Solvable Lie algebras with Borel nilradicals

Libor Šnobl, in collaboration with P. Winternitz

Department of Physics
Faculty of Nuclear Sciences and Physical Engineering
Czech Technical University in Prague

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Outline

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Classification of Lie algebras in general

Three classes of Lie algebras exist, namely

- semisimple, i.e. $\mathfrak{g}$ such that it has no nonvanishing commuting ideals

$$[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h},\ [\mathfrak{h}, \mathfrak{h}] = 0 \quad \Rightarrow \quad \mathfrak{h} = 0.$$

  All semisimple Lie algebras are well–known.

- solvable, i.e. $\mathfrak{g}$ such that the sequence of ideals $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}], \ldots, \mathfrak{g}^{(k+1)} = [\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}]$ vanishes for all $k > K$ for some $K \in \mathbb{N}$.
- Levi decomposable, i.e. semidirect sum of the form

$$\mathfrak{g} = \mathfrak{l} \ni \mathfrak{r},\ [\mathfrak{l}, \mathfrak{l}] = \mathfrak{l},\ [\mathfrak{r}, \mathfrak{r}] \subsetneq \mathfrak{r}, [\mathfrak{l}, \mathfrak{r}] \subset \mathfrak{r}, \qquad (1)$$

  where $\mathfrak{l}$ is a semisimple subalgebra and $\mathfrak{r}$ is the radical of $\mathfrak{g}$, i.e. its maximal solvable ideal.

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Levi decomposable algebras

We note that by virtue of Jacobi identities $\mathfrak{r}$ is a representation space for $\mathfrak{l}$ and that $\mathfrak{l}$ is isomorphic to a subalgebra of the algebra of all derivations of $\mathfrak{r}$.

These observations put a rather stringent compatibility conditions on possible pairs of $\mathfrak{l}, \mathfrak{r}$ and can be employed in the construction of Levi decomposable algebras out of the classifications of semisimple and solvable ones. E.g. many solvable algebras do not have any semisimple subalgebra of derivations and hence cannot appear as a radical in a nontrivial Levi decomposition (see J. Phys. A: Math. Theor. 43 (2010) 505202).

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Remark: Indecomposable vs. decomposable Lie algebras

Lie algebra $\mathfrak{g}$ is decomposable if it can be written as a direct sum of ideals. Such algebras should be explicitly decomposed into components that are further indecomposable

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \ldots \oplus \mathfrak{g}_k, \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \delta_{ij}\mathfrak{g}_i. \tag{2}$$

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Classification of solvable Lie algebras

No complete classification of solvable Lie algebras exists. There are two approaches to their partial classification: by dimension, or by structure.

The dimensional approach for real Lie algebras:

- dimension 2 and 3: Bianchi L 1918 *Lezioni sulla teoria dei gruppi continui finite di trasformazioni*, (Pisa: Enrico Spoerri Editore) p 550–557
- dimension 4: Kruchkovich GI 1954, *Usp. Mat. Nauk* **9** 59
- nilpotent up to dimension 6: Morozov V V 1958 *Izv. Vys. Uchebn. Zav. Mat.* **4** (5) 161–71
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Some potentially incomplete classifications are known for solvable Lie algebras in dimension 7 and nilpotent algebras up to dimension 8 (E.N. Safiulina, M.P. Gong, Gr. Tsagas, A.R. Parry).

It seems to be neither feasible, nor fruitful to proceed by dimension in the classification of Lie algebras $\mathfrak{g}$ beyond $\dim \mathfrak{g} = 6$. It is however possible to proceed by structure, i.e. to classify solvable Lie algebras with the nilradicals of a given type.

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Structural classification of solvable Lie algebras

- Heisenberg nilradicals: Rubin J and Winternitz P 1993 *J. Phys. A* **26** 1123–38,
- Abelian nilradicals: Ndogmo JC and Winternitz P 1994 *J. Phys. A* **27** 405–23,
- nilradicals of strictly upper triangular matrices: Tremblay S and Winternitz P 1998 *J. Phys. A* **31** 789–806,
- several classes of filiform nilradicals: Šnobl L and Winternitz P 2005 *J. Phys. A* **38** 2687–700, 2009 *J. Phys. A* **42** 105201, Ancochea JM, Campoamor–Stursberg R, Garcia Vergnolle L 2006 *J. Phys. A* **39** 1339–1355,
- and others in papers of R. Campoamor–Stursberg et al., L. Šnobl et al.; Y. Wang et al. ...

Some basic concepts

Any solvable Lie algebra $\mathfrak{s}$ has a uniquely defined nilradical $\mathrm{NR}(\mathfrak{s})$, i.e. maximal nilpotent ideal. Its dimension satisfies

$$\dim \mathrm{NR}(\mathfrak{s}) \geq \frac{1}{2} \dim \mathfrak{s}. \quad (3)$$

The derived algebra of a solvable Lie algebra $\mathfrak{s}$ is contained in the nilradical, i.e.

$$[\mathfrak{s}, \mathfrak{s}] \subseteq \mathrm{NR}(\mathfrak{s}). \quad (4)$$

A derivation D of a given Lie algebra $\mathfrak{g}$ is a linear map $D: \mathfrak{g} \to \mathfrak{g}$ such that for any pair x, y of elements of $\mathfrak{g}$

$$D([x, y]) = [D(x), y] + [x, D(y)]. \quad (5)$$

If an element $z \in \mathfrak{g}$ exists such that $D = \mathrm{ad}_z$, i.e. $D(x) = [z, x], \ \forall x \in \mathfrak{g}$, the derivation D is called inner, any other one is outer.

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Classification of solvable Lie algebras with the given nilradical

We assume that the nilradical $\mathfrak{n}$, $\dim \mathfrak{n} = n$ is known. That is, in some basis $(e_1, \ldots, e_n)$ we know the Lie brackets

$$[e_i, e_j] = \sum_k N_{ij}{}^k e_k. \qquad (6)$$

We wish to extend the nilpotent algebra $\mathfrak{n}$ to all possible indecomposable solvable Lie algebras $\mathfrak{s}$ having $\mathfrak{n}$ as their nilradical. Thus, we add further elements $f_1, \ldots, f_q$ to the basis $(e_1, \ldots, e_n)$ which together will form a basis of $\mathfrak{s}$. It follows from (4) that

$$\begin{aligned} [f_a, e_i] &= \sum_j (D_a)_i^j e_j, \; 1 \leq a \leq q, \; 1 \leq j \leq n, \\ [f_a, f_b] &= \sum_i \gamma_{ab}{}^i e_i, \; 1 \leq a, b \leq q. \end{aligned} \qquad (7)$$

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We have

- Jacobi identities between $(f_a, e_i, e_j) \Longrightarrow$ operators D_a are derivations of the nilradical $\mathfrak{n}$.
- Jacobi identities between $(f_a, f_b, e_i) \Longrightarrow {\gamma_{ab}}^i$ satisfy

$$[D_a, D_b] = \sum_i {\gamma_{ab}}^i \mathrm{ad}(e_i)|_{\mathfrak{n}} \quad (8)$$

  i.e. $\sum_i {\gamma_{ab}}^i e_i$ is determined up to element in the center $C(\mathfrak{n})$ of $\mathfrak{n}$
- Jacobi identities between $(f_a, f_b, f_c) \Longrightarrow$ bilinear compatibility conditions on ${\gamma_{ab}}^i$ and D_a.

Since $\mathfrak{n}$ is the maximal nilpotent ideal of $\mathfrak{s}$, no nontrivial linear combination of D_a can be a nilpotent matrix, i.e. they are linearly nil–independent (and consequently also outer). By Eq. (8) $[D_a, D_b]$ must be inner derivations.

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Isomorphic Lie algebras with the given nilradical

The resulting Lie algebra is isomorphic to the original one if we

1. add any inner derivation to D_a, i.e. we consider outer derivations modulo inner derivations,

$$D'_a = D_a + \sum_{j=1}^{n} r_a^j \,\mathrm{ad}(e_j)|_{\mathfrak{n}}, \qquad r_a^j \in \mathbb{F}. \tag{9}$$

2. perform a change of basis in $\mathfrak{n}$ such that the Lie brackets (6) are not changed,

$$D'_a = \Phi \circ D_a \circ \Phi^{-1}, \qquad \Phi \in \mathrm{Aut}(\mathfrak{n}) \subseteq GL(n, \mathbb{F}). \tag{10}$$

i.e. we consider only conjugacy classes of sets of outer derivations (modulo inner derivations)

3. change the basis in the space $\mathrm{span}\{D_1, \ldots, D_q\}$.

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Solvable extensions of Borel nilradicals

In this talk we shall concentrate on nilpotent Lie algebras $\mathfrak{n}$ that are isomorphic to the nilradicals of the Borel subalgebras of a complex simple Lie algebra (for the split real form see the paper J. Phys. A: Math. Theor. 45 (2012) 095202). Such nilpotent Lie algebra $\mathfrak{n}$ can be interpreted as the one consisting of all positive root spaces. We shall present general structural properties of all solvable extensions of $\mathfrak{n}$.

The motivation for such an investigation comes from the particular case of triangular nilradicals which are Borel nilradicals of simple Lie algebras $A_l = \mathfrak{sl}(l+1, \mathbb{F})$.

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Triangular nilradicals were investigated in Tremblay S and Winternitz P 1998 Solvable Lie algebras with triangular nilradicals. *J. Phys. A* **31** 789–806 where the dimension and structure of their solvable extensions were studied using explicit matrix calculations.

We shall show that for any Borel nilradical the same results concerning the structure of its solvable extensions holds. This simultaneous treatment is made possible by the fact that all outer derivations of these nilradicals are known, due to Leger G F, Luks E M 1974 Cohomology of nilradicals of Borel subalgebras. *Trans. Amer. Math. Soc.* **195** 305–316.

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Borel nilradicals

Let $\mathfrak{g}$ be a simple complex Lie algebra, $\mathfrak{g}_0$ its Cartan subalgebra, $l = \mathrm{rank}\,\mathfrak{g} = \dim \mathfrak{g}_0$. Let us denote by Δ the set of all roots, by Δ^+ the set of all positive roots and by $\Delta^S = \{\alpha_1, \ldots, \alpha_l\}$ the set of simple roots. Let $\mathfrak{g}_\lambda$ denote the root subspace of the root λ. Let S_β denote the Weyl reflection with respect to the root β,

$$S_\beta(\alpha) = \alpha - 2\frac{\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle}\beta, \qquad \alpha \in \Delta.$$

Every semisimple complex Lie algebra $\mathfrak{g}$ contains a unique (up to isomorphisms) maximal solvable subalgebra, its Borel subalgebra $\mathfrak{b}(\mathfrak{g})$. It contains the Cartan subalgebra and all positive root subspaces

$$\mathfrak{b}(\mathfrak{g}) = \mathfrak{g}_0 \dotplus \left(\dotplus \{\mathfrak{g}_\lambda | \lambda \in \Delta^+\} \right).$$

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Properties of Borel nilradicals

The properties of root systems imply that the Borel subalgebra is indeed a solvable subalgebra of $\mathfrak{g}$ with the nilradical

$$\mathrm{NR}(\mathfrak{b}(\mathfrak{g})) = \dot{+}\{\mathfrak{g}_\lambda | \, \lambda \in \Delta^+\}.$$

For the sake of brevity we shall call the nilpotent Lie algebra $\mathrm{NR}(\mathfrak{b}(\mathfrak{g}))$ the Borel nilradical (although it is not the nilradical of the simple Lie algebra $\mathfrak{g}$).

Properties of Borel nilradicals

Let

$$\mathfrak{g}_m = \dot{+}\{\mathfrak{g}_\lambda \,|\, \lambda = \sum_{i=1}^{l} m_i \alpha_i, \ \sum_{i=1}^{l} m_i \geq m\}.$$

The vectors e_α, $\alpha \in \Delta^S$ generate the entire $\mathrm{NR}(\mathfrak{b}(\mathfrak{g})) = \dot{+}\{\mathfrak{g}_\lambda | \lambda \in \Delta^+\}$ through commutators

$$[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] = \mathfrak{g}_{\lambda+\mu} \qquad \text{whenever} \quad \lambda, \mu, \lambda + \mu \in \Delta^+$$

and this implies that the ideals in the lower central series of the nilradical $\mathrm{NR}(\mathfrak{b}(\mathfrak{g}))$ of the Borel subalgebra are

$$(\mathrm{NR}(\mathfrak{b}(\mathfrak{g})))^m = \mathfrak{g}_m.$$

The center $\mathfrak{z}$ of $\mathrm{NR}(\mathfrak{b}(\mathfrak{g}))$ is one–dimensional and is spanned by e_λ where λ is the highest root of $\mathfrak{g}$, i.e. the only root such that no root $\lambda + \alpha, \quad \alpha \in \Delta^+$ exists. The center $\mathfrak{z}$ coincides with the last nonvanishing ideal in the lower central series.

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Outer derivations of Borel nilradicals

All derivations of the nilradical $\mathfrak{n} = \mathrm{NR}(\mathfrak{b}(\mathfrak{g}))$ were found by G. F. Leger and E. M. Luks and the result is as follows.

Proposition

Let $\mathfrak{g}$ be a complex simple Lie algebra of rank l, $\mathfrak{g}_0$ its Cartan subalgebra, $\Delta^S = \{\alpha_1, \ldots, \alpha_l\}$ the set of simple roots and $\mathfrak{n} = \mathrm{NR}(\mathfrak{b}(\mathfrak{g}))$. Then the algebra of derivations of the nilradical $\mathfrak{n} = \mathrm{NR}(\mathfrak{b}(\mathfrak{g}))$ of the Borel subalgebra of a complex simple Lie algebra $\mathfrak{g}$ satisfies

- $\mathfrak{Der}(\mathfrak{n}) = \mathfrak{Out}(\mathfrak{n}) \dot{+} \mathfrak{Inn}(\mathfrak{n})$,
- $\dim \mathfrak{Out}(\mathfrak{n}) = 2l$,
- $\mathfrak{Out}(\mathfrak{n}) = \mathrm{span}\{D_i, \tilde{D}_i \,|\, i = 1, \ldots, l\}$ *where the derivations $D_i, \tilde{D}_i$ are defined below.*

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Outer derivations of Borel nilradicals

Proposition (continued)

The derivations D_i act diagonally in the basis of $\mathfrak{n}$ consisting of positive root vectors e_α, $\alpha \in \Delta^+$

$$D_i(e_\alpha) = m_i e_\alpha, \qquad \alpha = \sum_{j=1}^{l} m_j \alpha_j \in \Delta^+.$$

$\tilde{D}_i$ are nilpotent outer derivations acting on simple root vectors

$$\tilde{D}_i(e_\beta) = e_\gamma, \quad \text{where} \quad \gamma = S_{\alpha_i}(\lambda), \quad \text{if } \beta = \alpha_i, \qquad (11)$$
$$= 0, \quad \text{if } \beta = \alpha_j,\ j \neq i.$$

The action of $\tilde{D}_i$ on e_α, $\alpha \in \Delta^+ \backslash \Delta^S$ follows from the definition of a derivation (5).

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Outer derivations of Borel nilradicals

For the sake of brevity, we shall write $S_i(\lambda)$ instead of $S_{\alpha_i}(\lambda)$ and introduce nonnegative integer constants s_i

$$S_i(\lambda) = \lambda - s_i \alpha_i.$$

We notice that for $\mathfrak{g} = A_l$ only two constants s_i, namely s_1 and s_l, are nonvanishing and equal to one; for all other simple algebras only one s_i is nonvanishing and turns out to be equal to 1 or 2.

It can be easily deduced that for any simple complex Lie algebra $\mathfrak{g}$ the derivations $\tilde{D}_i$ of the algebra $\mathrm{NR}(\mathfrak{b}(\mathfrak{g}))$ give zero whenever they act on e_β, $\beta \in \Delta^+ \backslash \Delta^S$.

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Outer derivations of Borel nilradicals

Let us assume from now on that $l > 2$. Then we always have $S_i(\lambda) \notin \Delta^S$ for all $i = 1, \ldots, l$ and consequently

$$\tilde{D}_i \circ \tilde{D}_j(e_{\alpha_k}) = 0 \qquad (12)$$

for every $\alpha_k \in \Delta^S$. The Leibniz property (5) allows us to conclude that equation (12) must hold for any $\alpha \in \Delta^+$, i.e. we have

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The derivations D_i obviously commute among each other and act diagonally on $\tilde{D}_j$,

$$[D_i, \tilde{D}_j] \in \mathrm{span}\{\tilde{D}_j\}. \tag{13}$$

To conclude, under the assumption that l is greater than 2, the $2l$ outer derivations $D_i, \tilde{D}_i$ span a subalgebra $\mathfrak{Out}(\mathrm{NR}(\mathfrak{b}(\mathfrak{g})))$ of the Lie algebra of all derivations $\mathfrak{Der}(\mathrm{NR}(\mathfrak{b}(\mathfrak{g})))$. This algebra can be further decomposed into a semidirect sum of an l–dimensional Abelian ideal spanned by the nilpotent derivations $\tilde{D}_i$ and an l–dimensional Abelian subalgebra spanned by D_i.

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Solvable extensions of the Borel nilradicals $\mathrm{NR}(\mathfrak{b}(\mathfrak{g}))$

Let us now study the structure of any solvable Lie algebra with the nilradical $\mathrm{NR}(\mathfrak{b}(\mathfrak{g}))$, $l = \mathrm{rank}\,\mathfrak{g} > 2$.

From the fact that there are only l linearly nilindependent derivations D_i in $\mathfrak{Out}(\mathrm{NR}(\mathfrak{b}(\mathfrak{g})))$ we conclude that the maximal number of nonnilpotent basis elements in any solvable Lie algebra $\mathfrak{s}$ with the nilradical $\mathrm{NR}(\mathfrak{b}(\mathfrak{g}))$ is l. One algebra with this number of nonnilpotent basis elements is already known, namely the Borel subalgebra $\mathfrak{b}(\mathfrak{g})$ of the simple Lie algebra $\mathfrak{g}$. Is it the only one?

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Extensions of maximal dimension $q = l$

Let us assume that we have a solvable Lie algebra $\mathfrak{s}$ with the nilradical $\mathfrak{n} = \mathrm{NR}(\mathfrak{b}(\mathfrak{g}))$ and $l = \mathrm{rank}\,\mathfrak{g}$ nonnilpotent basis elements f_i. They define l outer linearly nilindependent derivations $\hat{D}^i$ such that $\hat{D}^i = \mathrm{ad}(f_i)|_{\mathfrak{n}}$. Using the transformation (9) we may choose the basis vectors f_i so that

$$\hat{D}^i = D_i + \sum_{j=1}^{l} \omega^i_j \tilde{D}_j$$

where $D_i, \tilde{D}_j$ are the derivations defined before.

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Because $\hat{D}^i$ lie in the subalgebra $\mathfrak{Out}(\mathrm{NR}(\mathfrak{b}(\mathfrak{g})))$ of $\mathfrak{Der}(\mathrm{NR}(\mathfrak{b}(\mathfrak{g})))$ and at the same time $[\hat{D}^i, \hat{D}^j] \in \mathfrak{Inn}(\mathrm{NR}(\mathfrak{b}(\mathfrak{g})))$ must hold, we find that

$$[\hat{D}^k, \hat{D}^j] = 0. \quad (14)$$

This requirement together with $[D_i, \tilde{D}_j] \in \mathrm{span}\{\tilde{D}_j\}$ in turn implies

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$$0 = \omega_i^j [D_k, \tilde{D}_i] + \omega_i^k [\tilde{D}_i, D_j] \in \mathrm{span}\{\tilde{D}_i\}. \qquad (15)$$

for every $i, j, k = 1, \ldots, l$ such that $k \neq j$ (no summation over i). For any given i we can find $\tilde{i}$ such that $[\tilde{D}_i, D_{\tilde{i}}] \neq 0$. Consequently, the value of $\omega_i^{\tilde{i}}$ together with the root system specifying the Lie brackets $[D_k, \tilde{D}_i]$ completely determines all ω_i^j for $j \neq \tilde{i}$. Altogether, we still have one undetermined parameter $\omega_i^{\tilde{i}}$ for each $i = 1, \ldots, l$. Next, we show that one can eliminate these parameters through a suitable choice of automorphism in equation (10).

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Extensions of maximal dimension $q = l$

The essence of the argument is that for each $i = 1, \ldots, l$ we can find $\hat{D}^{\tilde{i}}$ which transforms nontrivially under the conjugation by $\exp(t_i \tilde{D}_i)$, i.e. the inner automorphism

$$\begin{aligned} D_j &\rightarrow D_j + t_i[\tilde{D}_i, D_j], \\ \tilde{D}_j &\rightarrow \tilde{D}_j, \end{aligned} \tag{16}$$

$$\hat{D}^j = D_j + \sum_{k=1}^{l} \omega^j_k \tilde{D}_k \rightarrow D_j + t_i[\tilde{D}_i, D_j] + \sum_{k=1}^{l} \omega^j_k \tilde{D}_k.$$

due to $[\tilde{D}_i, D_{\tilde{i}}] \neq 0$. We use it to set $\omega^{\tilde{i}}_i = 0$ after the transformation. Equation (15) then implies that after the transformation all $\omega^j_i = 0$.

Extensions of maximal dimension $q = l$

Therefore we have found that our derivations $\hat{D}^j$ can be brought to the form

$$\hat{D}^j = D_j$$

through a conjugation by a suitable automorphism $\tilde{\Phi}$ of $\mathrm{NR}(\mathfrak{b}(\mathfrak{g}))$.

Extensions of maximal dimension $q = l$

Next, we show that we can always accomplish

$$[f_i, f_j] = 0. \tag{17}$$

We have

$$[f_i, f_j] = \gamma_{ij} e_\lambda, \qquad \gamma_{ij} = -\gamma_{ji}$$

which is the preimage of the relation $[\mathrm{ad}(f_i)|_{\mathfrak{n}}, \mathrm{ad}(f_j)|_{\mathfrak{n}}] = 0$. It can be shown that by a suitable transformation of the form

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To sum up, we have found that for any complex simple Lie algebra $\mathfrak{g}$ such that $\operatorname{rank} \mathfrak{g} > 2$ the maximal solvable Lie algebra with the nilradical $\mathrm{NR}(\mathfrak{b}(\mathfrak{g}))$ is unique and isomorphic to the Borel subalgebra $\mathfrak{b}(\mathfrak{g})$ of $\mathfrak{g}$.

We notice that the same is true also when $\operatorname{rank} \mathfrak{g} = 1$ or $\operatorname{rank} \mathfrak{g} = 2$, i.e. $\mathfrak{g} = \mathfrak{sl}(2), \mathfrak{sl}(3), \mathfrak{so}(5)$ or G_2.

Thus, we have proven the following theorem:

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Thus, we have proven the following theorem:

Solvable extensions of Borel nilradicals of maximal dimension

Theorem

Let $\mathfrak{g}$ be a complex simple Lie algebra, $\mathfrak{b}(\mathfrak{g})$ its Borel subalgebra and $\mathfrak{n} = \mathrm{NR}(\mathfrak{b}(\mathfrak{g}))$ the nilradical of $\mathfrak{b}(\mathfrak{g})$. The solvable Lie algebra with the nilradical $\mathrm{NR}(\mathfrak{b}(\mathfrak{g}))$ of the maximal dimension $\dim \mathfrak{n} + \mathrm{rank}\, \mathfrak{g}$ is unique and isomorphic to the Borel subalgebra $\mathfrak{b}(\mathfrak{g})$ of $\mathfrak{g}$.

Solvable extensions of Borel nilradicals of non–maximal dimension

A similar analysis can be performed also for non–maximal solvable extensions. In this case we have derivations

$$\hat{D}^a = \sum_{j=1}^{l} \left(\sigma_j^a D_j + \omega_j^a \tilde{D}_j \right), \qquad a = 1, \ldots, q \qquad (18)$$

representing the elements f_a in the adjoint representation of $\mathfrak{s}$ on $\mathfrak{n}$, $\hat{D}^a = \mathrm{ad}(f_a)|_{\mathfrak{n}}$. The $q \times l$ matrix $\sigma = (\sigma_j^a)$ must have maximal rank, i.e. q, in view of the nilindependence of $\hat{D}^a$. However we can no longer set σ_j^a equal to the Kronecker delta δ_j^a as was the case for $q = l$. This leads to cumbersome complications. Therefore, we shall only present the resulting theorems whose proofs can be found in our paper.

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Non–maximal extensions

Theorem

Any solvable extension $\mathfrak{s}$ *of the nilradical* $NR(\mathfrak{b}(\mathfrak{g}))$ *by* q *nonnilpotent elements* f_a, $a = 1, \ldots, q \leq \mathrm{rank}\,\mathfrak{g}$ *is defined by* q *commuting derivations* $\hat{D}^a$ *and a constant* $q \times q$ *antisymmetric matrix* $\gamma = (\gamma_{ab})$*. The derivations* $\hat{D}^a$ *determine the Lie brackets*

$$[f_a, e_\alpha] = \hat{D}^a(e_\alpha), \qquad a = 1, \ldots, q, \ \alpha \in \Delta^+$$

and take the form

$$\hat{D}^a = \mathrm{ad}(f_a)|_{\mathfrak{n}} = \sum_{j=1}^{l} \left(\sigma_j^a D_j + \omega_j^a \tilde{D}_j \right), \qquad a = 1, \ldots, q,$$

where $\sigma = (\sigma_j^a)$, $a = 1, \ldots, q$, $j = 1, \ldots, l$ *has the rank* q.

Non–maximal extensions

Theorem (continued)

For any given value of k all parameters ω_k^a are equal to zero when the condition

$$\sum_{j=1}^{l} \sigma_j^a \lambda_j - \sigma_k^a (1 + s_k) \neq 0 \qquad (19)$$

is satisfied for at least one $a \in \{1, \ldots, q\}$. The condition (19) is always satisfied for at least q values of the index k, i.e. there are at most $l - q$ values of k such that some of the parameters ω_k^a are nonvanishing.

Non–maximal extensions

Theorem (continued)

The matrix $\gamma = (\gamma_{ab})$ *defines the Lie brackets*

$$[f_a, f_b] = \gamma_{ab} e_\lambda, \qquad a, b = 1, \ldots, q.$$

When

$$\sum_{j=1}^{l} \lambda^j \sigma_j^a \neq 0$$

holds for at least one $a \in \{1, \ldots, q\}$*, the constants* γ_{ab} *are all equal to* 0*, i.e.*

$$[f_a, f_b] = 0.$$

Non–maximal extensions

We remark that the conditions in the theorem are sufficient, i.e. any set of constants σ_j^a, ω_j^a and γ_{ab} satisfying the properties listed in the theorem gives rise to a solvable extension of the nilradical $NR(\mathfrak{b}(\mathfrak{g}))$. On the other hand, the description presented in the theorem is not unique, i.e. different choices of σ_j^a, ω_j^a and γ_{ab} may lead to isomorphic algebras. As already noted, we may replace the derivations $\hat{D}^a$ by any linearly independent combination of them thus changing all the parameters σ_j^a, ω_j^a and γ_{ab}. Also we may employ the scaling automorphisms to change the values of ω_j^a and γ_{ab}.

Non–maximal extensions

By virtue of indecomposability of the Borel nilradicals, all solvable Lie algebras described in the theorem are indecomposable.

We notice that the statements of both theorems for the particular case $\mathfrak{g} = A_l$ are the same as the results proven by Tremblay and Winternitz.

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Dimension $n_{NR}+1$ solvable extensions of the Borel nilradicals

Theorem

Any solvable extension of the nilradical $NR(\mathfrak{b}(\mathfrak{g}))$ by one nonnilpotent element is up to isomorphism defined by a single derivation

$$\hat{D} = \mathrm{ad}(f_1)|_{\mathfrak{n}} = \sum_{j=1}^{l} \left(\sigma_j D_j + \omega_j \tilde{D}_j \right)$$

chosen so that the first nonvanishing parameter σ_j is equal to one. ω_k vanishes whenever $\sum_{j=1}^{l} \sigma_j \lambda_j - \sigma_k (1+s_k) \neq 0$. At most $l-1$ parameters ω_k are nonvanishing. They are all equal to 1 over the field of complex numbers. Over the field of real numbers they are equal to ± 1 and all parameters ω_k with $s_k = 0$ have the same sign.

Conclusions

- We have briefly reviewed our current knowledge concerning the classification of Lie algebras.
- We have introduced the structure of Borel nilradicals and their derivations.
- We have shown that a solvable extension of a given Borel nilradical of maximal dimension is unique and coincides with the corresponding Borel subalgebra.
- We have reviewed the results for the non–maximal case.

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Thank you for your attention