INTRODUCTORY SCHOOL
ON STRING THEORY

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TRIESTE 2002

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Introduction to Supersymmetry (1st lecture)

I. Spinor Representations

Poincaré Group in D-dimensions

$$X^{\mu} \longrightarrow X'^{\nu} = \Lambda^{\mu}{}_{\nu} X^{\nu} + b^{\mu}, \quad b \in \mathbb{R}^{D}, \quad \Lambda \in SO(1,D-1)$$

Generators (we assume hermitian)

$M_{\mu\nu} = -M_{\nu\mu}$ generate $SO(1,D-1)$

P_{μ} —"— translations

$$[M_{\mu\nu}, M_{\lambda\sigma}] = i\,(\eta_{\nu\lambda} M_{\mu\sigma} + \ldots \qquad)$$

$$[P_{\mu}, P_{\nu}] = 0$$

$$[P_{\mu}, M_{\lambda\sigma}] = i\,(\eta_{\mu\lambda} P_{\sigma} - \eta_{\mu\sigma} P_{\lambda})$$

elements of group $\exp\left(\frac{i}{2}\,\theta^{\mu\nu} M_{\mu\nu}\right)$

Super Symmetry

enlarge Poincaré by spinor elements Q (Note: P vector, M 2nd rank tensor)

Q.. spinors in D-dimensions

Physical Implication

states $|\text{bosons}\rangle$ $|\text{fermions}\rangle$

tensorial operators maintain the statistics, spinor ops. map bosons to fermions and vice versa $Q|B\rangle = |F\rangle$ $Q|F\rangle = |B\rangle$

Spinors in D-dimensions

Γ_μ-matrices $\{\Gamma_\mu, \Gamma_\nu\} = 2\eta_{\mu\nu}$

$\Rightarrow$ $\Sigma_{\mu\nu} = \frac{i}{4}[\Gamma_\mu, \Gamma_\nu]$ satisfy

$$[\Sigma_{\mu\nu}, \Sigma_{\lambda\sigma}] = i(\eta_{\lambda\nu}\Sigma_{\mu\sigma} + 3 \text{ terms})$$

.. spinor representation of Lorentz group

$SO(D), D=2n, \eta_{\mu\nu} = \delta_{\mu\nu}$ $\Rightarrow$ $\Gamma_\mu^+ = \Gamma_\mu$

$b_i = \frac{1}{2}(\Gamma_i + i\Gamma_{n+i})$ $b_i^+ = \frac{1}{2}(\Gamma_i - i\Gamma_{n+i})$

$\Rightarrow \{b_i, b_j^+\} = \delta_{ij}$ $\{b_i, b_j\} = 0 = \{b_i^+, b_j^+\}$

$\Rightarrow$ space of fermionic states

Clifford vacuum $b_i|\Omega\rangle = 0$ $i = 1,..,n$

span $\{|\Omega\rangle, b_i^+|\Omega\rangle, b_i^+ b_j^+|\Omega\rangle, .. \}$

dim $= 2^n$ = dimension of Γ-matrices and $\Sigma_{\mu\nu}$

Ex: Construct Γs in $D=2$ and $D=4$

$D = 2n+1$ $\Gamma_{2n+1} \equiv i^\alpha \Gamma_1 .. \Gamma_{2n}$ α .. $\Gamma_{2n+1}^+ = \Gamma_{2n+1}$

Ex: $\Gamma_{2n+1}^2 = 1$ (for suitable α) $\{\Gamma_{2n+1}, \Gamma_\mu\} = 0$ $\mu = 1,..,2n$

Spinor repr. of $SO(2n)$ and $SO(2n+1)$ have both dim $= 2^n$

~~For~~ .. <u>Dirac spinors</u> (2^n-component objects)

For $D = 2n$ Dirac spinor repr. is reducible

~~D=2n+1~~ $\{\Gamma_{2n+1}, \Gamma_\mu\} = 0 \Rightarrow [\Gamma_{2n+1}, \Sigma_{\mu\nu}] = 0$

$\Gamma_{2n+1}^2 = 1$ $\Rightarrow$ eigenvalues of $\Gamma_{2n+1} = \pm 1$

$\text{tr}\,\Gamma_{2n+1} = 0 \Rightarrow$ # of +1 eigenvalues = # of −1 eigenvalues

$\Gamma_{2n+1}\psi = +\psi$

$$\Rightarrow \Gamma_{2n+1} \exp(\tfrac{i}{2}\theta^{\mu\nu}\Sigma_{\mu\nu})\psi = e^{\frac{i}{2}\theta^{\mu\nu}\Sigma_{\mu\nu}}\Gamma_{2n+1}\psi$$
$$= e^{\frac{i}{2}\theta^{\mu\nu}\Sigma_{\mu\nu}}\psi$$

eigensubspaces of Γ_{2n+1} are invariant under $e^{\frac{i}{2}\theta^{\mu\nu}\Sigma_{\mu\nu}}$

elements of —"—————— .. <u>Weyl</u> or <u>chiral spinors</u>

Dimension of Weyl spinor $\frac{1}{2}2^n = 2^{n-1}$

<u>Charge conjugation matrix C</u> product of one half of Γs

e.g. $C = \Gamma_{n+1} .. \Gamma_{2n}$

$\psi_c = C\psi^*$ (euclidean)

$= C\bar{\psi}^T$ (Lorentz)

if ψ_c as a repr. is unitarily equiv. to ψ .. spinor is called self-conjugate, otherwise it is called complex

in some dimensions it can be choosen real .. Majorana spinors

If we $\overset{\text{can}}{\vee}$ choose $e^{\frac{i}{2}\theta^{\mu\nu}\Sigma_{\mu\nu}}$ to be orthogonal $\Rightarrow$ self-conjugate = real e.g. $SO(4)$

if not .. pseudoreal

e.g. $SO(3)$ $\Gamma_1, \Gamma_2, \Gamma_3 = \sigma_1, \sigma_2, \sigma_3$ cannot be chosen real

$SO(p,q)$ .. add i to corresp. Γs $\Rightarrow$ diff. realities

Reality

$$\left.\begin{array}{lll} SO(2n) & \text{real} & n = 0 \bmod 4 \\ & \text{pseudoreal} & n = 2 \bmod 4 \\ & \text{complex} & n = 1,3 \bmod 4 \end{array}\right\} \text{Weyl spinors (irreps)}$$

$$\begin{array}{lll} SO(2n+1) & \text{real} & n = 0,3 \bmod 4 \\ & \text{pseudoreal} & n = 2,1 \bmod 4 \end{array}$$

$$\left.\begin{array}{lll} SO(1,2n-1) & \text{real} & n = 1 \bmod 4 \\ & \text{pseudoreal} & n = 3 \bmod 4 \\ & \text{complex} & n = 0,2 \bmod 4 \end{array}\right\} \text{Weyl spinors}$$

$$\begin{array}{lll} SO(1,2n) & \text{real} & n = 0,1 \bmod 4 \\ & \text{pseudoreal} & n = 2,3 \bmod 4 \end{array}$$

Examples: $D=2,\ n=1$ $\quad SO(1,1)$ real

$D=4,\ n=2$ $\quad SO(1,3)$ complex

$D=6,\ n=3$ $\quad SO(1,5)$ pseudoreal

$D=8$ $\quad SO(8)$ real

$SO(1,7)$ pseudoreal

$D=10,\ n=5$ $\quad SO(1,9)$ real $\quad 2^{n-1} = 16\ \underbrace{\text{real components}}_{\text{Majorana-Weyl}}$

~~D1~~

$D=11,\ n=5$ $\quad SO(1,10)$ real $\quad 2^n = \cancel{342}\ 32$

$D=10$ Dirac spinor $\quad 2^5 = 32$ components

$$\Gamma_{11} = \Gamma_0 \cdots \Gamma_9 \qquad \left.\begin{array}{l} \Gamma_{11}\psi = \psi \\ \Gamma_{11}\psi = -\psi \end{array}\right\} \text{subspaces of Weyl spinors, 16 components each}$$

in general $\quad \psi = \frac{1+\Gamma_{11}}{2}\psi + \cancel{1+\Gamma_{11}}\,\frac{1-\Gamma_{11}}{2}\psi = \psi_+ + \psi_-$

SUSY algebra

extension of Poincaré algebra $\quad M_{\mu\nu},\ P_\mu,\ Q_\alpha$

$\nearrow \mu = 0,1,..,D-1$ $\qquad \uparrow$ spinorial index $\alpha = 1,..,2^n$ or 2^{n-1}

$[M_{\mu\nu}, Q_\alpha] = \frac{i}{2}(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta,\ (\sigma \equiv \Sigma)$ transformation of a spinor under rotations & boosts

$[P_\mu, Q_\alpha] = 0$ $\qquad$ invariance under translations

$\{Q_\alpha, Q_\beta\} =$ "$i(\sigma^\mu)_{\alpha\beta} P_\mu$" $+ \ldots$

$D=4$ $\quad$ Weyl spinors are complex $\quad Q_\alpha^A \quad \alpha = 1,2,\ (Q_\alpha^A)^+$

$A = 1,..,N$ $\quad$ N-extended supersymmetry

$N=1$ $\quad$ ~~single~~ simple SUSY

$\{Q_\alpha^A, (Q_\beta^B)^+\} = 2\,\sigma^\mu_{\alpha\beta} P_\mu \delta^A_B$ $\qquad \sigma^0 = 1,\ \sigma^i$ Pauli matrices

$\{Q_\alpha^A, Q_\beta^B\} = \varepsilon_{\alpha\beta} X^{AB},\ X^{AB} = -X^{BA}$ $\qquad \varepsilon_{12} = -\varepsilon_{21} = 1$

X^{AB} commute with everything... central charges

Bosonic string (1st and 2nd lecture)

1. Quantization of the free bosonic string in $R^{1,D-1}$ (D=26)
(old covariant & light-cone quantization)
2. Spectrum of massless states
3. Partition function / one-loop free energy
(4. Strings moving in nontrivial backgrounds
5. Circle compactification)

$$S = -\frac{1}{4\pi\alpha'}\int \sqrt{h}\, h^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X_\mu \, d^2\sigma \qquad X^\mu(\sigma,\tau)\ \mu=0,\dots,D-1$$

$$\alpha,\beta=0,1 \qquad \sigma=(\sigma^0,\sigma^1)=(\tau,\underset{\substack{\cap \\ \langle 0,2\pi\rangle}}{\sigma})$$

$$\eta_{\mu\nu} = \mathrm{diag}\,(-1,1,\dots,1)$$

$$[\alpha'] = (\text{length})^2 \qquad \tfrac{1}{\alpha'} \text{ string tension}$$

Invariance of S

reparametrization of worldsheet $\xi^\alpha(\sigma,\tau)$ infinitesimally

$$\delta X^\mu = \xi^\alpha \partial_\alpha X^\mu \qquad \delta h^{\alpha\beta} = \xi^\gamma \partial_\gamma h^{\alpha\beta} - \partial_\gamma \xi^{(\alpha} h^{\gamma\beta)}$$

Weyl $\delta h^{\alpha\beta} = \omega h^{\alpha\beta}$

Global symmetry $\delta X^\mu = \omega^{\mu\nu} X_\nu + a^\mu$

Ricci tensor in 2D $\overset{\mu a}{\text{total}}$ derivative ⇒ no dynamics for h

$$-4\pi\alpha' \frac{\delta S}{\sqrt{h}\,\delta h^{\alpha\beta}} = T_{\alpha\beta} = 0$$ variation of S w.r.t. h, i.e. eq of motion for h

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} h_{\alpha\beta}\, h^{\alpha'\beta'} \partial_{\alpha'} X^\mu \partial_{\beta'} X_\mu$$

⇒ $h^{\alpha\beta} T_{\alpha\beta} = 0$ identically ⇒ $\underline{T_{\alpha\beta} \text{ traceless}}$ property of conformally invariant systems

Other formulation $S \sim -\frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{-\det(\partial_\alpha X^\mu \partial_\beta X_\mu)}$ <u>Nambu-Goto action</u>
classically equivalent since $h_{\alpha\beta} \sim \partial_\alpha X^\mu \partial_\beta X_\mu$

$h_{\alpha\beta}$ symmetric 2×2, 3 gauge degrees of freedom ⇒ <u>by gauge fixing</u> locally
$$h_{\alpha\beta} = \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

⇒ eq. of mot. $\Box X^\mu = 0 \quad (-\partial_\tau^2 + \partial_\sigma^2)\, X^\mu = 0$
for closed strings

open strings ⇒ boundary terms $\int_A d^2\sigma\ \partial_\alpha(\sqrt{h}\, h^{\alpha\beta} \partial_\beta X^\mu \delta X_\mu)$
$$= \int_{\partial A} d\sigma_\alpha\, \varepsilon_{\alpha\beta} (\sqrt{h}\, h^{\beta\gamma} \partial_\gamma X^\mu \delta X_\mu)$$
$$= \int d\tau \left[\partial_\sigma X\cdot\delta X\big|_{\sigma=2\pi} - \partial_\sigma X\cdot\delta X\big|_{\sigma=0}\right]$$

⇒ two boundary conditions

<u>Neumann</u> $\partial_\sigma X^\mu|_\pi = \partial_\sigma X^\mu|_0 = 0$

<u>Dirichlet</u> $X^\mu|_{\pi,0}$ fixed ⇒ $\delta X^\mu|_{\pi,0} = 0$

$$\Box X^\mu = 0 \Rightarrow X^\mu(\sigma,\tau) = X_R^\mu(\sigma^-) + X_L^\mu(\sigma^+) \qquad \sigma^\pm = \tau \pm \sigma$$

since $\Box = 4\partial_+\partial_-$ $\quad \partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$

$\eta_{++} = \eta_{--} = 0 \quad \eta_{+-} = \eta_{-+} = -\frac{1}{2}$

$T_{\alpha\beta} = 0 \quad T_{00}, T_{11}, \overset{=\dot X X'}{T_{01}} \quad \eta^{\alpha\beta}T_{\alpha\beta} = 0 \Rightarrow T_{11} = T_{00} = \frac{1}{2}(\dot X^2 + X'^2)$

$T_{++}, T_{--} \quad T_{++} = \partial_+ X^\mu \partial_+ X_\mu = \frac{1}{2}(T_{00} + T_{01})$

$T_{--} = \partial_- X^\mu \partial_- X_\mu = \frac{1}{2}(T_{00} - T_{01})$

Tracelessness $\Leftrightarrow T_{+-} = T_{-+} = 0 \quad (\Leftrightarrow \eta^{\alpha\beta} T_{\alpha\beta} = 0)$

$\partial_\alpha T^{\alpha\beta} = 0 \Rightarrow \partial_- T_{++} = 0 = \partial_+ T_{--} = 0$

$\Rightarrow T_{++} = T_{++}(\sigma^+), \quad T_{--} = T_{--}(\sigma^-)$

$\Rightarrow$ ∞-set of conserved quantities $\quad \partial_-\big(f(\sigma^+)T_{++}(\sigma^+)\big) = 0 \ldots$

Residual gauge freedom $\quad \xi^\pm = \xi^\pm(\sigma^\pm) \wedge \omega = 1\Big/\frac{\partial\xi^+\partial\xi^-}{\partial\sigma^+\partial\sigma^-}$

Closed string $\quad X_R^\mu = \frac{x^\mu}{2} + \frac{\alpha'}{2}p^\mu(\tau - \sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq0}^{\infty}\frac{\alpha_n^\mu}{n}e^{-in(\tau-\sigma)}$

$X_L^\mu = \frac{x^\mu}{2} + \frac{\alpha'}{2}p^\mu(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq0}\frac{\tilde\alpha_n^\mu}{n}e^{-in(\tau+\sigma)}$

$X_R^\mu + X_L^\mu$ periodic in $\sigma \in \langle 0, 2\pi\rangle$

Quantisation $\quad \big[X^\mu(\sigma,\tau), \Pi^\nu(\sigma',\tau)\big] = i\eta^{\mu\nu}\delta(\sigma-\sigma')$

$\Pi^\nu = \frac{\delta S}{\delta \dot X^\nu} \Rightarrow [X^\mu(\sigma,\tau), \dot X^\nu(\sigma',\tau)] = 2i\pi\alpha'\delta(\sigma-\sigma')\eta^{\mu\nu}$

$\Rightarrow [x^\mu, p^\nu] = i\eta^{\mu\nu} \quad [\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+n,0}\eta^{\mu\nu}$

$[\tilde\alpha_m^\mu, \tilde\alpha_n^\nu] = m\delta_{m+n,0}\eta^{\mu\nu}, \quad [\alpha_m^\mu, \tilde\alpha_n^\nu] = 0$

(using $\delta(\sigma-\sigma') = \sum_n \frac{1}{2\pi}e^{in(\sigma-\sigma')}$)

sometimes $\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}\,p^\mu = \tilde\alpha_0^\mu$

Hermiticity $\quad (\alpha_n^\mu)^\dagger = \alpha_{-n}^\mu \quad (\tilde\alpha_n^\mu)^\dagger = \tilde\alpha_{-n}^\mu$

$\Rightarrow$ Hilbert space $\equiv$ Fock space $\quad \{|0,k^\mu\rangle\} \quad p^\mu|0,k\rangle = k^\mu|0,k\rangle$

$\alpha_{-n_1}^{\mu_1}\cdots\alpha_{-n_p}^{\mu_p}|0,k^\mu\rangle$ etc. (α_{-n}^μ, $n>0$ creation ops)

problem $\quad [\alpha_m^0, \alpha_n^0] = -m\,\delta_{m+n,0}$ whereas $[\alpha_n^i, \alpha_m^j] = m\delta_{n+m,0}\delta^{ij}$

$\Rightarrow$ negative norm states $\quad$ (if $\|\alpha_{-1}^i|0,k\rangle\|^2 > 0$ then $\|\alpha_{-1}^0|0,k\rangle\|^2 < 0$)

Open strings $\quad$ boundary conditions $\Rightarrow$

Neumann $\quad X^\mu = x^\mu + 2\alpha'\boxed{p^\mu}\tau + i\sqrt{2\alpha'}\sum_{n\neq0}\frac{\alpha_n^\mu}{n}\cos(n\sigma)e^{-in\tau}$

$\Rightarrow \partial_\sigma X|_{\pi,0} = 0 \quad$ momentum

Dirichlet $\quad X^\mu = x_1^\mu + \boxed{(x_2^\mu - x_1^\mu)}\frac{\sigma}{\pi} + \sqrt{2\alpha'}\sum\frac{\alpha_n^\mu}{n}\sin(\sigma n)e^{-in\tau}$

$\Rightarrow X^\mu(\sigma=0) = x_1^\mu \quad X^\mu(\sigma = \cancel{2}\pi) = x_2^\mu$

stretching factor

$\Rightarrow$ standard commutation relations etc.

Conserved quantities

$T_{++}(\sigma_+) = \frac{1}{4}(\dot X^2 + X'^2 + 2\dot X X') \quad T_{--}(\sigma_-) = \frac{1}{4}(\dot X^2 + X'^2 + (-2)\dot X X')$

$T_{++} + T_{--} = \frac{1}{2}(\dot X^2 + X'^2) \quad T_{++} - T_{--} = \dot X X'$

$\Rightarrow H = \int_0^{2\pi} d\sigma\, \frac{1}{2}(\dot X^2 + X'^2) \quad P = \int_0^{2\pi} d\sigma\, \dot X \cdot X'$

Fourier mode expansion of T_{++}, T_{--}

$L_m = \frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma\, e^{im\sigma}\, T_{--}(\sigma_-)$

$$\tilde{L}_m = \frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma\, e^{im\sigma}\, T_{++}(\sigma_+)$$

using $\frac{1}{2\pi}\int_0^{2\pi} d\sigma \exp(\pm ik\sigma) = \delta_{k,0}$ one finds $L_m = \frac{1}{2}\sum_{n=-\infty}^{+\infty} \alpha^\mu_{m-n}\,\alpha_{n\mu}$

$$\tilde{L}_m = \frac{1}{2}\sum_{n=-\infty}^{+\infty} \tilde{\alpha}^\mu_{m-n}\,\tilde{\alpha}_{n\mu}$$

$\Rightarrow H = L_0 + \tilde{L}_0$ & $P = L_0 - \tilde{L}_0$

$$L_0 = \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha^\mu_{-n}\,\alpha_{n\mu}$$

L_0 not normally ordered, $[\alpha^\mu_m, \alpha^\nu_{-n}] = n\eta^{\mu\nu}$

$\Rightarrow$ after normal ordering $L_0 = \frac{\alpha_0^2}{2} + \underbrace{\sum_{n=1}^{\infty} \alpha^\mu_{-n}\alpha_{n\mu}}_{\text{normal ordered}} + \underbrace{D\sum_{n=1}^{\infty} n}_{-a\ ..\ \text{indefinite constant}}$

$$\tilde{L}_0 = \frac{\alpha_0^2}{2} + \sum \tilde{\alpha}^\mu_{-n}\,\tilde{\alpha}_{n\mu} + (a)$$

in the following we write $L_0 = \frac{\alpha_0^2}{2} + \sum_{n=1}^{\infty}\alpha^\mu_{-n}\alpha_{n\mu}$, $\tilde{L}_0 = \frac{\alpha_0^2}{2} + \sum_{n=1}^{\infty}\tilde{\alpha}^\mu_{-n}\tilde{\alpha}_{n\mu}$

i.e. $L_0 - a = \frac{\alpha_0^2}{2} + N - a$, $\tilde{L}_0 - a = \frac{\alpha_0^2}{2} + \tilde{N} - a$

? Fix a ? "(intercept)"

$[L_m, L_n] = (m-n)\, L_{m+n} + \delta_{m,-n}\cdot((A(m)))$ number-valued function (it's due to normal ordering)

? How to determine $A(m)$? $(A(-m) = -A(m)$ from antisymmetry of $[\,,\,])$

Jacobi identity $[L_n,[L_m,L_k]] + [L_m,[L_k,L_n]] + [L_k,[L_n,L_m]] = 0$

$\Rightarrow A(m) = c_1 m^3 + c_2 m$ (together with antisymmetry up to transl. of L_0 etc.)

$m, n = 0, \pm 1$ $A(m) = 0$ $(\Leftarrow [L_1, L_{-1}] = 2L_0)$

$\Rightarrow c_1 = -c_2$

$\langle 0|[L_2, L_{-2}]|0\rangle \Rightarrow c_1 = \frac{D}{12}$

$\Rightarrow [L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12}(m^3 - m)\,\delta_{m+n,0}$

open $\{L_0, L_{+1}, L_{-1}\}$ generate subalgebra $sl(2,\mathbb{R})$

$A(m)$ ... central charge, connected with conformal anomaly

Gauge still not fully fixed ! $\sigma^+ \to f^+(\sigma^+)$, $\sigma^- \to f^-(\sigma^-)$

$\omega = \ldots$

If we put T to zero on physical states as in classical situation

$\Rightarrow L_m|\text{phys}\rangle = \tilde{L}_m|\text{phys}\rangle = 0, \forall m\in\mathbb{Z}$ not possible because of $A(m)$

We proceed as in QED, only fix the positive modes

$\Rightarrow L_m|\text{phys}\rangle = \tilde{L}_m|\text{phys}\rangle = 0, \forall m > 0$

$(L_0 - a)|\text{phys}\rangle = (\tilde{L}_0 - a)|\text{phys}\rangle = 0$

↪ i.e. $\frac{1}{4}\alpha' p^2 + N - a = 0$ $p^2 = -M^2$ mass of the state on-shell

$\Rightarrow M^2 = \frac{4}{\alpha'}(N-a) = \frac{4}{\alpha'}(\tilde{N} - a)$

also $(L_0 - \tilde{L}_0)|\text{phys}\rangle = 0 \Leftrightarrow$ total momentum along the string vanishes $((N-\tilde{N})|\text{phys}\rangle = 0)$

open strings .. L_m only, constraints $L_m|\text{phys}\rangle = 0$ $m > 0$, $\underbrace{(L_0 - a)|\text{phys}\rangle = 0}_{\Downarrow\ M^2_{\text{open}} = \frac{1}{\alpha'}(N-a)}$

The light-cone gauge

$X^0, X^1, .., X^{D-1} \longrightarrow X^+ = (X^0 + X^{D-1})\frac{1}{\sqrt{2}}$, $X^- = \frac{X^0 - X^{D-1}}{\sqrt{2}}$, $X^1, .., X^{D-2}$

$X\cdot Y = -X^+Y^- - X^-Y^+ + X^iY^i$ $i = 1, .., D-2$ transverse coords

$X^+ = -X_-$, $X^- = -X_+$

$\sigma^\pm \to f^\pm(\sigma^\pm)$ $\Rightarrow \Box f^\pm = 0$ and also $\Box X^\mu = 0$

$\Rightarrow$ identify $X^+ = X^+_L + X^+_R \overset{!}{=} x^+ + p^+\tau + \cancel{\text{oscillators}}$

(by suitable redefinition of $\sigma_\pm$) $\tau = \sigma_+ + \sigma_-$

$$T_{++} = \partial_+ X^\mu \partial_+ X_\mu = -2 \underbrace{\partial_+ X^+}_{p_+} \partial_+ X^-_m + \partial_+ X^i \partial_+ X^i = 0$$

=> we can solve the contraint

$$\Rightarrow \quad \partial_+ X^- = \frac{1}{2p_+}\left(\partial_+ X^i \partial_+ X^i\right) \Rightarrow \alpha^-_m = \text{function}(\alpha^i_m)$$

The remaining independent oscillators are only the transverse ones α^i_m, $i = 1, .., D-2$

=> no negative norm states but Lorentz covariance is hidden

$$L_0 = -2 \frac{\alpha_0^+ \alpha_0^-}{2} + \frac{1}{2}\alpha_0^i \alpha_0^i + N - a \overset{!}{=} 0 \qquad M^2 = (2p^+p^- - \vec{p}^{\,2}) \overset{!}{=} 2(N-a)$$

mass-shell condition

$\left(\begin{matrix}\alpha_m^+ = 0 \text{ redefinition} \\ \Rightarrow \alpha_n^+ \alpha_m^- = 0\end{matrix}\right)$ where $N = \sum_{n=1}^{\infty} \alpha^i_{-n} \alpha^i_n \quad i = 1, .., D-2$

Bosonic string (3rd lecture)

Note: $P|phys\rangle = (L_0 - \bar{L}_0)|phys\rangle$ level matching condition

Spectrum of an open string (closed string similarly)

$(\alpha' = \frac{1}{2})$

$$\boxed{M^2 = 2(N-a)} \qquad N = \sum_{\substack{n=1 \\ i}}^{\infty} \alpha^i_{-n} \alpha^i_n \qquad i = 1, .., D-2$$

$|0,k\rangle \qquad \alpha^i_{-1}|0,k\rangle \qquad i$ .. polarization => D-2 degrees of freedom

$\Rightarrow M^2 = 2(1-a)$

because of number of degrees of freedom and the fact that only a massless vector has D-2 degrees of freedom we have to fix $\boxed{a = 1}$

$\Rightarrow N = 0$ no oscillator $\Rightarrow M^2 = -2$ tachyon

$N = 1$ 1 —"— $\Rightarrow M^2 = 0$ vector

$N > 1$ $\Rightarrow M^2 > 0$ massive states

Condition of Lorentz invariance: after computation of commutators of Lorentz alg. and requesting that it closes => fixed D = 26 - (rather complicated)

Other explanation $\quad -a = (D-2)\,\frac{1}{2}\sum_{n=1}^{\infty} n$

$\zeta(s)$ Riemann ζ-function $\quad \zeta(s) = \sum_{n=1}^{\infty} n^{-s}$

$\exists$ analytic continuation $\Rightarrow \zeta(-1) = -\frac{1}{12}$

$$\Rightarrow -a = -\frac{D-2}{12}$$

$$a = 1 \Rightarrow \boxed{D = 26}$$

Covariant gauge

Physical states $|\psi\rangle \qquad L_n|\psi\rangle = 0 \quad n > 0 \quad (L_0 - a)|\psi\rangle = 0$

together with $\quad \bar{L}_n|\psi\rangle = 0 \quad (\bar{L}_0 - a)|\psi\rangle = 0$ for closed strings

Spurious states (Null states) physical & $|\psi'\rangle = L_{-n}|\psi\rangle$

for some $n > 0$ and $|\psi\rangle$ physical

physical states have zero scalar product with null states

=> null states decouple, we define $|\psi_1\rangle \cong |\psi_2\rangle$

$\Leftrightarrow \quad |\psi_1\rangle - |\psi_2\rangle$ is a null state

$N=1: |\psi\rangle = \varepsilon_\mu(k)\,\alpha^\mu_{-1}|0,k\rangle$ vector

?physical state? $L_n|\psi\rangle = 0$

$L_1|\psi\rangle = 0 \Rightarrow k_\mu \varepsilon^\mu = 0 \qquad (L_1 \sim p_\mu \alpha^\mu_1)$

How to remove gauge degrees of freedom

$\lambda L_{-1}|0,k\rangle$ a spurious state, should be physical $\Rightarrow L_1(L_{-1}|0,k\rangle) = 0$

$\Rightarrow \quad k^2 = 0$

$\Rightarrow \varepsilon^\mu \underset{\overset{\uparrow}{\text{for } k^2=0}}{\sim} \varepsilon^\mu + \lambda k^\mu$ $\Rightarrow$ from original D degrees two are removed

by $\underbrace{\varepsilon^\mu k_\mu = 0}_{\Rightarrow 1}$ and $\underbrace{\varepsilon^\mu \sim \varepsilon^\mu + \lambda k^\mu}_{\Rightarrow 1}$ $\Rightarrow \boxed{D-2 \text{ phys. states}}$

Closed string $\Rightarrow$ tensor product of left- and right-moving oscillators

$\alpha' = \frac{1}{2}$ $\qquad N = \tilde{N} = 0 \qquad M^2 = -8$

$N = \tilde{N} = 1 \qquad M^2 = 0$ massless $\alpha^i_{-1}\tilde{\alpha}^j_{-1}|0,k\rangle$

.. $(D-2)^2$ degrees of freedom

... reducible repr. of $SO(D-2)$

(graviton $\longrightarrow$) symmetric traceless tensor h_{ij} $\qquad \frac{(D-2)(D-1)}{2} - 1$

antisymmetric tensor b_{ij} $\qquad \frac{(D-2)(D-3)}{2}$

(dilaton $\longrightarrow$) scalar $\qquad \phi$

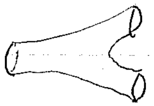
Note: $\underline{a}$ again fixed as before by requirement that $M^2=0$ for these states (since the space fits the repr. of $SO(D-2)$ not $SO(D-1)$).

again null states $\Rightarrow h_{\mu\nu} \sim h_{\mu\nu} + k_\mu \xi_\nu + k_\nu \xi_\mu$

$b_{\mu\nu} \sim b_{\mu\nu} + k_{[\mu} a_{\nu]}$

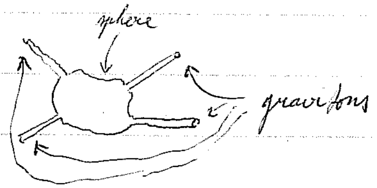
$\Rightarrow$ action for $h_{\mu\nu}$ should be of Einstein type (to be invariant w.r.t above mentioned transf.), similarly $b_{\mu\nu}$ should be a gauge field, i.e. action should depend on $H_{\mu\nu\varrho} = \partial_\mu b_{\nu\varrho}$ + cyclic

Interactions


basic vertex

Feynman diagrams ~ Riemann surfaces e.g. one loop

e.g. 4-point function



tachyon $\quad |0,k\rangle \longrightarrow$ plane wave $e^{ik_\mu X^\mu(\sigma^+,\sigma^-)}$

$\alpha^\mu_{-1}\tilde{\alpha}^\nu_{-1}|0,k\rangle \longrightarrow \underbrace{\partial_+ X^\mu \partial_- X^\nu e^{-\text{''}-}}_{\text{vertex operators}}$

Euclidean worldsheet (Wick rotation)

$\sigma^0 = -i\sigma^2 \quad \Rightarrow w = \sigma_2 + i\sigma_1 \quad \bar{w} = \sigma_2 - i\sigma_1$

$\Rightarrow X_\mu = x_\mu + p_\mu \mathrm{Re}\,\overset{w}{\cancel{z}} + \left(\sum \frac{\alpha_n}{n} e^{-nw} + \frac{\tilde{\alpha}_n}{n} e^{n\bar{w}}\right)$

where $z = e^w$

$\partial_z X^\mu \cong \sum \alpha^\mu_n z^{-n-1} \qquad \partial_{\bar{z}} X \sim \sum \tilde{\alpha}^\mu_n \bar{z}^{-n-1}$

Partition function

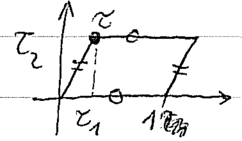
$$Z = \mathrm{tr}\, \underbrace{e^{-\beta H}}_{\sim\text{propagation in Euclidean time}} \overbrace{e^{i\vartheta P}}^{\sim\text{propagation in space direction}} \qquad H = L_0+\tilde{L}_0 \underset{\uparrow\, 2a}{-2}, \quad P = L_0-\tilde{L}_0$$

$$\beta = 2\pi\tau_2 \quad \vartheta = 2\pi\tau_1, \quad \tau_2 > 0, \ -\tfrac{1}{2} \le \tau_1 \le \tfrac{1}{2}$$

$$\Rightarrow Z = \mathrm{tr}\, e^{-2\pi\tau_2(L_0+\tilde{L}_0-2)}\, e^{2\pi i\tau_1(L_0-\tilde{L}_0)} = \mathrm{tr}\, q^{L_0-1}\, \bar{q}^{\tilde{L}_0-1}$$

$$q = e^{2\pi i\tau}, \quad \tau = \tau_1+\tau_2 i, \quad \tilde{q} = e^{-2\pi i\bar{\tau}}$$

Modulus of a torus τ (torus ⇐ trace, i.e. ends of cylinder identified)



$$L_0 = p^2+N = p^2 + \sum \alpha^i_{-n}\alpha^i_n \quad \text{from } i$$

$$\Rightarrow Z = q^{-1}\bar{q}^{-1}\, \mathrm{tr}\, q^{L_0}\bar{q}^{\tilde{L}_0} = q^{-1}\bar{q}^{1}\cdot\left(\prod_n \frac{1}{1-q^n}\right)^{24}\left(\prod_n\frac{1}{1-\bar{q}^n}\right)^{24}\cdot\frac{1}{\tau_2^{13}}$$

note $[\alpha_{n},\alpha_{m}] = n\,\delta_{..} \Rightarrow \alpha_{-n}\alpha_n$ not exactly number operator

$$\mathrm{tr}\, q^{\alpha_{-n}\alpha_n} = \sum_{k=0}^{\infty} q^{nk} = \frac{1}{1-q^n}, \text{ the same for tildas } (\tilde{L}_0)$$

$$e^{2\pi i\tau p^2 - 2\pi i\bar{\tau}p^2} = e^{-4\pi i\tau_2 p^2}, \quad \int d^{26}p\, e^{-4\pi i\tau_2 p^2} \sim \frac{1}{\tau_2^{13}}$$

Dedekin η-function $\eta = q^{\frac{1}{24}}\prod_n(1-q^n)$

$$\Rightarrow Z = \frac{1}{\tau_2^{13}}\left(\frac{1}{\eta}\frac{1}{\bar{\eta}}\right)^{24}$$

1-loop **Free energy** in QFT of a system of particles $F = \sum_i \int_0^\infty \frac{d\alpha}{\alpha}\, e^{-\alpha(p^2+m_i^2)}$

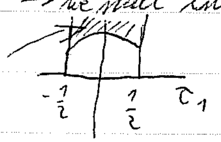
$$\Rightarrow F = \int_W \frac{d\tau_1\, d\tau_2}{\tau_2}\, \frac{1}{\tau_2^{13}}\left(\frac{1}{\eta}\frac{1}{\bar{\eta}}\right)^{24}$$

1-loop free energy of a bosonic string

modular invariance τ: $\left.\begin{array}{l}\tau \to \tau+1 \\ \tau \to -\frac{1}{\tau}\end{array}\right\}$ generates $SL(2,\mathbb{Z})$ $= \{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \mid ad-bc=1, a,b,c,d \in \mathbb{Z}\}$

$\eta(\tau+1) = \text{phase}\,\eta(\tau)$, $\eta(-\frac{1}{\tau}) = \sqrt{\tau}\,\eta(\tau)$

⇒ F invariant under $SL(2,\mathbb{Z})$ ⇒ we shall integrate over a fundamental region (W)



⇒ no ultraviolet divergencies (can be proved to all orders)

Introduction to Supersymmetry (2nd lecture)

Representations of SUSY algebra

First consider Poincaré $\{M_{\mu\nu}, P_\lambda\}$

First Casimir ops. $P^2 = P_\lambda P^\lambda = m^2$

$$W_\mu = \tfrac{1}{2}\varepsilon_{\mu\nu\lambda\sigma}P^\nu M^{\lambda\sigma} \Rightarrow \text{Casimir } W^2 = W_\mu W^\mu$$

Massive states $m^2 > 0$ ⇒ rest frame $P_\mu = (m,0,0,0) \Rightarrow W_0 = 0$,

$$W_i = \tfrac{1}{2}\varepsilon_{i0jk}\, m\, M^{jk} = m S_i,$$

$$S_i = \tfrac{1}{2}\varepsilon_{ijk}M^{jk} \text{ spin operator}$$

$$[S_i, S_j] = i\varepsilon_{ijk}S_k \quad SU(2) \text{ algebra}$$

$$W^2 = m^2 S^2 = m^2 s(s+1), \ s = 0, \tfrac{1}{2}, 1, \ldots$$

$\Rightarrow$ states characterized by $|m, s, s_3\rangle \quad |s_3| \leq s$

Massless $P^2 = 0$ frame $P_\mu = (\omega, 0, 0, \omega)$

$$W_0 = -W_3 = \omega M^{12} \qquad W_1 = \ldots \qquad W_2 = \ldots$$

W_0, W_1, W_2 generate the Euclidean group in $\mathbb{R}^2$

Representations characterized by $|\lambda\rangle \quad \lambda = 0, \pm \frac{1}{2}, \pm 1, \ldots$

<u>SUSY</u> (for simplicity $N = 1$)

$P_\mu P^\mu = m^2$ is still Casimir, W^2 is not

$$C_{\mu\nu} = B_\mu P_\nu - B_\nu P_\mu = -C_{\nu\mu} \qquad B_\mu = W_\mu - \frac{1}{4}(Q_\alpha)^+(\sigma_\mu)^{\alpha\beta} Q_\beta$$

$C^2 = \frac{1}{2} C_{\mu\nu} C^{\mu\nu}$ is a Casimir

<u>Massive</u> $P_\mu = (m, 0, 0, 0) \qquad W_0 = 0 \qquad B_0 = -\frac{1}{4} Q_\alpha^+ Q_\alpha$

$$\Rightarrow C_{ij} = 0, \quad C_{i0} = -m B_i = -m\left(m S_i - \frac{1}{4} Q_\alpha^+ (\sigma_i)^{\alpha\beta} Q_\beta\right)$$

$B_i = m J_i$ where $[J_i, J_j] = i\epsilon_{ijk} J_k$

$C^2 = m^2 j(j+1) \qquad j = 0, \frac{1}{2}, 1, \ldots$

$\Rightarrow$ states $|m, j, j_3\rangle$

Construct Clifford vacuum $|\Omega\rangle = Q_1 Q_2 |m, j, j_3\rangle \Rightarrow Q_\alpha |\Omega\rangle = 0 \quad \leftarrow Q_\alpha^2 = 0$

$$\{Q_\alpha, Q_\beta^+\} = 2\sigma^\mu_{\alpha\beta} P_\mu = 2\sigma^0_{\alpha\beta} P_0 = 2m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{\alpha\beta}$$

$b_\alpha \equiv \frac{1}{\sqrt{2m}} Q_\alpha \Rightarrow \{b_\alpha, b_\beta^+\} = \delta_{\alpha\beta}$

$b_\alpha |\Omega\rangle = 0 \Rightarrow \quad |\Omega\rangle \qquad b_\alpha^+ |\Omega\rangle \qquad b_1^+ b_2^+ |\Omega\rangle$

spin $\quad j \qquad j + \frac{1}{2} \qquad j$

$j = 0 \Rightarrow \quad 0 \qquad \frac{1}{2} \qquad 0$

scalar ϕ $\quad$ spinor ψ $\quad$ pseudoscalar χ $\Big\}$ massive Wess-Zumino multiplet

if $j > 0 \Rightarrow$ massive states with spin ≥ 1

There is no relativistic local renormalizable field theory containing massive particles with spin ≥ 1. ($\Rightarrow j > 0$ is not interesting)

In all SUSY theories, the # of physical on-shell states bosonic and the # of fermionic states is the same. (e.g. $1+1$ [illegible] vs 2 [illegible] [illegible])

<u>Massless states</u> $P_\mu = (\omega, 0, 0, \omega) \qquad N = 1 \quad (\Rightarrow X = 0)$

$$\{Q_\alpha, Q_\beta^+\} = 2\sigma^\mu_{\alpha\beta} P_\mu = 2\omega \begin{pmatrix} 1+1 & 0 \\ 0 & \underset{0}{\underset{\shortparallel}{(1-1)}} \end{pmatrix}_{\alpha\beta}$$

$$\Rightarrow \{Q_2, Q_2^+\} = 0 \underset{\substack{\uparrow \\ \text{positive norm states}}}{\Rightarrow} Q_2 \equiv 0$$

$\Rightarrow \{Q_1, Q_1^+\} = 1 \qquad \{b_1, b_1^+\} = 1$

$\Rightarrow \quad |\Omega\rangle \qquad b^+|\Omega\rangle$ in relativistic theories also similar

$\lambda \qquad \lambda + \frac{1}{2}$ multiplet with $-\lambda, -\lambda - \frac{1}{2}$ should be present

<u>Examples</u>: $\lambda = 0 \quad 0, \frac{1}{2}, -\frac{1}{2}, 0$ „massless WZ"

$$\lambda = -1 \quad -1, \underbrace{-\tfrac{1}{2}, +\tfrac{1}{2}}_{\psi}, \underset{A_\mu}{1} \quad \text{„vector multiplet"}$$

$$\lambda = \tfrac{3}{2} \quad \underbrace{\tfrac{3}{2}, -\tfrac{3}{2}}_{\text{gravitino } \psi_\mu}, \underbrace{2, -2}_{\text{graviton } G_{\mu\nu} = G_{\nu\mu}} \quad \text{graviton multiplet}$$

Extended SUSY $\quad N \neq 1 \quad X=0$

massive $\quad P_\mu = (m,0,0,0) \qquad \{Q_{\alpha A}, Q^+_{\beta B}\} = 2m \begin{pmatrix}1 & \\ & 1\end{pmatrix}_{\alpha\beta} \delta_{AB}$

$b_{\alpha A} = \frac{1}{\sqrt{2m}} Q_{\alpha A} \qquad \{b_{\alpha A}, b^+_{\beta B}\} = \delta_{\alpha\beta}\, \delta_{AB}, \quad A = 1,..,N$

$N=2 \qquad b_{\alpha A}|\Omega\rangle = 0$

$|\Omega\rangle, \; b^+_{\alpha A}|\Omega\rangle, \; b^+_{\alpha_1 A_1} b^+_{\alpha_2 A_2}|\Omega\rangle, \; b^+_{\alpha_1 A_1} b^+_{\alpha_2 A_2} b^+_{\alpha_3 A_3}|\Omega\rangle, \; b^+_{\alpha_1 A_1} .. b^+_{\alpha_4 A_4}|\Omega\rangle$

1 spin ~~[illegible]~~ s state, 4 spin s+1/2 states, 3 spin s+1 states + 1 spin s state, 4 spin s+1/2 states, 1 spin s state

again problem ~~renormalibility~~ renormalisability (for N > 1 and any s)

massless $\quad P_\mu = (\omega,0,0,\omega) \qquad \{Q_{\alpha A}, Q^+_{\beta B}\} = 2\omega \begin{pmatrix}2 & 0 \\ 0 & 0\end{pmatrix}_{\alpha\beta} \delta_{AB}$

$\Rightarrow Q_{2A} = 0 \qquad \{Q_{1A}, Q^+_{1A}\} = \cancel{8}\, \delta_{AB} \cdot 4\omega$

$\Rightarrow$ N creation operators $\; b^+_A = \frac{1}{2\sqrt{\omega}} Q^+_{1A}, \quad \{b_A, b^+_B\} = \delta_{AB}$

Examples: $N=2$

$\lvert\Omega\rangle$	$b^+_A\lvert\Omega\rangle$	$b^+_1 b^+_2\lvert\Omega\rangle$
λ	$\lambda+\frac{1}{2}$	$\lambda+1$

1,2 .. A indices, not spin

$\lambda=0 \quad 0 \quad \frac{1}{2} \quad 1$

$\lambda=-1 \quad \cancel{4}-1 \quad -\frac{1}{2} \quad 0$

$\Big\}$ $\underbrace{\text{2 scalars, 2 fermions and one vector}}$ vector multiplet of $N=2$

$\lambda = -\frac{1}{2}$ .. $N=2$ hypermultiplet

$N=4 \quad \lambda=-1 \qquad N=4$ vector multiplet

	$\lvert\Omega\rangle$	$b^+_A\lvert\Omega\rangle$	$b^+_{A_1} b_{A_2}\lvert\Omega\rangle$	$b^+_{A_1}..b^+_{A_3}\lvert\Omega\rangle$	$b^+_{A_1}..b^+_{A_4}\lvert\Omega\rangle$
~~of~~ spin λ	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
# of states	1	4	6	4	1

fermions 4+4

1+6+1 bosons
(6 scalars, 1 vector)

$N=8$ limit? $\qquad$ for $N>8$ $\exists$ states with spin (helicity) bigger than 2 in every multiplet, we don't know any such theory with finite number of particles with helicity > 2 which is consistent.

$\lambda = -2$

$\lvert\Omega\rangle$	$\lambda=-2$	1
$b^+_A\lvert\Omega\rangle$	$\lambda=-\frac{3}{2}$	8
$b^+_{A_1} b^+_{A_2}\lvert\Omega\rangle$	$\lambda=-1$	28
$b^+_{A_1} b^+_{A_2} b^+_{A_3}\lvert\Omega\rangle$	$\lambda=-\frac{1}{2}$	56
$b^+_{A_1}..b^+_{A_4}\lvert\Omega\rangle$	$\lambda=0$	70
	$\lambda=\frac{1}{2}$	56
	$\lambda=1$	28
	$\lambda=\frac{3}{2}$	8
	$\lambda=2$	1

$\underbrace{G_{\mu\nu} \text{ graviton}, \; \psi^i_\mu, \; i=1,..,8 \text{ gravitinos}, \; A^{[ij]}_\mu \text{ vector}, \; \lambda^{[ijk]} \text{ spinor}, \; \phi^{[ijk]} \text{ scalar}}$

Supermultiplet of $N=8$ supergravity

Bosonic string $\quad$ (4th lecture)

Path integral formulation

$$\mathcal{Z} = \int \mathcal{D}h_{\alpha\beta}\, \mathcal{D}X^\mu \; e^{-\frac{1}{4\pi\alpha'}\int \sqrt{h}\, h^{\alpha\beta}\, \partial_\alpha X^\mu\, \partial_\beta X_\mu\, d^2\sigma}$$

Faddeev - Popov

pick reference metric $\hat{h}_{\alpha\beta}$, then $h_{\alpha\beta}$ (general metric) obtained by variation of $\hat{h}$, $\delta h_{\alpha\beta} \propto \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha + \varphi h_{\alpha\beta}$

$[\mathcal{D} h_{\alpha\beta}] = \mathcal{D}\varphi \mathcal{D}\xi$ (Jacobian)

=> auxiliary fermionic variables (ghosts) with integer spin

ξ are vector fields promoted to ghosts

in euclidean signature $c^z, c^{\bar{z}}$ ghosts $\left(\begin{matrix} c^z \partial_z \text{ vector field} \\ b_{zz}(dz)^2 \text{ 2-form} \end{matrix} \right)$

$b_{zz}, b_{\bar{z}\bar{z}}$ antighosts

$$=> J = \int \mathcal{D} b_{zz} \mathcal{D} c^z \mathcal{D} b_{\bar{z}\bar{z}} \mathcal{D} c^{\bar{z}}\, e^{-\int (b_{zz} \partial_{\bar{z}} c^z + b_{\bar{z}\bar{z}} \partial_z c^{\bar{z}})}$$

=> det of a diff. operator, similarly after integration over Xs

Dependence of det on φ $\quad e^{-\frac{D-26}{24\pi}\int (\partial_\alpha \varphi) d^2\sigma}$

one wants that it doesn't depend on φ => $D=26$ in path-integral formulation

Spectrum

BRS charge $Q_{BRST}^2 = 0$ for $D=26$

physical spectrum is the cohomology of Q_{BRST}

Q_{BRST} written in terms of ghosts and stress-energy tensor

$c(z) = \Sigma c_n z^{-n-1}$ -> tachyon $c_1 \tilde{c}_1 |0,k\rangle$ => one can understand that $\overset{\text{intercept}}{a} = 1$

c has dimension $[\text{mass}]^{-1}$ whereas $[\partial_z X]$ has dim. $[\text{mass}]$

Strings moving in nontrivial background

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \underbrace{\sqrt{h}\, h^{\alpha\beta}}_{\text{can be gauged to flat metric}} \underbrace{G^{\mu\nu}(X)}_{\Rightarrow \text{no longer free theory}} \partial_\alpha X_\mu \partial_\beta X_\nu$$

is renormalizable action (because of two derivatives)

$G^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}(X)$ =>

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \left(\partial_\alpha X^\mu \partial^\alpha X_\mu + h_{\mu\nu}(x) \underbrace{\partial_\alpha X^\mu \partial^\alpha X^\nu}_{\text{can be considered}} \right)$$

a vertex op. for the graviton

We may add antisym. tensor & dilaton

$$S = -\frac{1}{4\pi\alpha'} \int dz\, d\bar{z} \Big[[G_{\mu\nu}(x) + i B_{\mu\nu}(x)]\, \partial_z X^\mu \partial_{\bar{z}} X^\nu + \alpha' R^{(2)}_{z\bar{z}} \underbrace{\Phi(X)}_{\text{dilaton}} \Big] \sqrt{h}\, h^{\alpha\beta}$$

if Φ constant => $\frac{1}{4\pi}\int R^{(2)} = 2 - 2g$ $\quad g$ genus of Riemann surface

$g=0$  $g=1$  $g=2$ 

if $\langle \Phi \rangle = \Phi_0$ => Φ_0 determines the loop expansion parameter of the string theory (string coupling constant)

=> Conformal invariance in general broken by nontrivial G, B, Φ

β-function $\quad \beta = \frac{d(\text{coupling})}{d\mu} \overset{!}{=} 0$ (conformal invariance)

1-loop results : 0/0

$H_{\mu\nu\rho} = \partial_\mu H_{\nu\rho} + \text{cyclic}$

$$\beta^G_{\mu\nu} = \alpha' R_{\mu\nu} + 2\alpha' \nabla_\mu \nabla_\nu \Phi - \frac{\alpha'}{4} H_{\mu\rho\sigma} H_\nu{}^{\rho\sigma} + \mathcal{O}(\alpha'^2)$$
$$\beta^B_{\mu\nu} = -\frac{\alpha'}{2} \nabla^2 H_{\lambda\mu\nu} + \alpha' (\nabla^2 \Phi) H_{\lambda\mu\nu} + \mathcal{O}(\alpha'^2)$$
$$\beta^\Phi = -\frac{\alpha'}{2} \nabla^2 \Phi + \alpha' (\nabla^2 \Phi) + \alpha' (\nabla\Phi)^2 - \frac{1}{24} \alpha' H^2 + (D-26)$$

$\Rightarrow$ requesting conformal invariance leads to $\beta^G_{\mu\nu} = \beta^B_{\mu\nu} = \beta^\Phi = 0$

$\Rightarrow$ equations of motion for the background, they can be also derived from the action (effective action)

$$S_{\text{spacetime}} \propto \int d^{26}X \sqrt{-G}\, e^{-2\Phi} \left(R - \frac{1}{12} H^2 + 4(\nabla\Phi)^2\right)$$

$G_{\mu\nu}$ = sigma model metric

$\rightarrow G^E_{\mu\nu}$: $G_{\mu\nu} = e^{-\Phi/6} G^E_{\mu\nu}$ Einstein metric

$$\Rightarrow S \propto \int d^{26}X \left(R^{(E)} - \frac{1}{12} e^{-\Phi/2} H^2 + \frac{1}{6}(\nabla\Phi)^2\right) \sqrt{-G^E}$$

Obvious solutions: (1) $G_{\mu\nu} = \eta_{\mu\nu}$, $B_{\mu\nu} = 0$, $\Phi = \text{const.}$ (trivial)

(2) $G_{\mu\nu} = \eta_{\mu\nu}$, $B_{\mu\nu} = 0$, $\underbrace{D \neq 26 \wedge \Phi \sim c X^0}_{\text{linear dilaton backgrounds}}$ (c compensates $(D-26)$)

Compactification to a circle (in a flat space)

$X^{25} = X^{25} + 2\pi R n$, $n \in \mathbb{Z}$ R - radius

$\Rightarrow$

$$X^{25} = x_{25} + \alpha' \frac{n}{R} \tau + m R \sigma + \overset{\text{as before}}{\boxed{\text{oscillators}}}$$

$\sigma \to \sigma + 2\pi k \Rightarrow X^{25} \to X^{25} + 2\pi m R k$ O.K.

n .. momentum m .. winding

$$X^{25}_L = \frac{1}{2}\left(\alpha' \frac{n}{R} + mR\right)(\tau - \sigma) + \text{oscillators} + \frac{1}{2} x^{25}$$
$$X^{25}_R = \frac{1}{2}\left(\alpha' \frac{n}{R} - mR\right)(\tau + \sigma) + \dots\dots$$
$$p_{L/R} = \sqrt{\frac{\alpha'}{2}} \frac{n}{R} \pm \sqrt{\frac{1}{2\alpha'}}\, mR$$

$$L_0 = \frac{1}{2} p_L^2 + N + \overbrace{\frac{p^2}{2}}^{\text{momenta corresp. other coordinates}} - 1$$
$$\tilde{L}_0 = \frac{1}{2} p_R^2 + \underbrace{\tilde{N}}_{\text{full } N, \tilde{N}} + \frac{p^2}{2} - 1$$

in non-compact case $L_0 - \tilde{L}_0 = 0 \Rightarrow N = \tilde{N}$

now $L_0 - \tilde{L}_0 = \frac{1}{2}(p_L^2 - p_R^2) + N - \tilde{N} \overset{!}{=} 0$

$\Rightarrow \boxed{N - \tilde{N} = n \cdot m}$ level-matching for compactified string

spectrum resembles Kaluza-Klein-like theory

T-duality $R \to \frac{\alpha'}{R}$ $\underset{\text{winding}}{m} \leftrightarrow \underset{\text{momenta}}{n}$

– symmetry of the bosonic string theory, holds also in perturbation theory accompanied by a transf. of dilaton:

$$\int d^{26}X\, e^{-2\Phi}\, \text{Ricci}^{26} = \int d^{25}X\, R\, e^{-2\Phi}\, \text{Ricci}^{25}$$

$\Rightarrow$ invariance of the physical 25-dim coupling enforces $\frac{\alpha'}{R} e^{-2\Phi'} = R e^{-2\Phi}$

i.e. $\Phi' = \Phi - \frac{1}{2} \ln \frac{R^2}{\alpha'}$

Superstrings (1st lecture)

$$S = -\frac{1}{2\pi}\int d^2\sigma\,\left(\partial_\alpha X^\mu \partial^\alpha X^\nu \eta_{\mu\nu} - i\bar\psi^\mu \rho^\alpha\partial_\alpha\psi^\nu\eta_{\mu\nu}\right) \qquad \left(\alpha' = \frac{1}{2}\right)$$

$\eta_{\mu\nu} = \mathrm{diag}(-1,1,..,1)$ $\quad d^2\sigma = d\sigma\, d\tau \quad 0 \le \sigma \le \pi$

$\eta_{\alpha\beta} = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$ $\quad -\infty < \tau < \infty$

$\mu = 0,1,..,D-1$ $\quad \psi^\mu$ Majorana fermions (vector-valued in the target space), ρ^α Γ-matrices

i.e.

NSR formulation - SUSY on the worldsheet $\quad \{\rho^\alpha,\rho^\beta\} = -2\eta^{\alpha\beta}$

$$\rho^0 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \qquad \rho^1 = \begin{pmatrix}0 & i\\ i & 0\end{pmatrix}$$

Free theory (of free fermions and free bosons)

$$\boxed{\delta X^\mu = \bar\epsilon\psi^\mu \qquad \delta\psi^\mu = -i\rho^\alpha\partial_\alpha X^\mu\epsilon}$$

SUSY transformation ~~$\delta\psi^\mu = \epsilon X^\mu \quad \delta X^\mu = -i\rho^\alpha\partial_\alpha X^\mu\epsilon$~~

ϵ is a constant Majorana spinor

using $\bar\chi\psi = \bar\psi\chi$ $\quad (\Leftarrow \bar\chi\psi = \chi^T\rho^0\psi)$

$\bar\chi\rho^\alpha\psi = -\bar\psi\rho^\alpha\chi$

one finds e.g. $\delta\bar\psi^\mu = i\bar\epsilon\rho^\alpha\partial_\alpha X^\mu$

$\Rightarrow$ Noether conserved currents:

$$\epsilon = \epsilon(\sigma,\tau) \rightarrow \delta S = \sim\frac{1}{2}\int\partial_\beta\bar\epsilon\,(\underbrace{\rho^\alpha\rho^\beta\partial_\alpha X^\mu\psi_\mu}_{2J^\beta})$$

$$\Rightarrow Q = \int_0^\pi d\sigma\cdot J^0$$

$$\Rightarrow [Q\bar\epsilon, X] = \bar\epsilon\psi$$

Equations of motion (EoM) $\boxed{\partial_\alpha\partial^\alpha X^\mu = 0}$ $\quad \boxed{i\rho^\alpha\partial_\alpha\psi^\mu = 0}$

Prop. of supercurrent $\rho_\alpha J^\alpha = \frac{1}{2}\underbrace{\rho_\alpha\rho^\beta\rho^\alpha}\partial_\beta X^\mu\psi_\mu = 0$ $\quad$ i.e. $\boxed{\rho_\alpha J^\alpha = 0}$

$$-\rho^\beta\rho_\alpha\rho^\alpha - 2\eta_\alpha{}^\beta\rho^\alpha = -2\rho^\beta + 2\rho^\beta = 0$$

on-shell (using eqn. of mot.) $\boxed{\partial_\alpha J^\alpha = 0}$

$$\left(\Leftarrow \partial_\alpha J^\alpha = \partial_\alpha\left(\frac{1}{2}\rho^\beta\rho^\alpha\partial_\beta X^\mu\psi_\mu\right) = \frac{1}{4}\underbrace{\{\rho^\beta,\rho^\alpha\}}_{-2\eta^{\beta\alpha}}\overbrace{\partial_\alpha\partial_\beta X^\mu}^{=0}\psi_\mu \right.$$
$$\left. + \frac{1}{2}\rho^\beta\underbrace{\rho^\alpha\partial_\beta X^\mu\partial_\alpha\psi_\mu}_{=0} = 0\right)$$

Energy-momentum $\quad T_{\alpha\beta} = \partial_\alpha X^\mu\partial_\beta X_\mu + \frac{i}{4}\bar\psi^\mu\rho_\alpha\partial_\beta\psi_\mu + \frac{i}{4}\bar\psi^\mu\rho_\beta\partial_\alpha\psi_\mu$

$- (\text{trace})$

$$\Rightarrow T_\alpha{}^\alpha = 0 \qquad \partial_\alpha T^{\alpha\beta} = 0 \text{ on-shell}$$

The action is already partially gauge fixed ($h = \eta$), without gauge fixing the action is

$$S = \frac{1}{2\pi}\int d^2\sigma\, e\left(h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu - i\bar\psi^\mu\rho^\alpha\nabla_\alpha\psi_\mu + \right.$$
$$\left. i\bar\chi_\alpha\rho^\alpha\rho^\beta\psi^\mu\partial_\beta X_\mu + \frac{1}{2}\bar\psi_\mu\psi^\mu\bar\chi_\alpha\rho^\beta\rho^\alpha\chi_\beta\right)$$

$e: e^a_\alpha\eta_{ab}e^b_\beta = h_{\alpha\beta}$ $\quad e = \det|e^a_\alpha| = \sqrt{|\det h_{\alpha\beta}|}$ $\quad \rho^\alpha = e^\alpha_b\rho^b$

χ_α 2D gravitino, non-dynamical

$\nabla_\alpha\psi^\mu = \partial_\alpha\psi^\mu + \frac{1}{4}\omega_\alpha{}^{ab}\rho_{ab}\psi$ $\quad \rho_{ab} = \frac{1}{2}[\rho_a,\rho_b]$

vielbein

Bosonic symmetries (4 bosonic parameters) $2\ \xi$'s, λ, Λ^{ab} (only one indep. component)

Diffeom. transf.

$$\delta X^\mu = \xi^\alpha \partial_\alpha X^\mu$$

$$\delta h_{\alpha\beta} = \xi^\gamma \partial_\gamma h_{\alpha\beta} + \partial_\alpha \xi^\gamma h_{\gamma\beta} + \partial_\beta \xi^\gamma h_{\alpha\gamma}$$

$$\delta \psi^\mu = \xi^\alpha \partial_\alpha \psi^\mu$$

Weyl symmetry

$$\delta h_{\alpha\beta} = \lambda h_{\alpha\beta}$$

Lorentz symmetry

$$\delta \psi = -\frac{1}{4} \Lambda^{ab} \rho_{ab} \psi$$

Fermionic symmetries (4 fermionic parameters) $2\ \varepsilon$'s, $2\ \eta$'s : **local SUSY**

$$\delta X^\mu = \bar{\varepsilon} \psi^\mu \qquad \delta \psi^\mu = -i \rho^\alpha \varepsilon (\partial_\alpha X^\mu - \bar{\psi}^\mu \chi_\alpha)$$

local fermionic symmetry

$$\delta \chi_\alpha = i \rho_\alpha \eta$$

Fix gauge: $e_\alpha^a = \delta_\alpha^a$, $\chi_\alpha = 0$ => previous form of action

Note: $\frac{\delta S}{\delta \chi^\alpha} \sim J^\alpha$ $\qquad \frac{\delta S}{\delta h^{\alpha\beta}} \sim T_{\alpha\beta}$ (before gauge fixing) (i.e. originally more complicated, (after gauge fixing) we obtain the eq. before)

=> constraints $J^\alpha = 0$, $T_{\alpha\beta} = 0$ (χ, h ~ Lagrange multipliers)

Light-cone coordinates

$$\sigma_\pm = \tau \pm \sigma \qquad \partial_\pm = \partial / \partial \sigma_\pm$$

=> EoM $\boxed{\partial_+ \partial_- X^\mu = 0}$ and see the next page

Digression: Dirac eq. $i\rho^\alpha \partial_\alpha \psi = 0$

$$i\rho^\alpha \partial_\alpha = \begin{pmatrix} 0 & \partial_0 - \partial_1 \\ -\partial_0 - \partial_1 & 0 \end{pmatrix} = 2 \begin{pmatrix} 0 & \partial_- \\ -\partial_+ & 0 \end{pmatrix}$$

$$\Rightarrow 0 = \begin{pmatrix} 0 & \partial_- \\ -\partial_+ & 0 \end{pmatrix} \psi = \begin{pmatrix} 0 & \partial_- \\ -\partial_+ & 0 \end{pmatrix} \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} = \begin{pmatrix} \partial_- \psi_+ \\ -\partial_+ \psi_- \end{pmatrix}$$

=> EoM $\boxed{\partial_- \psi_+ = 0 = \partial_+ \psi_-}$

$$\bar{\rho} = -\rho^0 \rho^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \bar{\rho} \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} = \begin{pmatrix} -\psi_- \\ +\psi_+ \end{pmatrix} \text{ chirality}$$

Constraints

$$T_{++} = \partial_+ X \cdot \partial_+ X + \frac{i}{2} \psi_+ \cdot \partial_+ \psi_+ \overset{!}{=} 0$$

$$T_{--} = \partial_- X \cdot \partial_- X + \frac{i}{2} \psi_- \cdot \partial_- \psi_- \overset{!}{=} 0$$

$$J_+ = \psi_+ \cdot \partial_+ X \overset{!}{=} 0$$

$$J_- = \psi_- \cdot \partial_- X \overset{!}{=} 0 \qquad \cdot \equiv (\ )^\mu (\ )_\mu$$

action in light-cone coords

$$S = -\frac{1}{2\pi} \int d\sigma_+ d\sigma_- \left(2\partial_+ X \cdot \partial_- X - i(\psi_- \partial_+ \psi_- + \psi_+ \partial_- \psi_+) \right)$$

Boundary conditions

closed superstrings X periodic

open superstrings $X' = \partial_\sigma X \big|_{\text{boundary}} = 0$ (or $X = \text{const}\big|_{\text{boundary}}$)

But fermions $(\psi_+ \delta\psi_+ - \psi_- \delta\psi_-)\big|_{\text{boundary}} = 0$

→ ψ periodic Ramond R / antiperiodic Neveu-Schwarz NS

→ $\psi_+(0,\tau) \pm \psi_-(0,\tau) = 0 \ \wedge$ $\psi_+(\pi,\tau) \pm \psi_-(\pi,\tau) = 0$

R and NS conditions don't mix in free theory => R, NS sectors

irrelevant

in open string: we fix the overall phase $\psi_+(0,\tau) = \psi_-(0,\tau)$

and have 2 possibilities : $\psi_+(\pi,\tau) = \psi_-(\pi,\tau)$ R

$\psi_+(\pi,\tau) = -\psi_-(\pi,\tau)$ NS

At closed string 2 sets of bosonic oscillators $X^\mu = f_+(\sigma_+) + f_-(\sigma_-)$

=> by diff. conditions on left and right-movers we have

<u>4 sectors</u> $\boxed{NS\text{-}NS,\ NS\text{-}R,\ R\text{-}NS,\ R\text{-}R}$ (we allow spinor to be $\psi(2\pi,\tau) = \pm\psi(0,\tau)$ $+ .. R$, $- .. NS$ as in repr. of $SO(3)$))

<u>Oscillator expansion</u>

<u>Closed ~~Open~~ superstrings</u>

$$X^\mu = x^\mu + p^\mu\tau + i\left(\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-2in(\tau\cancel{+\sigma}-\sigma)} + \sum_{n\neq 0}\frac{\tilde{\alpha}_n^\mu}{n}e^{-2in(\tau+\sigma)}\right)$$

$$R:\quad \psi_-^\mu = \sum_{n\in\mathbb{Z}} d_n^\mu e^{-2in(\tau-\sigma)}$$

$$\psi_+^\mu = \sum_{n\in\mathbb{Z}} \tilde{d}_n^\mu e^{-2in(\tau+\sigma)}$$

$$NS:\quad \psi_-^\mu = \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-2ir(\tau-\sigma)},\quad \psi_+^\mu = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \tilde{b}_r^\mu e^{-2ir(\tau+\sigma)}$$

by different combinations we have R-R, .., NS-NS

<u>Open superstrings</u> $X^\mu = x^\mu + p^\mu\tau + i\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-in\tau}\cos n\sigma$

$$R:\quad \psi_-^\mu(\sigma,\tau) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d_n^\mu e^{-in(\tau-\sigma)},\quad \psi_+^\mu(\sigma,\tau) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d_n^\mu e^{-in(\tau+\sigma)}$$

$$NS:\quad \psi_-^\mu(\sigma,\tau) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-ir(\tau-\sigma)},\quad \psi_+^\mu(\sigma,\tau) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-ir(\tau+\sigma)}$$

$0 \le \sigma \le \pi$, coefficients in exp are periodic in $\langle -\pi,\pi\rangle$ => we extend ψ ~~symmetrically~~ to $\langle -\pi, 0)$ and then expand the redefined $\hat{\psi}$ and consequently restrict them to $\langle 0,\pi\rangle$

$$\hat{\psi}(\sigma,\tau) = \begin{cases}\psi_+(\sigma,\tau), & 0 \le \sigma \le \pi\\ \psi_-(-\sigma,\tau), & 0 \ge \sigma \ge -\pi\end{cases}$$

<u>Introduction to Supersymmetry</u> (3rd lecture)

<u>Central charges</u> in $N > 1$ SUSY algebra

Reminder : $N=4$ SUSY $Q_{\alpha A}$, $A = 1,..,4$

$$\{Q_{\alpha A}, Q^+_{\beta B}\} = 2\sigma^\mu_{\alpha\beta}P_\mu\delta_{AB}$$

massive states $P_\mu = (m,0,0,0)$ rest frame

$$\{Q_{\alpha A}, Q_{\cancel{\beta B}}\} = 2m\,\delta_{\alpha\beta}\,\delta_{AB}$$ 8 harm. oscillators

$$Q_{\alpha A}|\Omega\rangle = 0$$

$$N=8 \Rightarrow |\Omega\rangle,\ Q^+_{\alpha A}|\Omega\rangle,\ \dots,\ Q^+_{\alpha_1 A_1}\dots Q^+_{\alpha_8 A_8}|\Omega\rangle$$

Δ $\Delta+1/2$ $\Delta+1\dots$

for any s, incl. $s=0$, we have massive states with spin $\geqslant 2$.

There is no renormalizable QFT with elementary spin 2 or higher states.

Massless states of $N=4$ $\quad A_\mu, \phi^{[ij]}, \psi^i, i=1,..,4$ (vector multiplet)

$\quad\quad\quad\quad\quad\quad\quad\quad\quad 1 \quad 6 \quad \cdot \quad 4$

<u>Second fact</u>: Construct a $SU(2)$ gauge theory using massless $N=4$ vector multiplets

$$\mathcal{L} = -\frac{1}{4g^2} \operatorname{tr} F_{\mu\nu} F^{\mu\nu} + \operatorname{tr} \nabla_\mu \phi \nabla^\mu \phi - V(\phi) + \text{fermionic terms}$$

e.g. $\phi^{[ij]} = \phi_a^{[ij]} T^a \quad T^a$.. basis of adjoint repr. of $SU(2)$

$A_\mu = A_{\mu a} T^a \quad\quad \nabla_\mu \phi = \partial_\mu \phi + [A_\mu, \phi]$

<u>Third fact</u>: Spontaneous symmetry breaking $\quad \langle \Phi \rangle \neq 0 \quad$ Higgs-Kibble

$\Rightarrow$ some of the vector fields become massive (Higgs mechanism)

symmetry $SU(2)$ is spontaneously broken to $U(1)$ but SUSY is not broken $\Rightarrow$ We shall have an $N=4$ $U(1)$ gauge multiplet involving massive vector field

Note: SSB doesn't broke renormalizability

! BUT: we don't have a suitable massive supermultiplet in $N=8$ for renormalizability

$\Rightarrow$ <u>Need for central charges</u>, these may cure the problem.

$$\{Q_{\alpha A}, Q_{\beta B}\} = \varepsilon_{\alpha\beta} X_{AB}$$

In massless case X_{AB} not allowed $\quad P$ mass dimension of $1 \Rightarrow Q$s have mass dim. $1/2 \Rightarrow X$ should have mass dim $1 \Rightarrow$ not allowed

also $P_\mu = (\omega, 0, 0, \omega) \quad\quad Q_{2A} = 0 \Rightarrow X = 0$

<u>Massive states</u> (we for simplicity assume N even, otherwise an extra term occurs)

Zumino theorem: $\exists$ unitary matrix U such that $X_{AB} \rightarrow X'_{AB} = U_A{}^C U_B{}^D X_{CD}$

where
$$X' = \begin{pmatrix} Z_1 \varepsilon_{ab} & & 0 \\ & Z_2 \varepsilon_{ab} & \\ 0 & & \ddots \; Z_{\frac{N}{2}} \varepsilon_{ab} \end{pmatrix} \quad \begin{aligned} &a,b = 1,2 \\ &\varepsilon_{12} = +1 \\ &\quad = -\varepsilon_{21} \end{aligned}$$

without proof

def. indices $A = (a, r) \quad\quad a = 1,2 \,;\, r = 1,..,N/2$

$$\Rightarrow \{Q_{\alpha a r}, Q_{\beta b s}\} = \varepsilon_{\alpha\beta} \underset{\uparrow \text{ maybe } \varepsilon_{ab}}{\delta_{ab}} \delta_{rs} Z_r$$

$$P_\mu = (m,0,0,0) \Rightarrow \{Q_{\alpha a r}, Q^+_{\beta b s}\} = 2m\, \delta_{\alpha\beta} \delta_{ab} \delta_{rs}$$

$$b_{\alpha r \pm} = \frac{1}{2}(Q_{\alpha 1 r} \pm Q^+_{\alpha 2 r}) \quad\quad \alpha = 1,2$$

$$\Rightarrow \{b_{\alpha r \pm}, b^+_{\beta s \pm}\} = \delta_{\alpha\beta} \delta_{rs} (\overset{\text{maybe } 2m}{m} \pm Z_r)$$

$-$ case $\Rightarrow \langle \; | \; \rangle$ should be positive $\Rightarrow m - Z_r \geqslant 0 \Rightarrow \boxed{Z_r \leq m}$

1) $Z_r < m \Rightarrow 2\left(\frac{N}{2} + \frac{N}{2}\right) = 2N$ oscillators as in $X_{AB} = 0$ case.

$\quad$ - not particulary interesting

2) $Z_r = m \quad \{b_{\alpha r -}, b^+_{\beta s -}\} = 0 \Rightarrow$ the positive norm space condition eliminates them $b_{\alpha r -} \equiv 0$

$\Rightarrow$ "BPS-states" ... only N oscillators $b_{\alpha r+}$ $\Rightarrow 2^N$ states

$$\{ b_{\alpha r+}, b^+_{\beta s+} \} = 2m\, \delta_{\alpha\beta}\, \delta_{rs} \qquad r,s = 1,..,N/2$$

$\Rightarrow$ massive $N=4$ supermultiplets which are shorter than the ones when $X=0$

- after SSB the massive multiplets should be short and this eliminates massive particles with spin ≥ 2.

$D=11$

?M-theory? $\longrightarrow$ low energy limit $D=11$ supergravity

Yesterday $N=8$ in $D=4$ spectrum $G_{\mu\nu}$, ψ^i_μ, $A_\mu^{[ij]}$, $\lambda^{[ijk]}$, $\phi^{[ijkl]}$

$i=1,..,8$ (times 2 .. helicity) 8 56 70

a bound on number of dimensions since we want to ~~not time~~ reobtain the $D=4$ multiplets by ~~compactification or~~ dimensional reduction

in $D=11$ spinors real, 32 components $\longrightarrow Q_\alpha^+ = Q_\alpha$

$\alpha\beta = 1,..,32$: $\{Q_\alpha, Q_\beta\} = \underbrace{(C\Gamma^\mu)_{\alpha\beta}}_{\text{symmetric in } \alpha\beta} P_\mu$ (+plus eventually other terms)

, in suitable formulation $C = \Gamma^0$

massless states $P_\mu = (\omega, \omega, 0, .., 0)$ invariant under SO(9)

$$\{Q_\alpha, Q_\beta\} = \omega(1+\Gamma^0\Gamma^1)_{\alpha\beta} \qquad (\Gamma^0\Gamma^1)^2 = 1,\ \mathrm{tr}\,\Gamma^0\Gamma^1 = 0$$

$\Rightarrow$ sixteen $+1$ eigenvectors, 16 -1 eigenvectors

$$\Rightarrow \quad 1+\Gamma^0\Gamma^1 = \begin{pmatrix} 1+1 & 0 \\ 0 & \underbrace{1-1}_{0} \end{pmatrix} \Rightarrow Q_\alpha = 0 \quad \alpha = 17,..,32$$

$$\boxed{\{Q_\alpha, Q_\beta\} = 2\omega\, \delta_{\alpha\beta}}, \quad \alpha,\beta = 1,..,16$$

$$b_i = \frac{1}{\sqrt{2\omega}}(Q_i + iQ_{8+i}) \qquad b_i^+ = \frac{1}{\sqrt{2\omega}}(Q_i - iQ_{8+i})$$

$$\Rightarrow \boxed{\{b_i, b_j^+\} = \delta_{ij}} \qquad i,j = 1,..,8$$

$\Rightarrow$ in a same way as in $N=8$ $D=4$ we generate 128 Bose and 128 Fermi states (considering both chiralities)

Superstrings (2nd lecture)

We have oscillators α_n^μ, b_n^μ, d_n^μ (for open strings)

Moments $$L_m = \frac{1}{\pi}\int_{-\pi}^{\pi} d\sigma\, e^{im\sigma}\, \hat{T} \qquad \hat{T} = \begin{cases} T_{++}(\sigma), & 0 \leq \sigma \leq \pi \\ T_{--}(-\sigma), & -\pi \leq \sigma \leq 0 \end{cases}$$

$$= \frac{1}{\pi}\int_0^\pi d\sigma \left(e^{im\sigma} T_{++}(\sigma) + e^{-im\sigma} T_{--}(\sigma)\right)$$

$$\Rightarrow L_m = \frac{1}{2}\sum_k \alpha_{m-k} \cdot \alpha_k + \frac{1}{2}\sum_k \left(k + \frac{1}{2}m\right) d_{-k}\, d_{m-k} \text{ in R sector}$$

$$L_m = \frac{1}{2}\sum_k \alpha_{m-k} \cdot \alpha_k + \frac{1}{2}\sum_{r \in \mathbb{Z}+\frac{1}{2}} \left(r + \frac{1}{2}m\right) b_{-r} \cdot b_{m+r} \text{ in NS sector}$$

$$\hat{J} = \begin{cases} J_+, & \pi \geq \sigma \geq 0 \\ J_-, & -\pi \leq \sigma \leq 0 \end{cases}$$

$$m \in \mathbb{Z}: \quad F_m = -\frac{\sqrt{2}}{\pi}\int_{-\pi}^{\pi} d\sigma\, e^{im\sigma}\, \hat{J} = \sum_k \alpha_{-k} \cdot d_{m+k} \quad \text{in (R)}$$

$$r \in \mathbb{Z}+\tfrac{1}{2}: \quad G_r = \frac{\sqrt{2}}{\pi}\int_{-\pi}^{\pi} d\sigma\, e^{ir\sigma}\, \hat{J} = \sum_k \alpha_{-k} \cdot b_{r+k} \quad \text{in (NS)}$$

L_n, F_n, G_r are conserved ⇒ don't depend on time

⇒ constraints $L_n = 0, \quad F_n = 0, \quad G_r = 0$

Canonical quantisation $\left[\dot{X}^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\right] = -i\pi\,\delta(\sigma-\sigma')\,\eta^{\mu\nu}$

$\{\psi^\mu_A(\sigma,\tau), \psi^\nu_B(\sigma',\tau)\} = \pi\,\delta(\sigma-\sigma')\,\eta^{\mu\nu}\delta_{AB}$

$$\Rightarrow \quad [\alpha^\mu_m, \alpha^\nu_n] = m\,\delta_{m+n,0}\,\eta^{\mu\nu} \qquad \text{using } \frac{1}{2\pi}\int_{-\pi}^{\pi} d\sigma\, e^{i(n-n')\sigma} = \delta_{n,n'}$$

$$\{b^\mu_r, b^\nu_s\} = \eta^{\mu\nu}\delta_{r+s,0} \quad (NS)$$

$$\{d^\mu_m, d^\nu_n\} = \eta^{\mu\nu}\delta_{n+m,0} \quad (R)$$

reality cond. $\Rightarrow (\alpha^\mu_m)^+ = \alpha^\mu_{-m}$

$(b^\mu_r)^+ = b^\mu_{-r}$

$(d^\mu_m)^+ = d^\mu_{-m}$

Vacuum $|0\rangle$:

$\alpha^\mu_n|0\rangle = 0, \quad \forall n \geq 0$

$(\alpha^\mu_0 = p^\mu)$

$|0,k\rangle \;..\; \alpha^\mu_0|0,k\rangle = k^\mu|0,k\rangle \quad \left(|0,k\rangle \approx \overset{\text{vacuum w.r.t. oscillator}}{|0\rangle} \otimes e^{ikX} \text{ if we represent } p^\mu = -i\frac{\partial}{\partial x_\mu}\right)$

$b^\mu_r|0\rangle = 0$

$d^\mu_n|0\rangle = 0 \quad n > 0 \quad \left(\{d^\mu_0, d^\mu_0\} = \eta^{\mu\nu} \Rightarrow n > 0\right)$

d^μ_0 behave like Γ matrices (up to rescaling)

⇒ vacuum must be a representation of Γ-matrix algebra

⇒ vacuum in ~~NS~~ R sector is a spacetime fermion .. $|\phi\rangle^{a}$ ← spinor index

Normal ordering of L_n, F_n, G_r e.g. $F_n = \sum :\alpha_{-k}\cdot d_{n+k}:$ etc.

Creation ops. $\alpha^{\mu+}_n \quad n > 0$

$b^{\mu+}_r \quad r > 0$

$d^{\mu+}_m \quad m > 0$

Super Virasoro algebra

NS $[L_n, L_m] = (n-m)L_{n+m} + \frac{D}{8}(n^3-n)\,\delta_{n+m,0}$

$[L_n, G_r] = \left(\frac{1}{2}n - r\right)G_{n+r}$

$\{G_r, G_s\} = 2L_{r+s} + \frac{D}{2}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}$

R $[L_n, L_m] = (n-m)L_{n+m} + \frac{D}{8}n^3\,\delta_{n+m,0}$

$[L_n, F_m] = \left(\frac{1}{2}n - m\right)F_{n+m}$

$\{F_n, F_m\} = 2L_{n+m} + \frac{D}{2}n^2\,\delta_{n+m,0}$

(total centre charge $\frac{c}{12}$, $c = \overset{\text{boson}}{D} + \overset{\text{Majorana spinors}}{\frac{1}{2}D} = \frac{3}{2}D$)

Note on BRS quantisation ghosts corresp. to bosonic transf. .. usual anticommuting bosons

— // — to fermionic —//— .. commuting ghosts

$$+\frac{1}{\pi}\int_0^\pi d\sigma\,(b_{++}\partial_- c^+ + c^+\partial_- b_{++}) + \frac{1}{\pi}\int_0^\pi d\sigma\,(\beta_+\partial_-\gamma^+ + \gamma^+\partial_-\beta_+)$$

b .. weight $\lambda = 2$ $\qquad\beta$ .. weight $\lambda\,\frac{3}{2}$

$c_b = -2(6\lambda^2 - 6\lambda + 1) = -26 \qquad c_\beta = 2(6\lambda^2 - 6\lambda + 1) = 11$

$\Rightarrow \quad c = \frac{3}{2}D - 26 + 11 = \frac{3}{2}D - 15 \Rightarrow c = 0 \text{ if } \boxed{D = 10}$

↑ complete central charge

Problem with negative norm states $\| \xi_\mu \alpha^\mu_n |0\rangle \|^2 \sim \xi_\mu \cdot \xi^\mu$

We select Fock space states which satisfy the following physicality condition

(NS) $L_n|\phi\rangle = 0 \quad , \forall n>0$

$L_0|\phi\rangle = a_{NS}|\phi\rangle$ $\quad$ a_{NS} because of normal ordering

(NS) $G_r|\phi\rangle = 0 \quad , \forall r>0$

(R) $L_n|\phi\rangle=0, \forall n>0$ $\quad L_0|\phi\rangle = a_R|\phi\rangle \quad F_n|\phi\rangle=0, \forall n\geq 0$

normal ordering in F_n is insignificant

but $2L_0=\{F_0,F_0\} \Rightarrow \boxed{a_R=0}$

since $L_0|\phi\rangle = F_0^2|\phi\rangle = 0$

Mass operator

$L_0 = \frac{1}{2}\sum_{k\in\mathbb{Z}} :\alpha_{-k}\cdot\alpha_k: + \frac{1}{2}\sum_{k\in\mathbb{Z}} k :d_{-k}\cdot d_k:$ in R sector

i.e. $L_0 = \frac{1}{2} p^2 + \sum_{k>0}\alpha_{-k}\cdot\alpha_k + \sum_{k>0} k\, d_{-k}\cdot d_k$ where $p^2=\alpha_0^\mu\alpha_{0\mu}$

$\Rightarrow M^2=-p^2$ gives e.g. in NS sector

$$M^2 = -2a_{NS} + \sum_{k>0}\alpha_{-k}\cdot\alpha_k + \sum_{r>0} r\, b_{-r}\cdot b_r$$

$\Rightarrow M^2|0\rangle = -2a_{NS}|0\rangle$

Light-cone quantization

Residual gauge symmetry (⇐ ∞ number of conserved quantities)

$X^\pm = \frac{X^0\pm X^{D-1}}{\sqrt{2}} \qquad \psi^\pm = \frac{\psi^0\pm\psi^{D-1}}{\sqrt{2}}$

We fix

$X^+(\sigma,\tau) = x^+ + p^+\tau$, to be consistent with SUSY we have

$\psi^+(\sigma,\tau)=0$

$$\partial_+X\cdot\partial_+X = -2\underbrace{\partial_+X^+}_{p^+}\partial_+X^- + \partial_+X^i\cdot\partial_+X^i + \dots$$

$$\Rightarrow \partial_+X^- = \frac{1}{2p^+}\sum_i\left(\partial_+X^i\cdot\partial_+X^i + \frac{i}{2}\psi_+^i\cdot\partial_+\psi_+^i\right)$$

$$\psi^- = \frac{1}{p^+}\sum_i \psi_+^i\partial_+X^i$$

$\Rightarrow$ only transverse degrees of freedom are independent

α_n^i, d_m^i, b_r^i

but Lorentz symmetry is broken explicitly, [illegible] may be effectively restored on quantum level

Compute the LG algebra in terms of transverse oscillators and check whether it is closed.

$$M^{\mu\nu} = \underbrace{\int_0^\pi d\sigma\left(X^\mu X^\nu - X^\nu X^\mu + \overset{\text{fermionic part}}{\dots}\right)}$$

quadratic expression in oscillators & $\alpha_n^i, \underbrace{d_m^i, b_r^i}_{\text{one or the other depending on the sector}}$

$[M^{\mu\nu}, M^{\rho\sigma}] = \dots\dots$

There is only one trouble $0 \overset{!}{=} [M^{i-}, M^{j-}] = \frac{1}{p_+^2}\sum_{k=1}^{\infty} \Delta_k\left(\alpha_{-k}^i\alpha_k^j - \alpha_{-k}^j\alpha_k^i\right)$

all other Ms commute as they should

where $\Delta_k = \left(\frac{D-10}{8}\right) + \frac{1}{k}\left(2a_{NS} - \frac{D-2}{8}\right)$

$\Rightarrow$ we find that must be $D=10$, $a_{NS}=\frac{1}{2}$

Superstrings (3rd lecture)

Open string case:

Spectrum (in $D=10$) creation op. $\alpha_n^{\mu +}$, $b_r^{\mu +}$ resp. $d_n^{\mu +}$

constraints: $L_n|\phi\rangle = 0, n>0$ $L_0|\phi\rangle = \frac{1}{2}|\phi\rangle$ in (NS)

and $G_r|\phi\rangle = 0$ $r>0$

in R $L_n|\phi\rangle = 0, n>0, L_0|\phi\rangle = 0, F_m|\phi\rangle \geq 0$

light-cone quantisation ⇒ only transverse degrees of freedom survive

⇒ no negative norm states, we assume that this is true also in covariant quant. after imposing constraints

$$-M^2 = p^2 = \alpha_0^2 \qquad L_0 = \frac{1}{2}\alpha_0^2 + \underbrace{\sum_{k>0} \alpha_{-n}\cdot\alpha_n + \sum_{r>0} r\, b_{-r}\cdot b_r}_{\text{number op. } N_{NS}} \quad (NS)$$

$[N_{NS}, \alpha_{-n}^{\mu}] = n\,\alpha_{-n}^{\mu}, \ldots$

$$L_0 = \frac{1}{2}\alpha_0^2 + \underbrace{\sum_{n>0} \alpha_{-n}\cdot\alpha_n + \sum_{m>0} m\, d_{-m}\cdot d_m}_{\text{number op. } N_R} \quad (R)$$

$[N_R, \alpha_{-m}^{\mu}] = m\alpha_{-n}^{\mu}, [N_R, d_{-m}^{\mu}] = m d_{-m}^{\mu}$

$$\Rightarrow \quad M^2 = 2\left(-\tfrac{1}{2} + N_{NS}\right) \quad \text{resp. } M^2 = 2N_R$$

Level – eigenvalue of N

$\boxed{(NS)}$ $N_{NS} = 0$ $|0,k\rangle = |0\rangle_{NS}\, e^{ikx} \Rightarrow \alpha' M^2 = -\frac{1}{2}$

Note by reintroducing α' by dim. analysis (above is $\alpha' = \frac{1}{2}$)

$$\boxed{\alpha' M^2 = -\tfrac{1}{2} + N_{NS} \text{ resp. } \alpha' M^2 = N_R}$$

$M \sim \frac{1}{\sqrt{\alpha'}} \sim$ Planck mass, in (NS) is a tachyon

$N_{NS} = \frac{1}{2}$ $b_{-\frac{1}{2}}^{\mu}|0,k\rangle$, $\alpha' M^2 = 0$ massless vector state

$N_{NS} = 1$ $\alpha_{-1}^{\mu}|0,k\rangle$, $b_{-\frac{1}{2}}^{\mu} b_{-\frac{1}{2}}^{\nu}|0,k\rangle$, $\alpha' M^2 = \frac{1}{2}$ massive state

$\vdots$

$\boxed{(R)}$ $N_R = 0$ vacuum a fermion $|\phi^a\rangle$ (repr. of $\{d_0^{\mu}, d_0^{\nu}\} = \eta^{\mu\nu}$)

$\alpha' M^2 = 0$ massless spinor

$N_R = 1$ $\alpha_{-1}^{\mu}|\phi^a\rangle$, $d_{-1}^{\mu}|\phi^a\rangle$ $\alpha' M^2 = 1$ massive spinor

Constraints e.g. $G_{1/2}\, \xi_{\mu}\, b_{-\frac{1}{2}}^{\mu}|0,k\rangle = 0 \ldots\ldots \xrightarrow{\text{after computations}} k\cdot\xi = 0$

other constraints imply connection between k and ξs saturating the spacetime $(\mu,\nu,\ldots)$ indices

$F_0|\phi^a\rangle = 0$ $\quad F_0 = \ldots + d_{-1}\alpha_1 + \underbrace{\alpha_0 d_0}_{p\cdot\Gamma} + a_{-1}d_1 + \ldots$

$\Rightarrow \underbrace{p\cdot\Gamma|\phi^a\rangle = 0}$

Dirac equation, in generality $F_0|\psi\rangle = 0$ imposes a generalisation of Dirac equation

GSO projection

solves the tachyon problem, restores the SUSY in the spectrum and also removes the puzzle that $b_{-\frac{1}{2}}^{\mu}$ applied on a bosonic state again seems to be a boson

$$\boxed{P_{NS} = \frac{(-1)^{F_{NS}}+1}{2}}$$

$-(-1)^{F_{NS}} = (-1)^{\mathcal{B}}$ $\mathcal{B} = \sum_{r>0} b_{-r} \cdot b_r$

$\mathcal{B}$ counts the number of b's, anticommutes with b's $\{(-1)^{F_{NS}}, b_r^{\mu}\} = 0$

(NS) $N_{NS} = 0$ tachyon $(-1)^{F_{NS}} |0,k\rangle = -|0,k\rangle$

$\Rightarrow P_{NS} |0,k\rangle = 0$ mapped out

$N_{NS} = \frac{1}{2}$ $b^{\mu}_{-\frac{1}{2}} |0,k\rangle$ $(-1)^{F_{NS}} b^{\mu}_{-\frac{1}{2}} |0,k\rangle = b^{\mu}_{-\frac{1}{2}} |0,k\rangle$

$\Rightarrow P_{NS} b^{\mu}_{-\frac{1}{2}} |0,k\rangle = b^{\mu}_{-\frac{1}{2}} |0,k\rangle$ survives

$N_{NS} = 1$ projected out

to attain SUSY we need projection also in R-sector

(R) $(-1)^{F_R} = \Gamma_{11} (-1)^{\mathcal{D}}$ $\mathcal{D} = \sum_{n>0} d_{-n} \cdot d_n$

$\uparrow$ chirality matrix $\Gamma_0 \dots \Gamma_9$

we assume $\Gamma_{11} |\phi^a\rangle = |\phi^a\rangle$ (definite chirality of vacuum)

$\uparrow$ $d_0^{\mu} |\phi^a\rangle$ projected out because of anticommutator

$$\boxed{P_R = \frac{(-1)^{F_R}+1}{2}}$$

$N_R = 0$ survives

$N_R = 1$ $\alpha^{\mu}_{-1} |\phi^a, k\rangle$ survives, $d^{\mu}_{-1} |\phi^a, k\rangle$ projected out but $d^{\mu}_{-1} d^{\nu}_0 |\phi, k\rangle$ seems to survive (also it seems needed for SUSY at this level)

<u>Complete spectrum</u> after GSO projection

massless: physical degrees of freedom   massless states: little group SO(8)

(NS) $b^{\mu}_{-\frac{1}{2}} |0,k\rangle$ $b^{i}_{-\frac{1}{2}} |0,k\rangle$, $i = 1,..,8$ 8_v repr. of SO(8)

(R) $|\phi^a, k\rangle$ $|\phi^a, k\rangle$ $a = 1,..,8$ 8_s or 8_c –〃–

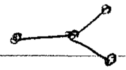
ad fermions   dim $= 2^{D/2} = 32$ (complex spinors)

Majorana condition $\Rightarrow$ 32 real components

Weyl –〃– (definite chirality) $\Rightarrow$ 16 real components

Dirac eqn. $\Gamma \cdot D |\phi\rangle = 0$ linear equations, they have 8 dim. nullspace $\Rightarrow$ $\boxed{\text{8 real components}}$

<u>Note</u>: SO(8)



$\Rightarrow$ three 8-dim. representations

$\Rightarrow$ Spectrum might be supersymmetric. (Complete check complicated.)

<u>Partition function</u> $f(q) \overset{!}{=} \mathrm{Tr}\, q^N = \sum_n d(n)\, q^n$

over phys. states   # of states at level n

$$f_{NS}(q):\ f_{NS}(q) = \frac{1}{2\sqrt{q}} \left(\prod_{k=1}^{\infty} \left(\frac{1+q^{k-\frac{1}{2}}}{1-q^k}\right)^8 - \left(\frac{1-q^{k-\frac{1}{2}}}{1-q^k}\right)^8 \right) \dots$$

$$= 8(1 + 16 q + \dots)$$

$$f_R(q) = 8 \prod_{k=1}^{\infty} \left(\frac{1+q^k}{1-q^k}\right)^8 = 8(1 + 16 q + \dots)$$

surprisingly the same

$\Rightarrow$ number of states bosonic and fermionic on each level match

Closed superstrings

left + right	$\overbrace{\alpha^\mu_n, b^\mu_n, d^\mu_n}^{+}$	$\overbrace{\tilde\alpha^\mu_n, \tilde b_n, \tilde d_n}^{-}$	
NS − NS	$L_0\lvert\phi\rangle = \frac{1}{2}\lvert\phi\rangle$	$\tilde L_0\lvert\phi\rangle = \frac{1}{2}\lvert\phi\rangle$	$L_0+\tilde L_0 = 1,\ L_0-\tilde L_0 = 0$
NS − R	$L_0\lvert\phi\rangle = \frac{1}{2}\lvert\phi\rangle$	$\tilde L_0\lvert\phi\rangle = 0$	$L_0+\tilde L_0 = \frac{1}{2},\ L_0-\tilde L_0 = \frac{1}{2}$
R − NS	$L_0\lvert\phi\rangle = 0$	$\tilde L_0\lvert\phi\rangle = \frac{1}{2}\lvert\phi\rangle$	$L_0+\tilde L_0 = \frac{1}{2},\ L_0-\tilde L_0 = -\frac{1}{2}$
R − R	$L_0\lvert\phi\rangle = 0$	$\tilde L_0\lvert\phi\rangle = 0$	$L_0+\tilde L_0 = 0,\ L_0-\tilde L_0 = 0$

in any sector $L_n\lvert\phi\rangle = 0 \ \forall n>0$, $\tilde L_n\lvert\phi\rangle = 0 \ \forall n>0$

similar conditions on $G_n, F_n, \tilde G_n, \tilde F_n$ in each sector

$$L_0+\tilde L_0 \sim \frac{1}{2}\alpha_0^2 + \frac{1}{2}\tilde\alpha_0^2 = \frac{p^2}{4} = -\frac{M^2}{4}$$

since $\alpha_0 = \tilde\alpha_0 = \frac{p}{2}$

$\Rightarrow$ NS − NS $\quad \frac{M^2}{4} = -1 + N_{NS} + \tilde N_{NS} \qquad N_{NS} = \tilde N_{NS} \quad (\Leftarrow L_0 - \tilde L_0 = 0)$

NS − R $\quad M^2/4 = -\frac{1}{2} + N_{NS} + \tilde N_R \qquad N_{NS} = \tilde N_{\cancel{NS}R} + \frac{1}{2}$

R − NS $\quad M^2/4 = -\frac{1}{2} + N_R + \tilde N_{NS} \qquad N_R = \tilde N_{NS} - \frac{1}{2}$

R − R $\quad M^2/4 = N_R + \tilde N_R \qquad N_R = \tilde N_R$

where

$$N_{NS} = \sum_{n>0} \alpha_{-n}\cdot\alpha_n + \sum_{n>0} n\, b_{-n}\cdot b_n \qquad \tilde N_{NS} = \widetilde{\ldots\ldots} = N_{NS}(\alpha\to\tilde\alpha, b\to\tilde b)$$

$$N_R = \sum_{n>0} \alpha_{-n}\cdot\alpha_n + \sum_{n>0} n\, d_{-n}\cdot d_n \qquad \tilde N_R = N_R(\alpha\to\tilde\alpha, d\to\tilde d)$$

again tachyon $\Rightarrow$ GSO projection on both sides separately

$$P_{NS} = \frac{(-1)^{F_{NS}}+1}{2} \qquad P_R = \frac{(-1)^{F_R}+1}{2} \qquad \tilde P_{NS} = \ldots \quad \tilde P_R = \ldots$$

$\Rightarrow$ completely SUSY spectrum

Introduction to Supersymmetry (4th lecture)

Clifford vacuum $b_i\lvert\Omega\rangle = 0$

How we assemble 128 bosons and 128 fermions into irreps SO(9)

Bosons: We should have a graviton G^t_{ij} (traceless) $\frac{9\times 10}{2} - 1 = 44$ mass-shell (physical) degrees of freedom $\searrow$ G_{MN} in $D=11$ (i,j physical degrees of freedom!)

remaining 84 degrees of freedom

A_{ijk} … 84 degrees of freedom … gauge field

$\hookrightarrow A_{MNP} = -A_{NMP} = \ldots$, gauging $A_{MNP} \approx A_{MNP} + \partial_{[M}A_{NP]}$, $A_{NP} = -A_{PN}$

$\Rightarrow$ Bosonic spectrum G_{MN}, A_{MNP}

Fermions: ψ_M gravitinos $\qquad$ SO(9) $\psi_i = \psi_i^t + \gamma_i\chi$, $\gamma^i\psi_i^t = 0$

spinor repr. has 16 components, vector index

$\Rightarrow$ times 9, eliminate trace $\Rightarrow 16\times(9-1) = \underline{128}$

$\Rightarrow$ nothing more, only gravitinos ψ_M

$D=11 \longrightarrow D=4$ reduction

$x^M = (x^\mu, y^m)$ $\quad \mu = 0,1,2,3 \quad$ ~~y~~ $m = 1,..,7$

Digression : <u>Action</u>

Chern-Simons term in D=1

$$S = \int d^{11}x \underbrace{\sqrt{-G}} \left\{ \frac{\beta}{\kappa^2} R + \alpha F_{MNPQ} F^{MNPQ} + \gamma \overbrace{\varepsilon^{M_1..M_{11}} F_{M_1..M_4} F_{M_5..M_8} A_{M_9 M_{10} M_{11}}} \right.$$

contains gravity => should be invariant w.r.t general coord. transf. $\quad$ + fermionic terms }

A gauge fields => in the action should appear $F_{MNPQ} = \partial_{[M} A_{NPQ]}$

SUSY $\Longrightarrow$ fixed params $\beta = -\frac{1}{4}$, $\alpha = \frac{1}{48}$, $\gamma = \frac{2\kappa^2}{(144)^2}$

$R_{MN} - \frac{1}{2} G_{MN} = T_{MN}$

special soln. $A_{MNP} = 0 \Rightarrow T_{MN} = 0 \Rightarrow \boxed{R_{MN} = 0}$ Ricci flat space

We assume $M_{11} = M_4 \times T_7$ $\qquad T_7 = S^1 \times .. \times S^1$

~~Minkowski~~

Generic field $\phi(x^\mu, y^m)$ $\quad (G_{MN}, A_{MNP}, \psi_M)$

Fourier expansion $\phi(x^\mu, y^m) = \sum \phi_{m_1..m_7}(x)\, e^{i m_1 y^1 + ... + i m_7 y^7}$

$\phi_{00..0}(x)$ indep. of y .. zero modes

Zero modes look like D=4 fields

$G_{MN}(x) \longrightarrow G_{\mu\nu}(x)$ $\quad \mu,\nu = 0,1,2,3 \quad m = 1,..,7$

~graviton in D=4. (carries 2 vector indices)

$\searrow G_{\mu m}(x)$ ~ 7 vector fields in D = 4

$\searrow G_{mn}(x) \sim \frac{7 \times 8}{2} = 28$ scalars

$A_{MNP}(x) \longrightarrow A_{\mu\nu\rho}(x)$ ~ nothing (no phys. degree of freedom because of little group SO(2))

$\searrow A_{\mu\nu m}(x)$ ~ 7 scalars $(\Leftarrow \partial_\lambda A_{\mu\nu m} = \varepsilon_{\lambda\mu\nu\sigma} \partial_\sigma \phi_m)$

$\searrow A_{\mu m n}(x) \sim \frac{7 \cdot 6}{2} = 21$ vectors

$\searrow A_{mnl}(x) \sim \frac{7 \cdot 6 \cdot 5}{2 \cdot 3} = 35$ scalars

=> vectors $7 + 21 = 28$

scalars $28 + 7 + 35 = 70$

Fermions $\psi_M(x) = \begin{cases} \psi_\mu^A(x) & A = (a,i) \quad a = 1,..,4, \quad i = 1,..,8 \\ \psi_m^A(x) & .\_\_ \end{cases}$

<u>D=10 from D=11</u>

$M_{11} = M_{10} \times S^1$

$G_{MN}(x) \longrightarrow G_{\mu\nu}(x)$ graviton in D=10

$\searrow G_{\mu 11}(x) \rightarrow A_\mu(x)$ in D=10

$\searrow G_{11\,11}(x) \rightarrow \phi(x)$ in D=10 .. dilaton

$A_{MNP}(x) \longrightarrow A_{\mu\nu\lambda}(x)$ gauge field in D=10

$\searrow A_{\mu\nu 11}(x) = B_{\mu\nu}(x)$

i.e. bosonic spectrum $\underbrace{(G_{\mu\nu}, B_{\mu\nu}, \phi)}_{\text{NS-NS}}, \underbrace{(A_{\mu\nu\lambda}(x), A_\mu)}_{\text{R-R}}$

bosonic spectrum of the massless states of type <u>IIA string theory</u>

Fermionic part : $\psi_M(x) = \begin{cases} \psi_\mu(x) & \mu = 0..,9 \quad \text{gravitino} = \underbrace{\frac{1+\Gamma_{11}}{2}\psi_\mu}_{\psi_{+\mu}} + \underbrace{\frac{1-\Gamma_{11}}{2}\psi_\mu}_{\psi_{-\mu}} \\ \psi_{11}(x) & \text{Majorana spinor} \end{cases}$

16 comp. spinor

$\psi_{11} = \frac{1+\Gamma_{11}}{2}\psi_{11}(x) + \frac{1-\Gamma_{11}}{2}\psi_{11}(x)$

$\Rightarrow$ 2 Majorana-Weyl spinors and 2 Majorana-Weyl gravitinos

.. fermionic massless spectrum of <u>II A string theory</u>

<u>SUSY algebra reduction</u> $D=11 \rightarrow D=10$

$$\{Q_\alpha, Q_\beta\} = (C\Gamma^M)_{\alpha\beta} P_M$$

$$Q = Q_+ + Q_- \qquad Q_\pm = \frac{1\pm\Gamma_{11}}{2} Q$$

$$\Rightarrow \{Q_{\alpha\pm}, Q_{\beta\pm}\} = \left\{\left(\frac{1\pm\Gamma_{11}}{2}\right) C\Gamma^M\right\}_{\alpha\beta} P_M$$

$$\{Q_{\alpha+}, Q_{\beta-}\} = 0$$

Possibilities:

1) take only one ~~Weyl~~ half $\quad Q_{\alpha-} = 0 \qquad \{Q_{\alpha+}, Q_{\beta+}\} = \left(\frac{1+\Gamma_{11}}{2} C\Gamma^M\right)_{\alpha\beta} P_M \cdot \delta^{ij}$

2) both $Q_{\alpha-}, Q_{\alpha+}$ $\Rightarrow$ 32 supercharges ~~with the same chirality~~ ~~$\Rightarrow$ chiral theory~~

massless states

bosonic $\quad (G_{\mu\nu}, B_{\mu\nu}, \phi) \Leftarrow$ NS-NS $\qquad \partial_{[\mu_5} A_{\mu_1 \dots \mu_4]} = \frac{1}{5!}\varepsilon_{\mu_1 \dots \mu_{10}} \partial^{[\mu_{10}} A^{\mu_6 \dots \mu_9]}$

$\quad (\chi, \tilde{B}_{\mu\nu}, A^+_{\mu\nu\kappa\lambda}) \Leftarrow$ R-R $\qquad A^+_{\dots}$ its field strengths is self-dual

fermionic $\quad \psi^i_\mu, \lambda^i \leftarrow$ $i=1,2$

$\quad$ have ~~both~~ both + chirality $\qquad$ have both − chirality

only known chiral $D=10$ anomaly free theory

.. massless spectrum of <u>II B string theory</u>

$N=1$ theory with 16 supercharges

Γ_μ are 32×32, $\exists$ repr. $\Gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \Gamma_i = \begin{pmatrix} 0 & \gamma_i \\ \gamma_i & 0 \end{pmatrix}, \Gamma_9 = \begin{pmatrix} 0 & \gamma_9 \\ \gamma_9 & 0 \end{pmatrix}$

γ^i symmetric 16×16 matrices, real, $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$,

$\gamma_9 = \gamma_1 \dots \gamma_8 \quad \gamma_9^2 = 1, \ \mathrm{Tr}\,\gamma_9 = 0$

$P^M = (\omega, \omega, 0, \dots, 0)$

$\Rightarrow \{Q_\alpha, Q_\beta\} = \omega(1+\gamma_9)_{\alpha\beta} = \omega\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \qquad \alpha, \beta = 1, \dots, 16$

$\Rightarrow b_i = \frac{1}{\sqrt{2\omega}}(Q_i + iQ_{4+i}) \qquad i = 1, \dots, 4$

$\Rightarrow$ 4 creation ops b_i^+ .. the same number as in $N=4$ $D=4$ theory

$\Rightarrow \quad 2^4 = 16$ states $\underbrace{8 \text{ bosons}}$ & $\underbrace{8 \text{ fermions}}$ if $|\Omega\rangle$ is a singlet of $SO(8)$

8_B: A_i $O(8)$ vectors $\quad \psi$ $O(8)$ spinor ~~field~~

$\Rightarrow$ vector multiplet in $D=10$ $\quad \underbrace{A_M, \psi}_{\text{vector multiplet}} \leftarrow$ Majorana Weyl spinor

$$S = \int d^{10}x \left\{ -\frac{1}{4} F_{MN} F^{MN} + \bar{\psi} \not{D} \psi \right\}$$

SUSY YM in $D=10$

reduction $\rightarrow$ $D=4$: $N=4$ SUSY YM in $D=4$

with scalar ~~potential~~ fields ($\sim A_M$) and quartic potential for them

if $|\Omega\rangle \in 8_v$ itself $\Rightarrow$ $8_v \otimes (8_v + 8_s) = 35 + 1 + 28 + \underbrace{8 + 56}_{\text{spinorial}}$

35: symm. tensor $G_{\mu\nu}$; 1: trace of sym tensor ϕ; 28: antisym tensor $B_{\mu\nu}$; 8: λ fermion; 56: gravitino

.. <u>$N=1$ supergravity in $D=10$</u>

One can put together gravity and vector multiplet, i.e. $\otimes$

$\Rightarrow$ $N=1$ SYM coupled to gravity

Anomaly cancellation $\Rightarrow G = SO(32)$ or $E_8 \times E_8$

(possibly ~~by~~ times powers of $U(1)$)

$G = SO(32)$ → heterotic string theory

↓

type I string theory

$G = E_8 \times E_8$

↓

heterotic string theory

Superstrings (4th lecture)

Closed superstrings

as before $N_{NS} = \sum_{n>0} \alpha_{-n} \cdot \alpha_n + \sum_{r>0} r\, b_{-r} \cdot b_r$ $\quad N_R = \sum_{n>0} (\alpha_{-n} \cdot \alpha_n + n\, d_{-n} \cdot d_n)$

$\tilde{N}_{NS} = (\tilde{\alpha}, \tilde{b})$ $\quad \tilde{N}_R = (\tilde{\alpha}, \tilde{d})$

NS-NS $\quad \frac{M^2}{4} = -1 + N_{NS} + \tilde{N}_{NS} \qquad N_{NS} = \tilde{N}_{NS}$

NS-R $\quad \frac{M^2}{4} = -\frac{1}{2} + N_{NS} + \tilde{N}_R \qquad N_{NS} = \tilde{N}_R + \frac{1}{2}$

R-NS $\quad \frac{M^2}{4} = -\frac{1}{2} + N_R + \tilde{N}_{NS} \qquad \tilde{N}_{NS} = N_R + \frac{1}{2}$

R-R $\quad \frac{M^2}{4} = N_R + \tilde{N}_R \qquad N_R = \tilde{N}_R$

GSO $\quad \widetilde{GSO}$

NS-NS $\Rightarrow$ $N_{NS} = \tilde{N}_{NS} = 0 \quad \frac{M^2}{4} = -1$ tachyon → out by GSO

$N_{NS} = \tilde{N}_{NS} = \frac{1}{2} \qquad b^{\mu}_{-\frac{1}{2}} b^{\nu}_{-\frac{1}{2}} |0, k\rangle \qquad \frac{M^2}{4} = 0 \qquad$ o.k. GSO

$\therefore \phi, G^{\mu\nu}, B^{\mu\nu}$ phys. content

$8_v \otimes 8_v = 1 + 28_a + 35$ under $SO(8)$

massive states ...

NS-R $\Rightarrow$ massless state $N_{NS} = \frac{1}{2}, \ \tilde{N}_{\cancel{NS}R} = 0$

$b^{\mu}_{-\frac{1}{2}} |\tilde{\psi}^a\rangle \qquad M^2 = 0 \qquad$ GSO O.K

ferm. vacuum (before denoted by $|\widetilde{\phi^a}\rangle$)

$8_v \otimes 8_s = 8_c + 56_c$

Note: $8_i \qquad i = v, s, c$ (vector, spinor, conjugate spinor)

$8_i \otimes 8_i = 1 + 28_a + 35_i \qquad$ a antisymmetric tensor

$8_i \otimes 8_j = 8_k + 56_k \qquad k: \ \epsilon_{ijk} = 1$

R-NS $\Rightarrow \qquad 8_s \otimes 8_v = 8_c + 56_c$

R-R $\Rightarrow \qquad N_R = \tilde{N}_R \qquad |\psi^a\rangle \otimes |\tilde{\psi}^b\rangle$

Note: $(-1)^{F_R} = \Gamma_{11} (-1)^{\mathcal{D}} \qquad (-1)^{\tilde{F}_R} = \pm \Gamma_{11} (-1)^{\tilde{\mathcal{D}}}$

overall sign unimportant but the relative sign ± is important

If the chirality is the same (+) $\Rightarrow$ II B

— " — opposite (−) $\Rightarrow$ II A

$\Rightarrow$ $\underline{II}B$ $\quad 8_v \otimes 8_v = 1 + 28_a + 35 \quad \phi, G_{\mu\nu}, B_{\mu\nu}$

$8_v \otimes 8_s = 8_c + 56_c$ } chiral theory: 2× dilatino (of the same chirality) } 2× gravitino

$8_s \otimes 8_v = 8_c + 56_c$

$8_s \otimes 8_s = 1 + 28 + 35$

$\longrightarrow \phi' \quad B'_{\mu\nu} \quad D^{(+)}_{\mu\nu\lambda g}$ (self-dual 4-form)

$\underline{II}A$ $\quad 8_v \otimes 8_v = 1 + 28_a + 35 \quad \phi, B_{\mu\nu}, G_{\mu\nu}$

$8_v \otimes 8_s = 8_c + 56_c$ dilatino + gravitino } nonchiral theory

Here is the difference { $8_c \otimes 8_v = 8_s + 56_s$ —"— with opposite chirality

$8_c \otimes 8_s = 8_v + 56_v$

$A_\mu \quad C_{\mu\nu\lambda}$

the number of physical components $\binom{8}{\#\text{ of indices}}$ in the case of forms

e.g. $\binom{8}{2} = 28$, $\binom{8}{1} = 8$

Summary: $\boxed{\text{type I}}$ $\quad A^a_\mu, \ \psi_a \quad a = 1, .., 496$ dim of adjoint repr.

gauge field gaugino

(anomaly free) consistency $\Rightarrow$ gauge group $SO(32)$

low energy spectrum .. SYM $SO(32)$

Not unitary $\Leftarrow$ one-loop computations $\Rightarrow$ poles corresponding to close strings

e.g.  either a 1-loop of an open string or a propagation of a closed string from left to right

Chan - Paton ~~[illegible]~~ factors add the same charge to the both ends $\Rightarrow$ unoriented strings, i.e. invariant w.r.t $\sigma \to \pi - \sigma$

$\underline{II}A$ hasn't this symmetry since the massless multiplet is $(8_v + 8_s) \otimes (8_v + 8_c)$

unoriented $\underline{II}B$ $\quad \alpha_n \leftrightarrow \tilde{\alpha}_n$ equivalence

in $\underline{II}B$ we have $(8_v + 8_s) \otimes (8_v + 8_s)$

One must add close strings $\left(\frac{\underline{II}B}{Z_2}\right)$ to maintain unitarity

$\Rightarrow$ a massless spectrum of SYM $SO(32)$ + SUGRA ($\phi, B'_{\mu\nu}, G_{\mu\nu}$ + superpartners)

($B_{\mu\nu}$ is antisymmetric $\Rightarrow$ killed by $\alpha_n \leftrightarrow \tilde{\alpha}_n$ i.e. $\mu \leftrightarrow \nu$ whereas $B'_{\mu\nu}$ coming from $8_s \otimes 8_s$ survives because of spinorial nature, analysis is close to the one on page 57

massless spectrum

$\underline{II}A \longrightarrow$ SUGRA $\quad$ (R-R : $A + C_3$)

$\underline{II}B \longrightarrow$ SUGRA $\quad$ (R-R : $\phi', B'_2, D^{(+)}_4$)

chiral anomalies in $\underline{II}B$: arise from chiral fermions but get exactly cancelled by anomaly from $D^{(+)}_{\mu\nu\lambda\sigma}$

also note that 8_c and 56_c have in fact opposite chirality

$\phi, B_{\mu\nu}, G_{\mu\nu}$ can be coupled to a string via

$$S = +\frac{1}{2\pi}\int d^2\sigma \sqrt{h}\,\big(h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu} + \varepsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu} + \alpha \, \Phi \, R^{(2)}\big)$$

How other fields couple to a string ?

Vertex operator

conformal invariance $\to$ vanishing β-function $\to$ action for background

equiv. description using scattering amplitudes

vertex op $\sim \int d\sigma d\tau : e^{ikX(\sigma,\tau)} :$ $\quad$ for tachyon

$\underbrace{\theta_{\mu\nu} h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu e^{ikX}}$ $\quad$ for gravitons

—"— $|0\rangle \underset{\tau}{=} \theta_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1}|0,k\rangle$ ~~[illegible]~~ ok.

euclidean time $\tau = -\infty$

in path-integral (forgetting B, ϕ)

$$G_{\mu\nu} = \eta_{\mu\nu} + \chi_{\mu\nu}(X)$$

$$e^{-S} = e^{-S_0}\left(1 - \frac{1}{2\pi}\int d^2\sigma \sqrt{h}\, \underbrace{h^{\alpha\beta}\chi_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu}_{\text{vertex operator}} + \dots\right)$$

we identify $\chi_{\mu\nu} \equiv \Theta_{\mu\nu} e^{ikx}$ =>

=> inserting the vertex operator ~ correction corresp. to replacement of the flat metric by $G_{\mu\nu}$

e.g. amplitudes for scattering of 2 gravitons into gravitons are computed using 4 vertex ops. ~ graviton.

? vertex operators representing A, C_3

R-R potentials ← bispinors

$$|\psi^a\rangle \otimes |\tilde{\psi}^a\rangle$$

$|\psi^a_R\rangle$ can be generated $|\psi^a_R\rangle = \mathcal{S}^a |0\rangle_{NS}$

$\mathcal{S}^a$ are fermionic ops

the simplest object constructed from a bispinor

$$\mathcal{S}^T_a \left(C\Gamma^{[\mu_1 \dots \mu_n]}\right)^{ab} \tilde{\mathcal{S}}_b\, G_{[\mu_1 \dots \mu_n]}\, e^{ikx}$$

$$C^{-1}\Gamma^\mu C = \Gamma^{\mu T}, \quad C^T = C \qquad C\Gamma_{11} = -\Gamma_{11}C$$

where $\Gamma^{[\dots]}$ is a completely antisymmetrised product of Γs

$$\text{II A} \qquad \Gamma_{11}\mathcal{S}^a = \mathcal{S}^a \qquad \Gamma_{11}\tilde{\mathcal{S}}^a = -\tilde{\mathcal{S}}^a$$

$$\text{II B} \qquad \Gamma_{11}\mathcal{S}^a = \mathcal{S}^a \qquad \Gamma_{11}\tilde{\mathcal{S}}^a = \tilde{\mathcal{S}}^a$$

$$\mathcal{S}^T\Gamma_{11}\left(C\Gamma^{[\dots]}\right)\mathcal{S}\, G_{[\dots]} = (-1)^{n+1}\mathcal{S}^T C\Gamma^{[\dots]}\Gamma_{11}\tilde{\mathcal{S}}\, G_{[\dots]}$$

By the insertion of $\overset{\text{to } \mathcal{S}_a}{\Gamma_{11}}$ and commuting it to $\tilde{\mathcal{S}}_a$ we have:

in the II A case $\dots \neq 0$ if $(-1)^n = 1$ i.e. n even

in the II B case $\dots \neq 0$ if $(-1)^{n+1} = 1$ i.e. n odd

$$F_0, \tilde{F}_0 \Big|_{\text{phys}} = 0$$

in the lowest order (linear) approx. $F_0 \sim a_0 d_0$, $\tilde{F}_0 \sim \tilde{a}_0\tilde{d}_0$

d_{-1}, d_1 etc give the massive states

$F_0 \sim p\cdot\Gamma$, $\tilde{F}_0 \sim p\cdot\tilde{\Gamma}$

$\Gamma_\Delta, \tilde{\Gamma}_\Delta$ act only on $\mathcal{S}$ resp. $\tilde{\mathcal{S}}$

=>

$$C\Gamma^{[\dots]}\, p_\nu \Gamma^\nu G_{[\dots]} = 0 \quad (\text{times } \mathcal{S}^T_a \dots \tilde{\mathcal{S}}_b \text{ etc.})$$

$$\Gamma^{\nu T}\Gamma^{[\dots]}\, p_\nu G_{[\dots]} = 0$$

$$\Gamma^{[\mu_1, \dots, \mu_n]}\Gamma^\mu = (-1)^n \Gamma^{[\mu\mu_1 \dots \mu_n]} + (-1)^{n+1} n\, \eta^{\mu[\mu_1}\Gamma^{\mu_2 \dots \mu_n]}$$

$$\Gamma^\mu \Gamma^{[\mu_1 \dots \mu_n]} = \Gamma^{[\mu\mu_1 \dots \mu_n]} + n\,\eta^{\mu[\mu_1}\Gamma^{\mu_2 \dots \mu_n]}$$

=> 2 indep. combinations $\quad p_{[\mu} G_{\mu_1 \dots \mu_n]} = 0 \qquad p^\mu G_{[\mu\mu_1 \dots \mu_n]} = 0$

in middle repr. $\quad \partial_{[\dots} G_{\dots]} = 0 \qquad \partial^\mu G_{[\mu \dots]} = 0$

$$dG = 0 \qquad d{*}G = 0$$

$G = dC$ => Bianchi $\qquad$ EoM

=> vertex operators contain not the potential itself but its curvature ... i.e. not the minimal coupling

Actions for II A and II B are completely fixed by SUSY => the consideration of vertex ops - is needed for it.

$$S = \frac{1}{2}\int d^{10}x \sqrt{-g}\, e^{-2\Phi}\left[R + (\nabla\phi)^2 - \frac{1}{12}H^2\right] - \frac{1}{8}\int d^{10}x \sqrt{g}\left(F_2^2 + \frac{1}{12}F_4'^2\right)$$

$$- \frac{1}{4}\int F_4 \wedge F_4 \wedge B + (\text{fermions}))$$

$$H \equiv dB \qquad F_2 \equiv dA \qquad F_4 = dC_3 \qquad F_4' = \overset{dC_3}{dC_3} + A \wedge H$$

$$H^2 \equiv H \wedge {}^*H$$

$e^{\Phi} = g_s$ if Φ is constant .. stable vacuum => g_s string vacuum

---

Remark from discussion : the action $S \sim \int \sqrt{h}\, h^{\alpha\beta}\, \partial X\, \partial X\, G$ is written in local coords., it can be invariantly interpreted as a pullback of the metric G from M^{10} to worldsheet $S \sim \int \phi^*(G)$

Heterotic String (1st lecture)

$$\left.\begin{matrix} \text{IIA} \\ \text{IIB} \end{matrix}\right\} \text{closed superstrings} \qquad 32 \text{ SUSY charges} \qquad \begin{matrix} \text{non-chiral} \\ \text{chiral} \end{matrix}$$

But also $N=1$ $D=10$ SUSY theories

.. 16 supercharges

vector multiplet (A_μ, ψ) transf. in adjoint repr. of some gauge group G

\+ gravity multiplet

=> coupled SYM & SUGRA

in general anomalies, for special choice of G they cancel

$$\boxed{G = E_8 \times E_8} \qquad \boxed{G = SO(32)}$$

Can these theories be realized in the context of string theory ?

Type I open + closed unoriented strings => $G = SO(32)$

But what about $E_8 \times E_8$?

=> Heterotic string Gross, ~~Harvey~~ Harvey, Rhom, Martinec

Each GSO projection → 16 SUSYs :

Fermionic string $X^\mu(\sigma,\tau)$, $\psi^\mu(\sigma,\tau) \xrightarrow[\text{projection}]{\text{GSO}}$ 16 spacetime SUSYs

We allow fermions only in left-moving sector $X^\mu(\sigma+\tau), \psi^\mu(\sigma+\tau)$, in the right-moving sector only bosons $X^\mu(\sigma-\tau)$

Critical dimensions : left sector $D=10$, right sector $D=26$ } How can we reconcile this ?

=> $X^\mu(\sigma,-\tau)$, $X^A(\sigma-\tau)$, $A=1,..,16$

Mode expansion for $X^A(\tau-\sigma)$

$$X^A(\tau-\sigma) = x^A + p^A(\tau-\sigma) + \sum_{n\neq 0} \frac{\alpha_n^A}{n} e^{in(\tau-\sigma)}$$

whereas

$$X^\mu = X_+^\mu(\tau+\sigma) + X_-^\mu(\tau-\sigma) = x^\mu + p^\mu\tau + \text{oscillators}$$

Boundary conditions $X^\mu(\sigma+2\pi,\tau) = X^\mu(\sigma,\tau)$

but we have $X^A(\sigma+2\pi,\tau) = X^A(\sigma,\tau) - 2\pi p^A$

$\Rightarrow X^A$ must behave like angles .. 16 circles $(S^1)^{16} = T^{16}$

$\Rightarrow p^A$ must be quantised: $p^A \in \Gamma_{16}$ 16-dim. lattice

Remark: $T^d = R^d / \Gamma_d$ $X' \equiv X \Leftrightarrow X' - X \in \Gamma_d$

Γ_d generated by d vectors e_i $\vec{x} \in \Gamma_d \Leftrightarrow \exists \underset{\mathbb{Z}^n}{\underset{\cap}{\vec{n}}} : \vec{x} = n_i \vec{e}_i$

$\Gamma_{16} \cdots \vec{e}_1, \dots, \vec{e}_{16}$

Do physical quantities depend on Γ_{16}?

Mass spectrum

Left: $L_0 = -\text{number}\, M^2 + \overset{\to \text{ oscillator \# operator}}{N} - a$ $a = \frac{1}{2}$ (NS) $a = 0$ (R)

Right $\bar{L}_0 = -\text{number} \cdot M^2 + \tilde{N} + \frac{1}{2}\sum_A (p^A)^2 \; \boxed{-1}$ bosonic string

↳ oscillator number operator (all 26 oscillators contrib.)

On physical states $L_0 = 0$ $\bar{L}_0 = 0$

Possible tachyon NS $N=0$ $M^2 \propto -\frac{1}{2}$

$\tilde{N} = 0$ $M^2 \propto -1$

} cannot be satisfied simultaneously

in general we may have $\vec{p}$ s.t. $\vec{p}^2 = 1 \Rightarrow$ both states are $M^2 \propto -\frac{1}{2}$

$\uparrow$ $\vec{p} = (p^A)$

$\Rightarrow$ we have a tachyon

GSO projection $\Rightarrow N=0$ state removed in NS,

the lowest value of N is $N = \frac{1}{2}$ in NS sector

$\Rightarrow$ the matching of left- and right-moving L_0, i.e. $L_0 = \bar{L}_0$ ✓ with together GSO projects out the tachyon state as unphysical

Massless spectrum

(NS) $N = \frac{1}{2}$ $b^i_{-\frac{1}{2}}|0\rangle_{NS}$ ← 8_v of SO(8) little group

in the light-cone gauge (easiest for counting the states)

$i = 1, \dots, 8$

Right ① $\tilde{N} = 1$ $\vec{p} = 0$ $\tilde{\alpha}^i_{-1}|0\rangle$, $\alpha^A_{-1}|0\rangle$ $8_v + 16 \times \overset{\text{scalars}}{\overset{\downarrow}{1}}$

② $\tilde{N} = 0$ $\vec{p}^2 = 2$

The whole state .. product of left & right

① $\Rightarrow$ a) $b^i_{-\frac{1}{2}}|0\rangle_{NS} \otimes \alpha^j_{-1}|0\rangle$ $8_v \otimes 8_v =$

= Symmetric traceless + Antisymmetric + trace

Graviton $G_{\mu\nu}$ $B_{\mu\nu}$ dilaton ϕ

a kind of universal sector

b) $b^i_{-\frac{1}{2}}|0\rangle_{NS} \otimes \alpha^A_{-1}|0\rangle$ .. 16 vectors 8_v of SO(8) ..... A^A_μ

massless

② $\Rightarrow$ $b^i_{-\frac{1}{2}}|0\rangle_{NS} \otimes |\vec{p}\rangle$, $\vec{p}^2 = 2$ .. 8_v of SO(8) if such $\vec{p}$ exist ... $A^{\vec{p}}_\mu$

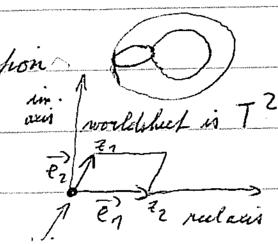
... massless gauge field

Remark: R-sector $\Rightarrow$ fermionic partners of the above given bosonic states

Is there any condition on Γ_{16} ?

<u>Modular invariance</u> of one-loop partition function

$$\frac{d^2\tau}{\tau_2}\, \mathrm{tr}\, q^{L_0}\, \bar{q}^{\bar{L}_0}$$



$$\mathbb{R}^2 \simeq \mathbb{C}$$

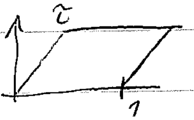
$\tau = \frac{z_1}{z_2}$ the result should depend on τ only

because of rotational symmetry ($\vec{e}_1 \parallel$ real ax.)

and conformal invariance ($|\vec{e}_1| = 1$)

any physical quantity, e.g. partition function

i.e. we have



~~movement along the time~~ evolution of the string $e^{2\pi i\, H\, \mathrm{Im}\,\tau + 2\pi i\, P\, \mathrm{Re}\,\tau}$

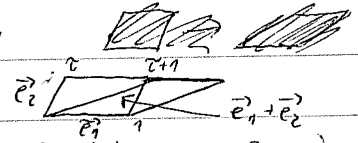
Wick rotation $\Rightarrow$ $e^{-2\pi\, H\, \mathrm{Im}\,\tau + 2\pi i\, P\, \mathrm{Re}\,\tau}$

$$= e^{2\pi i \tau L_0}\, e^{-2\pi i \bar{\tau} \bar{L}_0}$$

$$H = L_0 + \bar{L}_0 \qquad P = L_0 - \bar{L}_0$$

$\Rightarrow q = e^{2\pi i \tau}$

The same torus can be obtained also by $\tau + 1$



$$(\vec{e}_1, \vec{e}_2) \to (\vec{e}_1, \vec{e}_1 + \vec{e}_2)$$

$\Rightarrow$ any physical quantity should be invariant w.r.t $\tau \to \tau + 1$

in general $e_1' = m_1 e_1 + n_2 e_2 \qquad e_2' = m_1 e_1 + m_2 e_2$

$n_1, m_1, n_2, m_2 \in \mathbb{Z}$, the tori or lattices are the same

iff $\begin{pmatrix} n_1 & n_2 \\ m_1 & m_2 \end{pmatrix} \in SL(2, \mathbb{Z})$ (because of invertibility of transf. with integer coeffs.)

by rotation and conformal invariance we fix $e_1 = 1$ then $SL(2,\mathbb{Z})$

acts on τ : $\tau \to \frac{a\tau + b}{c\tau + d}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$

$$\bar{L}_0 = ?(p^\mu)^2 + \bar{N} + \frac{1}{2}\vec{p}^{\,2} - 1 \qquad \vec{p} \in \Gamma_{16}$$

$\Rightarrow$ Lattice contribution to the partition function

$$\left(\sum_{\vec{p} \in \Gamma_{16}} e^{-2\pi i \bar{\tau} \frac{1}{2} \vec{p}^{\,2}}\right) \times \underbrace{\text{rest of the contribution}}_{\text{fermions, left-movers etc.}}$$

$\bar{\tau} \to \bar{\tau} + 1 \Rightarrow \vec{p}^{\,2} \in 2\mathbb{Z}$ ... <u>even lattice</u>

$\bar{\tau} \to -1/\bar{\tau}$ by Poisson resummation one may show

$$(\Sigma \ldots) = \sum_{\vec{p}\,' \in \Gamma_{16}^*} \exp\left(i\, 2\pi \frac{1}{\bar{\tau}} \frac{1}{2} \vec{p}\,'^2\right)$$

where Γ_{16}^* is the dual lattice $\left(\Gamma_d^* = \{\vec{p}\,' \in \mathbb{R}^d \mid \vec{p}\,' \cdot \vec{p} \in \mathbb{Z}, \forall \vec{p} \in \Gamma_d\}\right)$

$\Rightarrow \Gamma_{16} = \Gamma_{16}^*$ $\quad$ Γ_{16} must be <u>even self-dual lattice</u>

<u>Theorem</u> : Even and self-dual lattice Γ_d exists iff d is a multiple of 8. Furthermore for $d = 8$ Γ_8 is E_8 root lattice, for $d = 16$ is $\Gamma_{16} = E_8 \times E_8$ root lattice or $\Gamma_{16} = SO(32)$ root lattice + spinor conjugacy class.

Remark: dim of a root lattice = dim of Cartan subalgebra

Massless states $\vec{p}^2 = 2$ $\sim$ nonzero roots of these algebras

1) $E_8 \times E_8$ Lie algebra

2) $SO(32)$ Lie algebra

Massless gauge fields

$b^i_{-\frac{1}{2}}|0\rangle_{NS} \otimes \tilde{\alpha}^A_{-1}|0\rangle$ 16 gauge fields

-- Cartan gauge fields (commuting)

$\vec{p}^2 = 2 \Rightarrow$ gauge fields corresp. to root vectors (nonzero).

altogether the whole algebra in a suitable basis (?Chevalley?)
we obtain written

D-branes (1st lecture)

Brief review $\hbar = c = 1$ (SISSA room 218)

extended objects (3-branes, 4-branes etc.) non visible in perturbation theory
$\sim$ extended solitons in QFT

low energy effective action in string theory .. 10 dim. SUGRAs
→ possible description of D-branes as classical solns of these actions

→ lectures by M. Blau

Contents
- D-branes
- compute the charge and mass
- D-branes and T-duality
- D-brane effective actions
- more general brane configurations
- unoriented strings: orientifold plane

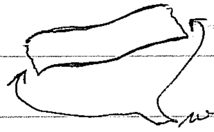
D-brane is an hyper-plane in a space-time were open strings can end

$$\boxed{S = \frac{1}{4\pi\alpha'}\int d\sigma d\tau \quad \partial_\alpha X \cdot \partial^\alpha X}$$
$\alpha = 1,2$

(+ fermionic part in the case of superstring)

$$\delta S = \frac{2}{4\pi\alpha'}\int d\sigma d\tau \ \delta\partial_\alpha X \cdot \partial^\alpha X$$
$$= \frac{1}{2\pi\alpha'}\Big\{\underbrace{\int d\sigma d\tau \ \partial_\alpha[\delta X \cdot \partial^\alpha X]}_{\text{boundary term}} - \int d\sigma d\tau \ \delta X \cdot \Box X\Big\}$$

$$\sim -\frac{1}{2\pi\alpha'}\int d\sigma d\tau \ \delta X \cdot \Box X + \frac{1}{2\pi\alpha'}\int d\tau \ \delta X \partial_\sigma X \Big|_{\partial\Sigma \, \binom{\sigma=\pi}{\sigma=0}}$$



we assume $\tau \to \pm\infty \Rightarrow$ not important

$\Rightarrow$ we have extra conditions in addition to EoM : $\Box X = 0$

$$\delta X \cdot \partial_\sigma X_{(\sigma=\pi)} = 0 \qquad \delta X \cdot \partial_\sigma X_{(\sigma=0)} = 0$$

either $\boxed{\partial_\sigma X\big|_{\sigma=(\pi,0)}=0}$, $\forall\tau$ Neumann boundary condition

or $\boxed{\delta X\big|_{\sigma=0}=0}$, $\forall\tau$ Dirichlet boundary condition

Most generic soln. of EoM $X(\sigma,\tau)=X_0+\alpha' p\tau+q\sigma+i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq0}\left[\frac{\alpha_n}{n}e^{-n(\tau+i\sigma)}+\frac{\tilde{\alpha}_n}{n}e^{-n(\tau-i\sigma)}\right]$

$\underline{\partial_\sigma X\big|_{\sigma=0}=0} \Rightarrow q+i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq0}\left[\alpha_n e^{-n\tau}-\tilde{\alpha}_n e^{-n\tau}\right]=0,\ \forall\tau$

$\Rightarrow \boxed{q=0,\ \alpha_n=\tilde{\alpha}_n}$, $\forall n$ presumably in euclidean time τ

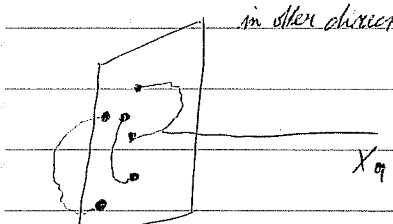
p unconstrained $\Rightarrow$ strings freely moving in $D=10$ spacetime

$\underline{\delta X\big|_{\sigma=0}=0}$ $X\big|_{\sigma=0}=$ constant $=X_0$ Dirichlet b.c.

$\Rightarrow \boxed{p=0,\ \tilde{\alpha}_n=-\alpha_n}$

D = 10 in the direction X^9 we assume the Dirichlet b.c.

in other directions —//— the Neumann b.c.



we assume $X_9=x_0$ as $(X_9|_{\sigma=0}=x_0=X_9|_{\sigma=\pi})$

time direction must be Neumann, otherwise not dynamical theory !

$\mu=0,1,\dots,p$ Neumann b.c

$i,j,k=\underbrace{p+1,\dots,9}_{9-p}$ coords Dirichlet (μ,i) .. M,N

Different b.c. in diff. directions break $SO(1,9)$ to $SO(1,p)\times SO(p-9)$

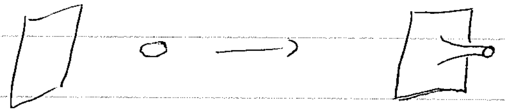
$\underbrace{X^9=x_0,\ X^8=x_1,\ \dots,\ X^{p+1}=x_{9-p}}$

$\Rightarrow$ **Dp-brane** hyperplane fixed by

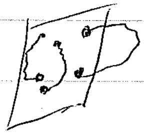
Moving Dp-branes $X(\sigma=0,\tau)=x^0(\tau)$

Interaction between moving D-branes and closed strings

e.g.



Excitation of D-brane .. attached open strings



Bosonic strings $D_{25},\dots,\boxed{D_{-1}}$

singular – $\underbrace{\text{all dir. Dirichlet}}_{\text{incl. time direction}}$ – instantons – we will not study them

$\underline{II\,A}$
$\underline{II\,B}$ $D_9,\dots,D_{-1}$

$\underline{I}$ a subset of branes $\underset{\uparrow}{D_9}$ D_5 D_1

Neumann open strings

other are not allowed due to $\Omega: \sigma \to \pi - \sigma$

Heterotic strings

SO(32) no D-branes since left and right moving degrees

$E_8 \times E_8$ of freedom are completely different

in the following we consider $\boxed{II A, II B}$ (bosonic strings not important for description of nature)

II A: (R-R) C_1 (1-form) C_3 (3-form)

II B: (R-R) C_0, C_2, $C_4^{(+)}$ $F_5 = dC_4^{(+)} = {}^*F_5$

perturbative states cannot carry charge (vis Superstrings) (minimal coupling not admissible) under R-R fields, whereas D-branes do carry charge under these fields

$\Rightarrow$ in II A only even D-branes D_0, D_2, D_4, D_6, D_8

II B only odd D-branes $D_{-1}, D_1, D_3, D_5, D_7, D_9$

(for argument see lecture notes)

$$\partial_\sigma X\Big|_{\partial\Sigma} = 0 \qquad \delta X\Big|_{\partial\Sigma} = 0$$

complex coordinate

$$z = \tau + i\sigma \qquad \bar{z} = \tau - i\sigma$$

$$\Box X = 0 \Leftrightarrow \partial\bar{\partial} X = 0 \Rightarrow X(z,\bar{z}) = \underset{\text{holomorphic}}{X_L(z)} + \underset{\text{antiholomorphic}}{X_R(\bar{z})}$$

$$\Rightarrow (N) \quad \partial_\sigma X\Big|_{\partial\Sigma} = 0 \Leftrightarrow \boxed{\partial X_L(z)\Big|_{\partial\Sigma} = \bar{\partial} X_R(\bar{z})\Big|_{\partial\Sigma}}$$

$$(D) \quad \delta X\Big|_{\partial\Sigma} = 0 \Leftrightarrow \partial_\tau X\Big|_{\partial\Sigma} = 0 \Leftrightarrow \boxed{\partial X_L\Big|_{\partial\Sigma} = -\bar{\partial} X_R\Big|_{\partial\Sigma}}$$

Worldsheet fermions

$$S_\psi = -\frac{i}{4\pi}\int d\tau\, d\sigma\, \bar{\Psi}^M \rho^\alpha \partial_\alpha \Psi_M, \quad M = 0, .., 9$$

Ψ Majorana fermions, their dimension $=2$ ρ^0, ρ^1

$$\bar{\Psi}^M = \Psi^{M+}\rho^0 = (\Psi^t)^M \rho^0$$

We choose

$$\rho^0 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$\delta S_{\bar{\Psi}^N} \sim \int d\tau\, d\sigma\, \delta\bar{\Psi}^N \rho^\alpha \partial_\alpha \Psi_N + \int d\tau\, \delta\bar{\Psi}^N \rho^1 \Psi_N\Big|_{\sigma=0}^{\sigma=\pi}$$

$$\Psi = \begin{pmatrix} \psi \\ \tilde{\psi} \end{pmatrix} \Rightarrow \delta\bar{\Psi}^N \rho^1 \Psi_N\Big| \sim \delta\psi^N \psi_N - \delta\tilde{\psi}^N \tilde{\psi}^N\Big|_{\sigma=0} = 0$$

$$— \text{''} — \Big|_{\sigma=\pi} = 0$$

ψ SUSY partner of X_L, $\tilde{\psi}$ SUSY partner of X_R

We require $\psi^N(\sigma=0) = \pm\tilde{\psi}^N(\sigma=0)$, $\psi^N(\sigma=\pi) = \pm\tilde{\psi}^N(\sigma=\pi)$

Lorentz invariance on the worldsheet allows to choose the phase

at $\sigma=0$ to $\psi^N(\sigma=0) = \psi^N(\sigma=0)$ and the same condition (i.e. either + (R) or (−) (NS) simult. $\forall N=0,..,9$)

for all N at the point $\sigma=\pi$ in the Neumann case

Neumann & Dirichlet:

$$\psi^\mu(\sigma=0) = \pm\tilde{\psi}^\mu(\sigma=0) \qquad \psi^i(\sigma=0) = \pm\tilde{\psi}^i(\sigma=0)$$

$$\psi^\mu(\sigma=\pi) = \pm\tilde{\psi}^\mu(\sigma=\pi) \qquad \psi^i(\sigma=\pi) = \pm\tilde{\psi}^i(\sigma=\pi)$$

we can fix $\psi^\mu|_0 = \tilde{\psi}^\mu|_0$ and $\psi^\mu|_\pi = \underset{\uparrow \text{ indep. of } \mu}{\pm} \tilde{\psi}^\mu|_\pi$

We obtain Dirichlet conditions by $\tilde{\psi} \to \tilde{\psi}$ from Neumann one (because of SUSY etc)

$$\Rightarrow \quad \psi^\mu|_0 = \tilde{\psi}^\mu|_0 \qquad \psi^i|_0 = -\tilde{\psi}^i|_0$$

$$\psi^\mu|_\pi = \pm\tilde{\psi}^\mu|_0 \qquad \psi^i|_\pi = \mp\tilde{\psi}^i|_\pi$$

NS or R R or NS

Classical solutions of Supergravity (1st lecture)

SUGRA Solitons

Why we should be interested in them?

What behaviour we should expect?

How to find them?

GR: physics → assumptions on symmetries ⇒ simplification of Einstein's eq.
⇒ they are solvable

Example: Reissner - Nordström M, Q static charged star

$$ds^2 = -\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)dt^2 + \left(\quad \text{—}''\text{—} \quad\right)^{-1} dr^2 + r^2 d\Omega^2$$

$M > Q$ would radiate in quantum theory - unstable

$\boxed{M=Q}$ stable (critical) $ds^2 = -\left(1-\frac{Q}{r}\right)^2 dt^2 + ...$

$R = r - Q$ $R = 0$ horizon

$$ds^2 = -\left(1+\frac{Q}{R}\right)^{-2} dt^2 + (\quad)^{+2}(dR^2 + R^2 d\Omega^2)$$

in fact SUSY solution (i.e. a kind of BPS state)

spherically symmetric

$1+\frac{Q}{R}$ harmonic

Solitons - static finite energy solutions of nonlinear eqs.,
localized in space which cannot be obtained as perturbation of trivial solns.
- behave like particles

or finite energy density strings, membranes, vortices, domain walls

$g \to 0$? $M \sim \frac{1}{g^n}$ ⇒ not visible in ~~the~~ weak-coupling limit

If we can trust this formula at strong coupling (usually not) then solitons might become the lightest objects in the theory.

SUSY with $N>1$ & central charges => short multiplets (BPS states)

– definite relation between mass and central charges $M=|Z|$

One assumes that $M=Z$ holds in both ~~these~~ weak and strong coupling (otherwise bigger multiplets would appear from nowhere).

SUSY solutions of SUGRA

SUSY: preserve some part of SUSY

Procedure:

We start with some bosons B and fermions F: $\delta B \sim F,\ \delta F \sim B$ (SUSY)

1step $F=0$ only bosonic solus => $\delta B = 0$

2step $\delta F \overset{!}{=} 0$ for some SUSY params – usually 1st order differential equation

equation: might be easy to show

3step check EoM usually already satisfied because of the constraints above

Differential forms

p-form $A \longleftrightarrow A_{[\mu_1 \cdots \mu_p]}$

exterior derivative $dA \longleftrightarrow \partial_{[\mu} A_{\mu_1 \cdots \mu_p]}$

$$\Rightarrow d(dA) = 0, \text{ i.e } d^2 = 0$$

Hodge duality op. $*$ $\quad *A \longleftrightarrow \epsilon^{\mu_1 \cdots \mu_D} A_{\mu_1 \cdots \mu_p}$

$$*d*A \longleftrightarrow \nabla^{\sigma} A_{\sigma\mu_1 \cdots \mu_{p-1}}$$

Maxwell equations $d*F = \underbrace{J}_{\text{3-form}}$

Bianchi $dF = 0 \Leftarrow F = dA$

Examples:

Electric & magnetic charges in $D=4$

A gauge field, electrically charged particle $S \sim \int J^{\sigma} A_{\mu} = \int_{\text{worldline}} A_{\mu} \dot{x}^{\mu} dt$ $\quad \sim \delta^{(3)}(..)$

$\Rightarrow S \sim \int_{\text{worldline}} A$ such coupling leads to

$Q_E = \int_{S^2_\infty} (*F)$ electric charge

$? \; Q_M = \int F \neq 0 \; ?$ $\quad \tilde{A} : d\tilde{A} = *F$

=> we need something that couples to $\tilde{A}$, naturally couples to a point particle (in $D=4$)

The same in $D=5$

$$\underset{1}{A} \longrightarrow \underset{2}{F} \longrightarrow \underset{3\text{ - form}}{*F} \quad \Rightarrow \tilde{A} \cdot d\tilde{A} = *F \quad \tilde{A} \text{ 2-form}$$

=> $\tilde{A}$ couples (naturally) to a ~~membrane~~ string $\int_{V_2} \tilde{A}$ .. a monopole string

$D=6 \Rightarrow \tilde{A}$ couples to a membrane

$C^{(p+1)}$ $(p+1)$-form in d dimensions naturally couples to a p-brane V_{p+1} where:

0-brane is a particle

1-brane is a string

2-brane is a membrane

⋮

$F^{(p+2)} = dC^{(p+1)}$ $\qquad$ $\ast F^{(p+2)}$ $d-p-2$ form

$$Q_E = \int_{S_\infty^{d-p-2}} \ast F \qquad Q_M = \int_{S_\infty^{p+2}} F$$

dual potential $\tilde{C}$ $d-p-3$-form couples to $(d-p-4)$-brane

In strings forms arise quite naturally e.g. NS B field

$H = dB$ 3-form $\Rightarrow$ $\ast H$ 7-form $\Rightarrow$ potential 6-form $\Rightarrow$

~~dual to NS~~ dual object is NS 5-brane $Q_m = \int_{S_\infty^3} H \neq 0$

· II A (RR) $C^{(1)}$, $C^{(3)}$ and dual forms $C^{(5)}$, $C^{(7)}$

one has no reason to choose C or its dual

couples to RR 0-brane, RR 2-brane, RR 4-brane, RR 6-brane

II B (RR) $C^{(0)}$ (axion) $\quad C^{(2)} \quad C^{(4)} \quad dC^{(4)} = \ast dC^{(4)}$

$C^{(0)}$: couples to instantons (after Wick rotation) (−1)-brane

$C^{(2)}$: string charged under RR. not fundamental! strings

$C^{(4)}$: 3-brane (non-singular object – no δ function because of self-duality)

Type II SUGRA action

Note: Chern-Simons terms don't contribute to solitons ⇒ omitted

$$S_{\mathrm{II}\,A\vee B} = \int d^{10}x \sqrt{-g_s}\Big\{e^{-2\Phi}\Big[R(g_s) + 4\underbrace{(d\phi)^2}_{g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi} + \frac{1}{12}(dB)^2\Big]$$

$$- \frac{1}{2}\sum_{p} \frac{1}{(p+2)}\left(F^{(p+2)}\right)^2\Big\} \quad \text{(in string frame)}$$

p ← odd or even
A B

new metric $g_{MN,s} = e^{\Phi/2} g_{MN,E}$ $\qquad$ (Einstein frame)

$$S_{\mathrm{II}A\vee B} = \int d^{10}x \sqrt{-g_E}\left\{R(g_E) - \frac{1}{2}(d\phi)^2 - \frac{e^{-\Phi}}{12}(dB)^2 - \frac{1}{2}\sum_p \frac{1}{(p+2)} e^{\frac{(3-p)\phi}{2}}\left(F^{(p+2)}\right)^2\right.$$

simplification: we keep only $\boxed{g_{\mu\nu},\ C^{(p+1)},\ \Phi}$

$$S = \int d^{10}x \sqrt{-g_E}\left\{R(g_E) - \frac{1}{2}(d\phi)^2 - \frac{1}{2}\frac{1}{(p+2)} e^{\frac{(3-p)\phi}{2}}\left(F^{(p+2)}\right)^2\right.$$

$$\Rightarrow \text{EoM} \quad R_{\mu\nu} \sim T_{\mu\nu}, \quad \underbrace{\nabla_\mu\left(e^{\frac{(3-p)\phi}{2}} F^{\cdots}\right) = 0}, \quad \Box\phi = (F^2)$$

(+ δ-function "source" $\sim T_p$ .. p-brane tension = mass density of the p-brane)

We impose symmetries and supersymmetries

⇒ Ansatz: flat $p+1$-dim. world volume ⇒ Poincaré invariance in $(p+1)$-dim.

\+ spherical symmetry in the transverse space

$$\Rightarrow ds^2 = e^{2A(r)} d\vec{x}^2 + e^{2B(r)} d\vec{y}^2, \quad d\vec{y}^2 = dr^2 + r^2 d\Omega_{8-p}$$

we assume only $C_{01\ldots p}(r) = e^{C(r)}$ nonvanishing

$\delta F \overset{!}{=} 0 \qquad \delta_\varepsilon \psi_\mu = \nabla_\mu \varepsilon + F^{\cdot\cdot} \Gamma_{\cdot\cdot} \varepsilon = 0$

$\delta_\varepsilon \lambda = \partial_\mu \phi \, \Gamma^\mu \varepsilon + \ldots = 0$

(see lecture notes)

$\Rightarrow$ A, B linear functions of C

EoM $\Rightarrow$ $e^{-C(r)}$ is harmonic function of r

(p~~+1~~)-brane $\Rightarrow$ transverse space $9-p$ $\Rightarrow$ harmonic functions $\sim \frac{1}{r^{7-p}}$

behaviour in infinity $\Rightarrow$ $H_p(r) = 1 + \frac{Q_p}{r^{7-p}}$, $Q_p \sim G_N T_p$

Newton's const.

$[G_N] = cm^8$, $[T_p] = cm^{1-p}$

$\Rightarrow [Q_p] = cm^{7-p}$

$\Rightarrow$ the solution in the ~~Einstein~~ String frame

$$\boxed{\begin{aligned} ds^2 &= H_p^{-\frac{1}{2}}(r)\, d\vec{x}^{\,2} + H_p^{\frac{1}{2}}(r)\, d\vec{y}^{\,2} \\ C^{(p+1)}_{01\ldots p} &= H_p^{-1} - 1 \\ e^{2\Phi} &= H_p^{\left(\frac{3-p}{2}\right)} \end{aligned}}$$

Exact soln of EoM carrying RR-charge

for $P = 3$ completely without singularities, otherwise singularities

---

Heterotic strings (2nd lecture)

rank $E_8 \times E_8$ = rank $SO(32) = 16$ (dim of Cartan subalg.)

$A_\mu^{\vec{p}}$ should be charged under A_μ^I

Examples: $SU(2)$ J_1, J_2, J_3 $[J_a, J_b] = i\varepsilon_{abc} J_c$

$h = [J_3]_{\mathbb{R}}$ a maximal set of commuting generators

$J_\pm = J_1 \pm i J_2 \Rightarrow [J_\pm, J_3] = \pm J_\pm$

$\Rightarrow$ J_+ is carrying the charge $+1$, J_- the charge -1

charges $\sim$ roots (their values on corresp. elements of ~~the~~ algebra)

$SU(3)$ 3×3 traceless hermitian matrices

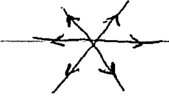
$h = \left\{ \begin{bmatrix} a & & 0 \\ & b & \\ 0 & & c \end{bmatrix}, a+b+c = 0 \right\}$

$= \left[J_3 ~~\alpha_3~~ = \frac{1}{2}\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}, J_8 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & -2 \end{pmatrix} \right]_{\mathbb{R}}$

Combinations of the remaining generators $J_{\vec{\alpha}}$

$[J_3, J_{\vec{\alpha}}] = \underline{\underline{\alpha_1}} J_{\vec{\alpha}} \qquad [J_8, J_{\vec{\alpha}}] = \underline{\underline{\alpha_2}} J_{\vec{\alpha}}$

$\vec{\alpha} = (\alpha_1, \alpha_2)$ root vectors

6 roots  all of the same lengths

In fact all the roots have equal length for $SU(n)$, $SO(2n)$, E_6, E_7, E_8 — <u>simply-laced algebras</u>

We have algebras with rank $= 16 \Rightarrow$ 16 commuting generators J_I, the others $J_{\vec{p}}$, $\vec{p} \in \mathbb{R}^{16}$, $[J_I, J_{\vec{p}}] = p^I J_{\vec{p}}$

<u>Proof that $A_\mu^{\vec{p}}$ are really charged</u>

<u>either</u> $S = (-)\int \mathrm{tr} F_{\mu\nu} F^{\mu\nu}$ $\quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$

$\Rightarrow$ cubic interaction $\quad \mathrm{tr}\, \partial_{[\alpha} A_{\nu]} [A_\mu, A_\nu]$

quartic —"— $\quad \mathrm{tr}\, [A_\mu, A_\nu]^2$

$\Rightarrow$ compute 3- and 4-point functions – need CFT techniques

<u>or</u> Vertex for graviton or antisymmetric tensor

$\eta^{ab} \partial_a X^\mu \partial_b X^\nu e^{ikx}$ $\quad \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu e^{ikx}$

worldsheet metric

$\Rightarrow$ by similarity argument A_μ^I vertex is $\partial_+ X^i \partial_- X^I e^{ikx}$

interaction with background field – known in background of $G_{\mu\nu}$ or $B_{\mu\nu}$

$$S \propto \int \eta\, \partial X \partial X\, G + \epsilon\, \partial X \partial X\, B$$

$$G_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$\int \mathcal{D}X e^{-S} = \int \mathcal{D}X e^{-S^0} \big(1 + \underbrace{h_{\alpha\beta} \partial_a X^\alpha \partial_b X^\beta \eta^{ab} + \dots}_{\text{vertex op}}\big) \cancel{A}$$

$\Rightarrow$ background gauge field will interact via $\partial_+ X^\mu \partial_- X^I \cdot A_\mu^I(X)$

$X^I = x^I + p^I(\tau - \sigma)$ + oscillator

we plug this into the action before (and forget oscillator) $\rightarrow S \propto \dots +$

$$+ \underbrace{\int d\tau\, A_{I\mu}(x)\, p^I \dot{x}^\mu}$$ (and assume tiny, pointlike string i.e. ignore σ-dependence)

$\Downarrow$

p^I is a charge w.r.t. A_I (see QED $eA\dot{x}$)

(p^I can be taken out of integral as a constant)

<u>Charge is conserved</u> ($\Leftarrow$ conservation of momentum) etc.

<u>Numbers of freedom</u>: in NS 8 bosonic, in R 8 fermionic } for each I $\quad$ massless $N=1$ vector $D=10$ supermultiplet $\sim$ massless $N=4$ vector $D=4$ supermultiplet

<u>Toroidal compactification of the heterotic string</u>

$$D = 10 \xrightarrow{T^d} D = 10 - d \qquad \underset{x}{M}^{10-d} \times \underset{y}{T}^d$$

$M, N = 0, \dots, 9 \quad \mu\nu = 0, \dots, 9-d \quad i, j = 1, \dots, d$

Kaluza – Klein

massless modes $G_{MN} \longrightarrow G_{\mu\nu}$ graviton

$\longrightarrow G_{\mu i}$ vector (d of them)

$\longrightarrow G_{ij}$ $\frac{1}{2}d(d+1)$ scalars

$B_{MN} \longrightarrow B_{\mu\nu}$ antisym. tensor

$\longrightarrow B_{\mu i}$ vectors (d of them)

$\longrightarrow B_{ij}$ $\frac{1}{2}d(d-1)$ scalars

$$\begin{tikzcd} & A^{I,\vec{P}}_{\mu} \\ A^{I,\vec{P}}_{M} \arrow[ur] \arrow[dr] & \\ & A^{I,\vec{P}}_{i} \end{tikzcd}$$

gauge fields (adjoint repr. of $G < {SO(32) \atop E_8 \times E_8}$)

scalars ($d \times \dim G$ of them)

$\Rightarrow$ at all $(\dim G + 2d)$ vectors $\dim G = 496$ in both cases

Is something charged under these new $2d$ fields ? … point-like particles certainly not, see below

<u>In general</u> $G(x,y) = G^0(x) + \underbrace{\sum - e^{i\cdot y}}$ Fourier expansion ~~also~~ also for B, A

above and below we consider only $G(x)$ ($\to$ massive states)

$$S = \ldots + G_{\mu i}(x)\, \partial_a X^\mu \partial_b X^i \gamma^{ab} + B_{\mu i}(x)\, \partial_a X^\mu \partial_b X^i \epsilon^{ab}$$

<u>Point-like strings</u> in M^{10-d} $X^\mu(\tau, \cancel{\sigma})$

$$X^i(\tau,\sigma) = x^i + p^i\tau + \omega^i\sigma + \cancel{\text{oscill.}}$$

we forget these

$\Rightarrow$ in S we have $\int G_{\mu i}(x)\, \partial_\tau X^\mu \partial_\tau X^i = p^i \int G_{\mu i}(x)\, \partial_\tau X^\mu d\tau$

$+ \int B_{\mu i}(x)\, \partial_\tau X^\mu \partial_\sigma X^i = \omega^i \underbrace{\int B_{\mu i}(x)\, \partial_\tau X^\mu d\tau}$

cannot appear in theory of point particles because there is no winding in them (i.e. $\omega^i = 0$)

$\Rightarrow$ charges p^i, ω^i

$\Rightarrow$ In heterotic string theory already at the perturbative level all charges are present. !

($\Rightarrow$ we don't need branes in these theories for dualities, non-perturbative states in I or II A,B are mapped to perturbative states in heterotic theory)

Expectation values of scalar fields :

e.g. G_{ij} is a metric on T^d, it is flat $\Rightarrow$ the change of it doesn't matter ~~[illegible]~~ … <u>flat directions</u> .. these scalars may have any expectation value as we wish, they can be fixed to any constant value (there is no potential for them in S)

similarly if B_{ij} is constant $\Rightarrow$ the corresp. term in action is a total derivative $\to$ vanishes … also flat-directions

whereas some of $A_i^{I,\vec{P}}$ are not flat directions

because of $\operatorname{tr}[A_M, A_N]^2 \to \operatorname{tr}[A_i, A_j]^2$ potential

for A_i, only flat directions .. those along Cartan generators

.. $16 \times d$ flat directions

Total # of flat directions $\frac{1}{2}d(d+1) + \frac{1}{2}d(d-1) + 16d = d(16+d)$

$\Rightarrow$ space of trivial solns … space of inequivalent vacua … Moduli space of string vacua .. is $d(16+d)$-dimensional

~~If~~ $\langle A_i^I \rangle \neq 0$ in a generic point of moduli space $\Rightarrow$ all nonzero roots $A_i^{\vec{P}}$ become massive $\left(\sim \operatorname{tr}[\langle A_i^I\rangle A_j^{\vec{P}}]^2 \sim \underbrace{P_I^2 \langle A_i^I \rangle^2}_{\sim \text{mass}^2} A_j^{\vec{P}\,2}\right)$

i.e. all the charged fields become massive ($\Rightarrow$ SSB (?))

$\Rightarrow U(1)^{16+2d}$

D-branes (2nd lecture)

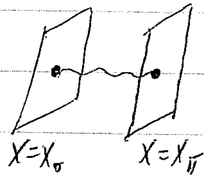
2 different planes in spacetime

described by $X(\sigma=0)=X_0$ $\quad X(\sigma=\pi)=X_\pi$

$$X(\tau,\sigma) = X + \alpha' g \tau + g\sigma + \text{oscillators}$$

$$X\Big|_{\sigma=0} \quad X+\alpha' g\tau + .. = X_0 \Rightarrow g=0,\ X=X_0$$

$$X\Big|_{\sigma=\pi} \quad X+\alpha' g\tau + g\pi + .. = X_\pi \Rightarrow g = \underbrace{\frac{X_\pi - X_0}{\pi}}_{\sim \text{ distance between planes}}$$



Quantisation

$$\begin{cases} (N) & \tilde\alpha_m = \alpha_m & \tilde\psi_r = \psi_r & r\in \mathbb{Z}\ (R) \\ (D) & \tilde\alpha_m = -\alpha_m & \tilde\psi_r = -\psi_r & r \in \mathbb{Z}+\frac{1}{2}\ (NS) \end{cases}$$

$$[\alpha_m^M, \alpha_m^N] = m\,\delta_{m+m,0}\,\eta^{MN} \qquad [\psi_r^M, \psi_r^S\} = \delta_{r+r,0}\,\eta^{MN}$$

$$H = L_0 + \bar L_0 = 2L_0 \qquad L_m = \oint \frac{dz}{2\pi i} z^{m+1} T_B(z)$$

$$T_B(z) = -\frac{1}{\alpha'}\partial X^M \partial X^N - \frac{1}{2}\psi^M \partial\psi_r$$

$$L_0 = \alpha' p^2 + \sum_{m=1}^{\infty} \alpha_{-m}^M \alpha_{M,m} + \sum_r r\,\psi_{-r}^M \psi_{M,r} \quad \text{if } X|_0 = X|_\pi$$

if $X|_0 \neq X|_\pi$ $\quad L_0 = \dots + \dfrac{(X_\pi - X_0)^2}{4\pi^2\alpha'}$

since $(\partial X)^2 \sim -\frac{1}{4}\dfrac{(X_\pi - X_0)^2}{\pi^2}$ and plug it into def. of L_0

$\Rightarrow$ such strings are always massive because of $\dfrac{(X_\pi - X_0)^2}{---}$ term

Computation of the RR-charge

usually from 4-point function i.e. from scattering .. not possible for D-branes, semiclassical treatment needed

Remark: Charge in QED

2 classical sources, quantum gauge fields between them

⊗ $\vec{x}=0$ $\qquad$ ⊗ $\vec{x}=\vec{x}_0$

$$S = -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu} + ie\int A_\mu J^\mu$$

$$J^\mu(x) = \delta_0^\mu\left[\delta^{(3)}(\vec{x}) + \delta^{(3)}(x - \vec{x}_0)\right]$$

the lowest approx. the 1-foton exchange

$$\mathcal{A} = \frac{(ie)^2}{2!}\int d^4x\, d^4y \ \langle A_\mu(x) J^\mu(x) A_\nu(y) J^\nu(y)\rangle$$

$$= \dots$$

$(\delta\text{-function})^2$ terms ..  ⊗ .. not interaction .. not interesting

$$= -e^2 \int dx^0 dy^0 \underbrace{G_{00}(x-y)}_{\int \frac{d^4p}{(2\pi)^4}\frac{-i\eta_{00}}{p^2 - i\epsilon} e^{-ip(x_0-y_0)}} + (\delta)^2 \text{ terms}$$

$$= -ie^2 \int dx_0 \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{e^{-i\vec{p}_i \cdot \vec{x}_i}}{\vec{p}^2 - i\varepsilon} = \underbrace{-ie^2 \overset{\text{time}}{\boxed{T}} \frac{1}{4\pi |\vec{x}_0|}}_{\text{Coulomb potential}}$$

We want a similar expression in the D-brane case

Firstly similar procedure for p-dim extended object

$$\int A_\mu dx^\mu = \int \tilde{A}_0(\tau) d\tau \qquad \tilde{A}_0(\tau) = A[x(\tau)] \frac{dx^\mu}{d\tau}$$

a pullback of a form

$$\int d^{p+1}\sigma \; \tilde{C}_{p+1} \qquad \tilde{C}_{\mu_0 \dots \mu_p} = C_{M_0 \dots M_p} \frac{\partial X^{M_0}}{\partial \sigma^{\mu_0}} \cdots \frac{\partial X^{M_p}}{\partial \sigma^{\mu_p}}$$

.. minimal coupling of an extended object to gauge field C_{p+1}

static sources $X^i(\tau) = 0$, $X^0(\tau) = f(\tau)$, in new coords $\tau'(\tau) = f(\tau)$

$\Rightarrow X^i(\tau) = 0$, $X^0(\tau) = \tau'$

$\Rightarrow \tilde{A}_0(\tau) = A_0$ ... static gauge

in the D-brane case $X^0 = \sigma^0, \dots, X^p = \sigma^p \Rightarrow \tilde{C}_{p+1} = C_{p+1}$

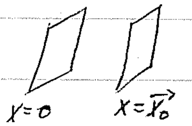
... static gauge in D-brane case

D = 10 p-brane

$$S = -\frac{1}{2(p+2)!} \int F_{M_0 \dots M_{p+1}} F^{M_0 \dots M_{p+1}} d^{10}X$$

$$+ iQ_p \int d^{p+1}\sigma \left[C_{0 \dots p}(0) + C_{0 \dots p}(\vec{X}_0) \right]$$

.. two static branes

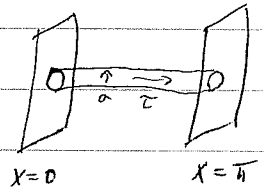


$$\Rightarrow \mathcal{A} = -iQ_p^2 \underbrace{V_{p+1}}_{\text{worldvolume of p-dim extended object}} \left(G^{(0)}_{9-p}(\vec{X}_0) \right) \text{ propagator } \sim \frac{1}{X^{9-p}}$$

We want to compute $\mathcal{A}$ in string theory and by comparison identify Q_p, $G^{(0)}_{9-p}$

String computation

Another interpretation



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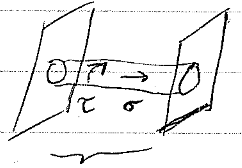


propagation of a closed string

an open string making a loop in time

(⇒ One may guess ~ gravity ~ ~ gauge field)

in our case easier computation

⇒ 1-loop partition function

Remark:

Partition function important in quant. gravity ~ energy of vacuum

~ cosmological const.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 + \frac{m^2}{2}\varphi^2$$

$$\mathcal{Z} = \int \mathcal{D}\varphi \, e^{-S} = \det(-\Box + m^2)^{-\frac{1}{2}} = e^{\mathrm{Tr} \log(-\Box + m^2)^{-\frac{1}{2}}}$$

$$\mathrm{Tr} \log(-\Box + m^2) = -\int_0^\infty \frac{dt}{t} \underbrace{\mathrm{Tr}\, e^{-t(-\Box + m^2)}}_{\text{resembles the usual partition function} \Rightarrow \text{hence the name}}$$

+ diverg. term indep. of m

$$\mathcal{Z} = \langle 0|0 \rangle_{1\text{-loop}} = e^{-E_0 \cdot T} \qquad -E_0 T = \frac{1}{2} \int_0^\infty \frac{dt}{t} \mathrm{Tr}\, e^{-t \boxed{H}}$$

$H = -\Box + m^2$

if more particles $\Rightarrow \sum_i (2j_i + 1)(-1)^{2j_i}$ ....

t ~ length of a loop $\Rightarrow t \to 0$ ~ ultraviolet divergences, will vanish in string theory

$\Rightarrow$ divergent energy of vacuum in usual QFT

in string case

$$Z_p(x_0) \overset{!}{=} \int_0^\infty \frac{dt}{t} \underset{\text{trace over all possible states}}{\boxed{\mathrm{Tr}}} \boxed{e^{-tH}} \qquad e^{-tH} = q^{L_0 - c}$$

c: zero point energy of bosons and fermions — each boson $-\frac{1}{24}$, each fermion in NS $-\frac{1}{48}$, in R $+\frac{1}{24}$

$$q = e^{-2\pi t}, \quad L_0 = \alpha' p^2 + \frac{(x_\pi - x_0)^2}{4\pi^2 \alpha'} + \sum \alpha^\mu_{-n} \alpha_{\mu,n} + \sum r \psi^\mu_{-r} \psi_{\mu,r}$$

$$\mathrm{Tr}_{\mathcal{H}}\, q^{L_0} = \mathrm{Tr}_{NS}\, q^{L_0(NS) + c_{NS}} \frac{(1 - (-1)^F)}{2} \ominus \mathrm{Tr}_R\, q^{+c_R + L_0(R)} \frac{(1 - (-)^F)}{2}$$

$\ominus$: since corresponds to fermions

3 different contributions:

- zero mode contribution
- bosonic contribution
- fermionic contribution

zero mode and bosonic contribution: universal

bosonic and fermionic contribution: (from worldsheet point of view

$$\int d^{p+1} x \int \frac{d^{p+1} p}{(2\pi)^{p+1}} e^{-2\pi t \left(\alpha' p^2 + \frac{x_0^2}{4\pi^2 \alpha'}\right)} = V_{p+1} (8\pi^2 \alpha' t)^{-\frac{p+1}{2}} e^{-\frac{t x_0^2}{2\pi^2 \alpha'}}$$

x_0: originally $x_\pi - x_0$

$p^2 = \vec{p}^{\,2} - p_0^2 \Rightarrow$ Wick rotation needed, then

---

<u>Classical solutions</u> (2nd lecture)

Properties of the found p-brane solution

$$ds^2 = H_p^{-1/2} d\vec{x}^{\,2} + H_p^{1/2} d\vec{y}^{\,2} \qquad \text{in the string frame}$$

$$e^{2\phi} = H_p^{(3-p)/2}$$

$$C_{01..p} = H_p^{-1} - 1 \qquad \text{where } H_p = 1 + \frac{Q_p}{r^{7-p}}, \; Q_p = c_p T_p G_N$$

<u>Einstein frame</u> $\Rightarrow \quad ds^2 = H_p^{(p-2)/8} d\vec{x}^{\,2} + H_p^{(p+2)/8} d\vec{y}^{\,2}$

other eqns. remain the same

IIA action in the Einstein frame

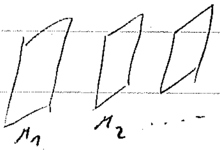
$$\sim e^{-\Phi} (dB)^2 + \ldots e^{\Phi} (dC^{(2)})^2$$

$\Rightarrow$ F1-string for $p = 1$, $\underbrace{\phi \to -\phi}$ or $g_s \to \frac{1}{g_s}$ from 1-brane F1

fundamental string is obtained by

Other possible harmonic functions

e.g. $H_p = 1 + \sum_i \frac{1}{|x - x_i|^{7-p}}$



Note: due to SUSY the gravitation cancels with repulsion due to other fields

$\Rightarrow$ configuration is in equilibrium for any x_i's

Remark on finding the solution

$$\delta F \overset{!}{=} 0$$

Dilatino .. purely algebraic condition $\delta\lambda = \partial_m \phi \, \Gamma^m \varepsilon + \frac{3-p}{4(p+2)!} \dots$

$$F_{m_1 \dots m_{p+2}} \Gamma^{m_1 \dots m_{p+2}} \varepsilon \overset{!}{=} 0$$

$\varepsilon \leftrightarrow \varepsilon'$ either by chirality or complex conj.

$\uparrow$ in II A (altogether Majorana spinor) $\uparrow$ in II B (altogether Weyl spinor)

assume $\phi = \phi(r)$ $\quad \Gamma^\mu \partial_\mu \phi \, \varepsilon + \overset{\text{const.}}{\sqrt{\ }} \partial_\mu C \, e^C \, \Gamma^\mu \Gamma^{0 \dots p} \varepsilon' = 0$

note: $\Gamma^\mu = e^{-A} \Gamma^a$ because of ansatz for metric

$\left(\{\Gamma^\mu, \Gamma^\nu\} \sim G^{\mu\nu} \text{ whereas } \underbrace{\{\Gamma^a, \Gamma^b\}}_{\text{the so-called } \Gamma\text{s in ON frame}} \sim \eta^{ab} \right)$

$$\Rightarrow \partial_\mu \phi \, \varepsilon + \frac{3-p}{4} \partial_\mu C \, e^{C - (p+1)A \cancel{\phi} + \phi} \underbrace{\Gamma_{\underline{0} \dots \underline{p}}}_{\text{in ON frame}} \varepsilon' = 0$$

$$\Rightarrow \partial_\mu \phi = \frac{3-p}{4} \partial_\mu C$$

$$C - (p+1)A + \phi = 0$$

$$\underbrace{\varepsilon + \Gamma_{01 \dots p} \varepsilon' = 0}_{\text{restricts SUSYs that are symmetry of the solution}}$$

RR-charge $\quad Q_E = \int_{S_\infty} * F^{(p+2)}$

Mass density (ADM) $\quad \boxed{Q_M = Q_E} \quad$ i.e. BPS $\boxed{M = Z}$

really BPS solution

T_p as function of g_s $\quad T_p(g_s)$

usually solitons $T \sim \frac{1}{g^2}$ $\quad$ e.g NS 5-brane

in string frame $\quad T_p(g_s) \sim \frac{1}{g_s}$ $\quad$ such behaviour is typical for open strings $\Rightarrow$ we have something like open strings in II A and II B closed string theory

$G_N \sim g_s^2$ (G_N Newton's constant) $\Rightarrow$

gravitation interaction $G_N T \sim 1$ for $T \sim \frac{1}{g^2}$

but $\quad G_N T \sim g_s \Rightarrow g_s \to 0 \Rightarrow G_N T \to 0 \Rightarrow$ should $\exists$ flat space description in this limit

Behaviour under T-duality

$R \leftrightarrow \frac{1}{R}$ $\quad$ winding modes $\leftrightarrow$ momentum modes

T-duality along x^p-direction

x^p periodic

$g_{pp} \rightarrow 1/g_{pp}$, transf. of other fields (e.g. $\tilde{G}_N$ should be G_N → scaling of dilaton)

=> p-brane ⟶ (p-1)-brane $\quad C_{01..p} \rightarrow C_{01..(p-1)}$

but H_p remains $H_p = 1 + \frac{Q_p}{r^{7-p}}$, it is not $H_{p-1} = 1 + \frac{Q_p}{(r')^{8-p}}$

=> the found (p-1)-brane is not localized but spread

in the x^p direction

vice versa y_1 T-duality → one has to "delocalize", i.e get rid

of y_1-dependence of r etc. then the T-dual is (p+1)-brane

Worldvolume dynamics of the RR 3-brane -- $N=4$ $U(1)$ SUSY Maxwell

F1 ending on a p-branes see lecture notes

<u>Finally</u>: D-branes - microscopic behaviour ⟷ RR-branes - macroscopic behaviour

<u>D=11 SUGRA</u>

bosonic field content (G_{MN}, C_{MNP}) $\quad M,N = 0,..,10$

(→ 10D IIA SUGRA $(G_{\mu\nu}, B_{\mu\nu}, \phi, A_\mu, C_{\mu\nu\lambda})$)

$\mu,\nu = 0,..,9$

$$\begin{tikzcd}
G_{MN} \arrow[d] \arrow[drr] \arrow[drrr] & C_{MNP} \arrow[d] \arrow[drrr] & & & \\
G_{\mu\nu} & B_{\mu\nu} & \phi & A_\mu & C_{\mu\nu\lambda}
\end{tikzcd}$$

Chern - Simons term

$$S_{11d} = \int d^{11}x \sqrt{G} \left[R(G) - \frac{1}{2\cdot 4!} (dC)^2 \right] + \underbrace{\int d^{11}x\, \varepsilon^{\mu_1 \cdots \mu_{11}} \partial_{\mu_1} C_{\mu_2 \cdots \mu_4} \partial_{\mu_5} C_{\mu_6 \cdots \mu_8} \cdot C_{\mu_9 \cdots \mu_{11}}}_{\int C\, dC\, dC}$$

$$\Big\downarrow$$

$$S_{10d} = \int (\ldots)$$

IIA in the string frame $\quad$ if we choose the fields so that :

$$ds_{11}^2 = e^{-2\phi/3} ds_{10}^2 + e^{4\phi/3} (dx^{10} - C^{(1)})^2$$

$$\text{and} \quad C = C^{(3)} + B\, dx^{10}$$

Electrically charged object $\quad \int_V C$ => V- membrane -- M2-brane

Magnetically -"- $\quad \int_V *C$ => 5-brane -- M5-brane

Similarly as in $D = 10$ we find

$$M2: \quad ds^2 = H^{-2/3} d\vec{x}^{\,2} + H^{1/3} \underbrace{d\vec{y}^{\,2}}_{8 \text{ coords}} , \quad H = 1 + \frac{Q_2}{r^6}$$

~~M5~~

$$M5: \quad ds^2 = H^{-1/3} d\vec{x}^{\,2} + H^{2/3} d\vec{y}^{\,2} \qquad H = 1 + \frac{Q_5}{r^3}$$

Relation between D = 10 and D = 11 solitons

- dim. reduction along worldvolume direction (because we need direction on which the metric doesn't depend)

  M2 ⟶ M2-brane wrapped over a longitudinal direction (a circle) ⟶ F1-string

  M5 ⟶ D4-brane

or: "delocalise" along a transverse direction and then reduce along it $\longrightarrow M2 \to RR$ 2-brane, $M5 \to NS$ 5-brane (there is no RR-brane in IIA)

What about D0 and D6 brane?

D6: $D=4$ Taub-NUT metric $\longrightarrow$ $D=11$ TN-6 brane, can be wrapped along transverse direction $\longrightarrow$ D6-brane

D0: charged under RR-1form $C^{(1)} = g_{\mu 10}$ ~ Kaluza-Klein modes, see later

---

D-branes (3rd lecture)

Computation of partition function continued

- bosonic contribution

Remark: In order to get rid of ghosts we should compute in light-cone gauge
$\Rightarrow$ summation of oscillators only $i=1,..,8$

$$\mathrm{Tr}_{\mathcal{H}_B}\, q^{\left(\sum_i \sum_n \alpha_{-n}\alpha_n\right)^N} \qquad N\alpha^i_{-n}|0\rangle = n\alpha^i_{-n}|0\rangle$$

fix i & n and consider 1 oscillator

$$\mathrm{Tr}\, q^{\alpha_{-n}\alpha_n} = \underset{\langle 0|.|0\rangle,}{\underset{\Uparrow}{1}} + \underset{\langle 0|\alpha_n|..|\alpha_{-n}|0\rangle,}{\underset{\Uparrow}{q^n}} + \underset{\langle 1..|(\alpha_{-n})^2|0\rangle}{\underset{\Uparrow}{q^{2n}}} + \dots + q^{n\cdot p} + \dots$$

$$= \frac{1}{1-q^n}$$

$$\Rightarrow \mathrm{Tr}_{\mathcal{H}_B}\, e^{\sum_i\sum_n \alpha^i_{-n}\alpha_{i,n}} = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^8} \Leftarrow i=1,..,8$$

$$C^{(NS)} = 8c_B + 8c_F^{(NS)} \qquad C^{(R)} = 8c_B + 8c_F^{(R)}$$

$$\Rightarrow \boxed{\mathrm{Tr}_{\mathcal{H}_B}\, q^{\sum_i\sum_n \alpha^i_{-n}\alpha_n - 8c_B} = q^{-\frac{1}{3}} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^8} = \left[q^{-\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n)^{-1}\right]^8}$$

- fermionic contributions

(NS) sector

$$\mathrm{Tr}_{NS}\, q^{\sum_i\sum_r r\psi^i_{-r}\psi^i_r + 8c_F^{(NS)}}$$

$$\mathrm{Tr}\, q^{r\psi^i_r\psi^i_{-r}} = 1 + q^r \qquad \left(\Leftarrow (\psi_{-r})^2 = 0\right)$$

$$\Rightarrow \boxed{\mathrm{Tr}_{NS}\, q^{\sum r\psi_{-r}\psi_r + 8c_F^{(NS)}} = \prod_{r=\frac{1}{2}}^{\infty} (1+q^r)^8\, q^{-\frac{1}{6}} \qquad (NS_+)}$$

$$\boxed{\mathrm{Tr}\, q^{\sum r\psi_{-r}\psi_r + 8c_F^{(NS)}} (-1)^F = \prod_{r=\frac{1}{2}}^{\infty} (1-q^r)^8\, q^{-\frac{1}{6}} \qquad (NS_-)}$$

(R) sector

$$\mathrm{Tr}_R\, q^{\sum_i \sum_n n\psi_{-n}^i \psi_n^i + 8c_F^{(R)}}$$

Note: ψ_0^i .. zero mode, doesn't change the Hamiltonian

$\psi_0^{i_1} .. \psi_0^{i_k}|0\rangle,\ k \leq 8$

$\Rightarrow$ degenerate vacuum

$$\mathrm{Tr}_R\left(q^{n\psi_{-n}\psi_n}\right) = 1 + q^n$$

R_+ $\Rightarrow$

$$\boxed{\mathrm{Tr}_R\, q^{\sum n\psi_{-n}\psi_n + 8c_F^{(R)}} = q^{\frac{1}{3}} \prod_{n=1}^{\infty} (1+q^n)^8 \cdot 16}$$

$$\psi_0^{\pm 1} = \frac{\psi_0^1 \pm i\psi_0^2}{\sqrt{2}}$$

$\vdots$

$$\psi_0^{\pm 4} = \frac{\psi_0^7 \pm i\psi_0^8}{\sqrt{2}}$$

$\Rightarrow$ we request (we may in any state act by ψ_0^i and get this)

$\psi_0^{+i}|0\rangle = 0$

but we don't have $\psi_0^{-i}|0\rangle = 0$

$\Rightarrow$ 16 states ~~of~~ of the same energy

degeneracy of vacuum $= 16$

R_-

$$\boxed{q^{+\frac{1}{3}} \prod_{n=1}^{\infty} (1-q^n)^8 \cdot \underset{\wedge}{0} = 0}$$

$1-4+6-4+1=0$ vacuum (zero-mode) contrib

<u>Modular functions</u> $\eta(\tau)$ $\quad \Theta_i(z,\tau)$

τ modular parameter of torus

we shall need $\eta(it)$, $\Theta_i(0, it)$

$$\eta(it) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n)$$ $\Rightarrow$ bosonic contrib $\dfrac{1}{\eta^8(it)}$

$NS_+ \ \ldots \left[\dfrac{\Theta_3(0,it)}{\eta(it)}\right]^4$ $\qquad R_+ \ \ldots \left[\dfrac{\Theta_2(0,it)}{\eta(it)}\right]^4$

$NS_- \ \ldots \left[\dfrac{\Theta_4(0,it)}{\eta(it)}\right]^4$ $\qquad R_- \ \ldots \left[\dfrac{\Theta_1(0,it)}{\eta(it)}\right]^4$

$\Rightarrow$ finally we have the product of zero-mode, bosonic and fermionic contribution

$$\boxed{Z_p(x) = i V_{p+1} (8\pi^2\alpha')^{-\frac{p+1}{2}} \int_0^{\infty} \frac{dt}{2t^{\frac{p+3}{2}}}\, e^{-\frac{tx^2}{2\pi\alpha'}} \frac{1}{\eta^{12}(it)} \cdot \left[\Theta_3^4(it) - \Theta_4^4(it) - \Theta_2^4(it)\right]}$$

Now we want to reinterpret $Z_p(x)$ as a propagation of closed string between two branes.

$$Z_p(x) = \sum_n -i\left(Q_p^{(n)}\right)^2 G_{9-p}^{(n)}(x)\, V_{p+1}$$ in analogy with QED

$\uparrow$ transverse propagators

$$G_d^{(m)}(x) = \int \frac{d^d p}{(2\pi)^d} \frac{e^{ipx}}{p^2+m^2} = \int \frac{d^d p}{(2\pi)^d} \int_0^{\infty} dl\, e^{-l(p^2+m^2)+ipx} =$$

$$= \int_0^{\infty} \frac{dl}{(4\pi)^{d/2}}\, l^{-\frac{d}{2}}\, e^{-\frac{x^2}{4l} - m^2 l}$$

$\Rightarrow$ By comparison we match terms containing x^2 $\rightarrow$ we need a change of variables $t = \frac{1}{l}$

$$Z_p(x) = iV_{p+1} (8\pi^2\alpha')^{-\frac{p+1}{2}} \int_0^{\infty} dl\, l^{-\frac{9-p}{2}}\, e^{-\frac{x^2}{2\pi\alpha' l}} \cdot M(il)$$

$$M(il) = \frac{1}{\eta^{12}\left(\frac{i}{l}\right)}\left[\ \cdot\ -\ \right]$$

We want to identify RR-charge of branes $\Rightarrow$ we have to identify all possible interactions (also gravity etc.) and be able to distinguish between them.

where $\theta_j(i\ell) \equiv \theta_j(0, i\ell)$

$M(\cancel{i\ell}\; i/\ell)$: $\theta_3(it) = \theta_3(\frac{i}{\ell}) = \sqrt{\ell}\ \theta_3(i\ell)$

$\theta_4(it) = \sqrt{2}\ \theta_2(i\ell)$

$\theta_2(it) = \sqrt{2}\ \theta_4(i\ell)$, $\eta(it) = \sqrt{\ell}\,\eta(i\ell)$

$$\Rightarrow M(i\ell) = \frac{1}{\eta^{12}(i\ell)} \left[\theta_3^4(i\ell) - \theta_2^4(i\ell) - \theta_4^4(i\ell) \right]$$

periodicity in time

$Tr_{NS+1} \sim \theta_3$ $\quad \psi(\sigma+2\pi) = -\psi(\sigma)$ $\quad \psi(\tau+2\pi) = -\psi(\tau)$

$Tr_R \sim \theta_2$ $\quad \psi(\sigma+2\pi) = \psi(\sigma)$ $\quad \psi(\tau+2\pi) = -\psi(\tau)$

$Tr_{NS(-1)F} \sim \theta_4$ $\quad \psi(\sigma+2\pi) = -\psi(\sigma)$ $\quad \psi(\tau+2\pi) = \psi(\tau)$

$\underbrace{(???)}$ somehow connected with GSO projection (free fermions antiperiodic in time, $(-)^F$ changes this)

$\Rightarrow$ for closed string $\tau \leftrightarrow \sigma$

$Tr_{NS+1} \sim NS-NS$

$Tr_R \sim$ —"— closed strings states

$Tr_{NS(-1)F} \sim R-R$

$\Rightarrow$ $\theta_2^4(i\ell)$ term is the important one for our purposes

$$\boxed{Z_p^{(RR)}(x) = -i V_{p+1} (8\pi^2\alpha')^{-\frac{p+1}{2}} \int_0^\infty d\ell \cdot \ell^{-\frac{9-p}{2}} e^{-\frac{x^2}{2\pi\alpha'\ell}} \frac{\theta_2^4(i\ell)}{\eta^{12}(i\ell)}}$$

<u>Heterotic string</u> (3rd lecture) & <u>String dualities</u> (1st lecture)

special subspaces of moduli space in which some charged states become massless and gauge group is enhanced from $U(1)^{16+2d}$

states charged under $G_{\mu i}$ -- Kaluza-Klein momentum modes (Fourier modes)

$B_{\mu i}$ -- stringy, winding numbers

<u>Stringy description</u> (without reference to K-K reduction)

$X^i = x^i + p^i t + w^i \sigma +$ oscillators

$X_L^i = \frac{x^i}{2} + \frac{1}{2} \overbrace{(p^i + w^i)}^{p_L^i} (t+\sigma) +$ —"—

$X_R^i = \frac{x^i}{2} + \frac{1}{2} \underbrace{(p^i - w^i)}_{p_R^i} (t-\sigma) +$ —"—

} d compactified coords.

in general $p_L^i \neq p_R^i$

$\omega^i = n_i R^i$ $\quad p^i = \frac{k_i}{R^i}$ quantized momenta

$X^I = x^I + \vec{p}_R (t-\sigma) +$ oscillators

<u>momenta</u> $(\vec{P}_L, \vec{P}_R)$ $\quad \vec{P}_R = (p^i, \vec{P}_R)$

~~10d~~ 16+d dimensional

<u>NS</u> $L_0 = -(\text{--}) M^2 + \frac{1}{2} \vec{P}_L^2 + N - \frac{1}{2}$

$\overset{\shortparallel}{-p^\mu p_\mu}$ $\quad \mu = 0, .., 9-d$

$\bar{L}_0 = -(\cdot) M^2 + \frac{1}{2} \vec{P}_R^2 + \bar{N} - 1$

GSO ⇒ $N \geq \frac{1}{2}$, $\cancel{N=\frac{1}{2}}$ $M^2 \propto \frac{1}{2}\vec{P}_L^2 + \dots$ $M=0$ $N=\frac{1}{2}$ $b^i_{-\frac{1}{2}}|0\rangle_{NS}$

$M^2 \propto \frac{1}{2}\vec{P}_R^2 + \bar{N} - 1$ $M=0$ $\wedge$ 1) $\vec{P}=0$

2) $\vec{P}_R^2=2$, $\bar{N}=0$
or
$\vec{P}_R=0$, $\bar{N}=1$

$(\vec{P}_L, \vec{P}_R) \in \Gamma_{d,16+d}$

Modular invariance $tr\, q^{L_0} \bar{q}^{\bar{L}_0}$

$$\sim \Big(\sum_{(\vec{P}_L,\vec{P}_R)\in\Gamma_{d,16+d}} q^{\frac{1}{2}\vec{P}_L^2} \bar{q}^{\frac{1}{2}\vec{P}_R^2}\Big) \cdot \text{(oscillators etc.)}$$

$q = e^{2\pi i\tau}$, modular invariance $\tau \to \tau+1$ $\tau \to -\frac{1}{\tau}$

$\Rightarrow e^{2\pi i\frac{1}{2}\vec{P}_L^2} e^{-2\pi i\frac{1}{2}\vec{P}_R^2}$ $\Downarrow$ $\boxed{\Gamma_{d,16+d} \text{ is self-dual}}$

w.r.t. the metric of sign.$(d,16+d)$

$$\Rightarrow \boxed{\vec{P}_L^2 - \vec{P}_R^2 \in 2\mathbb{Z}}$$

Theorem: $\Gamma_{\underset{+}{p},\underset{-}{q}}$ even & self-dual w.r.t. $(\overbrace{+,\dots,+}^{p},\overbrace{-,\dots,-}^{q}) \Longrightarrow$

1) exists if $p-q$ is a multiple of 8
2) if $p,q\neq 0$ then the lattice is unique
up to $SO(p,q)$ (or $O(p,q)$) transformation.

(Serre: A course in Arithmetics)

$SO(p) \subset SO(p,q)$ ⇒ rotation of $\vec{P}_L$ s only ⇒ physics doesn't change
(just a diff. selection of coords.)

and similarly $SO(q) \subset SO(p,q)$ doesn't change the physics
but boosts mixing p and q directions change the spectrum

⇒ $\boxed{\text{Moduli space (at least locally) is a coset space } SO(p,q)/O(p)\times O(q)}$

dim of moduli space $\frac{1}{2}(p+q)(p+q-1) - \frac{1}{2}p(p-1) - \frac{1}{2}q(q-1)$
$= \underline{\underline{p\cdot q}} = d(16+d)$ as we found
also before using KK

Generic points in moduli space ... all the charged states are massive
(because of boosts)

Remark: A^I_μ are called Wilson lines

The biggest non-Abelian group we can find for a special points of moduli space is $\underline{SO(32+2d)}$. (Proof possible in the string picture, not in KK)

BPS states

$b^i_{-\frac{1}{2}}|\vec{P}_L=0, \vec{P}_R^2=2\rangle$ + corresp. spinors .. short multiplet of theory with 16 Qs boost transf. $b^{i}_{-\frac{1}{2}} | \vec{P}_L \neq 0, \vec{P}_R \neq 0, \vec{P}_L^2 - \vec{P}_R^2 = -2 \rangle \Rightarrow$ mass $\propto |\vec{P}_L|$

but still short multiplet ... must carry central charge Z

$|Z| \sim$ mass $\sim |\vec{P}_L|$

<u>BPS condition</u> $N = \frac{1}{2}$ $\Rightarrow$ $8_B + 8_F$ degrees of freedom

$\Rightarrow \frac{1}{2}(\vec{P}_L^2 - \vec{P}_R^2) = \bar{N} - 1, \quad \bar{N} = 0, 1, 2, ..$

i.e. $\boxed{\vec{P}_L^2 - \vec{P}_R^2 = \textcircled{-2}, 0, 2, 4, ..}$

↑ only these can become massless at some point of modular space

$\vec{P}_L^2 - \vec{P}_R^2$	$\bar{N}$	# of short multiplets	
-2	0	1	
0	1	24	$(\alpha^a_{-1}\|0\rangle, \; a = 1,..,24)$
$+2$	2	$\frac{24\cdot25}{2} + 24 =$	$(\alpha^a_{-1}\alpha^b_{-1}\|0\rangle, \alpha^a_{-2}\|0\rangle)$
$+4$	3	$\frac{24\cdot25\cdot26}{2\cdot3} + 24^2 + 24$	$(\alpha_{-1}\alpha_{-1}\alpha_{-1}\|0\rangle, \alpha_{-2}\alpha_{-1}\|0\rangle, \alpha_{-3}\|0\rangle)$

Generating function for multiplicities $\frac{1}{q \prod_{n=1} (1-q^n)^{24}} = q^{-1}(1 + 24q + ...)$

<u>String dualities</u>

5 Superstring theories

32 SUSY charges IIA non-chiral IIB chiral

~~Hets~~ 16 SUSY charges Het $E_8 \times E_8$ Het SO(32) Type I SO(32)

Relations between them ?

Ex. (1) Compactification of IIA on a circle of radius r and IIB on a circle of radius $\frac{\alpha'}{r}$ $\Rightarrow$ the same 9-dim theory (we loose chirality)
... T-duality

(2) 11-dim SUGRA $\longrightarrow$ massless spectrum of IIA

(3) Het $E_8 \times E_8$ and SO(32) compactified on a circle $\Rightarrow$ lattice $\Gamma_{1,17}$ is unique $\Rightarrow$ the same theory .. again T-duality

(4) 10-dim low energy ~~[illegible]~~ effective action for a given gauge group is unique (due to SUSY) up to field redefinition
(massless states and 2-derivative terms only (higher derivatives suppressed like powers of Planck constant $\hbar$)

$\Rightarrow$ low energy connection between SO(32) Het and type I but completely lost for massive states

$G^h_{\mu\nu}, B^h_{\mu\nu}, \phi^h, F^h_{\mu\nu}$ ← heterotic SO(32) massless bosonic states

$G^I_{\mu\nu}, B^I_{\mu\nu}, \phi^I, F^I_{\mu\nu}$ ← type I —"—

$F^I_{\mu\nu}$ — from open string sector

G, B, ϕ from closed string sector IIB/Ω $G, \phi \subset$ NS-NS, $B \subset$ R-R sectors

$$S_h = \int d^{10}x \sqrt{G^h} \, \underline{\underline{e^{-2\phi^h}}} \left(R^h + (dB^h)^2 + (\partial\varphi^h)^2 + (F^h)^2 + \dots\right)$$

↑ ~ Euler characteristic of a sphere (sphere ← 0-loop calc.)

$$S_I = \int d^{10}x \sqrt{-G^I}\,\big[\underbrace{e^{-2\phi_I}(R^I + (\partial\phi^I)^2)}_{\text{NS fields}} + (dB^I)^2 + \underset{\text{open string} \to \text{disc}}{e^{-\phi^I}}(F^I)^2$$

NS fields

because R-R field

open string → disc

$\Rightarrow$ by comparison
$$\phi_h^h \to -\phi^I$$
$$G_{\mu\nu}^h \to e^{-\phi^I} G_{\mu\nu}^I$$
$$B_{\mu\nu}^h \to B_{\mu\nu}^I$$
$$F_{\mu\nu}^h \to F_{\mu\nu}^I$$

$(\Rightarrow)$: $\sqrt{G^h} \to e^{-5\phi^I}\sqrt{-G^I}$, $G_h^{\mu\nu}\partial_\mu\varphi^h\partial_\nu\varphi^h \to e^{\Phi_I} G_I^{\mu\nu}\partial_\mu\varphi^I\partial_\nu\varphi^I$

$(F^h)^2 \to e^{2\phi_I}(F^I)^2$, R ~~scales~~ scales like G^{-1}

2 inverse metrics in contraction

(Riemann tensor itself doesn't scale)

string coupling constant $g_{st}^h = e^{\langle\phi_h\rangle} \to e^{-\langle\phi_I\rangle} = \frac{1}{g_{st}^I}$

$\Rightarrow$ $\boxed{\text{strongly coupled type I} \longleftrightarrow \text{weakly coupled Het } SO(32) \text{ and vice versa}}$

$\boxed{\text{S-duality}}$ in contrast to T-duality which doesn't change the coupling

$\Rightarrow$ Not surprising that the massive modes don't agree (since they are in general unstable ~ coupling constant).

Classical solutions (3rd lecture)

$$ds_{11}^2 = e^{-2\phi/3}\, d^2s_{10} + e^{4\phi/3}(dx^{10} - C^{(1)})^2$$

F1: $ds^2 = H^{-1}d\vec{x}^2 + d\vec{y}^2$ (10 dim)

$e^{2\phi} = H^{+1}$

we put this into the equation above and expect to have

M2: $ds^2 = H^{-2/3}(d\vec{x}^2 + dx_{10}^2) + H^{1/3}d\vec{y}^2$ (11 dim)

really:

$ds_{11}^2 = H^{-1/3}(H^{-1}d\vec{x}^2 + d\vec{y}^2) + H^{-2/3}dx_{10}^2$ o.k

$G_{10,10} = e^{4\phi/3} = \underset{\text{radius}^2}{R_{10}^2} \Rightarrow \boxed{R_{10}^3 = g_s^2 l_p^3}$

Planck length in $d=11$

$l_p^2 = g_s^{2/3} l_s^2$ i.e. $\boxed{l_p^3 = g_s l_s^3}$

$\Rightarrow$ $\boxed{R_{10} = g_s l_s}$ $g_s \to 0 \Rightarrow$ approx. 10 dim theory because R_{10} is small

$g_s \to \infty \Rightarrow$ true 11-dimensional theory

$\Rightarrow \overset{?}{\exists}$ 11-dim Lorentz invariant theory ?

weak coupling → IIA string theory

strong coupling, low energies → 11 dim SOGRA

... M-theory conjecture

$T_{M2} \sim \frac{1}{\ell_p^3}$ wrap this on a circle along X_{10}

$$\Rightarrow T \sim \frac{g_s \ell_s}{\ell_p^3} = \frac{1}{\ell_s^2} = T_{F1} \quad \text{as it shall be}$$

wrap it on some other direction

$$T_{M2} = \frac{1}{\ell_p^3} \longrightarrow \frac{1}{g_s \ell_s^3} = T_{D2}$$

Similarly $\quad T_{M5} = \frac{1}{\ell_p^6} \longrightarrow T_{D4} \sim \frac{1}{g_s}$

$$\longrightarrow T_{NS5} \sim \frac{1}{g_s^2}$$

Compactified 11th dimension $\Rightarrow$ massive Kaluza-Klein modes

$$p \sim \frac{n}{R_{10}} \Rightarrow \underbrace{m_n^2 = \frac{n^2}{R_{10}^2} \text{ \& charged under } C^{(1)}}$$

is something like this in $\overline{II}A$

... D0-brane $\quad T_{D0} = \frac{1}{g_s \ell_s}$

no force between them $\Rightarrow T_{nD0} = \frac{n}{g_s \ell_s}$

$$\Rightarrow m_n^2 = \frac{n^2}{g_s^2 \ell_s^2}$$

a true test: there should be exactly one bound state of n D0-branes for every n (since KK modes are one per each n)

.. look on $U(1)$ SUSY QM and study bound states

.. rigorously known (hopefully) for $n = 2$, no proof for $n > 2$

Another topic:

Near horizon limits of non-dilatonic branes

$$\boxed{M2, M5, D_3}$$

Anti-de Sitter space $\quad$ spacetime of constant negative curvature

AdS embed in $\mathbb{R}^{2,d}$ $\quad (x^0, x^{d+1}, x^i, i = 1,..,d)$

$$\boxed{X_0^2 - \Sigma X_i^2 + X_{d+1}^2 = R_{AdS}^2} \quad AdS_{d+1} \text{ embedded into } \mathbb{R}^{2,d}$$

Poincaré coordinates:

$$\boxed{ds^2 = \frac{R_{AdS}^2\, dr^2}{r^2} + \frac{r^2}{R_{AdS}^2} \left(\eta_{\mu\nu}\, dx^\mu dx^\nu\right)}$$

$$\mu = 0, .., d-1$$

asymp. flat soln

$$D3: \quad ds^2 = H_3^{-1/2}(r)\, d\vec{x}^2 + H_3^{1/2}(r)\, d\vec{y}^2 , \quad H_3 = \boxed{1} + \frac{Q_3}{r^4}$$

$$Q_3 \sim N g_s \ell_s^4$$

$H_3 \rightarrow H_3 - 1 = \frac{Q_3}{r^4}$ also soln. of EoM

interpretation
- a) $N \rightarrow \infty \Rightarrow \frac{Q_3}{r^4}$ dominates
- b) $r \longrightarrow 0$ .. near-horizon limit
- c) particular limit of $N = 4$ SYM theory

$$\Rightarrow ds^2 = \underbrace{\frac{r^2}{Q^{1/2}} d\vec{x}^2 + Q_3^{1/2} \frac{dr^2}{r^2} + Q_3^{1/2} d\Omega_5^2}_{AdS_5 \times S^5}$$

$$R_{AdS} = Q_3^{1/4} \sim \ell_s (N g_s)^{1/4}$$

$$R_S = Q_3^{1/4}$$

... maximally SUSY, 32 supercharges
... enhancement of SUSY

$\Rightarrow$ D3 can be interpreted as a soliton ~~connecting~~ connecting two maximally SUSY solns., $\mathbb{R}^{1,9}$ and $AdS_5 \times S^5$

$M2 \to AdS_4 \times S^7$ (after some coord. transf.)

$$M5 \to \underbrace{AdS_7 \times S^4}_{\text{max. SUSY solns of 11d. SUGRA}}$$

Maximally SUSY solns.

11 dim	SUGRA	$\mathbb{R}^{10,1}$	$AdS_4 \times S^7$	$AdS_7 \times S^4$
10 dim	II B	$\mathbb{R}^{9,1}$	$AdS_5 \times S^5$	
	II A	$\mathbb{R}^{9,1}$	—〃—	

Are there any others?

1984 Kowalski - Glikman 11 dim pp-wave
2001 10d pp-wave in II B, II A

pp-waves plane-fronted gravitational waves with parallel rays

LC coords Minkowski $ds^2 = 2dx^+dx^- + \sum_i (dx_i)^2$

pp-wave $ds^2 = 2dx^+dx^- + \boxed{H(x^i, x^+)}(dx^+)^2 + \sum_i (dx_i)^2$ follows from Einstein eq.

$\frac{\partial}{\partial x^-}$ parallel null vector

the only non-vanishing element of Riemann tensor

$R_{+i-j} \sim \partial_i \partial_j H$

$R_{++} \sim \partial_i^2 H$, e.g. $H = A_{ij}(x^+)x^i x^j$, $\mathrm{Tr}A = 0$

$\Rightarrow$ vacuum soln of Einstein eq.

one may put into Einstein eq. only matter with nontrivial T_{++} only

Remark: $H(x^i, x^+) = A_{ij}(x^+)x^i x^j$ .. "exact gravitational waves"

$A_{ij} = -\delta_{ij}$ $F_{+1234} = F_{+5678} = 1$ $\Rightarrow$ max SUSY pp-wave of II B SUGRA

Isometries: $AdS_5 \times S^5$ $SO(4,2) \times SO(6)$ ... dim $= 30$

10d pp-wave non-~~simple~~ semisimple group also dim $= 30$

similar dimensional match for $D = 11$

Penrose limit

"Every spacetime has a pp-wave limit."

(1) preserves (at least) isometries
i.e. you don't loose Killing vectors
(2) you don't loose $SUSY_A$

$M, g_{\mu\nu}$ $\quad \gamma$ null geodesics $\quad$ a limit .. an observer boosting himself along this γ .. must be accompanied by large volume limit to be reasonable

$\Rightarrow$ pp-wave spacetime

in $AdS_5 \times S^5$ $\quad$ 2 possibilities $\quad$ 1) $\gamma \in AdS_5$

2) γ has a component along S^5

1) → Minkowski space

2) → IIB pp-wave

isometry groups are obtained by contraction of the original ones

and similarly for $AdS_4 \times S^7$, $AdS_7 \times S^4$

D-branes (4th and 5th lecture)

we write the field theory expression

$$A_p(X) = \sim i V_{p+1} \sum_n Q_p^{(n)\,2} \int_0^\infty \frac{dl}{(4\pi)^{\frac{9-p}{2}}}\, l^{-\frac{9-p}{2}}\, e^{-\frac{X^2}{4l} - m_n^2 l}$$

we make the rescaling $\pi\alpha' l \to 2l$ in stringy expression and expand

$$\frac{\Theta_2^4\left(\frac{4il}{\pi\alpha'}\right)}{\eta^{12}\left(\frac{2il}{\pi\alpha'}\right)} \underset{\text{expansion}}{=} \sum_{n=0}^\infty d_n e^{-\frac{4nl}{\alpha'}}$$

$\Rightarrow$ by comparison $\quad m_n^2 = \frac{4n}{\alpha'}$ $\quad \sim$ closed string states

$n=0$ .. massless states $\quad \Rightarrow$ RR-charge of a Dp-brane

$$\Rightarrow \boxed{Q_p = \sqrt{2\pi}\,(4\pi^2\alpha')^{\frac{p-3}{2}}}$$

Similarly exchange of NS-NS strings $\Rightarrow$ mass of a p-brane
(~ exchange of gravitons)

$$\Rightarrow T_p = Q_p \quad \text{i.e.} \quad \boxed{T_p = Q_p \frac{\sqrt{2}}{K_{10} g_s}} \qquad K_{10}^2 = 8\pi G_N$$

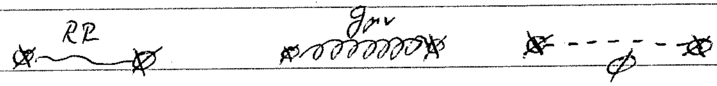
T_p very large at weak coupling $\Rightarrow$ solitonic behaviour

Remark: the full Z (both NS-NS & R-R) $\quad Z_p(X) \sim .. \int_0^\infty dl\, l^{-} M(il) = 0$

because $M(il)$ is trivially 0

$\Rightarrow$ interaction between two parallel d-branes vanishes

Interpretation



the same charge $\Rightarrow$ $\underbrace{\text{repulsive} \qquad \text{attractive}}_{\text{balance each other}}$ $\qquad$ also attractive

in closed string point of view
(i.e. exchange of a closed string)

The same cancellation is present at each ~~[illegible]~~ mass level.

This equilibrium configuration is due to SUSY since in the open-string picture (partition fction of an open string)

.. $\forall n \quad \underset{n}{\bigcirc} = 0$ at any order of perturbation theory

(due to bosonic - fermionic cancellations)

BPS state $|\phi\rangle$ if $\exists Q_{SUSY} : Q_{SUSY} |\phi\rangle = 0$

D-branes are $\left(\frac{1}{2}\right)$ BPS states

broken $\frac{1}{2}$ SUSY

Interpretation of charges

D_p a source of C_{p+1} (see eff. field theory in Classical Solns.)

$$\text{II A} \quad (D_0 \nearrow C_1,\ D_2 \nearrow C_3,\ D_4,\ D_6)$$

$$\text{II B} \quad (D_{-1})(D_1,\ D_3,\ D_5,\ D_7) \qquad D_{-1} \to C_0,\ D_1 \to C_2,\ D_3 \to C_4^{(+)}$$

under which fields are ↗↗ and ⟶ charged ?

E-M duality (Hodge duality)

e.g. ~~$^*C_3$ can be integrated~~ $C_3 \to F_4 \to (*F)_6 \to \cancel{d}\, C_5$

$\Rightarrow$ D_4 is charged under this C_5 $\Rightarrow$ D_4 is magnetically charged under C_3

Remark: E-M charges

$$Q_p = \cancel{\oint_{S^{p+2}} F_{p+2}} \int_{S^{D-p-2}} {}^*F_{p+2} \qquad \text{electric charge}$$

↑ in infinity

e.g. $D=4 \quad p=0$ $\qquad \vec{E} = \frac{e}{4\pi r^2}, \quad F_{0r} = E_r$

$$Q_0 = \int_{S^2} ({}^*F_{\vartheta\varphi})\, d\theta \wedge d\varphi$$

$$^*F_{\mu\nu\lambda\sigma} = \frac{\sqrt{\det g}}{2}\, \epsilon_{\mu\nu\lambda\sigma} F^{\lambda\sigma}$$

$$\Rightarrow (*F)_{\vartheta\varphi} = \sqrt{\det g}\, g^{rr} g^{00} F_{0r} = \frac{e}{4\pi} \sin\vartheta$$

$$\Rightarrow Q_0 = \int_{S^2} \frac{e}{4\pi} \sin\vartheta\, d\vartheta d\varphi = \underline{\underline{e}}$$

$$g_{D-p-4} = \int_{S^{p+2}} F_{p+2} \qquad \text{magnetic charge}$$

since E-M duality exchanges $\vec{E} \leftrightarrow \vec{B}$ i.e. $F \longleftrightarrow *F$

Ex: in $D=4,\ p=0$ $\qquad D-p-4=0$ again a particle

$D=10,\ p=1$ $\qquad D-p-4 = 5$ - brane .. a magnetic dual of a string

$\Rightarrow$ D_4 is a magnetic dual of D_2, D_6 magnetic dual of D_0

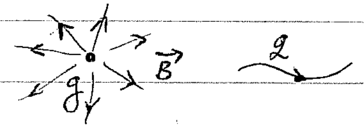
Similarly D_7 is a magn. dual of D_{-1}, D_5 is a magn. dual of D_1,

D_3 is a self-dual .. both magn. and electrically charged .. dyons

$C_4^{(+)}$: $F_5 \equiv {}^*F_5$ $\Rightarrow$ the same magn. and ~~charge~~ electric charge of D_3-brane

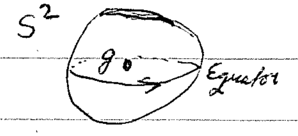
Dirac quantization condition

for a particle



$$\mathcal{L} = .. + iq\int A$$

on a sphere



$$\mathcal{L} = .. + iq\int_{\text{Equator}} A$$

$\Rightarrow$ in path-integral $e^{iq\oint_{E_q} A + \ldots}$

$$\oint_{E_g} A = \int_M B \qquad \partial M = E_g$$

M for instance the northern hemisphere M_N
or the southern — " — M_S
with correct orientation

wrong D
A not globally def.

$$\xrightarrow{e} \oint_{E_q} A = \int_{M_N} B = -\int_{M_S} B \Rightarrow e^{iq\oint A} = e^{iq\int_{M_N} B} = e^{-iq\int_{M_S} B}$$

$$\Rightarrow e^{iq\left(\int_{M_N} B + \int_{M_S} B\right)} = 1 \quad \text{i.e.} \quad e^{iq\int_{S^2} B} = 1$$

but $\int_{S^2} B = g \Rightarrow$ ~~q~~ $\boxed{q\cdot g \in 2\pi\mathbb{Z}}$ Dirac quantization condition

Generalization for an extended object

$$\boxed{Q_p\, g_{D-p-4} = 2\pi n}, \ n \in \mathbb{Z}$$

We have found $Q_p = \sqrt{2\pi}\,(4\pi^2\alpha')^{\frac{3-p}{2}}$

$$g_{D-p-4} = Q_{6-p}$$

$\Rightarrow$ should be $Q_p \cdot Q_{6-p} = 2\pi n$

$$\overset{\shortparallel}{\sqrt{2\pi}\,(4\pi^2\alpha')^{\cdots}\cdot\sqrt{2\pi}\,(\ldots)^{\cdots}} = 2\pi \Rightarrow \text{fulfilled with } \boxed{n = 1}$$

$\Rightarrow$ D-branes look like a fundamental quanta carrying the charge Q_p

T-duality and D-branes

II A on a circle $S^1 \times R^{1,8} \Rightarrow$ discrete momenta .. KK modes $p = \frac{n}{R}$

and also winding modes $\sim mR$

$\Rightarrow M_{IIA}^2 = \left(\frac{n}{R}\right)^2 + c_1 (mR)^2 + c_2 (N + \bar{N})$ depending on m, n

phys. states $L_0|\phi\rangle = \bar{L}_0|\phi\rangle \Rightarrow$ a cond. relating $m, n, N, \bar{N}$

$R \to \frac{\alpha'}{R}$, $m \Leftrightarrow n \Rightarrow$ spectrum is invariant .. T-duality transf.

one can also check that interactions are invariant $\to$ exact symmetry of theory

$$P_L = \frac{m}{R} + \frac{\alpha'}{R}n \qquad \xrightarrow{\text{T-duality}} \qquad X_L \to X_L \qquad P_L \to P_L \qquad \psi \to \psi$$

$$P_R = \frac{m}{R} - \frac{R}{\alpha'}n \qquad\qquad X_R \to -X_R \qquad P_R \to -P_R \qquad \tilde{\psi} \to -\tilde{\psi}$$

$\Rightarrow$ from II A on $S^1\times R^{1,8}$ we find II B on $S^2(\frac{\alpha'}{R}) \times R^{1,8}$ and vice versa

$\mathbb{I}A$ $\quad (-)^F |0\rangle_R = +|0\rangle_R \qquad (-)^F |\tilde{0}\rangle_R = -|\tilde{0}\rangle_R$

$\mathbb{I}B$ $\quad$ –"– $\qquad (-)^F |\tilde{0}\rangle_R = +|\tilde{0}\rangle_R$

$(-)^{\tilde{F}} \longrightarrow -(-)^{\tilde{F}}$ by $\tilde{\psi} \to -\tilde{\psi}$ since $(-)^{\tilde{F}} = \tilde{\psi}_0^1 \cdots \tilde{\psi}_0^8 (-1)^{\sum \tilde{\psi}_{-n}^i \tilde{\psi}_n^i}$
$\to -(-)^{\tilde{F}}$

Boundary cond. $\partial X_L = \bar{\partial} X_R$ (N) $\qquad \partial X_L = -\bar{\partial} X_R$ (D)
$\Rightarrow T \longrightarrow (N) \leftrightarrow (D)$

$\Rightarrow$ T-duality transf. D_p brane either to D_{p+1} or to D_{p-1}

1) $S^1 \perp D_p$-brane $\quad T: D_p \to D_{p+1}$-brane

2) $S^1 \parallel D_p$-brane $\quad T: D_p \to D_{p-1}$-brane

Low energy effective action describing dynamics in the D-branes

1 Dp-brane

a spectrum of open strings

NS-sector $\qquad |0\rangle$ tachyon $\to$ GSO projection

$$\frac{1 \mp (-1)^F}{2} |\phi_{phys}\rangle_{NS} = 0$$

$\psi_{-\frac{1}{2}}^M |0\rangle$ massless state

$\vdots$ massive state

little group of $p+1$-plane $\qquad \Rightarrow$ 1 massless vector $\overset{\nearrow U(1) \text{ group}}{\boxed{A_\mu}}$ $(M=\mu)$

& $9-p$ scalars $\quad (M=i)$

R-sector $\qquad \frac{1 \mp (-1)^F}{2} |phys\rangle = 0$

$\Rightarrow$ chiral spinors $\to$ on the $p+1$-plane a bunch of spinors ψ_i

$\Rightarrow$ on the D_p-brane we have SQED in $p+1$ dim trivially reduced from $D=10$ SQED

… 16 supercharges

N parallel branes $\qquad$ a stable configuration

$\Rightarrow$ Low-~~dim~~ energy effective action .. a sum of low-~~dim~~ energy effective actions for each brane (strings connecting branes become massive $(M)^2 \sim (\Delta X)^2 \Rightarrow$ integrated out, may contribute to renormalization of charges)

n-coincident D_p-branes .. give rise to symmetry enhancement (connecting strings become massless) .. $U(n)$ SYM that is the dim. reduction of $D=10$ $U(n)$ SYM to $p+1$-dim

$n=2$ we assume $U(2)$ gauge symmetry

$$\mathcal{L} = -\frac{1}{4g^2}\Big[\int d^{p+1}\sigma \, \mathrm{Tr}\Big[F^2 + 2\sum (D_\mu \varphi)^2 - \underbrace{\sum_{I,J} [\varphi^I, \varphi^J]^2}_{V(\varphi)}\Big\}$$

+ fermions

F, φ in adjoint repr. of $U(2)$

We assume φ = const. $\Rightarrow \frac{\partial V}{\partial \varphi} \overset{!}{=} 0 \Rightarrow \sum_J [\varphi^J, [\varphi^I, \varphi^J]] = 0$

$\Rightarrow \exists$ solns of the form $[\varphi^I, \varphi^J] = 0$, i.e. φ^I diagonal

$$\varphi^I = \mathrm{diag}\,(\lambda^I_1, .., \lambda^I_n) \qquad I = 1, .., 9-p$$

λ^I_α

↖ position of D-brane α [illegible], i.e. $9-p$ coords fixed

$\Longrightarrow n(9-p)$-dimensional space .. moduli space of vacua

$U(2)$ case we assume that only one φ has nonzero expectation values

$$\langle\varphi\rangle = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \text{ the other } \varphi_A \ \langle\varphi\rangle = 0$$

Quantum fluctuations $\Phi = \langle\varphi\rangle + \begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix}$

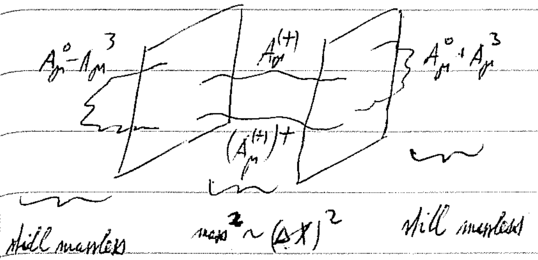
$$\mathrm{Tr}\,(D_\mu\varphi)^2 \sim [A_\mu, \langle\varphi\rangle]^2 \qquad D_\mu\varphi = \partial_\mu\varphi + [A_\mu, \varphi]$$

$$= (\lambda_1 - \lambda_2)^2 (A_\mu^{(+)})^\dagger A_\mu^{(+)}$$

where $A_\mu = \begin{pmatrix} A^0_\mu + A^3_\mu & A^{(+)}_\mu \\ (A^{(+)}_\mu)^\dagger & A^0_\mu - A^3_\mu \end{pmatrix}$

we interpret $\lambda_1 - \lambda_2$ as distance of the branes

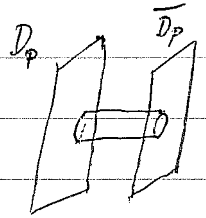
$\Rightarrow$ symmetry breaking $U(2) \to U(1) \times U(1)$



More generic configuration of D-branes

- we want stable solutions .. usually the ones with some SUSYs
static

anti-D-brane ... with opposite RR-charge ~ change of orientation of D-brane

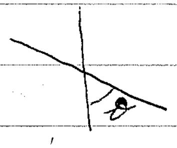


computation like before but sign flip $Q_p \to -Q_p$

=> attraction => non-static solution

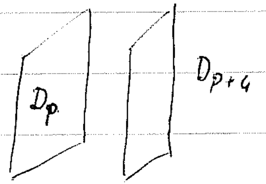
at certain separation a phase transition occurs => annihilation

2 D-strings



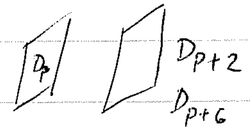
.. not stable ⇒ either annihilation or parallel but at some special angles unstable equilibria - still some SUSY like $\vartheta = \pi/2$

Other Ex:



.. stable but only 8 supercharges preserved
~~sos~~ .. $\frac{1}{4}$ BPS state

whereas



no [illegible] SUSY charges preserved → unstable

SUSY behaves like an attractor, non-SUSY solns. tend to SUSY solns.

other SUSY solns can be found by T-duality transf. from the ones given

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String dualities (2nd lecture)

Some massive states may be stable w.r.t S-duality

1) BPS states (preserve half supersymmetry)
2) non BPS states e.g.

Het SO(32) Γ_{16} contains lattice of SO(32) and spinor weight of SO(32) $\underbrace{(\frac{1}{2}, \dots, \frac{1}{2})}_{16 \text{ entries}}$

↳ states with $M^2 \propto (1+\tilde{N})$. ⇒ lengths 4 ⇒ $\vec{P}^2 = 4$

These spinors seem to be preserved but non BPS. They are preserved since the [illegible] spinor cannot decay.

ad BPS states

suppose compactification $10\text{-}d \xrightarrow{S^1} 9\text{-}d$

$G_{\mu\nu}, B_{\mu\nu}, A_\mu^J$

(for generic point in moduli space)

charges k .. KK momentum w .. winding

⇒ in heterotic perturbative theory we have all charged states

Type I $(G_{\mu\nu}^I, B_{\mu\nu}^I, A_{\mu I}^J)$ $(A_{\mu I}^{\vec{P}}$ massive for generic point$)$

KK momenta no charge .. we miss these charges

$B_{\mu\nu}$ 2-form, i.e. naturally couples to a 2-dim object .. D-string

then we have a complete match

Het SO(32) .. S^1 with radius r_9^h
Type I .. S^1 r_9^I

$$G_{99}^{h} = r_9^{h\,2} = e^{-\Phi_I} (r_9^I)^2$$

any scalar field with mass m_h KG eqn. $(G^{\mu\nu}\partial_\mu\partial_\nu - m_h^2)\Phi = 0$

i.e. $(e^{\Phi_I} G_I^{\mu\nu}\partial_\mu\partial_\nu - m_h^2)\Phi = 0$

$\Rightarrow \quad e^{\Phi_I}(G_I^{\mu\nu}\partial_\mu\partial_\nu - e^{-\Phi_I} m_h^2)\Phi = 0$

$\Rightarrow$ type I observer would see $m_I^2 = e^{-\Phi_I} m_h^2$

$\Rightarrow$ KK mode … $m_I = e^{-\Phi_I/2} m_h = e^{-\Phi/2}\, k/r_h = \cancel{\frac{e^{-\Phi_I/2}}{e^{-\Phi_I/2}}} \frac{k}{r_I} = \frac{k}{r_I}$

$\Rightarrow$ KK modes translate between themselves in the two theories

Winding … $m_I = e^{-\Phi_I/2} r_h = e^{-\Phi_I} r_I$ (for $\omega = 1$)

… really mass of a D-string $\frac{r_I}{g_{st}^I}$

<u>Remark</u>: single D-string in type I $\to$ low energy effective action on the world volume $= S^{het}_{Polyakov}$

$\Rightarrow$ match of multiplicities … maybe tomorrow

<u>Remark</u>: Type I closed strings $IIB/\Omega \xrightarrow[\text{T-duality}]{} IIB/Z_2\cdot\Omega$

Type I' (T-dual of I) _ _ _ _

$Z_2 : X_9 \to -X_9$ mirror reflection

open strings D9-branes $\searrow$ D8-branes

Connection between 11dim SUGRA, IIA, IIB

IIB self-dual, in fact ∞ number of symmetries

massless fields ϕ, $B \equiv B_N$, G NS-NS

$C_+^{(4)}$, $C^{(2)} \equiv B_R$, $C^{(0)} \equiv \chi$ R-R

2 scalars, 2 antisym. tensor

Effective action $\nexists$ action covariant & containing $C_+^{(4)}$

$\Rightarrow$ we assume $dC_+^{(4)} = 0$

define $\tau = \chi + ie^{-\phi} = \tau_1 + i\tau_2$, $\partial\mathcal{B} = \begin{pmatrix}\partial B_N \\ \partial B_R\end{pmatrix}$

$$S_{IIB} = \int d^{10}x \sqrt{-G}\left\{\tau_2^2 R + |\partial\tau|^2 + \partial\mathcal{B}^T \begin{pmatrix}|\tau|^2 & \tau_1 \\ \tau_1 & 1\end{pmatrix}\partial\mathcal{B}\right\}$$

+ Chern-Simons terms + fermions

-- $SL(2,\mathbb{R})$ symmetry

$\tau \longrightarrow \frac{a\tau+b}{c\tau+d}$ $\begin{pmatrix}a & b \\ c & d\end{pmatrix} \in SL(2,\mathbb{R})$

$\mathcal{B} \longrightarrow \begin{pmatrix}d & -c \\ -b & a\end{pmatrix}\mathcal{B}$

$G_{\mu\nu} \longrightarrow |c\tau+d|\, G_{\mu\nu}$

when one takes into account also $C_+^{(4)}$ (possible only on the level of EoMs, not for action) $\Rightarrow C_+^{(4)} \longrightarrow C_+^{(4)}$

This symmetry cannot survive on quantum level since winding and momenta are quantized

More precisely:

F1 couples to B_N — charge p

D1 string couples to B_R — charge q

$\Big\}$ altogether (p,q)-string

coupling $p\int B^N_{\mu\nu}\dot{X}^\mu + q\int B^R_{\mu\nu}\dot{X}^\mu$

$$\text{i.e.} \sim (p,q)\int\begin{pmatrix} B^N_{\mu\nu} \\ B^R_{\mu\nu}\end{pmatrix}\dot{X}^\mu \Rightarrow (p,q)\to(p,q)\begin{pmatrix} a & c \\ b & d\end{pmatrix}$$

$\Rightarrow$ quantum symmetry is at most $SL(2,\mathbb{Z})$

Conjecture: quantum symmetry is $SL(2,\mathbb{Z})$

$$\begin{tikzcd}
\text{11dim M-theory} \arrow[d, "S^1"', "\cancel{\text{IIA}}"] & & \\
\text{II}A \arrow[dr, "S^1"] & & \text{II}B \arrow[dl, "S^1"'] \\
& &
\end{tikzcd}$$

After some computation τ in IIB becomes the modulus of the torus $S^1\times S^1$ on which M-theory is compactified

$\Rightarrow SL(2,\mathbb{Z})$ symmetry of IIB

Tension for a system containing F1 and D1

$T_{(p,q)} \propto |p - q\tau|$ transforms (after taking into account the rescaling of metric) as it should

$$T(p,q) = \sqrt{(p-q\chi)^2 + e^{-2\phi}q^2}, \text{ if } \langle\chi\rangle = 0 \text{ then}$$
$$= \sqrt{p^2 + \frac{q^2}{g_{st}^2}}$$

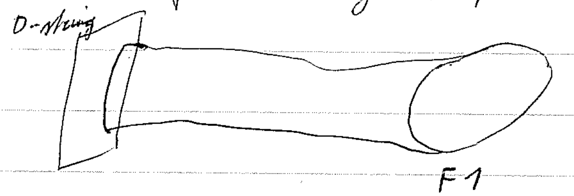
Example $p=q=1$ $\quad T(1,1) = \sqrt{1+\frac{1}{g_{st}^2}}$

if F1 is far away from D1 $\Rightarrow$ 1 resp. $\frac{1}{g_{st}}$

when we move them one to the other $\Rightarrow$ tension is smaller than the sum of $T(1,0)$ and $T(0,1)$

$\Rightarrow$ boundstate, binding energy

How can theories of closed strings IIA, IIB contain D-branes?



F1 string - may partially overlap with D1 - this part of the string doesn't in weak coupling contribute to action $\Rightarrow$ we effectively find an open string

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AdS/CFT correspondence (1st lecture)

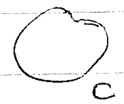
- type IIB superstring on $AdS_5\times S^5$ and $N=4$ $D=4$ SYM theory

1. Gauge fields or strings ?
2. Tests of the correspondence
3. Holographic renormalisation

YM in a confined phase ...

non-local observables e.g. Wilson loops

holonomy of a connection along a ~~this~~ curve



$$\langle W(C)\rangle = \frac{1}{N}\langle \mathrm{tr}\, \overset{\uparrow\text{path ordering}}{P} \exp i \oint A_\mu dx^\mu \rangle$$

$$W(C) \sim e^{-T \cdot \underset{\uparrow \text{area}}{A(\Sigma)}}$$

Σ .. minimal surface whose boundary is C

A. Polyakov .. string ansatz for confinement

$$\langle W(C)\rangle_{\text{string}} = \int \frac{\mathcal{D}X^\mu \mathcal{D}g}{\mathrm{Vol}} e^{-S[X,g]}$$

$$S[X,g] = \frac{1}{4\pi\alpha'}\int \sqrt{g}\, d^2\sigma\, g^{\alpha\beta}\partial_\alpha X \partial_\beta X$$

local Weyl invariance $g_{\alpha\beta} \to e^{2\phi} g_{\alpha\beta}$, ϕ Liouville mode

anomalous, only decouples in $D = 26$

$D \neq 26$ $(\partial X)^2 \to a^2(\phi)\, \partial X \partial X$

lack of zig-zag symmetry

unless Liouville mode is present

→ hint to the presence of a 5th coordinate → φ

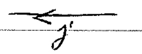
Large N SU(N)

$$A_\mu^a \Rightarrow A_\mu{}^i{}_j = A_\mu^a (T_a)^i{}_j$$

since adjoint repr. is contained in product of fundamental repres.

$$\underline{N} \times \underline{N}^* = \underline{(N^2-1)} + \underline{1}$$

$\Rightarrow$ $A_\mu{}^i{}_j$ may be represented by a double line

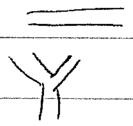


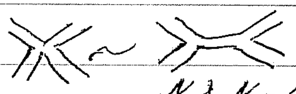
$$\langle A_\mu{}^i{}_j(x)\, A_\nu{}^l{}_k(y)\rangle = \delta_{\mu\nu}\, A(x-y)\left(\delta^i_k \delta^l_j - \frac{1}{N}\delta^i_j \delta^l_k\right)$$

$$S = \frac{N}{4\lambda}\int \mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) \qquad \lambda = g^2_{YM} N$$

$\Rightarrow$ propagator $\Rightarrow \frac{\lambda}{N}$

vertex $\Rightarrow \frac{N}{\lambda}$





$$\frac{N}{\lambda}\frac{\lambda}{N}\frac{N}{\lambda} = \frac{N}{\lambda}$$

loop $\Rightarrow N$

$\Rightarrow$ diagram with P propagators, V vertices, L loops is weighed by $A_{P,N,L} = N^L \left(\frac{\lambda}{N}\right)^P \left(\frac{N}{\lambda}\right)^V = N^{L-P+V} (\lambda)^{P-V}$

We keep λ fixed $N \to \infty$, $g_{YM} \to 0$

$\Rightarrow A \sim N^{\chi} \lambda^{P-V}$ $\chi = 2-2h$ = Euler characteristic of a surface triangulated by Feynman diagram

Large N limit -- only planar diagrams triangulating the sphere survive

$$A = \sum_{h=0}^{\infty} \sum_k a_k^{(h)} \lambda^k = \sum A_h(\lambda)$$

For other groups also other terms arise

+ fundamental boundaries $\chi = 2-2h-b$

+ symmetric / antisymmetric tensor irreps ~~cross~~ crosscaps $\chi = 2-2h-b-c$

 e.g. for $SU(N)$, $Sp(2N)$ $c = 0,1,2$

Exercise  $= 0$ by group theory

Maldacena conjecture ($\Leftarrow$ dynamics of D3-branes at low energies, i.e. near horizon)

D3-brane $\phi = g_s$ is constant

$$ds_{D3}^2 = \left(1+\frac{L^4}{r^4}\right)^{-\frac{1}{2}} \underbrace{dx\cdot dx}_{\substack{\text{4 coords incl. time} \\ x^\mu,\ \mu=0,..,3}} + \left(1+\frac{L^4}{r^4}\right)^{\frac{1}{2}} \underbrace{dy\cdot dy}_{y^i,\ i=1,..,6}$$

$r^2 = y\cdot y$

$L^4 = 4\pi g_s (\alpha')^2 \cdot N$ $N = \#$ of branes

$$\int_{S^5} F_5 = N$$

$r \to 0$ $\Rightarrow$ 1) gravitational redshift

2) $AdS_5 \times S^5$ metric $ds^2 = \frac{r^2}{L^2} dx\cdot dx + \frac{L^2}{r^2}(dr^2 + r^2 d\omega_5^2)$

$$= \cancel{\frac{r^2}{L^2} dx \cdot dx} \underbrace{\frac{L^2}{\varsigma}(dx\cdot dx + d\varsigma^2)}_{AdS} + \underbrace{L^2 d\omega_5^2}_{\substack{S^5 \\ \text{rad}=1}}$$

$\varsigma = \frac{L^2}{r}$

isometries $SO(4,2)$ $(AdS \sim SO(4,2)/SO(4,1))$

$SO(6)$ $(S^5 \sim SO(6)/SO(5))$

$$SO(4,2) \times SO(6) \subset \underbrace{SU(2,2/4)}$$

$N=4$ superconformal group in $D=4$

$\hookrightarrow$ $N=4$ SYM in $D=4$

We identify

$4\pi g_s = g_M^2$ $\vartheta_s = \langle \chi \rangle = \frac{\vartheta_{YM}}{2\pi}$

$\uparrow$ R-R axion

-- Holographic correspondence

Φ = string excitations with fixed boundary cond. on $\partial AdS_5 \sim \overline{M}_4$ $\swarrow J$

$$Z_{IIB}[\Phi(J)] = Z_{SYM}[J] = \int \mathcal{D}\sigma\, e^{-S[\sigma] + \int J\, \mathcal{O}_\Delta\, d^4x}$$

$\uparrow$ sources

gauge invariant local composite operators

$\mathcal{Z}_{\text{II}B}$ is difficult to compute due to RR-background

$L^2/\alpha' \gg 1$ $\quad L^2/\alpha' = \sqrt{g_s 4\pi N} = \sqrt{g_{YM}^2 N} = \sqrt{\lambda}$ 't Hooft coupling

$\lambda \gg 1$ .. type IIB SUGRA on $AdS_5 \times S^5$

⋮ $S_{\text{II}B}$ eff. action ---

CFT

Local QFT $\quad$ Poincaré symmetry $P_\mu, J_{\mu\nu}$

$$P_\mu = \int d^3x\, T_{0\mu}$$
$$J_{\mu\nu} = \int d^3x\, (x_\mu T_{0\nu} - x_\nu T_{0\mu})$$

$T_{\mu\nu}$ symmetric & conserved $\Rightarrow \frac{dP_\mu}{dt} = 0, \ \frac{dJ_{\mu\nu}}{dt} = 0$

traceless $T_\mu{}^\mu = 0 \Rightarrow D =$ dilation charge $= \int d^3x\, x^\mu T_{\mu 0}$

$K_\mu =$ conformal boosts $= \int d^3x\, (x^2 T_{\mu 0} - 2 x_\mu x^\nu T_{\nu 0})$

also conserved

$\Rightarrow$ conformal group $SO(d,2)$

$$\delta x^\mu = \underset{\substack{\uparrow \\ \text{transl.}}}{a^\mu} + \underset{\substack{\omega_{\mu\nu} = -\omega_{\mu\nu} \\ \text{rot \& boosts}}}{\omega^\mu_\nu x^\nu} + \underbrace{\underset{\substack{\nwarrow \\ \text{dilation}}}{\lambda x^\mu} + \underset{\substack{\uparrow \\ \text{special conformal boosts}}}{b^\mu x^2 - 2 b\cdot x\, x^\mu}}_{\text{change the metric e.g.}} = \xi^\mu$$

$$x^\mu = \Omega x^{\mu'} \Rightarrow ds^2 = \Omega^2 ds^2$$

$$\Rightarrow \quad \partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \frac{2}{d} (\partial \cdot \xi)\, \eta_{\mu\nu}$$

conformal Killing vector equations

Stand. repr. $\quad P_\mu = \partial_\mu \quad J_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu$

$D = - x^\mu \partial_\mu \quad K_\mu = x^2 \partial_\mu - 2 x_\mu x^\lambda \partial_\lambda$

$\Rightarrow [P_\mu, P_\nu] = 0, \quad [J_{\mu\nu}, P_\lambda] = \eta_{\nu\lambda} P_\mu - \dots$

$[J_{\mu\nu}, J_{\rho\sigma}] = \eta_{\nu\rho} J_{\mu\sigma}$

$[D, P_\mu] = P_\mu, \quad [D, K_\mu] = -K_\mu, \quad [D, J_{\mu\nu}] = 0$

$[K_\mu, K_\nu] = 0 \quad [P_\mu, K_\nu] = 2 J_{\mu\nu} + 2 \eta_{\mu\nu} D$

$L_{AB} \quad A, B = 0, \dots, d-1, d, d+1 \quad \eta_{AB} = (\eta_{\mu\nu}, +, -)$

$L_{\mu\nu} = J_{\mu\nu}, \ L_{\mu d} = P_\mu - K_\mu, \ L_{\mu d+1} = P_\mu + K_\mu$

$L_{d,d+1} = D$

$\Rightarrow [L_{AB}, L_{CD}] = \eta_{BC} L_{AD} + \dots$

Anomalies $\Rightarrow \langle T_\mu{}^\mu \rangle \sim \beta_i \langle \partial^i \rangle$ on quantum level

$SO(d,2)$ non-compact $\Rightarrow$ unitary IRREPS ∞-dim.

$[D, O_\Delta(x)] = (x\cdot\partial + \Delta)\, O_\Delta$

$[P_\mu, O_\Delta(x)] = \partial_\mu O_\Delta$

$$[J_{\mu\nu}, \mathcal{O}_\Delta(x)] = (x_\mu \partial_\nu - x_\nu \partial_\mu + S_{\mu\nu})\, \mathcal{O}_\Delta(x)$$
$$[K_\mu, \mathcal{O}_\Delta(x)] = (x^2 \partial_\mu - 2x_\mu\, x\cdot\partial - x^\nu S_{\nu\mu})\, \mathcal{O}_\Delta(x)$$

spin op. acting on Lorentz indices of $\mathcal{O}_\Delta(x)$ (if any)

$\Rightarrow \mathcal{O}_\Delta(x)$ a primary field

other fields (not satisfying these commut. rel.) .. descendants

$M^2 = P_\mu P^\mu$ no longer Casimir

interesting quantities $G(x_1, .., x_n) = \langle \mathcal{O}(x_1) .. \mathcal{O}(x_n) \rangle$

Ex: $\langle \mathcal{O}_\Delta^+(x)\, \mathcal{O}_\Delta(y) \rangle = \frac{a_\Delta}{(x-y)^\Delta}$ Δ-scaling dimension

for scalars .. fixed by conformal symmetry

also $\langle \mathcal{O}_{\Delta_1}(x_1)\, \mathcal{O}_{\Delta_2}(x_2)\, \mathcal{O}_{\Delta_3}(x_3) \rangle = \prod_{i<j} \frac{1}{(x_{ij})^{\Delta_i + \Delta_j - \Delta_{(i,j)}}}\, C_{\Delta_1 \Delta_2 \Delta_3}$

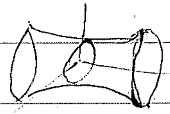
! $\mathbb{R}^d = R^+ \times S^{d-1}$

radial direction ~ time radial quantization

$|\mathcal{O}\rangle = \lim_{x\to 0} \mathcal{O}_\Delta(x) |0\rangle$ unique vacuum $D|0\rangle = 0$ etc.

.. operator - state correspondence e.g. $\mathbb{1} \longleftrightarrow |0\rangle$

AdS: Hyperboloid $X_0^2 - X_i^2 - X_d^2 + X_{d+1}^2 = L^2$



$X_0 = L \cosh\sigma \cos\tau$, $X_{d+1} = L \cosh\sigma \sin\tau$, $X_i = L \sinh\sigma \cdot n_i$; $\sum_{i=1}^{d} n_i^2 = 1$

$\Rightarrow ds^2 = \eta_{AB}\, dX^A dX^B \Big|_{\text{hyperboloid}} = L^2(-\cosh^2\sigma\, d\tau^2 + d\sigma^2 + \sinh^2\sigma\, d\Omega^2)$

$\sigma > 0$ $0 < \tau < 2\pi$ $\rightsquigarrow$ universal cover $-\infty < \tau < +\infty$
otherwise limited range of time

or Horospherical coords. $AdS_d = SO(d,2)/SO(d,1)$

$X_\mu = \frac{L}{\varrho} x_\mu$, $X_{d+1} = \frac{L^2 + \varrho^2 - x\cdot x}{\varrho}$, $X_d = \frac{\varrho^2 - L^2 - x\cdot x}{\varrho}$

$\Rightarrow ds^2 = \frac{L^2}{\varrho^2}(dx\cdot dx + d\varrho^2)$ conformally flat metric

.. Poincaré patch covered in these coords.

Euclidealization ~~Ad~~ $AdS^E_{d+1} = B^{d+1}$ (ball)

Cosmological constant $\Lambda = -\, d(d+1)/L^2$

String dualities (3rd lecture)

$a = d = 0 \qquad b = -c = 1 \qquad \tau \to -\frac{1}{\tau}$

if $\langle \chi \rangle = 0$ we have $e^{-\phi} \to e^{\phi}$

$\Rightarrow$ weak-strong coupling duality

$\Rightarrow (p,q) \to (q,-p) \qquad (1,0) \to (0,-1)$ } really S-duality

$(0,1) \to (1,0)$

$$\begin{tikzcd}
11\text{-dim} \arrow[d, "S^1"'] & & \\
\text{IIA} \arrow[d, "S^1"'] & & 10\text{-dim IIB} \arrow[dl, "S^1 \ldots \text{radius } R_9^B = \frac{1}{R_9^A} \text{ (up to } \alpha')"] \\
9\text{-dim IIA} \arrow[r, "T\text{-duality}", "\text{on } S^1"'] & 9\text{-dim IIB} &
\end{tikzcd}$$

higher KK modes ~ D0-branes of IIA

this circle is radius R_9^A

in IIB we have $\tau^B = \chi^B + i e^{-\Phi_B}$

in IIB: χ_B RR 0-form $\longrightarrow A_9$ RR 1-form in IIA

$e^{-2\Phi_B} \longrightarrow e^{-2\Phi_A} R_A^2$

$\Uparrow$ 9-dim $S \sim \int d^9x \, e^{-2\Phi_B} R_9^B \mathcal{R}^B_{9\text{-dim}}$ IIB

$S \sim \int d^9x \, e^{-2\Phi_A} R_9^A \mathcal{R}^A_{9\text{-dim}}$ IIA .

$G^B_{\mu\nu} \to G^A_{\mu\nu}$ → actions should be the same ($\Leftarrow$ duality)

i.e. $e^{-\Phi_B} \to e^{-\Phi_A} R_9^A$

Kinetic terms $\int d^9x \, R_B \sqrt{G} (\partial_\mu \chi) \partial_\nu \chi \, G^{\mu\nu}$ in IIB

$\int d^9x \, R_A \sqrt{G} \, \partial_\mu A_9 \partial_\nu A_9 G^{\mu\nu} \boxed{G_A^{99}}$ in IIA

$\sim \frac{1}{R_A^2}$

really transform one into another, similarly for ϕ

How to go from IIA to 11-dim ?

$$ds_{11}^2 = e^{-\frac{2}{3}\phi_A} (ds^A_{10})^2 + e^{\frac{4}{3}\phi_A} (dx_{10} + A_\mu dx^\mu)^2$$

here $\mu = 0, \ldots, 9$

i.e. $G^M_{\mu\nu} = e^{-2/3\,\phi_A} G^A_{\mu\nu} + e^{4/3\,\phi_A} A_\mu A_\nu$

$G^M_{\mu 10} = A_\mu e^{4/3\,\phi_A} \qquad G^M_{10 10} = e^{4/3\,\phi_A}$

$\Rightarrow \quad A_9 = \frac{G^M_{9 10}}{G^M_{10 10}} \qquad e^{-\Phi_A} = \frac{1}{(G^M_{10,10})^{3/4}}$

$\Rightarrow \quad e^{-\Phi_A} R_9^A = \ldots \qquad R_9^A = \sqrt{G^A_{99}}$

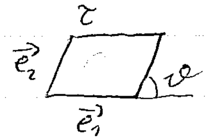
$$e^{\frac{2}{3}\phi_A}\left(G^M_{99} - e^{4/3\,\phi_A} (A_9)^2\right) = \frac{e^{\frac{2}{3}\phi_A}}{G^M_{1010}} \left(G_{99} G_{1010} - (G_{9,10})^2\right)$$

$$= \frac{1}{\sqrt{G^M_{1010}}} \left(\underbrace{G^M_{99} G^M_{10 10} - (G_{9,10})^2}_{\det G^M} \right)$$

$$\Rightarrow \boxed{\tau_B = \left(G^M_{9 10} + i \sqrt{\det G^M}\right) / G^M_{10 10}}$$

$\Rightarrow \tau_B$ is the same as the modulus of the compactification torus τ^M

since



$$\mathrm{Re}\,\tau = \frac{|\vec{e}_2|}{|\vec{e}_1|}\cos\vartheta = \frac{\vec{e}_1\vec{e}_2}{|\vec{e}_1||\vec{e}_1|}$$

$\uparrow$ normalization of torus

$$\mathrm{Im}\,\tau = \frac{|\vec{e}_2|}{|\vec{e}_1|}\sin\vartheta = \frac{1}{|e_1|}\sqrt{|\vec{e}_2|^2 - \frac{(\vec{e}_2\cdot\vec{e}_1)^2}{|\vec{e}_1|^2}}$$

$$= \frac{\sqrt{|\vec{e}_1|^2|\vec{e}_2|^2 - (\vec{e}_2\cdot\vec{e}_1)^2}}{|\vec{e}_1|^2}$$

$\vec{e}_1\cdot\vec{e}_2 = G_{9\,10}$ ($\Leftarrow$ compactification on $T = S^1\times S^1$, e_1 along one S^1, e_2 along 2nd S^1)

$\vec{e}_1{}^2 = G_{10}\,G_{10\,10}$ $\quad \vec{e}_2 = G_{99}$ $\Rightarrow$ by comparison $\tau_B = \tau^M$

$\Rightarrow$ we have in II B really a $SL(2,\mathbb{Z})$ symmetry (if M-theory exists).

one may also check transf. of BS

$$NS - B^B_{\mu\nu} \longrightarrow NS - B^A_{\mu\nu} \longrightarrow C_{\mu\nu 10}$$
$$R - B^B_{\mu\nu} \longrightarrow C^A_{\mu\nu 9} \longrightarrow C_{\mu\nu 9}$$

$\Rightarrow$ in M-theory the modular transf. $\overset{\text{really}}{\vee}$ mixes these $C_{\mu\nu 10}, C_{\mu\nu 9}$

Type I $\xrightarrow{\text{closed}}$ II B / Ω

unoriented $\xrightarrow{\text{open}}$

Massless spectrum

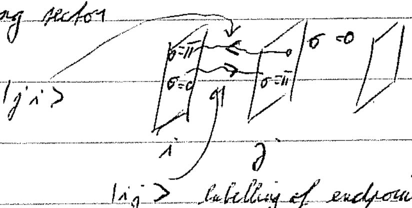
NS-NS	R-R
$G_{\mu\nu}$	χ
$B_{\mu\nu}$	$B^R_{\mu\nu}$
ϕ	$C^{(4)}_+$

vertex ops. $\quad b_{\mu\nu}\,\partial_+ X^\mu\,\partial_- X^\nu\,e^{ipx}$

$\Rightarrow$ $B_{\mu\nu}$ projected out in NS-NS

in R-R $\quad \chi, C_+^{(4)}$ project. out

open string sector



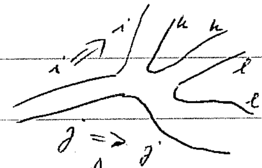
labeled by i ... N of them

$|ij\rangle$ labelling of endpoints

$\Rightarrow N^2$ sectors

a general state $\quad \lambda_{ij}\,\alpha^\mu_{-1}|ij\rangle \quad \lambda \in \mathrm{Mat}(N)$

Scattering amplitude



since the boundary must sit on one brane

$\Rightarrow$ scat amplitude $\propto \mathrm{tr}\,(\lambda^1\lambda^2\lambda^3\lambda^4)$

U(N) symmetry $\quad \lambda^{(k)} \to U\lambda^{(k)}U^{-1}$

$\Omega : |ij\rangle \to |ji\rangle \quad \overset{?}{\Rightarrow} \lambda \to \lambda^T$

U(N) symmetry $\Rightarrow$ we may allow $\lambda \xrightarrow{\Omega} \lambda' = \gamma\,\lambda^T\gamma^{-1}, \gamma \in U(N)$

$\Omega^2 = 1 \quad \Rightarrow \lambda'' = \gamma\lambda'^T\gamma^{-1} = \gamma\gamma^{-T}\,\lambda\,\gamma^T\gamma^{-1} \overset{!}{=} \lambda$

$\Rightarrow \gamma^T \gamma^{-1} \lambda = \lambda \gamma^T \gamma^{-1}$ $\Rightarrow$ since λ is arbitrary (general interactions)

$\Rightarrow \gamma^T \gamma^{-1} = \alpha \mathbb{1} \Rightarrow \gamma^T = \alpha \gamma \Rightarrow \alpha = \pm 1$ $(\Leftarrow (\gamma^T)^T = \gamma)$

$\Rightarrow$ either $\gamma^T = \gamma$ .. $\exists$ basis of indices $i, j, \dots$ s. that $\gamma = \mathbb{1}$

$\gamma^T = -\gamma$ .. $\exists$ - " - s.t. $\gamma = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$

$$\begin{tikzcd}
\lambda \arrow[r, "G"] \arrow[d, "\Omega"'] \arrow[dr, phantom, "?"] & U\lambda U^{-1} \arrow[d, "\Omega"] \arrow[dr] & \\
\gamma\lambda^T\gamma^{-1} \arrow[r, "G"] & U\gamma\lambda^T\gamma^{-1}U^{-1} \arrow[r, phantom, "\overset{!}{=}"] & \gamma(U^{-1})^T\lambda^T U^T\gamma^{-1}
\end{tikzcd}$$

We want it to commute

$\Downarrow$

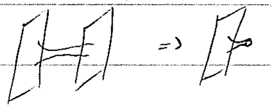
$[\lambda^T, U^T\gamma^{-1}U\gamma] = 0 \Rightarrow U^T\gamma^{-1}U\gamma = \alpha \mathbb{1}$

we put $U = 1 \Rightarrow \alpha = 1$

$\Rightarrow$ $U \in SO(N)$ in the case $\gamma = \mathbb{1}$ $\Rightarrow$ gauge group $SO(N)$ or $USp(N)$

$U \in USp(N)$ $\gamma^T = -\gamma$

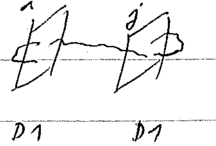
open strings $\Rightarrow$ tadpoles



tadpole cancellation $\rightarrow$ after computations $SO(32)$ gauge group

Type I D-1 string } the only D-branes

D-5



unoriented strings $\Rightarrow$ we must have $SO(N)$ or $USp(N)$

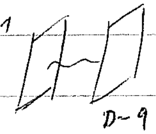
open strings ... D-9 branes (all cond. Neumann)

because of consistency D-9 SO projection because of tadpole cancellation

D-5 Sp

D-1 SO

also D-1



sector

$A_\mu, \mu = 0,1$ in Adj repr. of $O(N)$

$X_i, i = 2, \dots, 9$

Ω projection $\Rightarrow$ extra $-$ in X_i compared to A_μ

(D versus N)

$\Rightarrow X_i$ ... sym. tensor repr. of $O(N)$

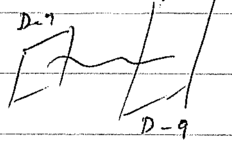
vacuum $SO(1,9)$ spinor, GSO proj. $\Rightarrow$ chirality chosen $(\Gamma^0\Gamma^1 \dots \Gamma^9)$

Ω acts as $\Gamma^2 .. \Gamma^9$ .. transverse $SO(8)$

$(SO(1,9) = SO(1,1) \times SO(8))$

$\Rightarrow_{\text{split}}^{\text{chir.}}$ $+ \;=\; + \quad 8_s$

$+ \;\cancel{(- \quad 8_c)}$ removed by Ω



⇒ mixed boundary cond. on $X^2, \ldots, X^9$

$$\partial_t X^i(\sigma=\pi)=0 \qquad \partial_\sigma X^i(\sigma=0)=0$$

$$\Rightarrow X = \sum \alpha^i_{n+\frac{1}{2}} e^{i(n+\frac{1}{2})t} \cos(n+\tfrac{1}{2})\sigma$$

⇒ half-integer moded X^i whereas X^μ are integer moded

fermions NS ψ^μ half-integer moded

ψ^i integer —"—

R ψ^μ integer moded

ψ^i half-integer moded

⇒ no massless states in NS at all

in R ⇒ 2-dim Clifford algebra d_0^0, d_0^1

… 2-dim spacetime fermion, GSO projection

⇒ 1 worldsheet fermion with definite chirality for each $a = 1, \ldots, 32$ ← $\Gamma^0\Gamma^1 = \pm$

⇒ The full content

		O(N)	SO(32)	Chirality SO(1,1)	Chirality SO(8)
	A_μ	adj	1		
	X^i	sym.	1		
fermions	S_{Adj}	adj	1	+	8_s
	S_{Sym}	symmetric	1	−	8_c
	ψ^{ia}	vector	vector (32)	+	1

$N=1$ single D-string

⇒ adj = 0 Sym = $\underline{1}$

$X^i, i=1,\ldots,8$ bosons

S_{Sym} … 8 fermions (left-moving)

ψ_i 32 right moving fermions

- central charges, can be written as 16 bosons

⇒ heterotic theory of D-string

O(1) symmetry $O(1) = \mathbb{Z}_2$ $\psi^a \to -\psi^a$ ⇒ GSO projection

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AdS / CFT correspondence (2nd lecture)

$N=4$ $D=4$ SYM

$$\mathcal{L} = -\frac{1}{2g^2}\mathrm{tr}\, F^{\mu\nu}F_{\mu\nu} + \frac{1}{2g^2} D_\mu\varphi^i D^\mu \varphi^i - \bar{\lambda}_A i \not{D} \lambda^A - \frac{1}{g^2}\left(\mathrm{tr}\, \varphi^i[\lambda^A,\lambda^B]\,\bar{\tau}_{iAB} + h.c\right) - \frac{1}{g^2}\mathrm{tr}[\varphi_i,\varphi_j][\varphi^i,\varphi^j] + \frac{\theta}{32\pi^2} F\tilde{F}^{\mu\nu}$$

φ^i	$i=1,\ldots,6$	$\underline{6}$	$SO(6) \sim SU(4)$	scalars
λ^A	$A=1,\ldots,4$	$\underline{4}$		fermions
A^μ		singlet		gauge potent.

R^{6N}/S^N $\quad \langle \varphi^i \rangle : \ [\varphi^i, \varphi^j] = 0$

$\langle \varphi^i \rangle = 0$ flat direction .. superconformal phase

$\langle \varphi^i \rangle \neq 0$ $\quad U(1)^N$ .. monopoles, dyons

$\Rightarrow$ S-duality group $SL(2, \mathbb{Z})$

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}, \quad \tau' = \frac{a\tau + b}{c\tau + d}$$

exactly superconformal invariant, no UV divergences

$$\beta_{1\text{-loop}} = -\frac{11}{3} C_A + \frac{2}{3} \sum_{\text{fermions}} l_2 (R_F) + \frac{1}{6} \sum_{\text{scalars}} l_2 (R_S)$$

$$= N \left(-\frac{11}{3} + \frac{2}{3} \cdot 4 + \frac{6}{6} \right) = 0$$

no global chiral anomalies

$SU(4)$ .. R-symmetry has no internal anomalies, θ dependent