\pm(g) = \partial_\pm\phi \cdot e^L \cdot T$$

and x_+, x_- are the light–cone coordinates on Minkowski $\mathbb{R}^2$ $\tau = x_+ + x_-$, $\sigma = x_+ - x_-$.

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The basic idea of Poisson–Lie T–duality

C. Klimčík and P. Ševera, Phys. Lett. B 351 (1995) 455.

Under certain conditions the equations of motion of the σ–model can be written as equations on

Drinfel'd double

$(G|\tilde{G})$ – Lie group D whose Lie algebra $\mathfrak{d}$ admits a decomposition $\mathfrak{d} = \mathfrak{g} \dot{+} \tilde{\mathfrak{g}}$ into a pair of subalgebras maximally isotropic with respect to a symmetric ad-invariant nondegenerate bilinear form $\langle\, .\,,.\, \rangle$.

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The matrices $E(g)$ for the dualizable σ–models are of the form

$$E(g) = [E_0^{-1} + \Pi(g)]^{-1}$$

where E_0 is a constant matrix,

$$\Pi(g) = b^t(g) \cdot a(g) = -\Pi(g)^t,$$

and $a(g), b(g)$ are submatrices of the adjoint representation of the subgroup G on the Lie algebra of the Drinfel'd double D.

The dual model

is obtained by the interchange

$$G \leftrightarrow \tilde{G}, \qquad \mathfrak{g} \leftrightarrow \tilde{\mathfrak{g}}, \qquad \Pi(g) \leftrightarrow \tilde{\Pi}(\tilde{g}), \qquad E_0 \leftrightarrow E_0^{-1}.$$

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The equations of motion of the dualizable σ–model can be formulated as the equations on the Drinfel'd double

$$\langle\, l^{-1}\partial_{\pm} l \,, \mathcal{E}^{\mp}\, \rangle = 0,$$

where $l = \tilde{h}g \in D$, $\tilde{h} \in \tilde{G}$, $g \in G$ and

$$\mathcal{E}^{+} = \mathrm{span}\left(T + E_0 \cdot \tilde{T}\right), \qquad \mathcal{E}^{-} = \mathrm{span}\left(T - E_0^t \cdot \tilde{T}\right)$$

are two orthogonal subspaces in $\mathfrak{d}$. (The unique decomposition $l = \tilde{h}g$ on D exists for a general Drinfel'd double only in the vicinity of the group unit. For the so–called perfect Drinfel'd double s it is defined globally and we shall consider only these.)

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R. von Unge, J. High En. Phys. 02:07 (2002) 014.

Main idea:

In general there are several decompositions (Manin triples) of a Drinfel'd double.

Let $\hat{\mathfrak{g}} \dot{+} \bar{\mathfrak{g}}$ be another decomposition of the Lie algebra $\mathfrak{d}$ into maximal isotropic subalgebras.The dual bases of $\mathfrak{g}, \tilde{\mathfrak{g}}$ and $\hat{\mathfrak{g}}, \bar{\mathfrak{g}}$ are related by the linear transformation

$$\begin{pmatrix} T \\ \tilde{T} \end{pmatrix} = \begin{pmatrix} K & Q \\ R & S \end{pmatrix} \begin{pmatrix} \hat{T} \\ \bar{T} \end{pmatrix}, \qquad (2)$$

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The duality of both bases (i.e. $\langle T_a, \tilde{T}^b \rangle = \delta_a^b$) requires

$$\begin{pmatrix} K & Q \\ R & S \end{pmatrix}^{-1} = \begin{pmatrix} S^t & Q^t \\ R^t & K^t \end{pmatrix}.$$

Besides that, the matrices K, Q, R, S are chosen in such a way that the structure constants of the Lie algebra $\mathfrak{d}$

$$\begin{aligned} [T_a, T_b] &= f_{ab}{}^c T_c, \\ [\tilde{T}^a, \tilde{T}^b] &= \tilde{f}^{ab}{}_c \tilde{T}^c, \\ [\tilde{T}^a, T_b] &= f_{bc}{}^a \tilde{T}^c - \tilde{f}^{ac}{}_b T_c \end{aligned}$$

transform to similar ones where the structure constants $f, \tilde{f}$ of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ are replaced by the structure constants $\hat{f}, \bar{f}$ of $\hat{\mathfrak{g}}$ and $\bar{\mathfrak{g}}$ and $T \to \hat{T},\ \tilde{T} \to \bar{T}$.

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The σ–model related by the Poisson–Lie T–plurality

to (1) is defined analogously to (1) but with

$$\begin{aligned}
\widehat{E}(\hat{g}) &= (\widehat{E}_0^{-1} + \widehat{\Pi}(\hat{g}))^{-1}, \\
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Relation between the classical solutions of the two σ–models is obtained from two possible decompositions of $l \in D$

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T–plurality transformation of extremal left–invariant fields

We derive the formulae for transformation of left–invariant fields evaluated on solutions of equations of motion (hence extremal).

This will enable us to get

- Transformation of boundary conditions for the classical solutions of the σ–models (a generalization of Cecilia Albertsson, Ronald A. Reid-Edwards, [hep-th/0606024]).
- Transformation of canonical variables of the σ–models.

We write the left–invariant field $l^{-1}\partial_+ l$ on the Drinfel'd double

$$\begin{aligned} l^{-1}\partial_+ l &= (\tilde{h}g)^{-1}(\partial_+(\tilde{h}g)) = L_+(g)\cdot T + \tilde{L}_+(\tilde{h})\cdot g^{-1}\tilde{T}g \\ &= L_+(g)\cdot T + \tilde{L}_+(\tilde{h})\cdot \left[b(g)\cdot T + a^{-t}(g)\cdot \tilde{T}\right] \end{aligned}$$

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and the definition of $E(g)$ we get

$$l^{-1}\partial_+ l = L_+(g) \cdot E(g) \cdot \left[E_0^{-1} \cdot T + \tilde{T}\right]. \qquad (3)$$

Similarly, from the decomposition $l = \bar{h}\hat{g}$

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the formula for transformation of the left–invariant fields under the Poisson–Lie T–plurality

$$\widehat{L}_+(\hat{g}) = L_+(g) \cdot E(g) \cdot \left[S + E_0^{-1} \cdot Q\right] \cdot \widehat{E}^{-1}(\hat{g}) \qquad (5)$$

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Transformation of canonical variables

The canonical momentum is found from the action (1)

$$\mathcal{P}_\mu = \frac{\partial \mathscr{L}}{\partial(\partial_\tau \phi^\mu)} = \frac{1}{2}\left(\mathcal{E}_{\mu\nu}\partial_- \phi^\nu + \mathcal{E}_{\nu\mu}\partial_+ \phi^\nu\right). \tag{6}$$

We shall use the momentum in local frame $\mathcal{P}_a = {v^L}_a{}^\mu(g)\mathcal{P}_\mu$, where $v^L = (e^L)^{-1}$ and write it as the column vector $\mathcal{P}$

$$\mathcal{P} = \frac{1}{2}\left(E(g) \cdot L_-^t(g) + E^t(g) \cdot L_+^t(g)\right).$$

We also define

$$L_\sigma = \frac{1}{2}\left(L_+(g) - L_-(g)\right).$$

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Using the similar expressions for the canonical variables after T–plurality transformation and the formulae for transformation of $L_\pm(g)$ (5) we find the generalization of the transformation found in T–duality case in K. Sfetsos, Nucl. Phys. B (Proc. Suppl.) 56 (1997) 302 and K. Sfetsos, Nucl.Phys. B 517 (1998) 549-566 to the T–plurality case, i.e.

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$$\begin{aligned} \widehat{\mathcal{P}} &= (Q^t \cdot \Pi(g) + S^t) \cdot \mathcal{P} + Q^t \cdot L_\sigma^t, \\ \widehat{L}_\sigma &= \mathcal{P}^t \cdot \Big[(S - \Pi(g) \cdot Q) \cdot \widehat{\Pi}(\hat{g}) + R - \Pi(g) \cdot K \Big] \\ &\quad + L_\sigma \cdot \Big(Q \cdot \widehat{\Pi}(\hat{g}) + K \Big) . \end{aligned} \tag{7}$$

Poisson–Lie T–plurality as canonical transformation

In order to show that (7) is really a canonical transformation we shall use the expressions for Poisson brackets of $\mathcal{P}_a$ and

$$\mathcal{J}^a = L^a_\sigma + \Pi(g)^{ab}\mathcal{P}_b, \qquad \mathcal{J} = L^t_\sigma + \Pi(g)\cdot\mathcal{P} \quad (8)$$

$$\{\mathcal{J}^a, \mathcal{J}^b\} = \tilde{f}^{ab}{}_c \mathcal{J}^c \delta(\sigma-\sigma'),$$
$$\{\mathcal{P}_a, \mathcal{P}_b\} = f_{ab}{}^c \mathcal{P}_c \delta(\sigma-\sigma'),$$
$$\{\mathcal{J}^a, \mathcal{P}_b\} = (f_{bc}{}^a \mathcal{J}^c - \tilde{f}^{ac}{}_b \mathcal{P}_c)\delta(\sigma-\sigma') + \delta^a_b \delta'(\sigma-\sigma').$$

These Poisson brackets are equivalent to the canonical ones

$$\{\mathcal{P}_\mu, \mathcal{P}_\nu\} = \{\partial_\sigma\phi^\mu, \partial_\sigma\phi^\nu\} = 0,$$
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We find that the Poisson brackets can be written in the compact form

$$\{\mathcal{Y}_\alpha, \mathcal{Y}_\beta\} = \mathcal{F}_{\alpha\beta}{}^\gamma \mathcal{Y}_\gamma \delta(\sigma - \sigma') + \mathcal{B}_{\alpha\beta}\delta'(\sigma - \sigma') \tag{10}$$

where $\alpha, \beta, \gamma = 1, \ldots, \dim \mathfrak{d}$,

$$\mathcal{Y} = \begin{pmatrix} \mathcal{P} \\ \mathcal{J} \end{pmatrix},$$

$\mathcal{F}_{\alpha\beta}{}^\gamma$ are structure constants of the Drinfel'd double and $\mathcal{B}_{\alpha\beta}$ are matrix elements of the bilinear form $\langle \,.\,, . \rangle$ in the basis T_a, $\tilde{T}^a$ of $\mathfrak{d}$.

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We rewrite the T–plurality transformation as

$$\begin{aligned} \widehat{\mathcal{P}} &= S^t \cdot \mathcal{P} + Q^t \cdot \mathcal{J}, \\ \widehat{\mathcal{J}} &= R^t \cdot \mathcal{P} + K^t \cdot \mathcal{J} \end{aligned} \tag{11}$$

which reminds of the transformation of the basis elements of the Drinfel'd double

$$\begin{pmatrix} \hat{T} \\ \bar{T} \end{pmatrix} = \begin{pmatrix} S^t & Q^t \\ R^t & K^t \end{pmatrix} \begin{pmatrix} T \\ \tilde{T} \end{pmatrix}.$$

Consequently, Poisson brackets (10) are form–invariant under the transformation (11). Therefore, the canonical Poisson brackets are invariant, i.e. (9) is transformed by Poisson–Lie T–plurality to

$$\begin{aligned} \{\widehat{\mathcal{P}}_\mu, \widehat{\mathcal{P}}_\nu\} &= \{\partial_\sigma \widehat{\phi}^\mu, \partial_\sigma \widehat{\phi}^\nu\} = 0, \\ \{\partial_\sigma \widehat{\phi}^\mu, \widehat{\mathcal{P}}_\nu\} &= \delta^\mu_\nu \delta'(\sigma - \sigma'). \end{aligned}$$

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The preservation of the Hamiltonian density

Finally, we compute the Hamiltonian density

$$\begin{aligned}\mathscr{H} &= \partial_\tau \phi^\mu \mathcal{P}_\mu - \mathscr{L} \\ &= \frac{1}{4}\left(L_-(g)\cdot E(g)\cdot L_-^t(g) + L_+(g)\cdot E(g)\cdot L_+^t(g)\right)\end{aligned}$$

where $L_\pm(g)$ are expressed in terms of $\mathcal{P}, L_\sigma$. Similarly,

$$\widehat{\mathscr{H}} = \frac{1}{4}\left(\widehat{L}_-(\hat{g})\cdot \widehat{E}(\hat{g})\cdot \widehat{L}_-^t(\hat{g}) + \widehat{L}_+(\hat{g})\cdot \widehat{E}(\hat{g})\cdot \widehat{L}_+^t(\hat{g})\right).$$

Using the transformation of $L_\pm(g)$ we find that the Hamiltonian density is preserved under Poisson–Lie T–plurality transformation, i.e. $\widehat{\mathscr{H}} = \mathscr{H}$.

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Thank you for you attention