SUMMER SCHOOL

" STRINGS, GRAVITY & COSMOLOGY "

JUNE 20^{th} - JULY 8^{th}, 2005

PERIMETER INSTITUTE

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Poppitz: $\boxed{\text{SUSY} \quad 1}$

www.physics.utoronto.ca/~poppitz/
SUSY-P1.pdf
SUSY-P1031.pdf

<u>SUSY</u> relates bosons & fermions

<u>Why</u>? (a) Particle theory - beyond SM → MSSM
GUTs
typically weakly coupled (not always)
(b) toy models for strong coupling
- dynamical SUSY breaking, confinement, dualities... both in 4D as well as $d = 2, 3, 5, 6$
(c) string theory
(d) mathematical results (index theorems,..)
(e) getting jobs (:-)

<u>SUSY QM</u>

SUSY oscillator

(a) bosonic oscillator b, b^+ $[b, b^+] = 1$

$$H = \omega_B (b^+ b + \tfrac{1}{2}) = \omega_B \{b^+, b\}$$

$$E_n = \tfrac{1}{2}\omega_B (m + \tfrac{1}{2})$$

(b) fermionic oscillator f, f^+ $f^2 = (f^+)^2 = 0$ $\{f, f^+\} = 1$

$$H = \omega_F (f^+ f - \tfrac{1}{2}) = \frac{\omega_F}{2} [f^+, f], \quad E_k = \omega_F (k - \tfrac{1}{2}), k = 0,$$

$$H_{SUSY} = H_B + H_F \Big|_{\omega_B=\omega_F} = \omega\,(b^+b + f^+f)$$

Energy levels (n,k) $\quad n=0,\dots \quad k=0,1$

$(3,0)\quad(2,1)$
$(2,0)\quad(1,1)$ $\quad\Rightarrow$ 1) ground state has zero energy
$(1,0)\quad(0,1)$
$(0,0)$ $\quad$ 2) all $E>0$ levels are doubly degenerate

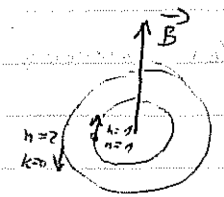
Q acts $\leftarrow$
Q^+ $\rightarrow$
on energy levels

SUSY is responsible for 1), 2) properties
(take $b^{(+)}, f^{(+)}$ to commute)

$Q = b^+ f \qquad Q^+ = bf^+$ - nilpotent $Q^2 = Q^{+2} = 0$

$H_{SUSY} = \omega\{Q, Q^+\} \qquad [H,Q] = [H,Q^+] = 0$

<u>Real system</u> 2D electron in B field along z axis with <u>$g=2$</u>



(another real system
tri-critical Ising model
realised as ... Helium)

in what follows $\boxed{\omega = 1}$

SUSY $\to$ all energy levels are ≥ 0
—— " —— with $E>0$ are doubly degenerate

$H = (Q+Q^+)^2 \ (= \{Q,Q^+\}) \qquad H|E\rangle = |E\rangle E$

$$\langle E|H|E\rangle = E\langle E|E\rangle = \langle E|(Q+Q^+)^2|E\rangle = \|(Q+Q^+)|E\rangle\|^2 \geq 0$$

$E=0$ if and only if $(Q+Q^+)|E\rangle = 0$

—— " —— $(Q-Q^+)|E\rangle = 0$

—— " —— $\underbrace{Q|E\rangle = Q^+|E\rangle = 0}_{\text{SUSY ground states}}$

is systems with unbroken SUSY $E_0 = 0$

Our system is <u>$N=2$ SUSY QM</u>

$Q_1 = Q + Q^+ \qquad Q_2 = i(Q - Q^+)$

$H = Q_1^2 = Q_2^2 \qquad \{Q_1, Q_2\} = 0 \qquad \{Q_i, Q_j\} = \delta_{ij} H \quad i,j =$

SO(2) automorphism acting on

<u>System with interactions</u>

$[p, x] = -i \qquad \psi(x) = \begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix}$

$H = \frac{1}{2}p^2 + \frac{1}{2}(W'(x))^2 + \frac{1}{2}\sigma_3 W''(x)$, $\quad W(x)$ superpotent

$Q_1 = \frac{1}{2}\sigma_1 p + \frac{1}{2}\sigma_2 W'(x) \qquad Q_2 = \frac{1}{2}\sigma_2 p - \frac{1}{2}\sigma_2 W'(x)$

$\{Q_i, Q_j\} = \delta_{ij} H$ (possibly up to 2$\cdot$)

states wt $\psi_2 = 0$ <u>fermionic</u>

$\psi_1 = 0$ <u>bosonic</u>

$F = \begin{pmatrix}1&0\\0&0\end{pmatrix} = \frac{1}{2}(\mathbb{1}+\sigma_3) \qquad [F,H]=0 \quad [F,Q]=Q \quad [F,Q$

define operator $(-1)^F$ $\quad +1$ on bosonic states

-1 on fermionic states

$(-1)^F H = H(-1)^F \qquad Q_i(-1)^F = -(-1)^F Q_i$

define $|B\rangle, |F\rangle$: $(-1)^F|B\rangle = |B\rangle, \quad (-1)^F|F\rangle = -|F\rangle$

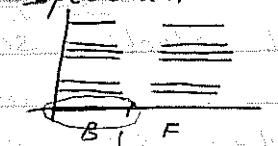
$\Rightarrow$ all eigenstates of H can be chosen as $|B\rangle, |F\rangle$

Q_i turns $|B\rangle$ into $|F\rangle$ and vice versa

$|E\rangle \quad H|E\rangle = |E\rangle E \qquad \langle E|E\rangle = 1$

def. $|E,1\rangle = \dfrac{Q_1|E\rangle}{\sqrt{E}}$ normalized eig state of same E with opposite $(-)^F$

General spectrum



$E=0$ state needs not to be paired
$E>0$ states are always paired

For a SUSY system with given H (i.e. W in QM case) can we say something about the properties of the ground state (i.e. $E_0 \overset{?}{=} 0$) without solving Schr. eqn?

$H|\psi\rangle = 0$ if $|\psi\rangle$ is SUSY ground state

some $Q_1^2 = Q_2^2 = H \Rightarrow H|\psi\rangle = 0$ iff $Q_1|\psi\rangle = 0$

i.e. 2nd order ODE $\to$ 1st order ODE

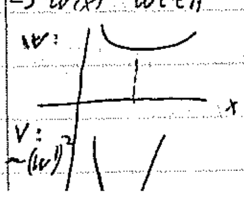
$$Q_1|\psi\rangle = 0 \Rightarrow \left(i\sigma_1 \frac{d}{dx} + \sigma_2 W(x)\right)\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix} = 0$$

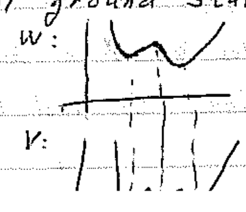
$$\Rightarrow \psi_0(x) = \begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix} = e^{\sigma_3 W(x)}\begin{pmatrix}c_1\\ c_2\end{pmatrix} = \begin{pmatrix}e^{W(x)}c_1\\ e^{-W(x)}c_2\end{pmatrix}$$

normalizable iff $W(x) \to +\infty$ as $|x| \to \infty$ $(c_1 = 0)$
OR $W(x) \to -\infty$ as $|x| \to \infty$ $(c_2 = 0)$ fast enough

i.e. $W(x)$ is "even at ∞"

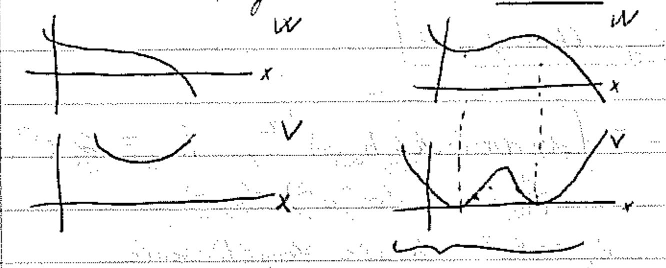
$\Rightarrow W(x)$ with SUSY ground state





here $Tr(-1)^F = 1$
(see Witten index below)

Without SUSY ground state (broken SUSY)



here T
(see belo

$\boxed{\text{even \# of minima of } V \text{ classically} \Rightarrow \text{in QM broken SUSY}}$

but on tree level unbroken SUSY, breaks non perturbatively

Witten index $\quad tr(-1)^F = \sum_E n_B(E) - n_F(E)$

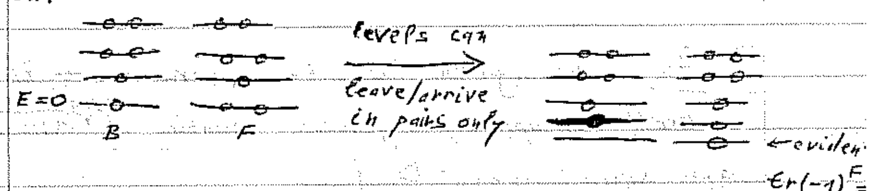
$= n_B(0) - n_F(0)$

$\Rightarrow tr(-1)^F \neq 0$ then SUSY is unbroken

$= 0$ non-conclusive, SUSY can break
e.g. $n_B(0) = 1 = n_F(0)$

Witten index topological - invariant under "continuous ~~change~~" deformations" of the theory

Ex.



Witten index in QFT e.g. SYM $SU(r)$ $\boxed{tr(-1)^F = r+1}$

Johnson: $\boxed{\text{String theory 1}}$

$S = -T \int d\tau d\sigma (-\det h_{ab})^{1/2}$, $h_{ab} = \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}$

$T = \frac{1}{2\pi\alpha'}$ $\quad (\sigma^1, \sigma^2) \equiv (\tau, \sigma)$

$l_s \sim \sqrt{\alpha'}$ basic length scale

$S = -\frac{1}{4\pi\alpha'} \int d\tau d\sigma (-\gamma)^{1/2} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$

Conformal gauge $\gamma_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} e^\phi = \eta_{ab} e^\phi$

... remnant conformal invariance

ϕ drops out classically

Polyakov action ... can include Einstein-Hilbert action in 2D

$\Rightarrow \underbrace{R_{ab} - \frac{1}{2}\gamma_{ab} R}_{\overset{\parallel}{0} \text{ in 2D}} = T_{ab}$

$\Rightarrow \overset{!}{0}$ constraint we impose on quantum level

D=26 comes from Lorentz invariance — ϕ really decouples, in general counting of (conformal) anomaly:

Each X gives 1 $\to c = D-1$

Faddeev-Popov ghosts $\to c = -26$

ϕ contributes $\to c = 1 + 3\left(\frac{26-D}{3}\right)$

$\underline{\qquad\qquad}$

$0 \Rightarrow$ without full Lorentz in spacetime we may have bosonic strings in any dimension

in background of G, B, ϕ fields (in world-sheet Euclidean $\Downarrow$ $\gamma_{mn} \to g_{mn}$)

$S = \ldots + \frac{1}{4\pi\alpha'} \int d^2\sigma\, g^{1/2} \alpha' \phi R$

if Φ is const. $\to g_s = e^\Phi$ weights every vertex in perturbation theory

$\left(\frac{1}{4\pi} \int R g^{1/2} d^2\sigma = 2 - 2\#\text{of handles}\right)$

Trace of stress tensor

$T^a_a = -\frac{1}{2\alpha'} \beta^G_{\mu\nu} g^{ab} \partial_a X^\mu \partial_b X^\nu - i \beta^B_{\mu\nu} \frac{\epsilon^{ab} \partial_a X^\mu \partial_b X^\nu}{2\alpha'} - \frac{1}{2} \beta^\Phi R \overset{!}{=} 0$

$\Rightarrow$ conformal invariance requires vanishing β-functions $\beta^G_{\mu\nu}, \beta^B_{\mu\nu}, \beta^\Phi$

Aspinwall: $\boxed{\text{Complex geometry 1}}$

- Differential geometry
- Algebraic geometry
  - computations in cohomology
- (may be) orbifolds, spectral sequences

Diff. geometry — familiar from GR

- any manifold looks locally like $\mathbb{R}^l$
- metric on manifold using local coords $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

$\to$ connections, curvatures etc.

Vector bundle $\pi : E \to M$ $\quad E, M$ manifolds

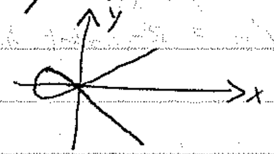
$\pi^{-1}(x) \simeq \mathbb{R}^k$, $\forall x \in M$

π is C^∞-map

$\forall x_0 \in M \ \exists \ U = U^\circ \ni x_0$ s.t. $\exists \varphi_U : \pi^{-1}(U) \xrightarrow[\text{diffeomorphism}]{} U \times \mathbb{R}^k$

Algebraic geometry

Ex: $y^2 = x^2 + x^3$ is a subspace of $\mathbb{R}^2$:



Connection of alg. geometry to physics?

1) Alg. geom. is often equiv. to diff. geom but easier to analyze
2) It might in some cases better represent "true physics" e.g. mirror symmetry

Alg. geom. is easy only if we work over an algebraically closed field — $\mathbb{C}$ but not $\mathbb{R}$

$\mathbb{C}$ often appears in connection with supersymmetry.

Diff. geometry of complex manifolds

A complex manifold of $\mathbb{C}$-dimension n is a manifold with open cover $\{U_\alpha\}$ and coord. maps $\varphi_\alpha : U_\alpha \to \mathbb{C}^n$ such that $\varphi_\alpha \circ \varphi_\beta^{-1}$ is holomorphic on $\varphi_\beta(U_\alpha \cap U_\beta)$.

$\Rightarrow$ complex manifold is also a real manifold of dimension $2n$ $(z_j = x_j + i y_j)$

Ex: • $\mathbb{C}^n$

• $\mathbb{P}^n$ (also $\mathbb{CP}^n$) $= \mathbb{C}^{n+1}\setminus\{0\} / \sim$

$[z_0, z_1, \ldots, z_n] \cong [\lambda z_0, \lambda z_1, \ldots, \lambda z_n]$, $\lambda \neq 0$

$U_k = \mathbb{P}^n - \{\text{all points with } z_k = 0\}$

$\varphi_k([z_0, \ldots, z_n]) = \left(\frac{z_0}{z_k}, \frac{z_1}{z_k}, \ldots, \widehat{\frac{z_k}{z_k}}, \ldots, \frac{z_n}{z_k}\right) \in \mathbb{C}^n$

transition functions

$\varphi_j \circ \varphi_k^{-1}(x^1, x^2, \ldots, x^n) = \left(\frac{x^1}{x^j}, \frac{x^2}{x^j}, \ldots, \widehat{\frac{x^j}{x^j}}, \ldots, \underbrace{\frac{1}{x^j}}_{k\text{-th posi}}, \ldots\right)$

Complex vector bundle $\pi : E \to M$ E, M manifolds (not necess

π is a C^∞-map such that

• $\pi^{-1}(x) \cong \mathbb{C}^k$, $\forall x \in M$ • $\forall x \in M \ \exists U = U^\circ \ni x$, C^∞-diffeomorphism $\varphi_U : \pi^{-1}(U) \to$

• $\pi \varphi_U^{-1}(x, v) = x$, $\forall v \in \mathbb{C}^k$

• $v \to \varphi_U^{-1}(x, v)$ for a fixed $x \in M$ yields isomorphism $\mathbb{C}^k \cong \pi$

k is the rank of bundle E

$k = 1$ ... E is a (complex) line bundle

• If U, V are open sets with trivializations φ_U, φ_V then $g_{UV}(x) = \varphi_U \circ \varphi_V^{-1}$ defines transition function

$g_{UV} : U \cap V \xrightarrow[\text{map}]{\text{differentiable}} GL(k, \mathbb{C})$

Clearly $g_{UV}\, g_{VU} = \mathbb{1}$, $g_{UV}\, g_{VW}\, g_{WU} = \mathbb{1}$ (*)

Conversely, given open cover $\{U_\alpha\}$ of M and C^∞-maps g_{UV} satisfying (*) then there exists a unique (up to change of coords.) complex vector bundle with these transition functions.

Ex: Given $E \to M$ ($F \to M$, $\{U_\alpha\}$ covering for both)

- $g_{UV}^{t\ -1}$ define a "dual bundle" E^* or $E^\vee$
- $E \oplus F$ defined by $\begin{pmatrix} g_{UV} & \\ & h_{UV} \end{pmatrix}$ (transf. fcns for F)

- $E \otimes F$ defined by $g_{UV} h_{UV}$
- $\Lambda^m E$ is antisym of $\underbrace{E \otimes E \otimes \dots \otimes E}_{m\text{-times}}$

Global section $\sigma: M \to E$: $E \underset{\sigma}{\overset{\pi}{\rightleftarrows}} M$, $\pi\sigma = 1$

Local section $\sigma: U \to E$, $U \subset M$, $\pi\sigma \equiv 1_U$

Holomorphic vector bundles

If M is a complex manifold and the transition functions $g_{UV}: U \cap V \to GL(k, \mathbb{C})$ are holomorphic then E is holomorphic vector bundle.

Similarly holomorphic sections

Tangent bundles

- Real space of directional derivatives $V^\mu \frac{\partial}{\partial x^\mu}$
  basis of $\mathbb{R}^n$ for each $x \in M$ is $\{\frac{\partial}{\partial x^\mu}\}$
- If we have a complex manifold of complex dimension n then $\{\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^j}\}$ form a basis for the fibre $\mathbb{R}^{2n}$ for a vector bundle.
- Define $T_{\mathbb{C}}$ to be the complex vector bundle with basis $\{\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^j}\}$ over each point - fibre is $\mathbb{C}^{2n}$
- If $\{\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^j}\}$ are basis vectors for each $\mathbb{C}^{2n}$-fibre then $\{\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^j}\}$ also form a basis.

If M has holomorphic transition fcns then $\frac{\partial}{\partial z^j}$ in one patch is only fcn of $\frac{\partial}{\partial z'^j}$ on other patch

$\Rightarrow$ $T_{\mathbb{C}} = T \oplus \overline{T}$

T .. holomorphic tangent bundle

$\overline{T}$ .. antiholomorphic —〃—

We may also define duals of tangent bundles using dx^j etc. as basis $\to$ cotangent bundle $T_{\mathbb{C}}^*$

Differential forms $T_{\mathbb{C}}^* = T^* \oplus \overline{T}^*$

$$\Lambda^k T_{\mathbb{C}}^* = \bigoplus_{p+q=k} (\Lambda^p T^* \oplus \Lambda^q \overline{T}^*)$$

global C^∞-sections of $\Lambda^k T_{\mathbb{C}}^*$ complexified k-form

—〃— $\Lambda^p T^* \oplus \Lambda^q \overline{T}^*$ (p,q)-form

not necessary holomorphic!

Let $A^{p,q}(M)$ be the space of C^∞-sections of $\Lambda^p T^* \oplus \Lambda$

Define $\partial: A^{p,q} \to A^{p+1,q}$ $\quad \overline{\partial}: A^{p,q} \to A^{p,q+1}$

$$\partial\Big(\sum_{j i_1 \dots \ell} a_{j i_1 \dots} dz^j \wedge dz^{i_1} \wedge \dots \wedge d\bar{z}^\ell\Big) = \sum_{s, j, i_1, \dots, \ell} \frac{\partial a_{j i_1 \dots \ell}}{\partial z^s} dz^s \wedge dz \dots$$

(similarly for $\overline{\partial}$)

$$d = \partial + \overline{\partial}$$

de Rham cohomology $H_{dR}^*(M) = \frac{\ker d}{\operatorname{Im} d}$ $\quad$ Betti # $b_k = \dim H_{dR}^k$

$H_{dR}^k = \frac{\ker d \text{ (on}}{\operatorname{Im} d \text{ (on}}$

Dolbeault cohomology $H_{\overline{\partial}}^{p,q}(M) = \frac{\ker \overline{\partial} \text{ (on } A^{p,q})}{\overline{\partial}(A^{p,q-1})}$

complex dimension

$h^{p,q} = \dim H^{p,q}_{\bar\partial}(M)$ Hodge numbers

Ex: Line bundle on $\mathbb{P}^n$ L_m, $m \in \mathbb{Z}$

U_j patch with $z_j \neq 0$

suppose $g_{U_j U_k}$ is given by $\left(\frac{z_j}{z_k}\right)^{-m} \in GL(1,\mathbb{C})$

E.g. $L_0 = \mathbb{P}^n \times \mathbb{C}$ trivial line bundle

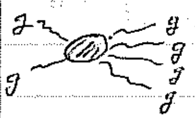
L_1 on $\mathbb{P}^1$

consider global sections which looks like 1 over U_0 then in U_1 this section looks like $1 \cdot \frac{z_0}{z_1}$. So this global section is never infinite but it has one zero at $z_0 = 0$

In fact any global section of L_m has m zeroes (might merge).

Cachazo: $\boxed{\text{Twistor String Theory \& Perturbative YM 1}}$

Perturbative QCD



Hope: New Physics in Background of "old" - QCD

\# of Feynman diagrams raises extremely with n ($n=6 \to 10^5$) $\Rightarrow$ need for new techniques

t'Hooft (1974) large N expansion in $U(N)$

$\lambda_t = g_{YM}^2 N$ large N, fixed λ_t

$\sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda)$ ← genus of Feyn. diag.

Maldacena (1997) AdS/CFT

$\lambda \gg 1$ Perturb. theory is out of reach

Twistor space (Penrose 1967)

complex analysis in 2D very useful (e.g. $\partial\bar\partial f = 0 \to f = g(z)$…)

can it be similarly useful in 4D (Euclidean…)

$|\vec{X}|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 = z\bar{z} + z'\bar{z}'$, e.g. $z = x_1 + i x$…, $z' = x_3 + i$…

inequivalent $\mathbb{C}$-structures $O(4)/U(2) \cong S^2 = \mathbb{CP}^1$

$\Rightarrow \mathbb{CP}^1$ bundle over S^4 (compactified $\mathbb{R}^4$) $\cong \mathbb{CP}^3$ = Twistor space

$\sigma^\mu = (\mathbb{1}, \vec\sigma)$ $\quad X_{a\dot a} = X^\mu (\sigma_\mu)_{a\dot a}$

$Z^I = (\underbrace{\lambda_1, \lambda_2}_{\mathbb{CP}^1}, \mu_1, \mu_2)$ $\quad (\mu)_{\dot a} = (X_{a\dot a}\lambda^a)$

ⓐ fix $Z^I \to$ null ray in Minkowski space

ⓑ fix $X_{a\dot a} \to \mathbb{CP}^1 \subset \mathbb{CP}^3$

Topological string theory

baby versions of string theory A-, B-model

B-model - target space super $\mathbb{CP}^3$

- finite # of fields

→ spectrum of $\mathcal{N}=4$ SYM

End of Intro

Helicity amplitudes gluons $g_i = (p_i^\alpha, \varepsilon_i^\alpha, a_i)$

polarization; color (adjoint rep. of…)

Spinor helicity formalism

$SO(3,1) \xrightarrow[\mathbb{C}]{} SL_L(2,\mathbb{C}) \times SL_R(2,\mathbb{C})$

Finite dim. IR $\quad (p \quad , \quad q)$

$(\frac{1}{2},0)$ - chirality spinors $\quad \lambda^a_a \quad a=1,2$

$(0,\frac{1}{2})$ + —"— $\quad \tilde{\lambda}^{\dot{a}}_{\dot{a}} \quad \dot{a}=1,2$

$SL(2,\mathbb{C})$ invariant tensors $\varepsilon_{ab}, \varepsilon_{\dot{a}\dot{b}}$

Lorentz invariants $\langle\lambda,\lambda'\rangle = \varepsilon_{ab}\lambda^a\lambda'^b$

$[\tilde{\lambda},\tilde{\lambda}'] = \varepsilon_{\dot{a}\dot{b}}\tilde{\lambda}^{\dot{a}}\tilde{\lambda}'^{\dot{b}}$

Vectors $(\frac{1}{2},\frac{1}{2})$ $\quad p^\mu$ bispinor $\quad p_{a\dot{a}} = p^\mu(\sigma_\mu)_{a\dot{a}}$

also $p_{a\dot{a}} = \lambda_a\tilde{\lambda}_{\dot{a}} + \lambda'_a\tilde{\lambda}'_{\dot{a}}$

$p^2=0 \iff p_{a\dot{a}} = \lambda_a\tilde{\lambda}_{\dot{a}} \quad (\Leftarrow p^\mu p_\mu = \det(p_{a\dot{a}}))$

$\lambda,\tilde{\lambda}$ is not unique $\quad (\lambda \to t\lambda, \tilde{\lambda} \to t^{-1}\tilde{\lambda})$

Minkowski $(+---)$ $\quad p$ real $\Rightarrow \lambda,\tilde{\lambda}$ both complex $\tilde{\lambda} = \pm\bar{\lambda}$

sign = future or past dir. null vector

$(++--)$ $\quad SO(2,2) \cong SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$

$\Rightarrow \lambda,\tilde{\lambda}$ real independent

$p^2=0=q^2 \qquad p_{a\dot{a}} = \lambda_a\tilde{\lambda}_{\dot{a}} \qquad q_{a\dot{a}} = \lambda'_a\tilde{\lambda}'_{\dot{a}}$

$2p\cdot q = \langle\lambda,\lambda'\rangle[\tilde{\lambda},\tilde{\lambda}']$

Gluon $(p_i^\mu, \varepsilon_i^\mu)$ $\quad$ helicity of i-gluon h_i

$\to (\lambda_i, \tilde{\lambda}_i, h_i) \qquad (p_{a\dot{a}})_i = \lambda_{ai}\tilde{\lambda}_{\dot{a}i}$

helicity $\quad \varepsilon^{(+)}_{i a\dot{a}} = \frac{\eta_a\tilde{\lambda}_{\dot{a}i}}{\langle\eta,\lambda_i\rangle} \qquad \varepsilon^{(-)}_{ia\dot{a}} = \frac{\lambda_{ai}\tilde{\eta}_{\dot{a}}}{[\tilde{\lambda}_i,\tilde{\eta}]}$

where $\eta,\tilde{\eta}$ are arbitrary reference spinors

Note: $\varepsilon^{(\pm)}_i$ satisfy $\varepsilon^{(\pm)}_i \cdot p_i = 0$

diff. choices of $\eta,\tilde{\eta}$ are gauge transf. $(\varepsilon_i \to \varepsilon_i + w$

Hint: $\Delta\eta = \alpha\eta + \beta\lambda$ (taking η,λ as basis)

Scattering amplitudes

$A(p_1,\dots,p_n) = F(p_i\cdot p_j)$ $\quad$ for scalar particles

spin $\quad A(\{p_i,\varphi_i\}) =$ multilinear in φ_i ($\Leftarrow$ Feyn...

$A(\{\lambda_i,\tilde{\lambda}_i,h_i\})$

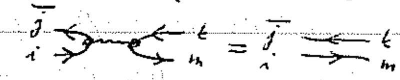
$(\lambda_i,\tilde{\lambda}_i) \to (t\lambda_i, t^{-1}\lambda_i) \Rightarrow A$ picks up t^{-2} if t^2

$$\Rightarrow \left(\lambda^a_i \frac{\partial}{\partial\lambda^a_i} - \tilde{\lambda}^{\dot{a}}_i \frac{\partial}{\partial\tilde{\lambda}^{\dot{a}}_i}\right)A(\{\}) = -2h_i A(\{\})$$

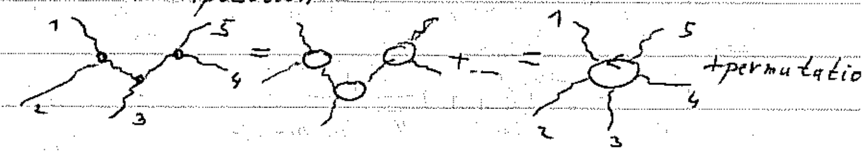
gluon $(p_i^\mu, \varepsilon_i^\mu, a_i)$ $\quad U(N)$ $\quad [T^a,T^b] = f^{abc}T$

$f^{abc} = tr(T^aT^bT^c) - tr(T$

$\sum_{a=1}^{\dim G}(T^a)_i^{\bar{j}}(T^a)_k^{\bar{m}} = \delta_i^{\bar{m}}\delta_k^{\bar{j}}$



Color decomposition

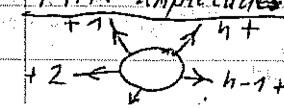


$\Rightarrow$ on tree level

$A_n(\{\lambda_i,\tilde{\lambda}_i,h_i,a_i\}) = g_{YM}^{n-2}(2\pi)^4\delta^{(4)}(\sum_{i=1}^n\lambda_i\tilde{\lambda}_i)\,tr(T^{a_1}\dots)$

$\cdot \underbrace{A(\{\lambda_i,\tilde{\lambda}_i,h_i\})}_{\text{color ordered partial amplitudes}}$ + inequivalent permu

MHV amplitudes (maximally helicity violating)



$1^+ = (\lambda_1,\tilde{\lambda}_1,h_1=+)$ etc.

$\mathcal{A}(1^+,2^+,\dots,n^+)=0$

$\mathcal{A}(1^+,2^+,\dots,j^-,\dots,n^+)=0$

$\mathcal{A}(1^+,2^+,\dots,i^-,i+1^+,\dots,j^-,j+1^+,\dots,n^+)=$

$= \dfrac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\dots\langle n-1\, n\rangle\langle n\, 1\rangle}$ $\quad \langle ij\rangle = \langle \lambda_i, \lambda_j\rangle$ (valid in Mink., $\tilde\lambda = \bar\lambda$)

Note: At tree level $A_{\mathcal{N}=4}(\text{gluons}) = A_{QCD}(\text{gluons})$

$\mathcal{A}(+,+,+,\dots,-,-,-) = \text{mess}(\langle ij\rangle, [ij])$

<u>Conformal invariance</u> of MHV amplitudes

Lorentz generators $J_{ab} = (\lambda_a \frac{\partial}{\partial\lambda^b} + \lambda_b \frac{\partial}{\partial\lambda^a})$

$J_{\dot a\dot b} = (\tilde\lambda_{\dot a}\frac{\partial}{\partial\tilde\lambda^{\dot b}} + \tilde\lambda_{\dot b}\frac{\partial}{\partial\tilde\lambda^{\dot a}})$

$P_{a\dot b} = \lambda_a\tilde\lambda_{\dot b}$

$\dim P = \dim 1 \Rightarrow \dim\lambda = \dim\tilde\lambda = \frac{1}{2}$

$[D,P]=iP \quad [D,K]=-iK \quad \dim K = -1 \Rightarrow$ guess $K_{a\dot a} = \frac{\partial^2}{\partial\lambda^a\partial\tilde\lambda^{\dot a}}$

Exercise $D = \lambda^a\frac{\partial}{\partial\lambda^a} + \tilde\lambda^{\dot b}\frac{\partial}{\partial\tilde\lambda^{\dot b}} + 2$

satisfies $[K_{a\dot a}, P^{b\dot b}] = (\delta_a^b J_{\dot a}^{\dot b} + \delta_{\dot a}^{\dot b} J_a^b + \delta_a^b\delta_{\dot a}^{\dot b} D)$

$\Rightarrow$

$A_{ij}^{MHV} = \delta^{(4)}\left(\sum_{k=1}^n \lambda_k\tilde\lambda_k\right)\dfrac{\langle ij\rangle^4}{\prod_{m=1}^n\langle m\ m+1\rangle}$ is conformally invariant

Note: $SO(4,2) \xrightarrow{\mathbb{C}} SO(6) \cong SU(4) \cong SL(4,\mathbb{C})$ acting on $\mathbb{C}^4$

how to get it - via twistor transf.

$\tilde\lambda_{\dot a} \to i\frac{\partial}{\partial\mu^{\dot a}} \qquad \mu_{\dot a} \to i\frac{\partial}{\partial\tilde\lambda^{\dot a}}$

$\Rightarrow$

$D \to \lambda^a\frac{\partial}{\partial\lambda^a} - \mu^{\dot a}\frac{\partial}{\partial\mu^{\dot a}} \qquad J_{ab} \to \lambda_a\frac{\partial}{\partial\lambda^b} + \lambda_b\frac{\partial}{\partial\lambda^a}$

$J_{\dot a\dot b} \to \mu_{\dot a}\frac{\partial}{\partial\mu^{\dot b}} + \mu_{\dot b}\frac{\partial}{\partial\mu^{\dot a}}$

$P_{a\dot a} \to \lambda_a\frac{\partial}{\partial\mu^{\dot a}} \qquad K_{a\dot a} \to \mu_{\dot a}\frac{\partial}{\partial\lambda^a}$

$\boxed{z^{\bar k} \overset{def}{=} \overline{z^k}}$

$\boxed{\text{Complex manifolds 2}}$

Ex: $T\mathbb{P}^1 = L_2$

$\boxed{\text{Note: } M \text{ is implicitly assumed to be compact.}}$

<u>Given a real manifold, is it a complex mfld?</u>

<u>Almost complex structure</u> on mfld of dim $2n$

J is a linear map on cotangent bundle

$dx^\mu = J^\mu{}_\nu dx^\nu$ such that $J^2=-1$

i.e. $J^\mu{}_\nu J^\nu{}_\xi = -\delta^\mu{}_\xi$. $\quad \mu,\nu = 1,\dots,2n$

On a complex manifold $Jdx^k = -dy^k$, $Jdy^k = dx^k$

$Jdz^k = i\,dz^k$, $Jdz^{\bar k} = -i\,dz^{\bar k}$

An almost complex structure defines a complex structure if it can be "integrated".

<u>Nijenhuis tensor</u> $N^\mu{}_{\nu\xi} = J^\lambda{}_\nu(\partial_\lambda J^\mu{}_\xi - \partial_\xi J^\mu{}_\lambda) - J^\lambda{}_\xi(\partial_\lambda J^\mu{}_\nu -$

$N^\mu{}_{\nu\xi} = 0 \Leftrightarrow$ almost cplx structure is complex s

E.g. S^2 is complex $\quad S^8$ has no almost cplx stru

S^6 has an obvious almost complex str. (octon

which is not a complex structure

<u>Metric</u> $\quad ds^2 = g_{\mu\nu}dx^\mu dx^\nu$

in complex coords $\quad ds^2 = g_{jk}dz^j dz^k + g_{j\bar k}dz^j dz^{\bar k} +$

Metric is <u>Hermitean</u> iff $g_{jk} = g_{\bar j\bar k} = 0$

$\Leftrightarrow \quad g_{\mu\nu}J^\mu{}_\xi J^\nu{}_\varsigma = g_{\xi\varsigma}$.

$\Rightarrow$ if metric is Hermitean then $J_{\mu\nu} \overset{def}{=} g_{\mu\xi}J^\xi{}_\nu = -$

$\Rightarrow$ It is natural to define 2-form

$J=\frac{1}{2}J_{\mu\nu}\,dx^\mu\wedge dx^\nu$ a real 2-form on M

$=\frac{i}{2}\,g_{j\bar k}\,dz^j\wedge d\bar z^{\bar k}$ a real (1,1)-form

A complex manifold with Hermitean metris has a <u>Kähler metric</u> if $dJ=0$, then J is a Kähler form. A complex manifold is <u>Kähler</u> iff it admits a Kähler metric.

For a Kähler manifold one can show that

(1) $H^k_{dR}(M,\mathbb{C})=\bigoplus_{p+q=k} H^{p,q}_{\bar\partial}(M)\;\left(\Rightarrow b_k=\sum_{p+q=k} h^{p,q}\right)$

(2) $\overline{H^{p,q}_{\bar\partial}(M)} = H^{q,p}_{\bar\partial}(M)\;\left(\Leftrightarrow h^{p,q}=h^{q,p}\right)$

Poincaré duality $h^{p,q}=h^{n-p,n-q}$

$\Rightarrow$

Hodge diamond

$$\begin{array}{ccccccc}
 & & & h^{0,0} & & & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
h^{n,0} & h^{n-1,1} & & \cdots & & h^{1,n-1} & h^{0,n} \\
 & h^{n,1} & & & & & \\
 & & & h^{n,n} & & &
\end{array}$$

$\leftarrow$ symmetry

symmetry

Algebraic geometry

An algebraic variety X is the intersection of m equations (homogeneous polynomials) in $\mathbb{P}^N$.

$f_1(z_0,z_1,\dots,z_N)=0$

$f_2(z_0,z_1,\dots,z_N)=0$

$\vdots$

$f_m(z_0,z_1,\dots,z_N)=0$

If $m=1$ X is called a hypersurface

If $\dim X=N-m$ (i.e. the generic case), X is called a "complete intersection"

In general $\dim X\geq N-m$

Ex: $\mathbb{P}^1\times\mathbb{P}^2\subset\mathbb{P}^5$

z_i hom. coords of $\mathbb{P}^1$, w_j of $\mathbb{P}^2$

$Y_{ij}\overset{def}{=}z_i w_j$, $Y_{00},\dots,Y_{12}$ hom. coords of $\mathbb{P}^5$

$\Downarrow$

$Y_{00}Y_{11}=Y_{01}Y_{10}$, $Y_{00}Y_{12}=Y_{02}Y_{10}$, $Y_{01}Y_{12}=Y_{02}Y_{11}$

defines $\mathbb{P}^1\times\mathbb{P}^2$ as alg. variety in $\mathbb{P}^5\Rightarrow$ not a complete inter.

$\mathbb{P}^N$ is Kähler (Fubini-Study metric

in U_0 $\underbrace{J=\frac{i}{2\pi}\partial\bar\partial\ln\left(1+\sum_{i=1}^{n}\left|\frac{z_i}{z_0}\right|^2\right)}_{\text{with apropriate indices } j\bar k}$

$\Rightarrow$ if smooth, algebraic varieties are Kähler manifolds

Kodaira embedding theorem

A compact complex manifold M is an algebraic variety iff it has a closed, positive (1,1)-form J such that $[J]\in H^2(M,\mathbb{Q})$. (N may be arbitr. large

(Positive (1,1)-form means that the induced Hermitean metric is positive definite.)

Sheaves

Let X be a topological space. A "presheaf" $\mathcal{F}$ does the following

a) associates an abelian group $\mathcal{F}(U)$ to each open set $U \subset X$ such that $\mathcal{F}(\emptyset) = 0$

b) if $U \subset V$ we define a restriction map (homomorphism) $r_{VU} : \mathcal{F}(V) \to \mathcal{F}(U)$

c) if $U \subset V \subset W$ then $r_{WU} = r_{VU} r_{WV}$

We use the notation $\sigma|_U$ for $r_{VU}(\sigma)$, $\sigma \in \mathcal{F}(V)$.

A presheaf is a <u>sheaf</u> iff

d) if $U, V \subset X$ and $\sigma \in \mathcal{F}(U)$, $\tau \in \mathcal{F}(V)$ such that $\sigma|_{U\cap V} = \tau|_{U \cap V}$ then there exists $\rho \in \mathcal{F}(U \cup V)$ such that $\rho|_U = \sigma$, $\rho|_V = \tau$

e) if $\sigma \in \mathcal{F}(U \cup V)$ and $\sigma|_U = 0 = \sigma|_V$ then $\sigma = 0$.

Examples:

- $\mathcal{F}(U) = \mathbb{Z}$ for any $U \neq \emptyset$, $r = \mathbb{1}$
  $\to$ presheaf, not a sheaf ($\Leftarrow$ $U = U_1 \amalg U_2$ ($= U_1 \cup U_2$, $U_1 \cap U_2 = \emptyset$)
  suppose $\sigma = 1 \in \mathcal{F}(U_1)$, $\tau = 2 \in \mathcal{F}(U_2)$ violates d)
- instead define sheaf $\mathbb{Z}$ so that
  $\mathcal{F}(U) = \mathbb{Z}^{\oplus (\#\text{disconnected components of } U)}$
  Similarly define "constant sheaves" $\mathbb{Q}, \mathbb{R}, \mathbb{C}$.
- If X is a manifold, $C^\infty(U)$ the group of differentiable functions over U under addition, r restriction of function ("$r = \mathbb{1}$") <u>sheaf of C^∞-functions</u>
- $\mathcal{A}^p$ sheaf of p-forms
- If X is a complex manifold
  $\mathcal{O}$ sheaf of holomorphic fcns
  $\mathcal{O}^*$ sheaf of nowhere zero holom. fcns (under
  $\mathcal{A}^{p,q}$ sheaf of (p,q)-forms
  $\mathcal{Z}^{p,q}$ —"— of $\bar\partial$-closed (p,q)-forms
  $\Omega^p = \mathcal{Z}^{p,0}$ —"— of holomorphic p-forms

To any holomorphic vector bundle we may associat the "sheaf of holomorphic sections" (sheaf on the base mf
E.g. $\mathcal{O}$ is the sheaf of holomorphic sections of trivial cplx line
On $\mathbb{P}^N$ we denote the sheaf of holom. sections of L_m by $\mathcal{O}(m)$.

Ex: <u>Skyscraper sheaf</u> $\mathcal{O}_x$ of a point $x \in X$
$\mathcal{O}_x(U) = \mathbb{C}$ if $x \in U$ $\quad = 0$ if $x \notin U$ $\quad (r = id)$

Ex: Suppose $X \subset W \to$ we may define $\mathcal{O}_X$ as a sheaf on W by
$\mathcal{O}_X(U)$ = holomorphic fcns on X in $X \cap U$, $U \subset W$

Ex If $x \in X$ define $\mathcal{I}_x$ as the sheaf of fcns on U which vanish
("ideal sheaf" of $x \in X$)

A sheaf which corresponds to sections of a holomorphic vector bundle is called "locally-free".
($\mathcal{I}_x, \mathcal{O}_X, \mathcal{O}_x$ are not locally free)

over the same X

Def: A *map of sheaves* $f: \mathcal{E} \to \mathcal{F}$ s.t. it satisfies diagram

$$\begin{array}{ccc} \mathcal{E}(U) & \xrightarrow{r_{UV}} & \mathcal{E}(V) \\ \downarrow f & & \downarrow f \\ \mathcal{F}(U) & \xrightarrow{\tilde{r}_{UV}} & \mathcal{F}(V) \end{array}$$

commutes for all $U, V \in X$.

Def: A *subsheaf* $\mathcal{E} < \mathcal{F} \Rightarrow \mathcal{E}(U) < \mathcal{F}(U)$, $\forall U < X$ and restriction maps for $\mathcal{F}$ act within $\mathcal{E}(U)$

If $f: \mathcal{E} \to \mathcal{F}$ is a sheaf map then $\ker(f)$, $\mathrm{im}(f)$ are subsheaves of $\mathcal{E}$ and $\mathcal{F}$, respectively.

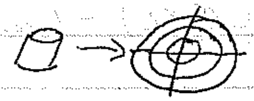
$\boxed{\text{String theory 2}}$

Supersymmetric strings

consider (in conformal gauge)

$$S = \frac{1}{4\pi\alpha'}\int d^2z \{\partial X^\mu \bar{\partial} X_\mu + \alpha'(\psi^\mu \bar{\partial}\psi_\mu + \tilde{\psi}^\mu \partial \tilde{\psi}_\mu)$$

$d^2\sigma = d\tau d\sigma$, open $0 \le \sigma \le \pi$, closed $0 \le \sigma \le 2\pi$ $\quad -\infty \le \tau \le \infty$ $\quad (\sigma^1, \sigma^2) = (\tau, \sigma)$

$\boxed{z = e^{\tau - i\sigma}}$ exponential map 

$\partial = \frac{\partial}{\partial z}$, $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$

For bosonic string we had

$X^\mu(z, \bar{z}) = X_L^\mu(z) + X_R^\mu(\bar{z})$

$X_L^\mu(z) = \frac{1}{2}x^\mu - i\left(\frac{\alpha'}{2}\right)^{1/2}\alpha_0^\mu \ln z + i\left(\frac{\alpha'}{2}\right)^{1/2}\sum_{m\neq 0}\frac{1}{m}\alpha_m z^{-n}$

$X_R^\mu(\bar{z}) = \frac{1}{2}x^\mu - i\left(\frac{\alpha'}{2}\right)^{1/2}\tilde{\alpha}_0^\mu \ln \bar{z} + i\left(\frac{\alpha'}{2}\right)^{1/2}\sum_{m\neq 0}\frac{1}{m}\tilde{\alpha}_m \bar{z}^{-n}$

x^μ is CM position, $p^\mu = \frac{1}{\sqrt{2\alpha'}}(\alpha_0^\mu + \tilde{\alpha}_0^\mu)$ is CM momentum

at one end, say π we are free to redefine $\tilde{\psi} \Rightarrow$ we can impose $\psi^\mu(\pi, \tau) = \psi(\pi, \tau)$ then at 0 we have either $\psi^\mu(0, \tau) = \tilde{\psi}^\mu(0, \tau)$ OR $\psi^\mu(0, \tau) = -\tilde{\psi}^\mu(0, \tau)$

Stress tensor (for superstrings)

$T_B(z) = -\frac{1}{\alpha'}\partial X^\mu \partial X_\mu - \frac{1}{2}\psi^\mu \partial \psi_\mu \qquad T_F(z) = \frac{2i}{\alpha'}\psi^\mu \partial$

$\bar{T}_B(\bar{z}) = \ \ldots \qquad \bar{T}_F(\bar{z}) = \frac{2i}{\alpha'}\tilde{\psi}^\mu$

Two sectors for fermions (*open strings* current

periodic ... *"Ramon"* sector

anti-periodic ... *"Nevea - Schwarz"* sector

$\psi^\mu(z) = \sum_n \psi_n^\mu z^{-r-1/2} \qquad n \in \mathbb{Z}$ (R), $r \in \mathbb{Z} + \frac{1}{2}$ (

Doubling trick used - we have $0 \le \sigma \le \pi$ for open string

$\rightarrow$ use a single ψ^μ ~~and recover~~ (σ, τ) on $0 \le \sigma \le 2\pi$

and recover $\tilde{\psi}(\sigma, \tau) = \psi(2\pi - \sigma, \tau)$.

Quantization - (anti) commutators

$\{\psi_r^\mu, \psi_s^\nu\} = \{\tilde{\psi}_r^\mu, \tilde{\psi}_s^\nu\} = \eta^{\mu\nu}\delta_{r+s} \qquad \boxed{\delta_k = \delta_{k,0}}$

$[\alpha_m^\mu, \alpha_n^\nu] = [\sim\ \sim] = m\delta_{m+n}\eta^{\mu\nu}$

$[x^\mu, p^\nu] = i\eta^{\mu\nu}$

So $T_B(z) = \sum_{m\in\mathbb{Z}}\frac{L_m}{z^{m+2}} \qquad T_F(z) = \sum_r \frac{G_r}{z^{r+3/2}}$

$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n}$

$\{G_r, G_s\} = 2L_{r+s} + \frac{c}{12}\left(4r^2 - \begin{smallmatrix}1 & NS \\ 0 & R\end{smallmatrix}\right)\delta_{r+s}$

$[L_m, G_r] = \frac{1}{2}(m - 2r)G_{m+r}$

$L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} :\alpha_{m-n}\cdot\alpha_n: + \frac{1}{4}\sum_r (2r - m) :\psi_{m-r}\cdot\psi_r: + a\delta_{m}$

$G_r = \sum_n :\alpha_n \cdot \psi_{r-n}:$

$c = D + \frac{D}{2}$ critical dimension $\boxed{D = 10}$

bosons ↓ NS ferms. ↓

a^2: NS z.p.e. $8(-\frac{1}{24})+8(-\frac{1}{48})=-\frac{1}{2}$

R z.p.e. $8(-\frac{1}{24})+8(\frac{1}{24})=0$ R ferms.

z.p.e. formula: boson $\frac{1}{2}\omega$ fermion $-\frac{1}{2}\omega$

where $\omega=\frac{1}{24}-\frac{1}{8}(2\vartheta-1)^2$, $\vartheta = 0$ integer modes, $\vartheta = 1/2$ half-int —"—

Physical states $|\psi\rangle$ $L_m|\psi\rangle=0$ $G_r|\psi\rangle=0$

$L_0=\frac{1}{2}\sum :\alpha_{-n}\cdot\alpha_n: + \frac{1}{4}\sum_n 2n :\psi_{-r}\cdot\psi_r: + a$

$\alpha_0\cdot\alpha_0 \sim p^2=-M^2$ $(\alpha_0^\mu=(2\alpha')^{1/2}p^\mu)$

$\Rightarrow$

$M^2=\frac{1}{\alpha'}\left[\sum_{n,r}(+\alpha_{-n}\cdot\alpha_n + r\psi_{-r}\psi_r) - \begin{matrix}1/2\\0\end{matrix}\right]$

Massless states e.g. $\psi_{-\frac{1}{2}}|0\rangle$

Starting point vacuum with momentum $p^\mu=k^\mu$

$|0,k\rangle$, $M^2=-1/2\alpha' \to$ tachyon field $T(X)$

$\psi^\mu_{-\frac{1}{2}}|0,k\rangle$ $M^2=0\to A^\mu(X)$ photon field

Choose that 0,1 directions don't contribute (light-cone gauge)

GSO projection

Define operator $(-1)^F$ ← fermion #

assign $(-1)^F=-1$ to the vacuum

$\Rightarrow$ project onto <u>even</u> fermion number

↳ long story ⟶ spacetime SUSY

$\Rightarrow$ $T(X)$ projected out, $A^\mu(X)$ survives

Classification of states in spacetime

according to $SO(8)\subset SO(1,9)$

$A^\mu(X)$ 8_v of $SO(8)$

Ramon sector

Notice $\{\psi_0^\mu,\psi_0^\nu\}=\eta^{\mu\nu}$, def. $\Gamma^\mu=\sqrt{2}\psi_0^\mu$

$\{\Gamma^\mu,\Gamma^\nu\}=2\eta^{\mu\nu}$ so there are $2^{10/2}=32$ states

massless. Lorentz can be built from Γs

$J_{\mu\nu}=-\frac{i}{4}[\Gamma_\mu,\Gamma_\nu]$. Let's construct

$S^{\mu\nu}=-\frac{i}{2}\sum_r[\psi^\mu_{-r},\psi^\nu_r]$. Choose the basis

$d_i^\pm=\frac{1}{\sqrt{2}}(\psi_0^{2i}\pm i\psi_0^{2i+1})$, $i=1,..,4$

$d_0^\pm=\frac{1}{\sqrt{2}}(\psi_0^1\mp\psi_0^0)$

d_j^+ creation operators, d_j^- annihilation ops.

Clifford algebra $\{d_i^+,d_j^-\}=\delta_{ij}$

$|s_0,s_1,s_2,s_3,s_4\rangle$ iS^{01} has eigenvalue s_0

$s_i=\pm 1/2$ S^{23} has eigenvalue s_1,

↳ 32 states $(d^-|-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\rangle=0$

Natural grading $32\to 16$ (even # of $+\frac{1}{2}$) + 16 (od

physical state condition

$G_0|\ \rangle=0$ $\alpha_0\cdot\psi_0\sim p_\mu\psi_0^\mu \to$ spacetime

$G_0=\alpha'^{1/2}(p_0\Gamma_0+p_1\Gamma_1) \Longrightarrow 2\alpha'^{1/2}p_1\Gamma_0(\frac{1}{2}-s_0)\overset{!}{=}0$

$\Rightarrow \boxed{s_0\overset{!}{=}0}$ (picking basis such that $p^0=p^1$)

$\Rightarrow 32\to 8_c + 8_s$ from physical state conditi

GSO: $\sum_{i=1}^4 s_i=0 \bmod 2$ (in one possible conventio

—kills 8_c ⟶ only 8_s survives in R sect

NS+R ⟶ $N=1$ $D=10$ SYM multiplet 8_s 8

$\boxed{\text{SUSY 2}}$

$H = \frac{1}{2}p^2 + \frac{1}{2}(W'(x))^2 + \frac{1}{2}\sigma_3 W''(x)$ acting on $\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix}$

$\sigma_3 = \begin{pmatrix}1&0\\0&-1\end{pmatrix} = \underbrace{\begin{pmatrix}0&1\\0&0\end{pmatrix}}_{\bar\psi}\underbrace{\begin{pmatrix}0&0\\1&0\end{pmatrix}}_{\psi} - \begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}$ $\quad \psi^2 = \bar\psi^2 = 0$

$\{\psi, \bar\psi\} = 1$

$\Rightarrow H = \frac{1}{2}p^2 + \frac{1}{2}(W'(x))^2 + \frac{1}{2}W''(x)\underbrace{(\bar\psi\psi - \psi\bar\psi)}$

$\{\psi,\bar\psi\} = 1 \Rightarrow \pi_\psi = i\bar\psi$ satisfies $\{\psi, \pi_\psi\} = i$ $\quad$ resembles Dirac Ham. (in d=1)

$= \frac{1}{2}p^2 + \frac{1}{2}(W'(x))^2 - \frac{i}{2}W''(\pi_\psi\psi - \psi\pi_\psi)$

$\dot\psi = \frac{\partial H}{\partial \pi_\psi} = -iW''(x)\psi \rightarrow \mathcal{L}_\psi = i\bar\psi\dot\psi - \frac{1}{2}W''(x)(\bar\psi\psi - \psi\bar\psi)$

$\Rightarrow$ $\quad$ we add total deriv. $\sim (\dot{\bar\psi}\psi)$

$\mathcal{L} = \frac{1}{2}\dot x^2 - \frac{1}{2}(W'(x))^2 + \frac{i}{2}(\bar\psi\dot\psi - \dot{\bar\psi}\psi) - \frac{1}{2}W''(x)(\bar\psi\psi - \psi\bar\psi)$

x, ψ $\quad$ 1 bosonic, 1 fermionic field

$\{\psi, \bar\psi\} = \psi^2 = \bar\psi^2 = 0 \quad \frac{\partial}{\partial\psi}\psi\chi = \chi, \ \frac{\partial}{\partial\chi}\psi\chi = -\psi$

$\{Q, \bar Q\} = -i2\frac{\partial}{\partial t} \qquad Q^2 = \bar Q^2 = 0$

superspace $(t, \theta, \bar\theta)$ $\qquad \bar\theta = (\theta)^+ \quad \theta^2 = \bar\theta^2 = 0$, $\{\theta, \bar\theta\} = 0$

$H = -i\frac{\partial}{\partial t}$

$Q = \frac{\partial}{\partial\theta} + i\bar\theta\frac{\partial}{\partial t} \qquad \bar Q = -\frac{\partial}{\partial\bar\theta} - i\theta\frac{\partial}{\partial t}$

$Q\bar Q + \bar Q Q = -\cancel{\frac{\partial}{\partial\theta}\frac{\partial}{\partial\bar\theta}} + \cancel{i\theta\frac{\partial}{\partial\theta}\frac{\partial}{\partial t}} - i\frac{\partial}{\partial t} + \cancel{\bar\theta\theta\frac{\partial^2}{\partial t^2}} - \cancel{i\bar\theta\frac{\partial}{\partial\bar\theta}\frac{\partial}{\partial t}}$

$+ \cancel{\frac{\partial}{\partial\theta}\frac{\partial}{\partial\bar\theta}} - i\frac{\partial}{\partial t} + \cancel{i\bar\theta\frac{\partial}{\partial\bar\theta}\frac{\partial}{\partial t}} + \cancel{\theta\bar\theta\frac{\partial^2}{\partial t^2}} - \cancel{i\theta\frac{\partial}{\partial\theta}\frac{\partial}{\partial t}}$

$= -2i\frac{\partial}{\partial t}$

functions on superspace

$f(t, \theta, \bar\theta) \overset{\text{def}}{=} f_0(t) + \theta f_1(t) + \bar\theta f_2(t) + \theta\bar\theta f_3(t)$

superfield (SF) $\quad$ lowest component $\quad$ highest comp.

may be bosonic/fermionic, real/complex

Real bosonic superfield $\quad \phi^+ = \phi$

$\phi(t, \theta, \bar\theta) = x(t) + \theta\bar\psi(t) - \bar\theta\psi(t) + \theta\bar\theta f(t)$

x, f real, ψ complex functions

SUSY transform of ϕ (definition of superfield)

$\delta\phi(t, \theta, \bar\theta) \overset{!}{=} (\epsilon Q - \bar\epsilon\bar Q)\phi = (\hat\delta_\epsilon + \hat\delta_{\bar\epsilon})\phi$

Grassmann params of SUSY, $\bar\epsilon = \epsilon^+$

$\delta\phi = (\epsilon Q - \bar\epsilon\bar Q)\phi = \delta x(t) + \theta\delta\bar\psi(t) - \bar\theta\delta\psi(t) + \theta\bar\theta\delta f(t)$

$\delta_\epsilon x = \epsilon\bar\psi \qquad \delta_\epsilon\bar\psi = -i\epsilon\dot x - \epsilon f$ $\quad$ | c.c. to get

$\delta_\epsilon\bar\psi = 0 \qquad \delta_\epsilon f = -i\epsilon\dot{\bar\psi}$ $\quad$ | $\bar\epsilon$ transfo

If ϕ is a SF, so is ϕ^2, ϕ^3 etc., in fact any $f(\phi)$ is a SF

ϕ_i many fields

$S = \int dt\, d\theta\, d\bar\theta\, \mathcal{F}(\phi_i(t, \theta, \bar\theta))$ $\qquad \mathcal{F}$ arbitrary fct

$\delta S = \int dt\, d\theta\, d\bar\theta\, (\mathcal{F}(\phi_i + \delta\phi_i) - \mathcal{F}(\phi_i)) =$

$= \int dt\, d\theta\, d\bar\theta\, (\epsilon Q - \bar\epsilon\bar Q)\phi_i \frac{\partial\mathcal{F}}{\partial\phi_i} =$

$= \int dt\, d\theta\, d\bar\theta\, (\epsilon Q - \bar\epsilon\bar Q)\mathcal{F}(\phi_i) = 0 \qquad (\int d\theta\theta = 1, \int d\theta = 0$

$\int dt \frac{\partial}{\partial t} = 0$

$S_W = \int dt\, d\theta\, d\bar\theta\, W(\phi(t, \theta, \bar\theta))$

$W(\phi) = W(x) + (\theta\bar\psi - \bar\theta\psi)W'(x) + \theta\bar\theta(fW'(x) + W''(x)\bar\psi\psi)$

$\Rightarrow$

$S_W = \int dt\,(-fW'(x) - W''(x)\bar\psi\psi)$

not exactly what we want

dimensions $\quad [t] = 1 \quad [\int dt\, \dot x^2] = 0 \quad [x] = -\frac{1}{2}$

$[\psi] = 0 \quad [\theta] = -\frac{1}{2} \quad \frac{1}{2} = [d\theta] = [\frac{\partial}{\partial\theta}]$

$[dt\, d\theta\, d\bar\theta] = 0 \qquad [\phi] = [x] = -\frac{1}{2}$

Superspace action must have dim 0 $\quad \frac{\partial}{\partial\theta}\phi\frac{\partial}{\partial\bar\theta}\phi$ has righ dimension but is not a superfield

$\delta\left(\frac{\partial}{\partial\theta}\phi\right)=\frac{\partial}{\partial\theta}(\delta\phi)=\frac{\partial}{\partial\theta}(\epsilon Q-\bar{\epsilon}\bar{Q})\phi\neq(\epsilon Q-\bar{\epsilon}\bar{Q})\frac{\partial}{\partial\theta}\phi$

because $\{\frac{\partial}{\partial\theta},Q\}\neq 0$

Goal

$\frac{\partial}{\partial\theta}\to D \quad \frac{\partial}{\partial\bar\theta}\to\bar{D}$ s.t. $\{D,Q\}=\{D,\bar{Q}\}=0$

$D=\frac{\partial}{\partial\theta}-i\bar\theta\frac{\partial}{\partial t} \quad \bar{D}=-\frac{\partial}{\partial\bar\theta}+i\theta\frac{\partial}{\partial t}$

$\bar{Q}=-\frac{\partial}{\partial\bar\theta}-i\theta\frac{\partial}{\partial t} \quad Q=\frac{\partial}{\partial\theta}+i\bar\theta\frac{\partial}{\partial t}$

$\{D,\bar{D}\}=-2i\frac{\partial}{\partial t}$

$\Rightarrow$ def. $S_{kin}=(\#)\underbrace{\int dt\,d\theta\,d\bar\theta\; D\phi\,\bar{D}\phi}=(\#)\int dt(f^2+\dot{x}^2)$

supersymmetric, real, no dimensionfull parameters

$D\phi=\bar\psi+\bar\theta(f-i\dot{x})-i\bar\theta\theta\dot{\bar\psi}$

$D\bar\phi=\psi+\theta(f+i\dot{x})+i\bar\theta\theta\dot\psi$

$S_{kin+W}=\int dt\,d\theta\,d\bar\theta\left(\frac{1}{2}D\phi\bar{D}\phi+W(\phi)\right)=$

$=\int dt\left(\frac{1}{2}\dot{x}^2+\frac{1}{2}f^2-fW'(x)-W''\bar\psi\psi+i\bar\psi\dot\psi\right)$

has no derivatives in S $\to$ auxiliary

$\int\mathcal{D}x\,\mathcal{D}f\,\mathcal{D}\psi\,\mathcal{D}\bar\psi$ in path integral

solve f from EoM $f=W'\Rightarrow\frac{1}{2}f^2-fW'=-\frac{1}{2}(W')^2$

$\to$ the action we started with

<u>Advantage of superfields</u> - allow to write most general SUSY action with a given SF content

e.g. with two derivatives & many ϕs

$S=\int dt\,d\theta\,d\bar\theta\left(\frac{1}{2}g_{ij}(\phi)D\phi^i\bar{D}\phi^j+W(\phi)\right)$

Generalizations

2d slight modifications (1,1) supersymmetry

$S=\int dt\,dx\,d\theta\,d\bar\theta\,(D\phi\bar{D}\phi\, g(\phi)+W(\phi))$

also to 3d ($\mathcal{N}=1$ in 3d)

<u>4D $\mathcal{N}=1$ theory</u>

- simplest (smallest amount of SUSY)
- in 4D, only $\mathcal{N}=1$ SUSY theories allow chiral fermions
- can dynamically break SUSY
  $\hookrightarrow$ can generate small scales (hierarchy problem)
- lot of interesting dynamics
  (confinement, chiral SUSY breaking, Seiberg dualities)

$\mathcal{N}=1$ algebra $\quad Q_\alpha,\quad \bar{Q}_{\dot\alpha}=(Q_\alpha)^+$

$SL(2,\mathbb{C})_{L\text{ or }R}$ indices

$\{Q_\alpha,\bar{Q}_{\dot\alpha}\}=2i\sigma^m_{\alpha\dot\alpha}P_m \quad m=0,1,2,3$ spacetime vec

$\{Q_\alpha,Q_\beta\}=\{\bar{Q}_{\dot\alpha},\bar{Q}_{\dot\beta}\}=0 \quad [P_m,Q_\alpha]=0$

Follow similar path $(X^m,\theta_\alpha,\bar\theta_{\dot\alpha})$, $\bar\theta_{\dot\alpha}=(\theta_\alpha)^+$

$Q_\alpha=\frac{\partial}{\partial\theta^\alpha}-i\sigma^m_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_m$

$\bar{Q}_{\dot\alpha}=-\frac{\partial}{\partial\bar\theta^{\dot\alpha}}+i\theta^\alpha\sigma^m_{\alpha\dot\alpha}\partial_m$

Reduction $\mathcal{N}=1$ in 4d $\xrightarrow[\partial_2=0]{\text{dim-red.}}\mathcal{N}=2$ in 3d $\xrightarrow[\partial_2=\partial_3=0]{}$ (2,2) in

Simplest representations of $\mathcal{N}=1$ 4d SUSY

1 $\mathbb{C}$ scalar + Weyl fermion

OR

Weyl fermion + massless vectors

$$F = f(x) + \theta^\alpha \varphi_\alpha(x) + \bar\theta_{\dot\alpha} \bar\chi^{\dot\alpha} + \theta^\alpha \theta_\alpha m(x) + \bar\theta_{\dot\alpha} \bar\theta^{\dot\alpha} n(x)$$
$$+ \theta^\alpha \sigma^m_{\alpha\dot\alpha} \bar\theta^{\dot\alpha} v_m + \theta^\alpha \theta_\alpha \bar\theta_{\dot\beta} \bar\lambda^{\dot\beta} + \bar\theta_{\dot\alpha} \bar\theta^{\dot\alpha} \theta^\beta \psi_\beta$$
$$+ \theta^\alpha \theta_\alpha \bar\theta_{\dot\beta} \bar\theta^{\dot\beta} d(x)$$

(using $\theta^\alpha \theta^\beta \sim \varepsilon^{\alpha\beta} \theta^j \theta_j$ etc.)

too many components → need to impose constraints

which (a) ~~SUSY~~ respect SUSY

(b) not involve diff eqns. in x^m

<u>one possibility</u> $F = F^+$, if F bosonic ⇒ vector supermultiplet

SUSY covariant derivatives

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i \bar\theta^{\dot\alpha} \sigma^m_{\alpha\dot\alpha} \partial_m \qquad D_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha \sigma^m_{\alpha\dot\alpha} \partial_m$$

$$\{D_\alpha, \bar Q_{\dot\beta}\} = \{D_\alpha, Q_\beta\} = 0, \{D_\alpha, P_\beta\} = 0, \{D_\alpha, \bar D_{\dot\alpha}\} = -2i\sigma^m_{\alpha\dot\alpha}\partial_m$$

⇒ <u>Chiral superfields</u> $\phi(x, \theta, \bar\theta)$

bosonic, complex s.t. $\boxed{\bar D_{\dot\alpha} \phi = 0}$

antichiral ϕ^+ $\quad D_\alpha \phi^+ = 0$

Twistor String Theory & Perturbative YM 2

Goal: relate top. string theory (B model on $\mathbb{CP}^{3|4}$ + N D5-branes + D1-instantons) and perturbative $N=4$ [illegible]

in $(++--)$ $\quad \lambda, \tilde\lambda$ real

$$SO(2,2) \xrightarrow[\text{conf.}]{} SO(3,3) \cong SL(4,\mathbb{R}) \text{ acts on } \mathbb{R}^4$$

Amplitude in twistor space

$$\tilde A(\{\lambda_j, \mu_j, h_j\}) = \int \prod_{i=1}^n \frac{d^2\tilde\lambda_i}{(2\pi)^2} e^{i[\mu_i, \tilde\lambda_i]} A(\{\lambda_j \tilde\lambda_j h_j\})$$

$$\Rightarrow \left(\lambda^a_i \frac{\partial}{\partial\lambda^a_i} + \mu^{\dot a}_i \frac{\partial}{\partial\mu^{\dot a}_i}\right) \tilde A(\ ) = (-2h_i - 2) \tilde A(\{\ \})$$

⇒ $\tilde A$ has degree $(-2h_i - 2)$ as homogeneous fcn on $\mathbb{C}^4_i$

$Z^I \sim t Z^I \quad \mathbb{CP}^3$ (or $\mathbb{RP}^3$)

$\tilde A$ sections of line bundle $\mathcal{O}(-2h_i - 2)$

MHV amplitudes $\quad (1^+, \dots, (i-1)^+, i^-, \dots, j^-, \dots n^+)$

$$\int \prod_{i=1}^n \frac{d^2\tilde\lambda_i}{(2\pi)^2} e^{i[\mu_i, \tilde\lambda_i]} \underbrace{\delta^{(4)}\left(\sum \lambda_\ell \tilde\lambda_\ell\right)}_{\int d^4x_{a\dot a}\, e^{ix_{a\dot a}\sum_{\ell=1}^n \lambda^a_\ell \tilde\lambda^{\dot a}_\ell}} \underbrace{\left(\frac{\langle ij\rangle^4}{\prod \langle k\, k+1\rangle}\right)}_{f(\lambda)}$$

$$= \int d^4x_{a\dot a}\, f(\lambda) \underbrace{\int \prod_{i=1}^n \frac{d^2\tilde\lambda_i}{(2\pi)^2} e^{i(\mu^{\dot a}_i \tilde\lambda_{\dot a} + x_{a\dot a} \lambda^a_i \tilde\lambda^{\dot a}_i)}}_{\prod_{i=1}^n \delta^{(2)}(\mu_{i\dot a} + x_{a\dot a}\lambda^a_i)}$$

⇒ MHVA vanishes unless $\mu_{i\dot a} + x_{a\dot a}\lambda^a_i$

$(++--)$ $\quad \mathbb{R}^4 \to \mathbb{RP}^3 \quad (\lambda_1, \lambda_2, \mu_1, \mu_2)$

$$Y_1 = \frac{\lambda_2}{\lambda_1} \quad Y_2 = \frac{\mu_1}{Y_1} \quad Y_3 = \frac{\mu_2}{Y_2}$$

⇒ ~~amp~~ localized on a line

Witten conjecture: L loops

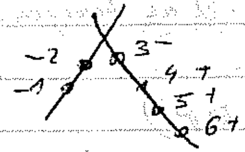
gluons q −helicity $\quad p$ +helicity

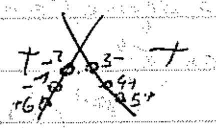
$A(1^+, \dots, i^-, j^-, k^-, \dots, n^+)$ are localized in twistor space on holomorphic curve of genus $g \le L$ and degree $d = q - 1 + L$

more minuses

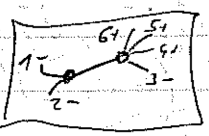
$A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$

Twistor





Spacetime



YM doesn't have 5 point vertices
→ new diagram expansion of YM
vertices — MHV amplitudes
propagators ?

$$A_L(1^-, 2^-, \hat{P}^+)\,\frac{1}{P^2}\,A_R(\hat{P}^-, 3^-, 4^+, 5^+, 6^+)$$

$P^2 \neq 0$ but $\hat{P}^2 = 0$

$$\underbrace{P_{a\dot{a}}}_{\hat{P}_{a\dot{a}}} = \lambda_a \tilde{\lambda}_{\dot{a}} + \lambda'_a \tilde{\lambda}'_{\dot{a}} \qquad \lambda_{\hat{P}} = \frac{P_{a\dot{a}}\tilde{\lambda}'^{\dot{a}}}{[\tilde{\lambda}, \tilde{\lambda}']}$$

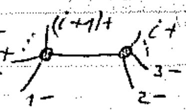
$$\frac{\langle 12\rangle^3}{\langle 2\hat{P}\rangle\langle \hat{P}1\rangle}\;\frac{1}{P^2}\;\frac{\langle \hat{P}3\rangle^3}{\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 6\hat{P}\rangle}$$

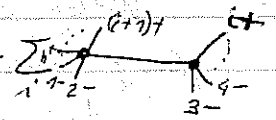
$\lambda_{\hat{P}} \equiv P_{a\dot{a}}\eta^{\dot{a}}$ $\quad \tilde{\lambda}' = \eta$ fixed but arbitrary η

($\frac{\tilde{\lambda}'^{\dot{a}}}{[\,,\,]}$ drops out)

In general:

$$A(1^-, 2^-, 3^-, 4^+, \dots, n^+) = \sum_{i}^{n^+} [\text{diagram}] + \sum_{i}^{i^+} [\text{diagram}]$$



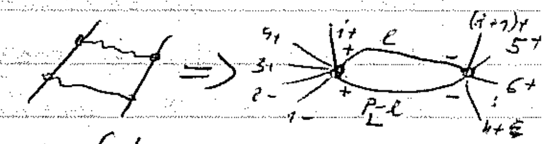


Is this correct ?

One can check that MHV diagrams produce correct singularities, all poles like $\frac{1}{\langle \hat{P}, 0\rangle}$ are spurious, drop out after Σ.

Loops ? now $\mathcal{N}=4$ SYM (at tree level doesn't matter)

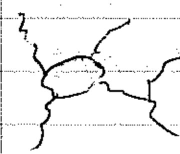
$d = g - 1 + L$
$= 2 - 1 + 1 = 2$



$$\sum_{i,h}\int d^4 l$$

tested to work

open problem - does it work for non-MHV 1-loop amplitud

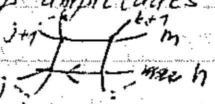


leading color ordered partial amplit

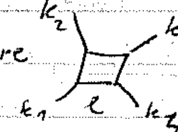
$$A^{1\text{-loop}}(\;) = \sum tr(T^{a_1}\dots T^{a_n})\,A(\;)$$
$$+\,tr(\;)\,tr(\;)\,A(\;)$$ ← lin. fc

What's known about 1-loop amplitudes?

$$A^{1\text{-loop}}(1, \dots, n) = \sum_{1<i<j<k<m} B_{ijkm}\,[\text{box}]$$



where



$$= \int d^{4-2\epsilon}\,\frac{1}{l^2(l+k_1)^2(l-k_1-k_2)^2(l+k_4)^2} \qquad \epsilon<0$$

are "scalar box integrals" and B_{ijkm} are rational functions of $\langle,\rangle, [,]$

each box has a unique natural singularity

2-loops and higher ABDK conjecture

define $M_n^{(L)}(\epsilon) = \frac{A_n^{(L)}(\epsilon)}{A_n^{tree}}$ $\quad\epsilon$ - IF regularization params

$$M_n^{(L)}(\epsilon) = \mathrm{Poly}(M_n^{(1)}, \dots, M_n^{(L-1)})$$

e.g. $M_n^{(2)}(\epsilon) = \frac{1}{2}(M_n^{1}(\epsilon))^2 + f(\epsilon)M_n^{1}(2\epsilon) - \frac{S}{4}h_4 + O(\epsilon)$

where $f(\epsilon) = \frac{\psi(1-\epsilon)-\psi(1)}{\epsilon}$, $\psi(x) = \frac{d}{dx}\ln\Gamma(x)$

Marolf: $\boxed{\text{QFT in curved } \cancel{\text{background}} \text{ spacetime} \quad 1}$

Jacobson hep-th/0308048

Ross hep-th/0502195

I. Covariant QFT without Poincaré group

II. Unruh effect

III. Hawking effect

QFT

free scalar field (real)

$$S = -\frac{1}{2}\int d^d x \sqrt{-g}\,(g^{ab}\nabla_a\phi\nabla_b\phi + m^2\phi^2)$$

$$\Rightarrow (\Box - m^2)\phi = 0$$

Solve using Green's functions

$$(\Box - m^2)\, G^{\pm}(x,y) = -\delta(x,y)$$

$G^+(x,y)$ vanishes $x^0 > y^0$ (advanced) $\neq 0$ in future light cone

$G^-(x,y)$ $x^0 < y^0$ (retarded) $\neq 0$ in past light cone

in curved space bi-scalar δ-fcn, i.e. satisfying

$$\int \sqrt{-g}\, f\,\delta(x,y)\,dx = f(y)$$

Initial value problem $\underline{\phi, \dot\phi \text{ given}}\ \Sigma$ $\quad \phi(x) = ?$

$$\phi(x) = -\int_{\Sigma = (\text{const},\cdot,\cdot,\cdot)} d^{d-1}y \left\{ (G^+ - G^-)(x,y)\frac{\partial}{\partial y^0}\phi(y) - \frac{\partial}{\partial y^0}(G^+ - G^-)(x,y)\,\phi(y) \right\}$$

Proof:

evidently $(\Box - m^2)(G^+ - G^-) = 0 \Rightarrow (\Box - m^2)\phi = 0$

initial conds: $[G^+ - G^-]\big|_\Sigma = [G^+(x,y) - G^+(y,x)]\big|_\Sigma = 0$

in Minkowski evidently, in curved also holds $G^+(x,y) = G^-(y,x)$

in Mink. using rotational symm., in general more complicated

$$\frac{\partial}{\partial y^0}(G^+ - G^-)\Big| = \delta^{(d-1)}(x,y) \Rightarrow \text{RHS}\big| = \phi(x)$$

$$\frac{\partial}{\partial x^0}\text{RHS}\Big|_{x\in\Sigma} = \dot\phi(x)$$ using $(\Box - m^2)G = -\delta$ and integration by parts "Q.E.D."

In general given two solutions f,g solns of EoM,

define $\boxed{f\circ g = -i\int_\Sigma d\Sigma \sqrt{h_\Sigma}\, f\, n\cdot\overleftrightarrow{\partial} g}$ "Klein-Gordon inner product"

h_Σ induced metric on Σ

n unit future pointing vector normal

$$f\, n\cdot\overleftrightarrow{\partial} g = f(n\cdot\partial)g - [(n\cdot\partial)f]\,g$$

$f\circ g$ doesn't depend on Σ

$(\Leftarrow \nabla^a(f\nabla_a g - g\nabla_a f) = 0$

Pf in Mink: $= \cancel{\partial_a f\partial^a g} + f\,\partial_a\partial^a g - \cancel{\partial_a g\partial^a f} - g\,\partial_a\partial^a f = f m^2 g - g m^2 f$

Def. $\tilde G(x,y) = G^+(x,y) - G^-(x,y)$

$\Rightarrow \phi(x) = -i\,\tilde G\circ\phi = \text{"}-i\,\tilde G(x,y)\circ\phi(y)\text{"}$ acts on 2nd argument of

$= -i\,\phi\circ\tilde G$

antisym. of $\tilde G$ and $\circ$ cancel each other

Calculate $[\phi(x),\phi(y)]$

in Mink $[\phi(\vec x,t),\phi(\vec y,t)] = 0 = [\dot\phi(\vec x,t),\dot\phi(\vec y,t)]$

canonical normalization $[\phi(\vec x,t),\dot\phi(\vec y,t)] = i\,\delta_\Sigma(\vec x,\vec y)$ bi-scalar δ-fcn

$[\phi(x),\phi(y)] = (-i)^2\,[(\tilde G\circ\phi)(x), (\phi\circ\tilde G)(y)] =$

in Mink., can be redone in curved

$$= \Big[\int_\Sigma dz_1 \big(\tilde G(x,z_1)\dot\phi(z_1) - \dot{\tilde G}(x,z_1)\phi(z_1)\big),$$

$$\int_\Sigma dz_2 \big(\phi(z_2)\dot{\tilde G}(z_2,y) - \dot\phi(z_2)\tilde G(z_2,y)\big)\Big] =$$

$$= i\int_\Sigma dz \big[\tilde G(x,z)\dot{\tilde G}(z,y) - \dot{\tilde G}(x,z)\tilde G(z,y)\big] = \tilde G\circ\tilde G = i\tilde G$$

using def of Green fcns it simplifies to ($\phi = -i \ldots$ form)

$$\Rightarrow [\phi(x),\phi(y)] = i\tilde G(x,y) = i[G^+(x,y) - G^-(x,y)]$$

Consider any solution $U(x)$. Define

$$\boxed{a_U = U \circ \phi = -\phi \circ U}$$

$[a_U, a_{U'}] = [U\circ\phi, \phi\circ U'] = U \circ \underbrace{[\phi, \phi']}_{i\tilde{G}(x,y)} \circ U' = U \circ U'$

suppose $U' = U^*$, $U \circ U^* < 0$, by scaling $U \circ U^* = -1$

$a_U^+ = (U\circ\phi)^+ = -U^* \circ \phi = -a_{U^*}$

$[a_U, a_U^+] = [a_U, -a_{U^*}] = -U\circ U^* = 1$

$\Rightarrow$ algebra of creation & annihilation ops. emerges

In general one can choose a basis $\mathcal{U} = \{U_n\}_{n\in\mathbb{Z}, n\neq 0}$ ← symplectic basis

such that $U_{-n} = U_n^*$ & $U_n \circ U_m = -\delta_{n,-m}\, \mathrm{sgn}(n)$.

Given $\mathcal{U}$ one can define "zero-particle state"

$$a_{U_n} |0\rangle_{\mathcal{U}} = 0, \quad \forall n > 0$$

Fock space

$$\mathrm{span}\{ a^+_{U_{n_1}} \cdots a^+_{U_{n_k}} |0\rangle_{\mathcal{U}} \}$$

(no symm. under time transl.)

Note: curved spacetime $\rightarrow$ no $H \rightarrow$ no vacuum as lowest energy state of H

"zero-particle state" sometimes called "$\mathcal{U}$-vacuum"

Different choices of basis

$\mathcal{U} = \{U_n\}$ $\quad$ $\mathcal{V} = \{V_n\}$ $\quad$ $U_n \circ U_m = V_n \circ V_m = -\delta_{n,-m}\,\mathrm{sgn}(n)$

$V_n = \sum_{m>0} \alpha_{nm} U_m - \beta_{nm} U_m^*$ because of normalization $\alpha^+\alpha - \beta^+\beta = \mathbb{I}$

$a_{V_n} = V_n \circ \phi = \sum_{m>0} (\alpha_{nm} a_{U_m} + \beta_{nm} a^+_{U_m})$

$|0\rangle_{\mathcal{V}}$: $a_{V_n} |0\rangle_{\mathcal{V}} = 0 \quad \forall n > 0$

$$\sum_{m>0} (\alpha_{nm} a_{U_m} + \beta_{nm} a^+_{U_m}) |0\rangle_{\mathcal{V}} = 0$$

$\Rightarrow |0\rangle_{\mathcal{U}} \neq |0\rangle_{\mathcal{V}}$ iff $\beta_{nm} \neq 0$

In fact $|0\rangle_{\mathcal{V}} = N \exp\left(-\frac{1}{2}\sum_{n,m} (\alpha^{-1}\beta)_{nm} a^+_{U_n} a^+_{U_m}\right) |0\rangle_{\mathcal{U}}$

String theory 3

Closed superstrings

$L_n |0\rangle = 0 = \bar{L}_n |0\rangle \quad H = L_0 + \bar{L}_0 \rightarrow m^2 = \frac{1}{\alpha'}(\ldots)$

$P_\sigma = L_0 - \bar{L}_0$ worldsheet momentum - generates translations on σ

$\rightarrow$ level matching $(L_0 - \bar{L}_0)|0\rangle = 0$

building spectrum of closed strings from 2 copies of open s

$(\overset{NS}{8_V} \oplus \overset{R}{8_S}) \otimes (\overset{NS}{8_V} \oplus \overset{R}{8_C})$ $\quad$ type IIA

$(8_V \oplus 8_S) \otimes (8_V \oplus 8_S)$ $\quad$ type IIB

type IIA

NS-NS $\quad 8_V \otimes 8_V = \overset{\Phi}{1} \oplus \overset{B_{\mu\nu}}{28} \oplus \overset{G_{\mu\nu}}{35}$ (also in IIB)

R-R $\quad 8_S \otimes 8_C = \overset{C_\mu}{8_V} \oplus \overset{C_{\mu\nu\kappa}}{56_T}$

(in IIB R-R $\quad 8_S \otimes 8_S = 1 \oplus \overset{C_{\mu\nu}}{28} \oplus \overset{C_{\mu\nu\kappa\lambda}\text{ self-dual}}{35_+}$)

NS-R $\quad 8_V \otimes 8_C = \overset{\lambda_\alpha}{8_S} \oplus \overset{\psi^\mu_{\dot\alpha}}{56_C}$ (λ_α resp. $\psi^\mu_{\dot\alpha}$)

R-NS $\quad 8_V \otimes 8_S = 8_C \oplus 56_S$ $\quad \lambda_{\dot\alpha}$ resp. ψ^μ_α

Type IIA has one sector each $\rightarrow$ (1,1) supersymmetry (chirality)

IIB has two $8_V \oplus 8_S$ sectors $\rightarrow$ (0,2) supersymmetry - is

These fields (just like in bosonic case) have low-energy effective actions

$$S_{IIA} = \frac{1}{2\kappa_0^2}\int d^{10}x\, (-G)^{1/2}\Big\{ e^{-2\Phi}\left[R + 4(\nabla\phi)^2 - \frac{1}{12}(H^{(3)})^2\right] - \frac{1}{4}(G^{(2)})^2 - \frac{1}{48}(G^{(4)})^2\Big\} - \frac{1}{4\kappa_0^2}\int B^{(2)}\wedge dC^{(3)}\wedge dC^{(3)}$$

$\underbrace{\text{field strength of } C_\mu \quad \text{field strength of } C_{\mu\nu\kappa}}$

NO $e^{-2\Phi}$ in front of R-R fields $\rightarrow$ they

$$S_{IIB} = \frac{1}{2\kappa_0^2}\int d^{10}x\,(-G)^{1/2}\Big\{[R + 4(\nabla\phi)^2 - \tfrac{1}{12}(H^{(3)})^2]e^{-2\Phi}$$
$$- \tfrac{1}{12}\left(G^{(3)} + c^{(0)} \wedge H^{(3)}\right)^2 - \tfrac{1}{2}(dC^{(0)})^2$$
$$- \tfrac{1}{48}\left(G^{(5)}\right)^2\Big\} + \frac{1}{4\kappa_0^2}\int \left(C^{(4)} + \tfrac{1}{2} B^{(2)} \wedge C^{(2)}\right) \wedge G^{(3)} \wedge H^{(3)}$$

where $G^{(5)} = dC^{(4)} + H^{(3)} \wedge C^{(2)}$ and one has to enforce $F^{(5)} = {}^*F^{(5)}$ by hand.

Go to Einstein frame using $\tilde{G}_{\mu\nu} = e^{\frac{\phi_0 - \phi}{2}} G_{\mu\nu}$

$\to 2\kappa^2 = 2\kappa_0^2 g_s^2 = (2\pi)^7 \alpha'^4 g_s^2 = 16\pi G_N$, $g_s \stackrel{!}{=} e^{\Phi_0}$

<u>Anomalies</u> (possibly chiral anomaly in IIB)

- look for anomalous variations of $\Gamma = \ln Z$

$$\delta \ln Z = \frac{c}{(2\pi)^{d/2}}\int \hat{I}_d \quad \text{under gauge transf.}$$

Define $F = dA + A^2 \quad (A^2 = A \wedge A)$

$R = d\omega + \omega^2 \quad (R = R^a{}_{b\mu\nu}\, dx^\mu \wedge dx^\nu)$

Descent relation from $\hat{I}_{d+2} = d\hat{I}_{d+1}$

$\hat{I}_{d+1}$ is not gauge invariant but $\delta \hat{I}_{d+1} = dI_d$

Ex: Chern-Simons 3-form

$$\omega_{3Y} = Tr(A \wedge F - \tfrac{1}{3} A \wedge A \wedge A) = Tr(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A)$$
$$d\omega_{3Y} = Tr(F \wedge F)$$
$$\hat{I}_{d+1} \qquad \hat{I}_{d+2}$$

$\delta A = d\Lambda + [A, \Lambda] \implies \delta\omega_{3Y} = Tr(d\Lambda\, dA) = d\omega_2$

where $\omega_2 = Tr(\Lambda\, dA)$

We need to worry about $\underbrace{8_c, 56_s}_{2\times}, \underbrace{35_+}_{1\times}$

$$I_{12}^{8_c} = -\frac{trR^6}{725760} - \frac{trR^4\, trR^2}{552960} - \frac{(trR^2)^3}{1327104}$$
$$I_{12}^{35_+} = 992(\;\;) - 448(\;\;) + 128(\;\;)$$
$$I_{12}^{56_s} = -495(\;\;) + 225(\;\;) - 63(\;\;)$$

$$\Rightarrow \boxed{2I_{12}^{8_c} + 2I_{12}^{56_s} + I_{12}^{35_+} = 0}$$

"miraculous" anomaly cancellation

Note: $N=1$ SYM $D=10$ multiplet $8_v \oplus 8_s$ with group (in repr. of dim

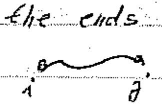
has anomaly $\hat{I}_{12}^{8_s}(R,F) = -Tr\frac{F^6}{1440} + \frac{TrF^4\,TrR^2}{2304} - \frac{TrF^2 TrR^4}{23040}$

$$- \frac{TrF^2(TrR^2)^2}{18432} + n\frac{trR^6}{725760} + n\frac{trR^4\, trR^2}{552960} + n\frac{(trR^2)^3}{1327104}$$

$\to$ by itself open strings anomalous, remedy $\to$

Chan-Paton factors

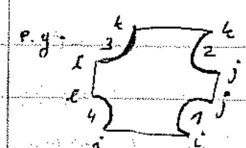
add non-dynamical degrees of freedom to the ends of open strings, promote $|k\rangle \to |ij;k\rangle$; in adjoint rep of $U(N)$, i.e. Hermitean



$i = 1, .., N$

This will give local $U(N)$ in spacetime (global on worldsheet). In any scattering amplitude

e.g.



there will be a factor

$$\lambda^1_{ij}\lambda^2_{jk}\lambda^3_{kl}\lambda^4_{li} = Tr(\lambda^1\lambda^2\lambda^3\lambda^4)$$

So $\lambda \to U\lambda U^{-1}$ invariance

Photon becomes $\lambda^a_{ij}\,\psi^\mu_{-1/2}\,|ij;k\rangle \sim A^a_\mu \ldots U(N)$ gauge

$\to$ anomalies will finally kill each other (hopefully :-))

Let's construct <u>unorientable strings</u>

open strings $\downarrow\Omega \quad \Omega: \sigma \to \pi - \sigma \quad \Omega: z \to -\bar{z}$

$\Omega: x^\mu \to x^\mu, \quad p^\mu \to p^\mu, \quad \alpha_n^\mu \to (-1)^n \alpha_n^\mu,$

$\psi_r^\mu \to (-1)^{r+\frac{1}{2}} \psi_r^\mu$

Project onto $\Omega = +1$ ($\to$ kills the photon $\psi^\mu_{-1/2}|0\rangle$ unless it acts on Chan-Paton fcts)

$\neq \lambda$: $\Omega\, \lambda_{ij}\, |k, ij\rangle \to \lambda'_{ij} |k, ij\rangle, \quad \lambda' = M \lambda^T M^{-1}$

but $\Omega^2 = 1 \overset{\text{Schur}}{\Longrightarrow} M (M^{-1})^T \propto \mathbb{1}$

(a) choose M symmetric $M = M^T = \mathbb{1}$

if $\lambda_{ij} \psi^\mu_{-\frac{1}{2}} |k, ij\rangle$ is to survive then $\lambda = -\lambda^T$

$\Rightarrow G = SO(N)$

(b) M antisymmetric $M = i \begin{bmatrix} 0 & \mathbb{1}_{N/2} \\ -\mathbb{1}_{N/2} & 0 \end{bmatrix} \Rightarrow \lambda = -M\lambda^T M$

$\Rightarrow G$ is $USp(N/2)$ or $Sp(N/2)$

Unoriented closed strings

$\Omega: \sigma \to -\sigma \quad z \to \bar{z} \quad \tilde{\psi}_n^\mu \leftrightarrow \psi_n^\mu$

Ω is a symmetry of $\mathbb{I}B$, project onto $\Omega = +1$

	$1\ \Phi$	$28\ B_{\mu\nu}$	$35\ G_{\mu\nu}$
NS-NS	$+1$	~~-1~~	$+1$
R-R	~~C, -1~~	$C_{\mu\nu}$, $+1$	~~$C_{\mu\nu\kappa\lambda}$, -1~~

only one dilatino-gravitino (lin. comb. of NS-R & R-NS) survives $\to$ again anomalous

- may be it will kill anomalies in open strings

So add unoriented open & unoriented closed strings

anomalies

- $(n - 496) \frac{\operatorname{tr} R^6}{(\ )}$ must vanish $\to \boxed{n = 496}$
- the rest $\underbrace{\text{will}}_{\text{involving only } F}$ vanish if either $G = E_8 \times E_8$ or $SO(32)$

$I_{12} = \frac{1}{3 \times 2^8} Y_4 X_8 \qquad Y_4 = \operatorname{tr} R^2 - \frac{1}{30} \operatorname{tr} F^2$ $\to$ indicates Chern-Simons

$X_8 = \left[\frac{\operatorname{Tr} F^4}{3} - \frac{(\operatorname{Tr} F^2)^2}{900} + (\ ) \right]$

How to cancel $Y_4 Y_8$? Green-Schwarz:

use $C_{(2)}$ in a clever way. Consider

$S_{GS} = \frac{1}{3 \times 2^6 (2\pi)^5 \alpha'} \int C_{(2)} \wedge X_{(8)}$

invariant under

$\delta A = dA + [A, \Lambda], \quad \delta\omega = d\theta + [\omega, \theta), \quad \delta C_{(2)} = d\lambda_{(1)}$

ask $C^{\text{not}}_{(2)}$ to transform $\delta C_{(2)} = \frac{\alpha'}{4}\left(\frac{1}{30} \operatorname{Tr}(\Lambda F) - \operatorname{tr}(\Theta\right.$

$\delta S_{GS} = \frac{1}{3 \times 2^6 (2\pi)^5} \int \frac{1}{30} (\operatorname{Tr}(\Lambda F) - \operatorname{tr}(\Theta R)) \wedge X_8$

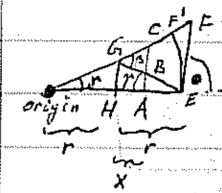
$\Rightarrow \hat{I}^{GS}_{12} = -Y_4 X_8$ that cancels the remaining anomaly $\to$ $\boxed{\text{anomaly free theory!}}$

Trodden: $\boxed{\text{Cosmology 1}}$

- cosmological spacetime
- dynamics & matter content (perfect fluid)
- the universe today + dark energy
- the early universe
- inflation

Cosmological spacetimes

assume homogeneity & isotropy



~~polar~~ sph. coords (r, θ, φ)

$d\sigma^2 = dr^2 + f^2(r)(d\theta^2 + \sin^2\theta\, d\varphi^2)$

$f(r) \sim r \quad r \to 0$

$EF \sim EF' = f(2r)\gamma = f(r)\beta \qquad AC = \gamma f(r+x) = AB + BC =$

$= \gamma f(r-x) + \alpha f(x$

$\Rightarrow \frac{df}{dr} = \frac{f(2r)}{2f(r)}$ & $f(r) \to r$ as $r \to 0$

$\Rightarrow f(r) = r, \sin(r), \sinh(r)$

$\Rightarrow$ most general hom & isot. metric in spacetime

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + f^2(r)(d\theta^2 + \sin^2\theta\, d\varphi^2)\right]$$

where f is $r, \sin(r)$ or $\sinh(r)$.

By redefining r FRW metric

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right]$$

$k \gtreqless 0 \Leftrightarrow \sin, r, \sinh$

Define conformal time $\tau(t) = \int^t \frac{dt'}{a(t')}$

$\Rightarrow ds^2 = a^2(\tau)\left[-d\tau^2 + \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right]$

Hubble parameter $H(t) \equiv \frac{\dot a}{a}$ ($\leftarrow \cdot$ w.r.t t, not τ)

Dynamics use Einstein eqns $G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$

need hom. & isotropic ansatz for matter content ($T_{\mu\nu}$) ... perfect fluid

$T_{\mu\nu} = (\rho + p) U_\mu U_\nu + p g_{\mu\nu}$

energy density — pressure (in the rest frame)

$U_\mu = (1,0,0,0) \Rightarrow T_{\mu\nu} = \left(\begin{array}{c|c} \rho & 0 \\ \hline 0 & p g_{ij} \end{array}\right)$ (ρ_i if more fluids)

E.g. $\rightarrow \boxed{H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\sum_i \rho_i - \frac{k}{a^2}}$ Friedmann eqn.

usually K scaled to $|K| = 1, 0$

& $\boxed{\frac{\ddot a}{a} + \frac{1}{2}\left(\frac{\dot a}{a}\right)^2 = -4\pi G \sum_i p_i - \frac{k}{2a^2}}$

Combine to give acceleration eqn

$\frac{\ddot a}{a} = -4\pi G \sum(\rho_i + 3p_i)$

Use Friedmann eqn. to define at any time

$\rho_c = \frac{3H^2}{8\pi G}$ critical density ~~($K \neq 0$)~~

If $\sum \rho_i = \rho_c$ then $K = 0$.

Define density parameter $\Omega_{tot} = \frac{\sum \rho_i}{\rho_c}$

$\Omega_{tot} > 1 \Leftrightarrow K = +1$, $\Omega_{tot} = 1 \Leftrightarrow K = 0$, $\Omega_{tot} < 1 \Leftrightarrow K = -$

Fractions of ρ_c $\Omega_i = \frac{\rho_i}{\rho_c}$

Matter EoM $\leftarrow$ energy-momentum conservation $\nabla_\mu T$

$\Rightarrow \boxed{\dot\rho + 3H(\rho + p) = 0}$

Need to supply an equation of state $p(\rho)$

(or at least $p = w\rho$, w... eqn. of state param., slowly va

Expectations 1) dust (clusters of galaxies) $\rho_m(a) \propto a^{-3}$

2) radiation $\rho_r(a) \propto a^{-4}$

$\frac{d\rho}{da} = -3(1+w)\frac{\rho}{a} \Rightarrow \rho(a) = \rho_0\left(\frac{a_0}{a}\right)^{3(1+w)}$

dust: pressureless matter $p_m = 0 \Rightarrow w_m = 0 \Rightarrow \rho \sim a^{-3}$

radiation: gas of photons $p_r = \frac{1}{3}\rho_r \Rightarrow w_r = \frac{1}{3} \Rightarrow \rho \sim a^-$

Cosmological constant – included into matter conten

$T^{(\Lambda)}_{\mu\nu} = -\frac{\Lambda}{8\pi G} g_{\mu\nu} \Rightarrow p_\Lambda = -\rho_\Lambda = -\frac{\Lambda}{8\pi G}$, $w_\Lambda = -$

Trick: Sometimes its useful to think of curvature as an energy component $\rho_{curv} = -\frac{3K}{8\pi G a^2}$, $p_{curv} = \frac{k}{8\pi G a^2}$, w_{curv}

Redshift $\frac{\lambda_{obs}}{\lambda_{emission}} = \frac{a_0}{a_{emission}} = 1 + z$ ← "redshift"

Flat universe ($K=0$)

If universe contains only one component of ρ_{tot} (or is dominated by one) then

$$a(t) = a_0\left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}} \quad \text{resp} \quad a(t) \propto e^{Ht}, \; w=-1$$

dust: $a(t) \propto t^{2/3}$ radiation $a(t) \propto t^{1/2}$

Note that $a \to 0$ as $t \to 0$ BIG BANG

Age of universe $t_0 = \int^{t_0} dt = \int_0^1 \frac{da}{a H(a)} = \frac{2}{3(1+w) H_0}$

this is $\sim \frac{1}{H_0}$ => age of universe is $\sim$ Hubble time $\frac{1}{H_0}$

Including curvature - work in conf. time

suppose $w=0 \Rightarrow$ $3(K+h^2) = 8\pi G \rho a^2$ $\quad h(\tau) = \frac{a'}{a}$

$K + h^2 + 2h' = 0$ $\quad (\;)' = \frac{d(\;)}{d\tau}$

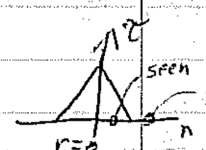
$$\Rightarrow h(\tau) = \begin{cases} \cot(\tau/2) \\ 2/\tau \\ \coth(\tau/2) \end{cases} \qquad a(\tau) = \begin{cases} 1-\cos(\tau) & 1 \\ \tau^2/2 & 0 \\ \cosh(\tau)-1 & -1 \end{cases} \; K$$

$$t(\tau) \propto \begin{cases} \tau - \sin(\tau) & +1 \\ \tau^3/6 & 0 \\ \sinh(\tau) - \tau & -1 \end{cases}$$

Horizons suppose an emitter E sends a light signal to an observer at $r=0$ (θ, φ const.)

in flat case radial null rays $\tau_0 - \tau = r \Rightarrow$

$\tau_0 - \tau_e = r_e$. Suppose τ_e bounded below by $\overline{\tau_e}$ (e.g. BIG BANG) $\Rightarrow \exists$ a maximal distance to which observer can see



.. particle horizon distance $r_{ph}(\tau_0) = \tau_0 - \overline{\tau_e}$

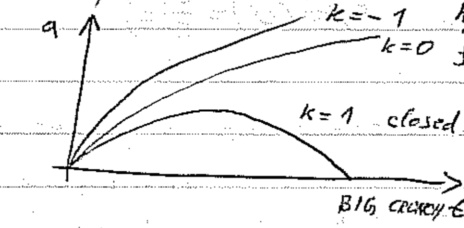
If τ_0 is bounded above by $\overline{\tau_0}$ $\to$ event horizon

$r_{eh}(\tau_e) = \overline{\tau_0} - \tau_e$

Convert to proper distances

$$d_H = a(\tau)\, r_{ph} = a(\tau)(\tau - \overline{\tau_e}) = a(t)\int_{t_e}^{t} \frac{dt'}{a(t')}$$

Geometry & destiny (dark energy)



if only radiation &

if cosmological constand or dark energy is present

$\to$ even spherical closed universe may expand fore

geometry no longer directly implies destiny

The universe today

$H_0 = 100\, h$ km/s/Mpc $\quad h = 0.71 \pm 0.06$

$\Omega_{Matter} \simeq 0.25$ baryons + dark matter

$\Leftarrow$ dynamics of galaxies, clusters & CMB

$\Omega_{baryon} \cong 0.05 < \Omega_{Matter}$ ($\Leftarrow$ nucleosynthesis)

Hubble law $v = Hd$ (+ corrections at large distanc

scale factor is increasing $\ddot{a} > 0$

$\to$ accelerating universe

CMB $\Rightarrow \Omega_{tot} = 1 \Rightarrow$ something missing

$\to$ dark energy $\Omega_{DE} \simeq 0.75$

Complex manifolds 3

Quotient sheaf

$\mathcal{E}, \mathcal{F}$. $\mathcal{E}/\mathcal{F} = ?$

obvious candidate $\mathcal{E}/\mathcal{F}(U) = \frac{\mathcal{E}(U)}{\mathcal{E}(\mathcal{F})}$

Ex: Let $\mathcal{O}$ be the sheaf of hol. fcns. (+)

and $\mathcal{O}^*$ — " — nowhere zero hol. fcns. (*)

$\mathcal{O} \xrightarrow{\exp(2\pi i -)} \mathcal{O}^*$ is a sheaf map

The cokernel of any map $\mathcal{E} \xrightarrow{f} \mathcal{F}$ is defined as $\mathcal{F}/Im(f) = coker(f)$

Let $X = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, z be the coord of $\mathbb{C}^*$,

ln z has branch cut → clearly z is in $\mathcal{O}^*(X)$ but z is not in $Im(\exp)$

$\Rightarrow z \in coker = \mathcal{O}^*/f(\mathcal{O})\ (X) = \mathcal{F}(X)$. Suppose $U \subset \mathbb{C}^*$

is a contractible open set. Then $z|_U$ is

in the image of $\exp(2\pi i -) \Rightarrow z|_U \sim 0$ in $\mathcal{F}(U)$

X can be written as a union of two contractible

open sets $\Rightarrow z$ violates "e)" sheaf condition

$\Rightarrow \mathcal{E}/\mathcal{F}$ presheaf, not sheaf. Can be fixed

by considering only "very small" contractible

open sets and then building up the whole

sheaf such that it satisfies sheaf definition.

– "sheafify" a presheaf (unique but technical)

($\Rightarrow$ in our example $\mathcal{E}/\mathcal{F} = 0$)

exists e.g. if X is manifold, for general topol. spaces some pathologies may arise

Čech cohomology

Let $\underline{\mathcal{U}} = \{\mathcal{U}_\alpha\}$ be an open cover of M. Define

$$\check{C}^0(\underline{\mathcal{U}}, \mathcal{F}) = \prod_\alpha \mathcal{F}(U_\alpha),$$

$$\check{C}^1(\underline{\mathcal{U}}, \mathcal{F}) = \prod_{\alpha<\beta} \mathcal{F}(U_\alpha \cap U_\beta),$$

$$\vdots$$

$$\check{C}^p(\underline{\mathcal{U}}, \mathcal{F}) = \prod_{\alpha_0<\alpha_1<\dots<\alpha_p} \mathcal{F}(\mathcal{U}_{\alpha_0} \cap \mathcal{U}_{\alpha_1} \cap \dots \cap \mathcal{U}_{\alpha_p}),$$

i.e. groups of Čech cochains. If $\sigma \in \check{C}^p(\underline{\mathcal{U}}, \mathcal{F})$,

let $\sigma_{\alpha_0 \dots \alpha_p} \in \mathcal{F}(\mathcal{U}_{\alpha_0} \cap \dots \cap \mathcal{U}_{\alpha_p})$. Coboundary operator is defined as follows

$$\delta_p : \check{C}^p(\underline{\mathcal{U}}, \mathcal{F}) \longrightarrow \check{C}^{p+1}(\underline{\mathcal{U}}, \mathcal{F})$$

$$(\delta_p \sigma)_{\alpha_0 \dots \alpha_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \sigma_{\alpha_0 \alpha_1 \dots \hat{\alpha_j} \dots \alpha_{p+1}} \Big|_{\mathcal{U}_{\alpha_0} \cap \mathcal{U}_{\alpha_1} \cap \dots}$$

σ is Čech cocycle iff $\delta\sigma = 0$,

— " — coboundary iff $\sigma = \delta\tau$.

It is easy to prove $\delta^2 = 0$. Therefore we have

cochain complex $0 \longrightarrow \check{C}^0 \xrightarrow{\delta_0} \check{C}^1 \xrightarrow{\delta_1} \check{C}^2 \to \dots$

Čech cohomology $\check{H}^p(\underline{\mathcal{U}}, \mathcal{F}) = \frac{\ker \delta_p}{Im\ \delta_{p-1}}$

depends on $\underline{\mathcal{U}}$ but if the open sets are "very small" then $\check{H}^p(\underline{\mathcal{U}}, \mathcal{F})$ turns out to be independent of $\underline{\mathcal{U}}$. This gives $\check{H}^p(X, \mathcal{F})$

$\check{H}^p(\underline{\mathcal{U}}, \mathcal{F}) = \check{H}^p(X, \mathcal{F})$ for all p if

$\check{H}^k(\mathcal{U}_{\alpha_0} \cap \mathcal{U}_{\alpha_1} \cap \dots \cap \mathcal{U}_{\alpha_n}, \mathcal{F}) = 0$ for all $k > 0$.

Ex: $\check{H}^p(\mathbb{P}^1, \mathcal{O}(k))$. In this case we may use $\mathcal{U}_0 = \mathbb{P}^1 \setminus \{[0,1]\}$ $\mathcal{U}_1 = \mathbb{P}^1 \setminus \{[1,0]\}$ as an open cover.

no triple intersect (or higher)

$\forall p \geq 2$ $\check{C}^p(\underline{\mathcal{U}}, \mathcal{O}(k)) = 0$, so

$\check{H}^p(\mathbb{P}^1, \mathcal{O}(k)) = 0 \quad \forall p \geq 2$.

$\check{H}^0(\mathbb{P}^1, \mathcal{O}(k))$ = holom. fcns on U_0 and U_1 which agree on overlap because $(\delta\sigma)_{01} = \sigma_1|_{01} - \sigma_0|_{01}$ -- global sections of $\mathcal{O}(k)$

On U_0 let $z = \frac{Z_1}{Z_0}$ be affine coordinate and let $z^m \in \mathcal{F}(U_0)$ be a local section. Clearly $m \geq 0$ (holomorphicity in U_0). We may define $\mathcal{O}(k)$ by $r_{0,01} = 1$, $r_{1,01} = z^k$ (compare with transition fcns of corresp. line bundle)

Let $w = z^{-1} = \frac{Z_0}{Z_1}$ be affine coord on $U_1 = \mathbb{C}$ and let w^l be a local section in $\mathcal{O}(k)(U_1)$

So the coboundary of this cochain is

$(\delta\sigma)_{01} = w^l|_{01} - z^m|_{01} = w^l z^k - z^m$. So we have a cocycle if $l = k - m$. So $0 \leq m \leq k$ for a global holom. section. m has $k+1$ values

(Laurent series)

These solutions form a basis for all global sections $\Rightarrow$ dim $\check{H}^0(\mathbb{P}^1, \mathcal{O}(k)) = k+1 \quad k \geq 0$

$= 0 \quad k < 0$

We may also write a global section as $\overbrace{Z_0^{k-m} Z_1^m}^{\text{homogeneous of degree } k}$ in homogeneous coords. We set $Z_0 = 1$ in U_0, $Z_1 = 1$ in U_1.

$\check{H}^1(\mathbb{P}^1, \mathcal{O}(k))$ suppose $(\sigma)_{01} = z^p$. If $p \geq 0$ then $\sigma = \delta\tau$ where $(\tau)_0 = -z^p$, $(\tau)_1 = 0$.

If $p \leq k$ then $\sigma = \delta\tau$ where $(\tau)_0 = 0$, $(\tau)_1 = w^{k-p}$

So if $p \geq 0$ or $p \leq k$ then σ is trivial in cohomology

$\Rightarrow$ $k < p < 0$ for a nontrivial element

$\check{H}^1(\mathbb{P}^1, \mathcal{O}(k)) = 0$ for $k \geq -1$

$= \mathbb{C}^{-k-1}$ for $k \leq -2$

In general: dim $\check{H}^p(\mathbb{P}^n, \mathcal{O}(k)) = 0$ if $p \neq 0$ & $p \neq n$

dim $\check{H}^0(\mathbb{P}^n, \mathcal{O}(k)) = \binom{n+k}{k}$

dim $\check{H}^n(\mathbb{P}^n, \mathcal{O}(k)) = \binom{-k-1}{-n-k-1}$

A global section of $\mathcal{O}(k)$ yields a map $\mathcal{O} \to \mathcal{O}(k)$

homogeneous fcns of Z_i's of degree k

similarly $\mathcal{O}(p) \to \mathcal{O}(k+p)$

<u>De Rham theorem</u> $H^p_{dR}(M) = \check{H}^p(M, \mathbb{R})$

<u>Dolbeault theorem</u> $H^{p,q}_{\bar{\partial}}(M) = \check{H}^q(M, \Omega^p)$

We know $H^2(\mathbb{P}^n, \mathcal{O}) \Rightarrow$ Hodge diamond of $\mathbb{P}$

<u>Exact sequences</u> $\to A^{-1} \xrightarrow{d_{-1}} A^0 \xrightarrow{d_0} A^1 \xrightarrow{d_1} A^2 \xrightarrow{d_2}$

is <u>complex</u> if $d_{j+1} d_j = 0$ and

<u>exact</u> if $\ker d_j = \operatorname{Im} d_{j-1}$, $\forall j$

<u>Short exact sequence</u> $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$

$\Rightarrow$ f injective, g onto (surjective)

"$C = B/A$"

Ex: $0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp(2\pi i\cdot)} \mathcal{O}^* \to 0$

is a short exact sequence of sheaves

Suppose we have $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0 \Rightarrow$

Snake lemma

$0 \to \check{H}^0(\mathcal{E}) \to \check{H}^0(\mathcal{F}) \to \check{H}^0(\mathcal{G}) \to \check{H}^1(\mathcal{E}) \to \check{H}^1(\mathcal{F})$
$\to \check{H}^1(\mathcal{G}) \to \check{H}^2(\mathcal{E}) \to \dots$ is exact.

Ex: Holomorphic tangent sheaf on $\mathbb{P}^n$ ... $\mathcal{T}_{\mathbb{P}^n}$

holomorphic tangent bundle on $\mathbb{C}^{n+1}$ .. fibre has a basis $\frac{\partial}{\partial z^j}$ for $j = 0, \dots, n$, i.e. a vector field is $f^j(z^k)\frac{\partial}{\partial z^j}$

These sections of hol. tang. bundle of $\mathbb{C}^{n+1}$ are invariant under $[z^0, \dots, z^n] \to [\lambda z^0, \lambda z^1, \dots, \lambda z^n]$ iff f^js are linear in z^k. A linear fcn in z^k is a section of $\mathcal{O}(1) \Rightarrow$

$$0 \to \mathcal{O} \xrightarrow{(z^0, \dots, z^n)} \mathcal{O}(1)^{\oplus(n+1)} \longrightarrow \mathcal{T}_{\mathbb{P}^n} \to 0$$

($\Leftarrow$ If $x^j = \frac{z^j}{z^0}$ are affine coords in U_0 then

$$\left.\begin{aligned} \frac{\partial}{\partial z^0} &= -\frac{z^j}{(z^0)^2}\frac{\partial}{\partial x^j} \\ \frac{\partial}{\partial z^j} &= \frac{1}{z^0}\frac{\partial}{\partial x^j} \end{aligned}\right\} \Rightarrow \sum_j z^j \frac{\partial}{\partial z^j} = 0$$
)

The short exact sequence defines $\mathcal{T}_{\mathbb{P}^n}$.

Define $\mathcal{T}_{\mathbb{P}^n}(k) = \mathcal{T}_{\mathbb{P}^n} \otimes \mathcal{O}(k)$. Then

$$0 \to \mathcal{O}(k) \to \mathcal{O}(k+1)^{\oplus(n+1)} \to \mathcal{T}_{\mathbb{P}^n}(k) \to 0$$

(Also, by dualization

$$0 \to \Omega^1_{\mathbb{P}^n} \to \mathcal{O}(-1)^{\oplus(n+1)} \to \mathcal{O} \to 0$$ Euler exact sequence)

$\Rightarrow$ $H^0(\mathcal{T}_{\mathbb{P}^n}) = \mathbb{C}^{n^2+2n}$ $\quad H^r(\mathcal{T}_{\mathbb{P}^n}) = 0$ for $r > 0$

Gates, Gusaru, Roček & Siegel "Superspace" 1000
Buchbinder

SUSY 3

Chiral superfields $\quad \phi(x^m, \theta_\alpha, \bar\theta_{\dot\alpha})$

$$\bar D_{\dot\alpha}\phi(x,\theta,\bar\theta) = 0$$

$$\bar D_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha\sigma^m_{\alpha\dot\alpha}\frac{\partial}{\partial x^m} = e^{-\hat X}\left(-\frac{\partial}{\partial\bar\theta^{\dot\alpha}}\right)e^{\hat X}$$

$$\hat X = -i\theta^\alpha\sigma^m_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_m$$

first solve $\frac{\partial}{\partial\bar\theta^{\dot\alpha}}\tilde\phi = 0 \Rightarrow$ no $\bar\theta$ components in $\tilde\phi$

$$\tilde\phi(x,\theta,0) = A(x) + \sqrt2\,\theta^\alpha\psi_\alpha(x) + \theta^\alpha\theta_\alpha F(x)$$

$$\Rightarrow\quad 0 = \underbrace{e^{-\hat X}\left(-\frac{\partial}{\partial\bar\theta^{\dot\alpha}}\right)e^{\hat X}}_{\bar D_{\dot\alpha}}\ \underbrace{e^{-\hat X}\tilde\phi(x,\theta,0)e^{\hat X}}_{\phi(x,\theta,\bar\theta)}$$

i.e. $\boxed{\begin{gathered}\phi(x,\theta,\bar\theta) = A(y) + \sqrt2\,\theta^\alpha\psi_\alpha(y) + \theta^\alpha\theta_\alpha F(y) \\ \text{where } y^m = x^m - i\theta^\alpha\sigma^m_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\end{gathered}}$

SUSY invariant action

- a function of ϕ, say $W(\phi)$ is chiral SF

then $\int d^4x\, F_W(\phi)$ is SUSY invariant

— so-called F-term, can be rewritten $\int d^4x \int d^2\theta\, W(\phi$,

Antichiral SF $\quad D_\alpha\phi^+ = 0$

$\phi^+(x,\theta,\bar\theta) = A^*(y^+) + \sqrt2\,\bar\theta_{\dot\alpha}\bar\psi^{\dot\alpha}(y^+) + \bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}F^*(y^+)$, $y^{+m} = x^m + i\dots$

$\phi^+\phi$ is a vector SF, hence $\int d^2\theta d^2\bar\theta\, \phi^+\phi = \int d^4\theta\, \phi^+\phi$ transforms into a total $\frac{\partial}{\partial x^m}$ under SUSY

the same holds for any real function $K(\phi^+\phi)$

$\Rightarrow$ 2 types of SUSY invariant terms in action

D-terms $\int d^4x\, d^4\theta\ K(\phi^+, \phi)$ $\quad$ K real ... Kähler pote

F-terms $\int d^4x\, d^2\theta\, W(\phi)$ W holomorphic superpotential

F^*-terms $\int d^4x\, d^2\bar\theta\, \overline{W}(\phi^\dagger)$

Dimensions $[K]=2$ $[\phi]=1$ $[W]=3$

$K = \underbrace{\phi^\dagger\phi}_{} + \frac{1}{M^2}(\phi^\dagger\phi)^2 \ldots$

renormalizable Kähler potential

$W = m\phi^2 + \lambda\phi^3$ (higher terms nonrenormalizable)

For a system of many fields $\{\phi^i, \phi^{+\bar j}\}$

$\mathcal{L}_{D\text{-term}} = K_{i\bar j}(A,A^*)\,\partial_m A^i \partial^m A^{*\bar j} + i K_{i\bar j}(A,A^*)\,\bar\psi^{\bar j}\bar\sigma^m \partial_m \psi^i$

$+ K_{i\bar j}(A,A^*) F^i F^{*\bar j}$ + others

$\mathcal{L}_{F\text{-term}} = F^i \frac{\partial W(A)}{\partial A^i} + \frac{\partial^2 W(A)}{\partial A^i \partial A^j}\psi^i\psi^j + h.c.$

Yukawa coupling

integrating out auxil. field F

$\Rightarrow V_{scalar} = \frac{\partial W}{\partial A^i}(K^{-1})^{i\bar j}\frac{\partial \overline{W}}{\partial \bar A^{\bar j}}$

- describes low-energy eff. theories of confining gauge dynamics (QCD)
- also low-energy eff. actions of compactified string theory

$N=1$ SUSY has $U(1)_R$ symmetry - automorphisms of SUSY algebra rotating $Q_\alpha \to e^{i\alpha}Q_\alpha$, $\bar Q_{\dot\alpha} \to e^{-i\alpha}\bar Q_{\dot\alpha}$

equivalently $\theta \to e^{i\alpha}\theta$, $\bar\theta \to e^{i\alpha}\bar\theta$

acts differently on different components of ϕ

$\to$ distinguishes particles & their superpartners

May or may not be a symmetry of full theory.

Consider a theory of chiral SF Q_L Q_R

$Q_L \to U Q_R$ $Q_R \to Q_R U^+$ $U \in SU(N)$ matrix

$W = m Q_R Q_L$ invariant under $SU(N)$

$K = Q_L^+ Q_L + Q_R Q_R^+$ $\Rightarrow$ renormalizable theory

Now we gauge $SU(N)$ $U = e^{i\omega^a(x)T^a}$

$Q_L \to U(x) Q_L$ is not SF $[\bar Q, U(x)] \neq 0$ (supercharge)

$\Rightarrow$ we promote U to SF chiral

$\omega^a \to \Lambda^a(x,\theta,\bar\theta)$. Then $Q_L \to e^{i\Lambda^a T^a} Q_L$ is a SF,

we also def. $Q_R \to Q_R e^{-i\Lambda^a T^a}$ $\Longrightarrow$ complexified SUSY gauge transf.

$\Rightarrow$ W is invariant

$Q_R^+ \to e^{i\Lambda^{a+}T^a} Q_R^+$ $Q_L^+ \to Q_L^+ e^{-i\Lambda^{a+}T^a}$

$K \to K' = Q_L^+ e^{-i\Lambda^{a+}T^a} e^{i\Lambda^a T^a} Q_L + Q_R e^{-i\Lambda^a T^a} e^{+i\Lambda^{a+}T^a} \ldots$

to cancel this we introduce vector SF in $SU(N)$

V: $e^V \to e^{i\Lambda^+ T} e^V e^{-i\Lambda T}$ so that

$K = Q_L^+ e^V Q_L + Q_R e^{-V} Q_R^+$ is invariant under SUSY gauge t…

i.e. V: $V \to V - i\Lambda\cdot T + i\Lambda^+ T$, "$\delta V = -i\Lambda + i\Lambda^+$"

V is a real SF, many components of V shift under gauge transf., e.g.

$V|_{lowest} = C(x)$ $\delta C \sim \mathrm{Im}\,\Lambda$, so one can choose

real scalar $\Lambda = \Lambda_{WZ}$ such that

many components of V vanish

- Wess-Zumino gauge

$V_{WZ} = -\theta\sigma^m\bar\theta v_m(x) + i\theta^2\bar\theta\bar\lambda - i\bar\theta^2\theta\lambda + \frac{1}{2}\theta^2\bar\theta^2 D(x)$

auxiliary

$V^2_{WZ} = -\frac{1}{2}\bar\theta^2\theta^2 v_m v^m$, $V^3_{WZ} = 0$

WZ gauge breaks SUSY so SUSY transf. must be accompanied by supergauge transf.

$$\phi^\dagger e^V \phi\big|_{\bar\theta^2\theta^2} = F^*F + A\Box A^* + i\partial_m\bar\psi\bar\sigma^m\psi + v_m\left(\tfrac{1}{2}\bar\psi\bar\sigma^m\psi + \tfrac{i}{2}A^*\overleftrightarrow{\partial}^m A\right) - \tfrac{i}{\sqrt2}\left(A\bar\lambda\bar\psi - A^*\lambda\psi\right) + \tfrac{1}{2}DA^*A + \tfrac{1}{2}v_m v^m A^*A$$

the rest correspond to covariant deriv. → Yukawa interaction squark-quark-gaugino

dim $V = 0$ ⇒

$\int d^4\theta\, V^2$ mass for v_m not invariant

$\underbrace{\xi_{FI}}_{\dim=2}\int d^4\theta\, V$ gauge invariant for $U(1)$

Kinetic term: superfields strengths

$W_\alpha = -\frac{1}{4}\bar D^2 D_\alpha V$ $\quad \bar W_{\dot\alpha} = \frac{1}{4}D^2\bar D_{\dot\alpha} V$

W_α are chiral SF → $\int d^4x\, d^2\theta\, W^\alpha W_\alpha$

$$\mathcal{L} = \frac{1}{4k}\int\frac{1}{g^2_{YM}}\,\mathrm{Tr}\, W^\alpha W_\alpha\, d^2\theta\, d^4x + \frac{1}{4k}\int\left(\frac{1}{g^2_{YM}}\right)^*\mathrm{Tr}\,\bar W_{\dot\alpha}\bar W^{\dot\alpha}\, d^2\bar\theta\, d^4x$$

+ matter / gauge part

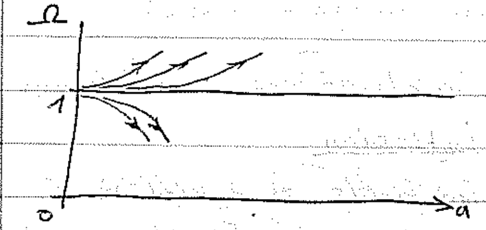
$\boxed{\text{Cosmology 2}}$

Problems of standard cosmology

$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \iff \Omega - 1 = \frac{k}{a^2H^2}$ $\quad \big|\frac{d}{da}$, use 2nd E.e.

$\Rightarrow \frac{d\Omega}{da} = (1+3w)\frac{\Omega(\Omega-1)}{a}$, where $p = w\rho$

dust: $w=0$, radn $w = \frac{1}{3}$ ⇒ in both cases $(1+3w) \neq 0$



to have $\Omega \neq 0$ today we need $0 \leq 1-\Omega \lesssim 10^{-60}$

⇒ extremely special initial conditions, no explanation for it in standard cosmology

"flatness problem"

Consider a photon along a radial trajectory

$([k=0],\ a_0 = 1)$ → radial null path $0 = ds^2 = -dt^2 + a^2(t)\,dr^2$

→ comoving distance travelled by photon between t_1, t_2

$\Delta_r = \int_{t_1}^{t_2}\frac{dt}{a(t)}$ (physical distance $\times a(t)$)

Assume matter domination ⇒ $a(t) = \left(\frac{t}{t_0}\right)^{2/3}$, $H(t) = \frac{2}{3}t^{-1}$

⇒ $\Delta_r = 2H_0^{-1}(\sqrt{a_2} - \sqrt{a_1})$ $\quad a_i \equiv a(t_i)$

Comoving horizon size at some time t^*

$r_{hor}(a_*) = 2H_0^{-1}\sqrt{a_*}$

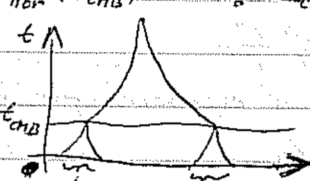
Physical horizon size $d_{hor}(a_*) = a_*\, r_{hor}(a_*) = 2H_*^{-1} = 2H$

⇒ in any almost flat cosmology with only radn and matter $d_{hor} \sim H^{-1}$ = (Hubble distance)$^{-1}$

CMB released at $a_{CMB} \sim \frac{1}{1200}$ ⇒ comoving distance between CMB release and us $\Delta_r = 2H_0^{-1}(1-\sqrt{a_{CMB}}) \sim 2H_0^{-1}$

while comoving ~~distance~~ horizon distance at t_{CMB}

$r_{hor}(a_{CMB}) = 2H_0^{-1}\sqrt{a_{CMB}} \sim 6\times10^{-2}H_0^{-1}$



horizon problem (homogeneity prob

correlated c

"Solution" if $\exists$ a period of cosmic evolution with $\frac{\ddot a}{a}>0$ – *inflation*

slowly-rolling scalar fields ϕ, $V(\phi)$

$$T_{\mu\nu} = (\nabla_\mu\phi)(\nabla_\nu\phi) - g_{\mu\nu}\left[\tfrac12 g^{\alpha\beta}(\nabla_\alpha\phi)(\nabla_\beta\phi)+V(\phi)\right]$$

for simplicity $k=0$ & assume homogeneity $\Rightarrow$

homog. scalar field behaves as a perfect fluid

$\rho_\phi = \frac12\dot\phi^2+V(\phi)$, $p_\phi = \frac12\dot\phi^2 - V(\phi)$

EoM: $\ddot\phi + 3\frac{\dot a}{a}\dot\phi + \frac{dV}{d\phi}=0$

Friedmann eq. $H^2=\frac{8\pi G}{3}\left[\frac12\dot\phi^2+V(\phi)\right]$

Accel. eq. $\frac{\ddot a}{a} = -\frac{4\pi G}{3}[\rho+3p]$

to get accelerated expansion we need $p<-\frac13\rho$

- easily satisfied by a scalar field if $\dot\phi^2 \ll V(\phi)$, then $p\sim-\rho$ (cf. cosmological constant)

Ad *flatness problem* $\frac{d\Omega}{da} = \underbrace{(1+3w)}_{\text{negative for } p<-\frac13\rho}\frac{\Omega(\Omega-1)}{a}$

$\rightarrow$ attraction towards $\Omega=1$ $\Rightarrow$ after short time $0<1-\Omega<10^{-60}$ $\Rightarrow$ since inflation not enough time to significantly change $\Omega\sim1$.

ad slow-roll approx. – neglect $\ddot\phi$ $\Rightarrow$ $\dot\phi \simeq -\frac{V'}{3H}$

$H^2\simeq \frac{8\pi G}{3}V(\phi)$

sufficient conditions for this are $|\epsilon|\ll1$, $|\eta|\ll1$

where the slow-roll parameters are

$\epsilon \equiv \frac{M_P^2}{2}\left(\frac{V'}{V}\right)^2$, $\eta = M_P^2\frac{V''}{V}$

It is useful to have a general expression for how much inflation takes place.

(size of universe goes like $e^{N(t)}$)

$\epsilon(\phi_{end})=1$

Number of e-foldings $N(t) = \ln\left(\frac{a(t_{end})}{a(t)}\right)$

$\Rightarrow N(t) = \frac{1}{M_P^2}\int_\phi^{\phi_{end}} \frac{V}{V'}\,d\phi$

Simple model (chaotic inflation)

$V(\phi)=\frac12 m^2\phi^2$ $\quad t=0: \rho\sim M_P^4$

slow-roll $\phi_0 \sim \frac{M_P^2}{m}\sim\frac{M_P}{\epsilon}$ $\quad \frac{m}{M_P}=\epsilon\ll1$

$H=\alpha\phi$ $\quad \alpha=\sqrt{\frac{4\pi}{3}}\,\epsilon$

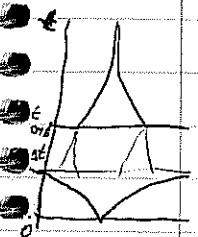
$\ddot\phi+3\alpha\phi\dot\phi+m^2\phi=0 \Rightarrow \phi=\phi_0-\beta t$ $\quad \beta=\frac{m^2}{3\alpha}$

slow-roll $\Leftrightarrow$ $\phi \gg \frac{\beta}{m}=\frac{M_P}{\sqrt{12\pi}}$ $\Rightarrow$ valid for $\Delta t\sim\frac{\phi_0}{\beta}\sim\frac{t}{\epsilon}$

... is an attractor solution

ad *horizon problem* $\quad a(t)=a(0)\exp\left[\frac{2\pi}{M_P^2}(\phi_0^2-\phi^2)\right]$

$a(\Delta t)\sim a(0)\exp\left(\frac{2\pi}{\epsilon^2}\right)$



Choose m such that $\epsilon<10^{-4}$ $\Rightarrow$ $a(\Delta t)\sim a(0)\cdot10^{2.7}$

proper distance L_P (Planck) at $t=0$ becomes 10^{10^8} cm at Δt

cf. horizon today 10^{28} cm $\rightarrow$ crucial difference between Hubble and horizon size

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$\boxed{\text{Complex manifolds 4}}$

We want to describe

Hodge diamond of the quintic CY 3-fold

Hypersurfaces in $\mathbb{P}^N$

Let $f(z^i)$ be a hom. function of degree d in $\mathbb{P}^N$

$f=0$ defines a smooth variety X if $df=0$ has no solut. on X (i.e. $f=df=0$ has no solns).

$0 \to \mathcal{O}_{\mathbb{P}^N}(-d) \xrightarrow{\cdot f} \mathcal{O}_{\mathbb{P}^N} \xrightarrow[\text{to } X]{\text{restrict}} \mathcal{O}_X \to 0$ any fcn on X can be

(f raises degree by d, $d-d=0$)

surj. $\Leftarrow$ {raised to fcn on neighborhood of X in $\mathbb{P}^N$.. enough for sheaves

If X is Calabi-Yau manifold ~~...~~ of complex dimension n then $\check{H}^n(X, \mathcal{O}_X) = \mathbb{C}$.

In our case $n = N-1$. From definition of $\check{H}$ one can see $\check{H}^n(X, \mathcal{O}_X) = \check{H}^n(\mathbb{P}^N, \mathcal{O}_X)$

$$0 \to \mathcal{O}_{\mathbb{P}^{n+1}}(-d) \to \mathcal{O}_{\mathbb{P}^{n+1}} \to \mathcal{O}_X \to 0$$

$$\begin{array}{llll}
H^0 & 0 \to & \mathbb{C} \to & \mathbb{C} \to \\
H^1 & 0 \to & 0 \to & 0 \to \\
\vdots & & & \\
H^n & 0 & 0 & \mathbb{C}^{\binom{d-1}{n+1}} \\
H^{n+1} & \mathbb{C}^{\binom{d-1}{n+1}} & 0 & 0
\end{array}$$

$\Rightarrow X$ is CY $\iff \binom{d-1}{n+1} = 1 \iff d = n+2$

n	d	
1	3	elliptic curve
2	4	K3 surface
3	5	Quintic 3-fold

$h^{p,q} = \dim H^q(X, \Omega^p_X)$

Adjunction formula $\quad W = \mathbb{P}^N$ ($= \mathbb{P}^4$ in our case)

$$0 \to T_X \to T_W|_X \to \mathcal{N} \to 0$$

normal sheaf of $X \subset W$

X hypersurf. $\Rightarrow \mathcal{N} = \mathcal{O}_W(5)|_X$ for quintic 3-fold

On a Calabi-Yau manifold of dimension n $\Lambda^n T_X^*$ is a trivial line bundle, denote corresp. sheaf of section $\Omega^t_X = \Lambda^t T^*_X$

$\Rightarrow \Lambda^p T_X \cong \Lambda^{n-p} T_X^* = \Omega^{n-p}_X$

$\dim H^2(X, \mathcal{O}_X) = h^{0,2}$ known

Now consider $H^2(X, T_X) = H^2(X, \Omega^2_X)$

Cohomology of $\mathcal{O}_W(5)|_X = \mathcal{O}_X(5)$

$$0 \to \mathcal{O}_W(-5) \to \mathcal{O}_W \to \mathcal{O}_X \to 0$$

$$\Rightarrow 0 \to \mathcal{O}_W \to \mathcal{O}_W(5) \to \mathcal{O}_X(5) \to 0$$

$$\begin{array}{llll}
H^0 & 0 \to \mathbb{C} & \to \mathbb{C}^{126} \to & \mathbb{C}^{125} \\
 & \to 0 & \to 0 \to & 0 \\
 & \to 0 & \to 0 \to & 0 \\
 & \vdots & \vdots & \vdots
\end{array}$$

($\Leftarrow \check{H}^0(\mathbb{P}^n, \mathcal{O}(d)) =$

$\check{H}^*(X, \mathcal{O}_X(5)) = \mathbb{C}^{125}, 0, 0, \ldots$

$0 \to \mathcal{O}_W(-5) \to \mathcal{O}_W \to \mathcal{O}_X \to 0$ tensor by T_W

$\Rightarrow 0 \to T_W(-5) \to T_W \to T_W|_X \to 0$

known $\check{H}^*(W, T_W) = \mathbb{C}^{24}, 0, 0, \ldots$

$$0 \to \mathcal{O}_W \to \mathcal{O}_W(1)^{\oplus 5} \to T_W \to 0$$

$$\Rightarrow 0 \to \mathcal{O}_W(-5) \to \mathcal{O}_W(-4)^{\oplus 5} \to T_W(-5) \to 0$$

$$\left.\begin{array}{lll}
H^0 & 0 & 0 \\
H^1 & 0 & 0 \\
H^2 & 0 & 0 \\
H^3 & 0 & 0 \\
H^4 & \mathbb{C} & 0
\end{array}\right\} \Rightarrow \begin{array}{l} 0 \\ 0 \\ 0 \\ \mathbb{C} \\ 0 \end{array}$$

$\check{H}^*(W, T_W(-5)) = 0, 0, 0, \mathbb{C}, 0$

$$0 \to \mathcal{T}_W(-5) \to \mathcal{T}_W \to \mathcal{T}_W|_X \to 0$$

$$\begin{array}{c|ccc}
H^0 & 0 & \mathbb{C}^{24} & \mathbb{C}^{24} \\
H^1 & 0 & 0 & 0 \\
H^2 & 0 & 0 & \mathbb{C} \\
H^3 & \mathbb{C} & 0 & 0 \\
H^4 & 0 & 0 & 0
\end{array}$$

$\Rightarrow \check{H}^*(X, \mathcal{T}_W|_X) = \mathbb{C}^{24}, 0, \mathbb{C}, 0, 0$

coming back

$$0 \to \mathcal{T}_X \to \mathcal{T}_{W|X} \to \mathcal{O}_X(5) \to 0$$

$$\begin{array}{c|cccc}
H^0 & 0 \to \cancel{\mathbb{C}}\,0 & \leftarrow \mathbb{C}^{24} & \xrightarrow{\text{injective}} \mathbb{C}^{125} \to \\
H^1 & \cancel{X}\,\mathbb{C}^{101} & 0 & 0 \\
H^2 & \mathbb{C} & \mathbb{C} & 0 \\
H^3 & 0 & 0 & 0 \\
H^4 & 0 & 0 & 0
\end{array}$$

without knowledge of map $\check{H}^0(X, \mathcal{T}_{W|X}) \to \check{H}^0(X, \mathcal{O}_X(5))$

we can directly prove

$$\dim H^0(X, \mathcal{T}_X) = \dim H^0(X, \Omega^2_X) = h^{2,0} = h^{0,2} = \dim H^2(X, \mathcal{O}_X) = 0$$

$\Rightarrow$ Hodge diamond of quintic 3-fold

$$\boxed{\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
1 & & 101 & & 101 & & 1 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}}
\Leftarrow
\begin{cases}
h^{2,1} = \dim \check{H}^1(X, \Omega^2) = \dim \check{H}^1(X, \mathcal{T}_X) = 101 \\
h^{2,2} = \dim \check{H}^2(X, \Omega^2) = \dim \check{H}^2(X, \mathcal{T}_X) = 1
\end{cases}$$

Deformations of complex structure

cmplx structure $J_\mu{}^\nu$ s.t. $J^\mu{}_\nu J_\rho{}^\nu = -\delta^\mu_\rho$ and $N^\mu_{\nu\rho} = 0$

In cplx coords $J^j{}_k = +i\delta^j_k$, $J^{\bar j}{}_{\bar k} = -i\delta^{\bar j}_{\bar k}$, $J^{\bar j}{}_k = J^j{}_{\bar k} = 0$

$J_\mu{}^\nu \to J_\mu{}^\nu + \underset{\text{infinitesimal}}{\tau_\mu{}^\nu} \Rightarrow J^2 = -1$ forces $\tau^j{}_k = \tau^{\bar j}{}_{\bar k} = 0$ leaving $\tau^j{}_{\bar k}$

Let $\tau = \tau^j{}_{\bar k}\, dz^{\bar k} \frac{\partial}{\partial z^j}$ (0,1)-form valued in T_X

then $N^\mu_{\nu\rho} = 0 \iff \bar\partial \tau = 0$.

If $\tau = \bar\partial v$ for some global section of T_X then

this change in cplx structure is equivalent to a holomorphic change of coordinates (i.e. trivial)

$\Rightarrow$

deformation of complex structure is to 1st order given by $\check{H}^1(X, \mathcal{T}_X)$. For CY it turns out that all 1st order deformations are integrable to true deformations

$\Rightarrow$

There are <u>101 independent deformations</u> of the complex structure of <u>quintic 3-fold</u>.

Another way of deriving this

- general 5th order homog. polynomial
  - $\to$ 126 terms $-$ 1 (scaling) $-$ 24 ($PGL(5,\mathbb{C})$ transformations of homog. coord.)
  - $= \underline{101}$

$\Rightarrow$ in this case all deformations are "polynomial" – this doesn't work e.g. for K3-surface

$$0 \to \mathcal{T}_X \to \mathcal{T}_{W|X} \to \mathcal{O}_X(4) \to 0 \qquad W = \mathbb{P}^3,\ d = 4,\ X = K3 \text{ surface}$$

$$\begin{array}{cccccc}
0 \to & 0 & \to & \mathbb{C}^{15} & \to & \mathbb{C}^{34} \to \\
 & \mathbb{C}^{20} & \to & 0 & \to & 0 \\
 & 0 & & 0 & & 0
\end{array}$$

$\dim \check{H}^1(X, \mathcal{T}_X) = 20$ versus $34 - 15 = 19$ quartic eqns

$\Rightarrow$ out of 20 deformations only 19 are polynomial

QFT in curved spacetime 2

$\mathcal{N} = (1 - |\alpha^{-1}\beta|^2)^{1/4} = |\alpha|^{-1/2} = \left(\sum \alpha_{nm}^{+} \alpha_{nm}\right)^{-1/4}$

$|0\rangle_u \to |0\rangle_v$ Bogoliubov transformation

so-called <u>Split of solutions into positive and negative frequencies</u>

Suppose on some Σ $[n\cdot\partial]\,U = i\omega\, U$

(i.e. "$U \sim e^{i\omega t}$") $\Rightarrow$ $[n\cdot\partial]U^* = -i\omega U^*$

$U \circ U^* = (-i)\int d\Sigma \sqrt{h_\Sigma}\, U (n\cdot\overleftrightarrow{\partial}) U^* =$

$= -2i\omega(-i)\underbrace{\int d\Sigma\sqrt{h_\Sigma}\, UU^*}_{>0} < 0$

$\Rightarrow a_U$ is annihilation op., $a_U^+ = -a_{U^*}$ is creation op.

Qualitative examples

FRW spacetime

No global time transl. symmetry $\to$ no energy

but local — " — ∂_t in regions $t<t_1$,

$t>t_2$ $\Rightarrow$ H_{early}, H_{late}

$[H_{early}, \phi(x)] = -i\frac{\partial}{\partial t}\phi$ $\quad x=(t,\vec{x}),\ t<t_1$

$[H_{late}, \phi(x)] = -i\frac{\partial}{\partial t}\phi$ $\quad t>t_2$

$\Rightarrow$ consider early sympl. basis $\mathcal{U}_{early}$ $\quad \partial_t U_n^{early} = i\omega_n^{early} U_n^{early}$

$\omega_n^{early} = -\omega_{-n}^{early}$ $\quad$ for $t<t_1$

$H_{early} = \sum_{n>0} \omega_n a_{n\,early}^{+} a_n^{early}$ $\Rightarrow$ if we expand ϕ in a_n^{early}, a

then $[H_{early}, \phi(x)] = -i\frac{\partial}{\partial t}$

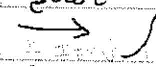
Similarly $\mathcal{U}^{late}$ $\quad \partial_t U_n^{late} = i\omega_n^{late} U_n^{late}$, —

In general $\boxed{\mathcal{U}^{late} \neq \mathcal{U}^{early}}$

by separation of variables $U(x,t) = \hat{U}(t)\hat{U}(x)$ Klein-Gordon

$\frac{\partial^2}{\partial t^2}\hat{U}(t) = -\omega^2(t)\hat{U}(t) + f(\ )\frac{\partial}{\partial t}\hat{U}(t)$

$\overset{early}{e^{i\omega t}} \Longrightarrow \overset{late}{e^{i\omega t}}$

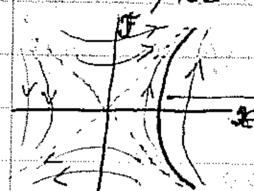


$e^{i\omega t}$

$e^{-i\omega t}$

$\Rightarrow a^+_{U_{early}} = \sum \# a^+_{U_{late}} + \sum \# a_{U_{late}} \Rightarrow \boxed{|0\rangle_{\mathcal{U}_{early}} \neq |0\rangle_{\mathcal{U}_{late}}}$

Rindler space — Minkowski space described by uniformly accelerated observers



$\underbrace{\mathcal{X}^2 - \mathcal{T}^2}_{\text{integral line of boost symmetry}} = \frac{1}{\alpha^2}$ ← proper acceleration

Rindler time transl. — transl. along int. lines of boost

Inertial observer — natural to use eigenmodes of $\frac{\partial}{\partial \mathcal{T}}$

Uniformly acc. observer — natural to use eigenmodes of Rindler time t

$ds^2 = -d\mathcal{T}^2 + d\mathcal{X}^2 = -\kappa^2 x^2 dt^2 + dx^2$ $\quad x = \sqrt{\mathcal{X}^2 - \mathcal{T}^2}$

$= -dv\,du$

$v = \mathcal{T} + \mathcal{X},\ u = \mathcal{T} - \mathcal{X}$ $\quad \tanh(\kappa t) = \frac{\mathcal{T}}{\mathcal{X}}$

κ.. if we have a particular unif. accel. observer at x_0

with $\alpha = \frac{1}{x_0}$ we put $\kappa = \frac{1}{x_0} = \alpha$

$X = e^{\kappa \varsigma} \rightarrow ds^2 = \kappa^2 e^{2\kappa\varsigma}(-dt^2 + d\varsigma^2)$

$V = t + \varsigma \quad U = t - \varsigma$ on R (right Rindler $\overset{\text{wedge}}{\text{spacetime}}$)

$\underbrace{V = -(t+\varsigma)}$ $U = -(t - \varsigma)$ on L (left — " —)

increase to future both in L, R

$ds^2 = -\kappa^2 e^{\pm 2\kappa(V-U)} \, dU dV \quad + \text{in } R, - \text{in } L$

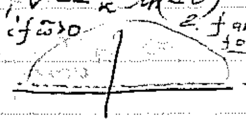
Massless K-G eqn.

R, L causally separated ⇒ can prescribe values independently on L, R

$$p_R = \begin{cases} e^{\pm i\omega V} = \exp(+\frac{i\omega}{\kappa}\ln \mathcal{V}) & \text{on } R \\ 0 & \text{on } L \end{cases}$$

using $\mathcal{V} = \pm e^{\pm\kappa V}$ (+ on R, − on L), $V = \pm\frac{1}{\kappa}\ln(\pm\mathcal{V})$

FT $\int d\mathcal{V} \, e^{i\tilde{\omega}\mathcal{V}} f$ if $\tilde{\omega} > 0$ & f analytic for $\operatorname{Im} f > 0$ $\rightarrow 0$



⇒ f negative frequency purely

f analytic in lower half-plane ⇒ f purely positive freq.

our $f = \mathcal{V}^{+\frac{i\omega}{\kappa}}$ is not analytic → mixture of + and − frequencies

$$p = \begin{cases} \exp(\frac{i\omega}{\kappa}\ln \mathcal{V}) & \text{for } \mathcal{V} > 0 \\ \exp(-\frac{\pi\omega}{\kappa} + \frac{i\omega}{\kappa}\ln(-\mathcal{V})) & \text{for } \mathcal{V} < 0 \end{cases}$$

is analytic in ~~upper~~ lower half plane ⇒ positive freq.

i.e.

$p = p_R + e^{-\pi\omega/\kappa} n_L \qquad n_L = \begin{cases} 0 & \text{on } R \\ \exp(\frac{i\omega}{\kappa}\ln(-\mathcal{V})) & \text{on } L \end{cases}$

$\downarrow \qquad \downarrow \qquad\qquad = e^{i\omega V}$ on L

$\alpha \sim 1 \qquad \beta = e^{-\pi\omega/\kappa}$

⇒

$|0\rangle_{Mink} = \exp\left(-\frac{1}{2}\sum_{mn}(\alpha^{-1}\beta)_{mn} \underbrace{a_n^+ a_m^+}_{\text{Rindler operators}}\right)|0\rangle_{Rindler}$

$= \exp\left(-\frac{1}{2}\sum_n e^{-\pi\omega/\kappa} \sum a^+_{n,R} a^+_{n,L}\right)|0\rangle_{Rindler}$

⇒ $|0\rangle_{Mink} = |0\rangle_{Rindler} + |2\text{-particle}\rangle + |4\text{-particle}\rangle + \ldots$

Physics

let us consider any observable $\mathcal{O}$ built from $\phi(x)$ on R

Ex 1: $\mathcal{O} = N_{\text{Right Rindler}}$

Ex 2: $\mathcal{O} = \int \phi(x) f(x) dx$ with $f(x) \neq 0$ only on R

Ex 3: $\mathcal{O} = \iint_{x\,y} \phi(x)\phi(y) f(x)\tilde{f}(y) \, d(\text{worldline } x) \, d(\text{worldline } y)$ on the uniformly accelerating worldline

$\langle\mathcal{O}\rangle_{Mink} = \iint \langle\phi(x)\phi(y)\rangle f(x)\tilde{f}(y) d(\,)d(\,)$

${}_{Mink}\langle 0|\mathcal{O}|0\rangle_{Mink} = {}_{Rindler}\langle 0|\exp(-\frac{1}{2}\alpha^{-1}\beta\, a a)\,\mathcal{O}\,\exp(-\frac{1}{2}\alpha^{-1}\beta a_R^+ a_L^+)|0\rangle$

by definition $[\mathcal{O}, a_L] = [\mathcal{O}, a_L^+] = 0$

${}_{Mink}\langle 0|\mathcal{O}|0\rangle_{Mink} = Tr_{\mathcal{H}}(\mathcal{O}\,|0\rangle_{Mink}\,{}_{Mink}\langle 0|) =$

$\mathcal{H} = \mathcal{H}_R \otimes \mathcal{H}_L$

$= Tr_{\mathcal{H}_R}(\mathcal{O}\varsigma)$ where $\varsigma = Tr_{\mathcal{H}_L}|0\rangle_{Mink}\,{}_{Mink}\langle 0|$

$\underbrace{\langle n_1, n_2, \ldots, n_k, \ldots | \varsigma | n_1', n_2', \ldots, n_k', \ldots\rangle}_{\text{occupation \# for R modes}} = \delta_{n_1 n_1'}\delta_{n_2 n_2'}\cdots\delta_{n_k n_k'}\, e^{-\frac{2\pi}{\kappa}\sum_j n_j}$

⇒ $\boxed{\varsigma = N\exp(-\frac{2\pi}{\kappa} H_{\text{Right Rindler}})}$ thermal state with $T =$

$\boxed{= \text{Unruh effect}}$

Tye: $\boxed{\text{String cosmology 1}}$

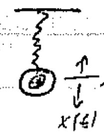
topic: Brane inflation: from superstrings to cosmic string

– see video (transparencies)

QFT in curved spacetime 3

$$H_{Right\ Rindler} = \sum_{\substack{Right\ Rindler \\ modes}} \omega\, a^+_{RR} a_{RR}$$

H_{RR} generates 'time translation' along integral curves of boosts, $\kappa = \alpha$ make it a unit time translation for particular uniformly accelerated observer



Physics: uniformly acc. observer carrying a particle detector - h.oscillator with charged weight

at each point we use local "rest" frame (inertial comoving frame) $\frac{\partial^2}{\partial \tau^2} X_{free} = -\omega_0^2 X_{free}(\tau)$

$\Rightarrow X_{free} = e^{-i\omega_0\tau} b^+ + e^{i\omega_0\tau} b$

become bosonic creation & annihilation osc.

We couple h.o. to $\phi(X)$

$$S_{interaction} = g \underbrace{\int X(\tau)\phi(\tau)\, d\tau}_{worldline\ of\ h.o.}$$ evaluated on worldline at proper time τ

in Heisenberg picture

$X_{int} = e^{-i\omega_0\tau} b^+(\tau) + e^{i\omega_0\tau} b(\tau)$

scattering operator $\Omega = e^{iS_{int}}$

$b^+(+\infty) = e^{-iS_{int}} b^+(-\infty) e^{iS_{int}} \simeq b^+(-\infty) + i[b^+(-\infty), g\int X\phi\, d\tau]$

$+ \underbrace{\mathcal{O}(g^2)}_{}$ to 2nd order suprisingly 0

in 1st order can be replaced by soln of free field of h.o. resp. ϕ

Suppose our state is $|0\rangle_{in}$ i.e. $b(-\infty)|0\rangle_{in} = 0$

a^+, a creation, annihilation ops. for ϕ

$a_{\substack{Minkowski \\ (inertial)}}(-\infty)|0\rangle_{in} = 0$ (we start in Mink. vacuum)

We want to find ${}_{in}\langle 0| N_{h.o.}(+\infty)|0\rangle_{in} = {}_{in}\langle 0| b^+(+\infty) b(+\infty)|0\rangle_{in}$

$= {}_{in}\langle 0| b^+(-\infty) \underset{\overset{\shortparallel}{0}}{b(-\infty)}|0\rangle_{in} + g\,\overset{\shortparallel}{(\quad)}$

$+ g^2 \cdot (\quad)$

(we replace only one b by order term $\Rightarrow$ the still kills $|0\rangle$ ...)

$g[b^+(-\infty), \int d\tau\, X_{free}(\tau)\phi_{free}(\tau)] = g\int d\tau\, e^{i\omega_0\tau}\phi(\tau) \leftarrow + \mathcal{O}(g^2)$ we neglect $S_{int} b^+(\tau) \sim b^+ ...$

$\Rightarrow$

${}_{in}\langle 0| N_{h.o.}(\infty)|0\rangle_{in} = g^2 \int d\tau \int d\tau'\, e^{i\omega_0\tau} e^{-i\omega_0\tau'}\, {}_{Mink}\langle 0|\phi(\tau)\phi(\tau')|0\rangle$

$\xrightarrow[\substack{obs.\ \tau = t_{Rindler} \\ (\kappa=\alpha)}]{uniformly\ acc.} g^2 \sum_{\substack{Right \\ Rindler\ modes}} {}_{Mink}\langle 0| a^+_{Rindler}\, a_{Rindler}|0\rangle_{Mink}$

FT picks up a^+ in ϕ, a in $\phi(\tau')$

$=$ thermal bath at $T = \frac{\alpha}{2\pi}$

Energy: acc. detector absorbs a "particle"

$|0\rangle_{Mink} \rightarrow a_{Rindler}|0\rangle_{Mink} = \# a^+_{Mink}|0\rangle_{Mink}$

$a_{Rindler} = \# a_{Mink} + \# a^+_{Mink}$

$\Rightarrow$ absorbtion in acc. frame is in fact emission of particle in Mink. spacetime

Local energy: $T_{ab}(x)$: ${}_{Mink}\langle 0| :T_{ab}(x): |0\rangle_{Mink} = 0$

$$[H_{RR}, \phi(x)] = \begin{cases} -i\frac{\partial}{\partial t_{Rindler}}\phi(x) & if\ x \in R \\ 0 & x \in L \end{cases}$$

H_{RR} is not normal order in Mink. $\Rightarrow {}_{Mink}\langle 0| H_{RR}|0\rangle_{Mink} \neq 0$

(but ${}_{Mink}\langle 0| \overbrace{H_{RR} - H_{LR}}^{\text{generates a boost - symm. of Mink.}} |0\rangle_{Mink}$ should be 0)

in other words $\left[\int_{t_R=0} :T_{ab}(x): n^a \xi^b , \phi(x)\right] = \begin{cases} -i\frac{\partial}{\partial t_R} & x\in R \\ -i\frac{\partial}{\partial t_R} & x\in L \end{cases}$

n^a: normal to $t_R=0$; $\xi^b \frac{\partial}{\partial x^b} = \frac{\partial}{\partial t_{Rindler}}$

similar to H_{RR} in R ~~..~~, i.e. by altering $\int$ to $\int_{on\,R}$ we kill and

$\int_{RR} T_{ab}$ must be the same up to a constant

$$H_{RR} = \int_{\substack{t_R=0 \\ on\ R\,(x>0)}} :T_{ab}(x): n^a\xi^b + C(\text{number})$$

$\Rightarrow {}_{Mink}\langle 0|H_{RR}|0\rangle_{Mink} = C$

$=$ energy of thermal bath ($\neq 0$)

Note: thermal bath is not transl. invariant

$ds^2 = (\underbrace{-k^2x^2}_{\text{redshift}} dt^2 + dx^2)$ .. thermal bath in a grav. field

sandwiching by ${}_R\langle 0|, |0\rangle_R$

$${}_{Rindler}\langle 0| \int_{\substack{t=0\\ on\,R}} :T_{ab}: n^a\xi^b |0\rangle_{Rindler} = -\left(\text{energy in a thermal bath in grav. field}\right)$$

(all osc. terms involve number ops which annihilate ${}_R\langle 0|$ or $|0\rangle_R$) ${}_R\langle 0|H_{RR}|0\rangle_R = 0 \Leftarrow$

generalization ${}_R\langle 0| :T_{ab}(x): |0\rangle_R = -(T_{ab})_{\text{thermal bath in grav. field}}$

$T_{00} \to -\infty$ as $\to$ horizon

$\Rightarrow |0\rangle_{Rindler}$ is not physically sound vacuum, T_{00} diverges on horizon, i.e. is singular

String cosmology 2,3

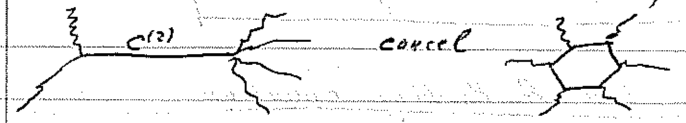
hep-th/010597 ~~hep~~ hep-th/0301240

hep-th/0308055 hep-th/0312020

String theory 4

$G_{(3)} = dC_{(2)} \to dC_{(2)} + \#(\omega_{3Y} - \frac{1}{30}\omega_{3L})$

This introduces tree level couplings



$\Rightarrow$ anomaly cancellation

T-duality & D-branes

compactifying on a circle

the momentum $p^\mu = \frac{1}{\sqrt{2\alpha'}}(\alpha_0^\mu + \tilde\alpha_0^\mu)$

$\sigma \to \sigma + 2\pi$ .... see transparencies

Worldvolume actions of Dbranes

$$S = \int d^{p+1}\xi \left\{ Tr\, F^2 \left(-\tfrac{1}{4} g_{YM}^2\right)\right\} e^{-\Phi^0} + \mu_p \int C_{(p+1)}$$

disc - tree level of open string

One may expect a generalization of Nambu-Goto action

$S = \tau_p \int d^{p+1}\xi \,(\det G_{ab})^{1/2}$ $\quad G_{ab} = G_{\mu\nu}\frac{\partial X^\mu}{\partial\xi^a}\frac{\partial X^\nu}{\partial \xi^b}$

both actions can be embedded in DBI action

$$S_p = -\tau_p \int d^{p+1}\xi\, e^{\Phi} \det{}^{1/2}\left(G_{ab}+B_{ab}+2\pi\alpha' F_{ab}\right) + \mu_p\int C_{(p$$

$\Phi = \Phi^0$ const., $B=0 \to g_{YM}^2 = \frac{1}{g_s}(\dots)$

like $()C_{(p-1)} + ()C_{(p-3)}$ $\to$ additional terms may be generat by wrapping on circles or other submanifolds of CY

Rey: $\boxed{\text{Advanced AdS/CFT 1}}$ (transparencies)

$\boxed{\text{QFT in curved } \cancel{\text{background}} \text{ spacetime 4}}$

Black holes & Hawking radiation

by analogy with Rindler space

Scharzschild BH

↑ .. time translation

Interesting observers:

1) approx. inertial observers in $\infty \Rightarrow$ generalized to static observers (moving along ↑)

2) freely falling observers (especially near horizon)

Coordinates

..... null surfaces

t Killing time

(= along ↑) ~ like Mink. time coordinate near ∞

extended as coords. along null surfaces

$\begin{cases} V = t + r \quad U = t - r, r \to \infty \text{ affine params. along } I^-, I^+ \\ V, U \text{ .. affine params. along } H^+, H^- \text{ (locally inertial coords for free falling obs. at horizon)} \end{cases}$

$\mathcal{V} = e^{\kappa V} \quad \mathcal{U} = e^{\kappa U}$ (after some computation :-))

3+1 Schw.-sch $\Rightarrow \kappa = \frac{1}{4\pi M}$, in general κ "surface gravity" of BH

Very similar to Rindler space description

(e.g. ϕ massless scalar, neglecting (θ, φ). $\partial_U \partial_V \phi = \partial_{\mathcal{U}} \partial_{\mathcal{V}} \phi = 0 \Rightarrow \phi = e^{i\omega V}$ vs $\phi = e^{i\omega \mathcal{V}}$)

but we should not – ↑ —— → alteration

Full 3+1 has scattering, but in certain regimes scattering small – high frequency (small wavelength compared to size of BH), neglible angular momentum w.r.t. BH

BH state: analogous state in Mink. space near the hori.

static $|0\rangle_{UV}$ ~ $|0\rangle_{Rindler}$

"Boulware state" + corrections $\mathcal{O}\left(\frac{\text{Riemann}}{[\text{distance to } H]^2}\right) \Rightarrow$ large black ho → small

singular on horizon, empty a

freely falling $|0\rangle_{\mathcal{U}\mathcal{V}}$ ~ $|0\rangle_{Mink}$

"Hartle-Hawking state" (HH) "thermal excitation" over lc

Analogy: star emits black body radiation at its surface but is scattered by gravit. field before reaching observer at ∞ – "grey body factors"

$\Rightarrow$ claim: "thermal excitation" have the right grey body factors

Physically ~~right~~ correct state:

from argument ${}_{UV}\langle 0| : T_{ab} : |0\rangle_{UV}$ diverges like $-(T_{ab})_{\text{thermal bath in grav. field}} \to -\infty$ a

initial conditions – at $I^-, H^- \Rightarrow |0\rangle_{UV}$ can never be investigated outside grav. field

$\Rightarrow |0\rangle_{\mathcal{U}\mathcal{V}}$ much more physically reasonable

${}_{HH}\langle 0| : T_{ab}(x) : |0\rangle_{HH} = (T_{ab})_{\text{th. bath in grav. field}} + \underbrace{{}_{Boulware}\langle 0| : T_{ab} : |0\rangle_{Boulware}}_{\to 0 \text{ as } r \to \infty}$

(in ∞ vacuum of approx. inertial obs

as viewed from ∞:

$\Rightarrow \boxed{\text{Temperature}_{HH} = \frac{\kappa}{2\pi}}$

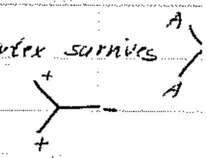
Note: Boulware state suffers from UV problem
HH state suffers from IR problem – thermal bath
→ positive energy → backreaction on geometry
— must be put in finite-size box
e.g. equilibrium in spherical mirror or thermal reservoir
in asymptotically flat space
Boulware state nice near I^- but
problems near H^-, HH state nice near H^-,
problems near I^-
⇒ <u>really nice state</u> "heterotic/hybrid ~~angular~~ state"
~ Boulware near I^-, HH near H^- — physically
reasonable for BH formed by star collapse
⟹ outward going particles, <u>negative</u>
flux of energy through H, positive
flux to ∞ ⇒ $\boxed{\text{black hole shrinks}}$

$\boxed{\text{Twistor string theory \& perturbative YM 3}}$

Twistor space description treats + & – helicity
particles in a very different way!
Can YM do the same thing? $S_{YM} = \frac{1}{g_{YM}^2}\int d^4x\, tr F_{\mu\nu}F^{\mu\nu}$
Canonical normalization of kinetic $\qquad F = dA + [A,A]$
terms for perturb. theory ⇒ g_{YM} goes into $g_{YM}[A,A]$
$g_{YM} \to 0$ ⇒ no interaction vertices
Trick $S = 2\int tr F^{+2} + \underbrace{tr(F^{-2} - F^{+2})}$, $F^{\pm} = (F \pm * F)$

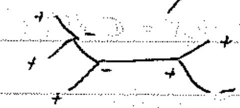
$S(A,G) = \int tr(G F^{+} - \overset{\varepsilon}{\overset{\|}{g_{YM}^2}}/2\, G^2)$
integrating out G gives back $\int d^4x\, tr F^{+2}$
only self-dual part of G matters
in limit $\varepsilon \to 0$ $\quad S_{\varepsilon=0} = \int tr(G(dA + A\wedge A))$ ⇒ GAA vertex survives

$\underset{h=-1}{G}$ —— $\underset{h=+1}{A}$ ⇒ $-$ —— $+$



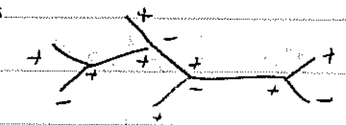
Perturbation theory

vanishes $(-+++)$



also vanishes

$\varepsilon \neq 0$ term εG^2 → $\underset{A}{+}$ —— $\underset{A}{+}$ propagator
⇒ new possibilities



$\neq 0$

$A(\overbrace{- \quad -}^{q}\overbrace{+ \quad +}^{n}) \sim \varepsilon^{q-1} G^q$

$U(1)$ transf. (not a symmetry) $\quad S \quad S_G = -4 \quad S_A = 0$
action $I = I_{-4} + I_{-8}\, q$
A violates S by $\Delta S = -4(d+1)$

Explanation – from twistor string theory
TFT $\quad \mathcal{N} = 2 \quad d = 2 \quad \underset{SO(2)_{Lor.}}{Q_{\alpha +}} \quad \underset{SO(2)_{internal}}{Q_{\beta -}}$

$[Q_{\alpha +}, Q_{\beta -}] = \gamma^{\mu}_{\alpha\beta} P_{\mu}$
$[J, Q_{\pm\alpha}] = \pm\frac{1}{2} Q_{\pm\alpha} \qquad [R, Q_{\alpha\pm}] = \pm\frac{1}{2} Q_{\alpha\pm}$
$Q = Q_{1+} + iQ_{+2} + Q_{-1} \pm iQ_{-2}$ ⇒ $Q^2 = 0$
spinor, not scalar $\quad$ for BRST need scalar
topological twist $\quad J' = J + R$ ⇒ Q scalar, $\tilde{T} = T_{\mu\nu} - \partial_{\mu} T_{\nu}$

Physical states

$Q\psi=0 \quad \varphi\sim\varphi' \Leftrightarrow \varphi-\varphi'=Q\chi$

Physical ops

$[Q,\mathcal{O}]=0 \quad \mathcal{O}\sim\mathcal{O}' \Leftrightarrow \mathcal{O}-\mathcal{O}'=\{Q,\nu\}$

$\delta_{g_{\mu\nu}} \langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle = \dots = 0$ metric independent

$\rightarrow$ topological theory

$\underline{\mathbb{CP}^{N-1} \text{ model}}$ $\quad z^I: \Sigma\rightarrow X=\mathbb{C}P^{N-1}$

$$I=\int d^2x\sqrt{\gamma}\left(\gamma^{\alpha\beta} g_{I\bar J}\frac{Dz^I}{Dx^\alpha}\frac{Dz^{\bar J}}{Dx^\beta} + \underset{\text{Lagr. mult.}}{D}\underbrace{(g_{I\bar J}z^I\bar z^{\bar J}-r)}_{\text{constraint}}\right)$$

$\mathbb{C}P^{N-1}: z^I\rightarrow\lambda z^I$

$\mathbb{CP}^{N-1|P}$ model

$(z^I,\psi) \quad I:1,..,M, A:1,..,P \quad U(1): \psi^A\rightarrow e^{i\alpha}\psi^A$

If X Kähler $\rightarrow$ $\mathcal{N}=2$ supersymmetry

Twist

R-symmetries $\quad (x,\theta,\bar\theta)$

Axial: $\theta^+\rightarrow e^{i\gamma}\theta^+ \quad \theta^-\rightarrow e^{-i\gamma}\theta^-$ } leads to B-model

Vector: $\theta^\alpha\rightarrow e^{i\gamma}\theta^\alpha$ } A-model

Why choose B?

A.. on any P

B.. only allows $P=M$ $\quad (\Rightarrow \mathbb{C}P^{3|4}\sim\mathcal{N}=4$ SYM$)$

classically R_{Axial} is symmetry. QM R_A is anomalous unless X is CY

$(\Leftarrow \psi\rightarrow e^{i\alpha}\psi \Rightarrow d\psi\rightarrow e^{-i\alpha}\psi$ for Gr. variables$)$

in B-model $\quad Q\sim\bar\partial \Rightarrow$ physical states $\sim$ cohomology

$(0,p)$-forms

Gauge group $\quad U(N)$

add N D5-branes wrapping

$\mathbb{C}P^3$ and holomorphic fermionic directions

it turns $\quad (0,p)$ for $p>1$ are ghosts for $(0,1)$-fo[rms]

Effective action

$$I_{Bos}=\int_X \Omega_{Bos}\wedge tr\left(A\bar\partial A+\tfrac{2}{3}A\wedge A\wedge A\right)$$

holomorphic Chern-Simons action

$\rightarrow$ SUSY version $\quad \mathcal{A}=d\bar z\,(A+\psi\chi+\psi\psi\phi+ +\varepsilon\,\psi\psi\psi\tilde\chi+\psi\psi\psi\psi\,G)$

$$I=\int\Omega_{SUSY}\wedge tr\left(\mathcal{A}\bar\partial\mathcal{A}+\tfrac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right)$$

ψ^A sections of $\mathcal{O}(1) \quad \mathcal{A}_{\bar z}\sim\mathcal{O}(0)$

	A	χ	ϕ	$\tilde\chi$	G
in	$\mathcal{O}(0)$	$\mathcal{O}(-1)$	$\mathcal{O}(-2)$	$\mathcal{O}(-3)$	$\mathcal{O}(-4)$
S:	0	-1	-2	-3	-4

Observables: correlation fcns $\quad \Omega_{Bos}, \tilde A, \vartheta$

amplitude in vec. space $\quad$ wave fcn

$\Omega\sim\mathcal{O}(4) \quad \tilde A\sim\mathcal{O}(-2-2h) \quad \vartheta\sim\mathcal{O}(\ell)$

$\Rightarrow \quad 4+(-2-2h)+\ell=0 \quad \ell$ implies helicity

$\Rightarrow \mathcal{N}=4$ super multiplet, correct R-charges under SU

but action has $S=-4 \Rightarrow$ doesn't give $\in I_{-8}$ piece

D-instantons

- Wrap holomorphic curves, have [illegible]

conjecture $S = -4(d+1)^2 - g)$ ← genus

MHV amplitudes from string theory

D1D5 strings $I_{D1-D5} = \int \beta (\bar{\partial} + A) \alpha$

β: D5-D1, transforms under $\bar{N}$; α: D1-D5, N; A: external gauge field (0,1)-form

$\Rightarrow A = \int \langle V_{\phi_1} \dots V_{\phi_n} \rangle$

altogether A_{MHV} can be expressed from String theory

Argyres: $\boxed{\text{SUSY gauge theory 1}}$

- $d=4$, non-perturbative

Outline

I. Non-renormalization theorems

II. $N=1$ SQCD : Seiberg duality, SCFT

time permitting:

III. $N=2$ SQCD : Seiberg-Witten theory, BPS states, supercharge

IV. $N=4$ SQCD : S-duality

Note: in QCD one has to guess low-energy behaviour respecting symmetries (because QCD strongly coupled)
→ there will be some amount of guesswork
then checking self-consistency

Non-renormalization theorems

A. $N=1$ review B. Holomorphy $\int d^2\theta\, W$

C. NR thm for chiral flds D. Review of YM

E. —"— for vector " F. Exactly marginal operators

A. $N=1$ review

chiral SF $\Phi(x,\theta) \sim \phi(x) + \theta\cdot\psi + \dots$ (complex, Weyl) $\bar{D}_{\dot\alpha}\Phi = 0$

vector SF, WZ gauge $V(x,\theta,\bar\theta) \sim \bar\theta\sigma^\mu\theta A_\mu + \bar\theta\theta\theta\lambda + \dots$ $(V = \bar{V})$

chiral field strength $W_\alpha(x,\theta) \sim \lambda_\alpha + (\sigma^{\mu\nu}\theta)_\alpha \mathcal{F}_{\mu\nu} + \dots$

$\left(W_\alpha = -\frac{1}{4}\bar{D}^2 e^{-V} D_\alpha e^V\right)$ $\mathcal{F}_{\mu\nu} = \frac{1}{2}(F_{\mu\nu} + i\tilde{F}_{\mu\nu})$, $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$

$\Rightarrow$ e.g. gauge kinetic term $\frac{i\tau}{16\pi}(\mathcal{F})^2 + c.c.$ $\tau = \frac{\theta}{2\pi} + i\dots$

gauge invariance

$e^{-V} \to e^{-i\bar\Lambda} e^{-V} e^{i\Lambda}$, $W_\alpha \to e^{-i\Lambda} W_\alpha e^{i\Lambda}$

SUSY Bianchi $D^\alpha W_\alpha = \bar{D}_{\dot\alpha}\bar{W}^{\dot\alpha}$

SUSY gauge ⇓ in action $-\frac{1}{4g^2}\mathrm{tr}F^2 + \dots$

$N=1$ SQCD $\mathcal{L}_{SQCD} = \int d^4\theta \left(\bar{Q}^i e^{-V} Q_i + \bar{\tilde{Q}}^i e^V \tilde{Q}^i\right)$

$+ \left(\int d^2\theta \frac{i\tau}{16\pi} \mathrm{tr}(W^2) + c.c.\right)$

gauge indices Q_{ia} "quark" χSF

$a = 1..N_c$ colour

$i = 1..N_f$ flavour .. N_f irrep (fund

$\tilde{Q}^{ia}$ "antiquarks" $\bar{N}_f$ irrep (antifundame

$(W_\alpha)^a{}_b$ adjoint repr. $(W_\alpha)^a{}_a = 0$

B. <u>Selection rules for superpotential terms</u>

<u>holomorphy of $\mathcal{W}$</u>

adding mass terms

$$\delta\mathcal{L}_{SQCD} = \int d^2\theta \; m_i Q_i \tilde{Q}^i + c.c.$$

implicitly contracting gauge indices a, b

<u>low-energy eff. action</u> – what conditions on it arise from $\mathcal{L}_{SQCD}$?

$$\mathcal{L}_{eff} = \int d^4\theta \; \mathcal{K}_{eff}(\Phi, \bar{\Phi}, m, \tau, \bar{m}, \bar{\tau})$$

$$+ \int d^2\theta \left[\underset{\text{chiral} \Rightarrow \text{no } \bar{\Phi}}{W_{eff}}(\Phi, m, \tau, \bar{m}, \bar{\tau}) + \tau_{eff}(\Phi, \tau, \bar{\tau}, m, \bar{m}) \, tr(W^2) \right] + c.c.$$

+ higher derivative terms

Note: <u>"generalized superpotential"</u> – anything integrated over $d^2\theta$, e.g. mass + kinetic term

One may consider τ a chiral SF (that might spoil renorm. but not SUSY) $\to \mathcal{W}_{eff}(\Phi, m, \tau)$, $\tau_{eff}(\Phi, \tau, m)$ ($\Leftarrow$ that's a guess about τ)

C. <u>χSF NR thm</u>

global R-symmetry: Q_α transforms in a nontrivial rep. of R

$N=1$ SUSY $U(1)_R$: $Q_\alpha \to e^{-i\varepsilon} Q_\alpha$, $\bar{Q}_{\dot\alpha} \to e^{i\varepsilon} \bar{Q}_{\dot\alpha}$

$\Rightarrow [R, Q_\alpha] = -Q_\alpha \qquad [R, \bar{Q}_{\dot\alpha}] = \bar{Q}_{\dot\alpha}$

R-charge of $Q_\alpha = -1 \Rightarrow Q_\alpha \sim \frac{\partial}{\partial\theta^\alpha} + \ldots \Rightarrow$ R-charge of $\frac{\partial}{\partial\theta^\alpha}$ is -1

R-charge of $d\theta$ is $R(d\theta) = 1 \Rightarrow$ to have $\mathcal{L}$ invariant we need $R(\mathcal{W}) = 2$, $R(\mathcal{K}) = 0$

Note: $R(\Phi) = r \qquad \Phi = \phi + \theta \cdot \psi + \ldots \Rightarrow \underbrace{R(\phi) = r, \; R(\psi) = r-1}_{\phi \to e^{ir\varepsilon}\phi, \; \psi_\alpha \to e^{i(r-1)\varepsilon}}$

R-charge is not unique

$\underset{\text{generates } R}{U(1)_R}$, some other symmetries $\underset{U_i}{U(1)_i}$ not touching R-ch

$\Rightarrow R' = R + \sum_i a_i U_i$, a_i arbitrary is also R-symm.

(Weinberg II $\Rightarrow$)

UV (microscopic) effective theory $\mathcal{W}_{\mu_0} = \mathcal{W}_{\mu_0}(\Phi_n)$, μ_0 UV

$\downarrow$

?? effective superpotential @ scale $\mu < \mu_0$ $\quad \mathcal{W}_\mu = \mathcal{W}_\mu(\Phi_n) = g$

assume same set of light degrees of freedom

Let's write instead of $\mathcal{W}_{\mu_0}$ using χSF Y

$Y \cdot \mathcal{W}_{\mu_0}(\phi_n) \Rightarrow$ R-symmetry

	$U(1)_R$
Y	$+2$
Φ_n	0

$\Rightarrow \mathcal{W}_\mu = \tilde{g}(Y, \Phi_n) \underset{\uparrow}{=} Y g(\Phi_n)$

R-symm. charge should be the same, i.e. +2 and n (except Y h...)

Now let $Y \to 0$ to get free theory $\Rightarrow$ we can (up to canonical scaling) get only $\boxed{g(\Phi_n) = \mathcal{W}_{\mu_0}(\Phi_n)}$

$\Rightarrow$ putting back $Y = 1$ we get

$$\mathcal{W}_\mu = \mathcal{W}_{\mu_0}(\Phi_n) \quad \text{as long as light degrees of freedom don't change.}$$

But higher-derivative ~~action~~ parts in $\mathcal{L}_{eff}$ may look like $\int d^2\theta f[\Phi, \bar{\Phi}$

and are renormalized, non-renormalizability applies only to leading superpotential term.

Wave fcn renormalization

$\mathcal{L}_{\mu_0} = \int d^4\theta \; Z_n \Phi_n \bar{\Phi}_n + \int d^2\theta \; W_{\mu_0}$ coupling

under assumption $W_{\mu_0} = \lambda \mathcal{O}$, $\mathcal{O} = \prod_n \phi_n^{r_n}$

scaling $\Delta = \sum_n r_n$

$\Rightarrow \boxed{W_{\mu} = \left(\frac{\mu_0}{\mu}\right)^{3-\Delta} \lambda \mathcal{O}}$

$[X] = -1$, $[\theta] = -\frac{1}{2}$, $[\frac{\partial}{\partial\theta}] = [d\theta] = \frac{1}{2}$

$[d^4x \, d^2\theta] = -3$, $[\mathcal{L}] = 0$

might be μ/μ_0 and similarly in RGE

~~due to wave fcn renormalization~~

rescale $\phi_n \to \phi_n^{can} = \sqrt{Z_n}\,\phi_n$ to get canonical normalization

$$W_\mu = \underbrace{\left(\frac{\mu_0}{\mu}\right)^{3-\Delta_n}\left(\prod_n Z_n^{-r_n/2}\right)\lambda}_{\lambda_n^{can}(\mu)} \mathcal{O}_n^{can}$$

$$\Rightarrow \boxed{\mu \frac{d\lambda_r^{can}}{d\mu} = \lambda_r^{can}(\mu)\left(\Delta - 3 - \tfrac{1}{2}\textstyle\sum r_n \gamma_n\right)}$$

where $\gamma_n = \frac{-d\ln Z_n}{d\ln\mu}$ RGE

(but we don't know how to compute γ) $\Rightarrow$ useless

An example theory with 2 χSFs

- UV $\mathcal{K}_{\mu_0} = \bar{\Phi}_1\Phi_1 + \bar{\Phi}_2\Phi_2$ $\quad W_{\mu_0} = \lambda \Phi_1 \Phi_2^2$

classically $\Rightarrow$ moduli space (space of vacua) $\mathcal{M}_{cl} = \{\phi_2 = 0, \phi_1 \text{ arbitrary}\} \cong \mathbb{C}$

(ϕ_1 flat direction) everywhere ϕ_1 massless

ϕ_2 has mass $\sim |\lambda \phi_1|$

Kähler term $\mathcal{K} \Rightarrow$ metric on $\mathcal{M}_{cl}$ $\quad ds^2_{\mu_0} = d\phi_1 \, d\bar{\phi}_1$ (flat)

flow $\to$ IR $\mu < \mu_0$

NR thm $\Rightarrow W_\mu = W_{\mu_0} \Rightarrow \mathcal{M}_{quantum} \simeq \mathcal{M}_{cl} \simeq \mathbb{C}$

1-loop renormalization of $\mathcal{K}_\mu$ (effective for Φ_1, integrating out Φ_2)

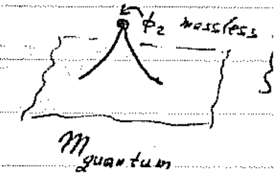
if $\mu <$ mass $\phi_2 < \mu_0$

$\Rightarrow$

$\mathcal{K}_\mu = \bar{\Phi}_1\Phi_1 - (\#)\,\bar{\Phi}_1\Phi_1 |\lambda|^2 \ln\left(\frac{\Phi_1}{\mu_0}\right)^2$ at 1-loop

$ds^2_\mu = \left(\partial_{\phi_1}\partial_{\bar{\phi}_1}\mathcal{K}_\mu\right) d\phi_1 \, d\bar{\phi}_1 = \left(-|\lambda|^2 \ln(\phi_1\bar{\phi}_1) + \text{const.}\right) d\phi_1$

cusp-like singularity at $\phi_1 = 0$



Singularity signals missing light degree of freedom – in this case ϕ_2 – approx. is invalid near $\phi_1 = \phi_2 = 0$

ϕ_2 cannot be integrated out when light!

Advanced AdS/CFT 3

Balasubramanian: Holography & Quantum foam 1

a.k.a AdS/CFT

Holographic principle

"In quantum theory of gravity the number of degrees of freedom scales with surface area, not volume."

$\Leftarrow$ Black holes thermodynamics

$$ds^2 = -\left(1-\frac{Gm}{r}\right)dt^2 + (\;)^{-1}dr^2 + r^2 d\Omega$$

BH: $dm = \kappa\, dA + \dots$ | $dE = TdS + \dots$ Thermodynamics

$\Delta A \geq 0$ | $\Delta S \geq 0$

κ const. on a horizon | T constant in equilibrium

$S_{BH} = \dfrac{A}{4G_N}$

$\Rightarrow$ suggestion $T \sim \kappa$

$\underline{S \sim A}$

is a fundamental property of gravity, not only BH

Generalization

(1) Rindler space has associated S

(2) de Sitter space $ds^2 = -\left(1-\frac{r^2}{\ell^2}\right)dt^2 + (\;)^{-1}dr^2 + r^2 d\Omega^2$

(3) Bousso: covariant bound.

$\dfrac{A}{4G_N} \geqslant$ total entropy passing through light sheet

(4) Inertial observers in SR + equivalence principle

$\Rightarrow G_{\mu\nu} = T_{\mu\nu} \Rightarrow$ Black hole entropy

Turn it around: instead families of locally acc. observers, assume $\delta E = T\delta S$

$\delta S = \dfrac{\delta A}{4G_N}$, A area of horizon, T local Unruh temp

$+ \dfrac{d\theta}{d\lambda} = R_{\mu\nu} k^\mu k^\nu \Rightarrow$ Einstein eqn. $G_{\mu\nu} = T_{\mu\nu}$

$\Rightarrow$ Laws of horizon thermodynamics

$\Leftrightarrow$ equation of motion

What it means? Gravity & spacetime are effective emerging concepts?

What are then atoms & microstates of smooth & singular spacetime geometries?

Plan: (1) AdS/CFT generalities & techniques

(2) Black holes in AdS

(3) SUSY ($\frac{1}{2}$ BPS) states of the theory

(4) Quantum foam

(1) AdS/CFT

Global AdS spacetimes

$$ds^2 = -\left(1+\frac{r^2}{\ell^2}\right)dt^2 + (\;)^{-1}dr^2 + r^2 d\Omega^2$$
$$= -\sec^2\rho\, dt^2 + \ell^2 d\rho^2 + \ell^2 \tan^2\rho\, d\Omega^2$$

Poincaré patch

$$ds^2 = \frac{-dt^2 + d\vec{x}^2 + d\xi^2}{\xi^2} \qquad \xi \to 0 \Rightarrow \text{boundary plane}$$

In string theory $AdS \times M = {}^{10d}_{11d}$

e.g. IIB $\Rightarrow AdS_5 \times S^5$ $\quad AdS_3 \times S^3 \times T^4$

M-theory $\Rightarrow AdS_4 \times S^7$ $\quad AdS_7 \times S^4$

Orbifold can replace S^d etc. – other generalizations poss

AdS/CFT correspondence

Field theory of strings $\int \mathcal{D}\phi\, e^{iS(\phi)} = \left\langle e^{i\int_{\partial M} \phi_0 \mathcal{O}} \right\rangle$

$\phi \to \phi_0$ on boundary

field

field $\phi \Leftrightarrow \mathcal{O}$ operator
boundary cond. $\phi_0 \Leftrightarrow \phi_0$ source
isometries $\Leftrightarrow$ global symmetries

Ex: $M = AdS_5 \times S^5 \Longleftrightarrow \mathcal{N}=4$ superconformal $SU(N)$
$SO(4,2) \times SO(6)$ gauge theory on ∂M

AdS $-u^2 - v^2 + x_1^2 + x_2^2 + \dots + x_d^2 = \ell^2$ has $SO(d,2)$ symm.

$\ell = (g_{YM}^2 \cdot N)^{1/4} \ell_s \qquad g_s = g_{YM}^2$
$g_s N$ 't Hooft coupling

Semiclassical limit $\frac{\ell}{\ell_p} \to \infty \quad \left(\sim \begin{matrix} \hbar \to 0 \\ \text{or} \\ N \to \infty \end{matrix} \right)$

saddle points of $S(\phi) \sim$ effective action
$e^{i S_{cl}(\phi_0)} = e^{i W[\phi]}$

$$\Rightarrow \frac{1}{n!} \frac{\delta^n S_{cl}(\phi_0)}{\delta \phi_0^n} \Big|_{\phi_0 = 0} = \langle \mathcal{O} \dots \mathcal{O} \rangle$$

Lorentzian signature: states, sources, S-matrices
many sol's approaching ϕ_0 at $\infty \Rightarrow$ there are many states

Ex: ϕ scalar of mass m $\quad \Box \phi = m^2 \phi$
Poincaré coords $ds^2 = \frac{\ell^2}{\xi^2}(-dt^2 + d\vec{x}^2 + d\xi^2)$
$\phi = e^{-i\omega t + i\vec{k}\vec{x}} \chi$

$$\xi^2 \partial_\xi^2 \chi + \xi \partial_\xi \chi - [(m^2 + \tfrac{d^2}{4}) + (k^2 - \omega^2)\xi^2] \chi = 0$$

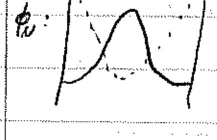
$$\Rightarrow \phi = e^{i\vec{k}\vec{x} - i\omega t} \xi^{d/2} [A J_\nu(|q|\xi) + \phi_0 Y_\nu(|q|\xi)]$$

$q^2 = \vec{k}^2 - \omega^2 \qquad |\xi| \to \infty \quad \phi \to A \xi^{2h_+} e^{ikx - i\omega t} + \phi_0 \xi^{2h_-} e^{\dots}$
$= \phi_N + \phi_{NN}$

$$2h_\pm = \frac{d}{2} \pm \frac{\sqrt{d^2 + 4m^2\ell^2}}{2} = \frac{d}{2} \pm \nu$$

ϕ_N have finite norm, can be quantized
ϕ_{NN} don't have —"— $\Rightarrow$ background

AdS.. effectively potential well for particles

ϕ_N:  effect of $\phi_{NN} \xrightarrow{\phi_0} S_{CFT} \to S_{CFT} + \int \phi_0 \mathcal{O}$
$\phi_N \Leftrightarrow$ state in CFT $\quad \mathcal{O}|0\rangle$
conformal dim $(\mathcal{O}) = \Delta = 2h_+$

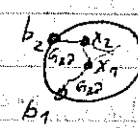
Note: CFT on Eucl. plane $ds^2 = dr^2 + r^2 d\theta^2$
$\mathcal{O}$ at origin, integrat[e] around

to get Lorentz $\quad r = e^t \Rightarrow ds^2 = e^{2t}(dt^2 + d\theta^2)$
$\to$ conf. transf $\quad dt^2 + d\theta^2$, Wick rotation
$\Rightarrow \Delta \sim r\partial_r \to \frac{\partial}{\partial t} = H$

Also: vacuum $|0\rangle$ is unique in AdS (the only state resp. all symmetr[ies])


prop. in from ∂M
$G_{B\partial}$
prop. up G_F

$G_F(b, \xi, b', \xi')$ Feynman propagator
$\lim_{\xi \to 0} \xi^{-2h_+} G_F(b, \xi, b', \xi') = G_{B\partial}$
Bulk-boundary propagator


inside bulk ordinary Feyn. diagram
$\langle \mathcal{O}(b_1) \mathcal{O}(b_2) \dots \mathcal{O}(b_n) \rangle =$

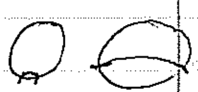
$$\int \left[G_{B\partial}[b_1, x_1]\; G_{B\partial}[b_2, x_2] \cdots G_F(x_1, x_2) \ldots \text{vertices} \ldots \right]$$

local info about spacetime hard to get

E.g. $\langle \mathcal{O}(b_1)\mathcal{O}(b_2)\rangle \sim \lim_{\xi, \xi' \to 0} G_F(b, \xi, b', \xi')$

$$\sim \int \mathcal{D}p\, e^{i\frac{m}{\hbar}\ell(p)} \sim \sum_g e^{i\frac{m}{\hbar}\ell_g}$$

geodesics



$\Rightarrow$ IR physics probes deep interior

Vacuum, states & transitions

$\langle s' | \mathcal{O} \cdot \mathcal{O} | s \rangle \qquad |s\rangle \sim \mathcal{O}_s(t \to -\infty)|0\rangle$

but with subtleties

Spectroscopy of AdS

$AdS_{d+1} \quad SO(d,2) \qquad \Box\phi = m^2\phi$ in irreps of $SO(d,2)$

$AdS_3 \qquad -u^2 - v^2 + x^2 + y^2 = \ell^2$

metric induced from $ds^2 = -du^2 - dv^2 + dx^2 + dy^2$

$\Rightarrow ds^2 = \ell^2[-\cosh^2\mu\, dt^2 + d\mu^2 + \sinh^2\mu\, d\theta^2]$

$SO(2,2) = SL(2,\mathbb{R})_R \times SL(2,\mathbb{R})_L$ R, L just indices

dual .. 2-dimensional CFT $\rightarrow$ conf. group is $SO(2,2)$ cp. with

$SL(2,\mathbb{R}) \qquad L_0, L_+, L_- \qquad [L_0, L_\pm] = \mp L_\pm, \; [L_+, L_-] = 2L_0$

Casimir $L^2 = \frac{1}{2}(L_+L_- + L_-L_+) - L_0^2$

$\{L_0, L_+, L_-\}_R$ as generators of isometry

Then $\Box\phi = m^2\phi \Longleftrightarrow -2(L_L^2 + L_R^2)\phi = m^2\ell^2\phi$

$L_0\phi = h\phi \qquad h$ conformal weight of ϕ

First highest weight state $L_{+L}\phi = L_{+R}\phi = 0$

$\Rightarrow L_{0L}\phi = L_{0R}\phi = h\phi$ and $h(h-1) = \frac{m^2\ell^2}{4}$ some ... here, some fa... of 2 ...

$\Rightarrow 2h_\pm = \frac{d}{2} \pm \frac{\sqrt{d^2 + 4m^2\ell^2}}{2}$ with $d = 2$

$\Rightarrow$

$$\phi_h^\pm = e^{i(2h_\pm)t} \frac{1}{(\cosh\mu)^{2h_\pm}}$$

Full basis $(L_{-L})^p (L_{-R})^q \phi_h = \phi_h^{p,q}$

Kane: $\boxed{\text{SUSY \& real world 1}}$

Why beyond SM? baryon asymmetry, dark matter, (mostly nonbar...) ν masses (new scale needed)

flavour – SM, MSSM can accomodate them but don't explain it, string theory might – may turn out to be probe on string theory

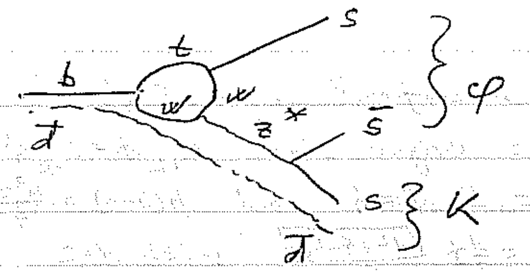
$\sin 2\beta$

recently $B \to \psi K \qquad 0{,}726 \pm 0{,}037$

but from $B \to \varphi K, \eta' K$ should be $0{,}43 \pm 0.0\ldots$

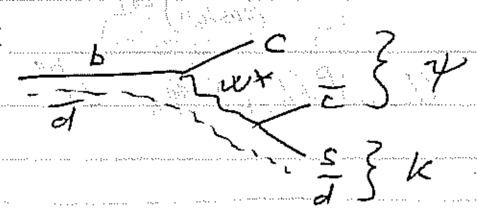
$\rightarrow$ if holds with improved data then 1st significant deviation from SM unexplicable by cosmology or short-distance (HE) physics

φK



sensitive to light superpartner loops

whereas ψK



(CP violating diagram)

What SUSY does/doesn't explain?

- assume soft masses ~ weak scale
  - $\longrightarrow$ stable hierarchy m_W/m_{Pl}
  - $\longrightarrow$ breaking of $SU(2)\times U(1)$ (on most of param. space)
  - $\longrightarrow$ gauge coupling unification (~~assuming~~ suggests theory can be described perturbatively up to unif. scale)

Problems in MSSM fine tuning

e.g. Λ not $\sim m^2_{susy}, m^2_{Pl}$

inflation

$\Omega_{DE} \sim \Omega_{DM} \sim \Omega_B$

superpartner masses

etc.

hep-ph/9709356

hep-ph/0312378 SUSY breaking

SUSY gauge theory 2

D. Vector multiplet, YM semi-classical gauge theory

$$\mathcal{L} = \int d^2\theta \, \frac{i\tau}{16\pi} \, tr(W^2) = -\frac{1}{4g^2} tr F^2 + \frac{\theta}{32\pi} tr(F\tilde{F})$$

1-loop running $\frac{1}{g^2(\mu)} = -\frac{b}{8\pi^2} \ln\left(\frac{|\Lambda|}{\mu}\right)^2$ ← strong coupling sc

where $b = \frac{11}{6} T(adj) - \frac{1}{3}\sum_a T(R_a) - \frac{1}{12}\sum_i T(R_i)$

in general — Weyl fermions — real scalars

$= \frac{3}{2} T(adj) - \frac{1}{2}\sum_i T(R_i)$

SUSY (SF formulation) — χSF

Normalization $SU(N_c)$ $T(N_c) = 1$, $T(adjoint) = 2$

$$\tau = \frac{\theta}{2\pi} - i\frac{4\pi}{g^2} = \frac{b}{2\pi i} \ln\left(\frac{\Lambda}{\mu}\right), \text{ where } \Lambda = |\Lambda| e^{i\theta/b}$$

chiral anomaly $U(1)_i$: chiral symm. $\underbrace{U_i[\psi_j]}_{\text{charge of } \psi_j \text{ under } U(1)_i} = \delta_{ij}$ $\psi_i \to e$

$\Rightarrow \theta \to \theta + \epsilon T(R_i)$ under $U(1)_i$

i.e. $U_i(\Lambda^b) = T(R_i)$

R-symmetry of $\mathcal{N}=1$ SQCD

	R	
Φ_i	2/3	$\Rightarrow R[\phi_i] = 2/3, R[\psi_i$
W_α	1	$\Rightarrow R[\lambda_\alpha] = 1$, $R[F_{\mu\nu}] = 0$

R-charge of $b = \frac{3}{2}(Tr(adj) - \frac{1}{3}\sum Tr(R_i))$

$\Rightarrow$ R-charge of Λ is $\frac{2}{3}$, i.e. $R[\Lambda] = 2/3$

scaling $D[\Lambda] = 1$

(In general SCFT $\Rightarrow D[\phi] = \frac{3}{2} R[\phi]$ $\leftarrow$ cSF)

E. ~~Non-renorm.~~ "Non-very-much renorm. thms"

UV scale $\mu_0 >> \Lambda$ (weak coupling)

$\downarrow$

IR scale $\mu < \mu_0$ $(\mu >> \Lambda)$

How does $\tau_{eff}(\mu, \Lambda, \phi_i) \equiv \tau_\mu(\Lambda, \phi_i)$ change with μ?

$\theta \simeq \theta + 2\pi \Rightarrow \tau_{\mu_0} \simeq \tau_{\mu_0} + 1$ the same physics

$\Rightarrow$ since disrete sym. cannot change continously $\tau_\mu \simeq \tau_\mu + 1$, since $\Lambda^b \to e^{2\pi i}\Lambda^b$

$\Rightarrow \tau_\mu = \frac{b}{2\pi i} \ln\left(\frac{\Lambda}{\mu}\right) + \underbrace{f(\Lambda^b, \phi_i)}_{\text{single valued in } \Lambda^b}$

invariant under $\theta \to \theta + 2\pi$

$\Lambda \to 0$ (weak coupling limit at fixed scale μ)

$\Rightarrow f$ involves only positive ($f \to 0$ as coupling $\to 0$) & integer (single valued) powers of Λ^b

$\Rightarrow f = \underbrace{\sum_{n=1}^{\infty} \Lambda^{nb} c_n(\phi_i)}_{\text{sum of instanton terms } \sim e^{-n\frac{8\pi^2}{g^2}} \text{ as } g \to 0}$

mass

Note: (SUSY $\to$ no $\bar{\Lambda}$, only $\Lambda \to$ argument works)

Now $W_\mu = W_\mu(\phi_i, \Lambda) = W_{\mu_0}(\phi_i) + \sum_{n=1}^{\infty} \Lambda^{nb} \omega_n(\phi_i)$

~~still~~ up to canonical rescaling

$\Rightarrow W_\mu$ not renormalized but may pick up instanton, i.e. nonperturbative, corrections

$\mathcal{L}_\mu = \int d^4\theta Z_i \bar{\phi}_i e^V \phi_i - \frac{1}{4}\int d^2\theta \left(\frac{1}{g^2_{(\mu)}} - \frac{i\theta}{8\pi^2}\right) tr(W^2) + \ldots +$

$\frac{dg^{-2}}{d\ln\mu} = \frac{b}{8\pi^2}$

instanton ~~fields~~ contribution hidden in ... as interacti

$\phi_i \to \phi_i^{can} = \sqrt{Z_i}\,\phi_i = e^{(-i\ldots\ln\sqrt{Z_i})}\phi_i \Rightarrow$

$\frac{1}{g^2} \to \frac{1}{g^2_{can}} = \frac{1}{g^2} - \sum_i \frac{\ln\sqrt{Z_i}\, T(R_i)}{8\pi^2}$

"holomorphic" "physical" coupling

$\Rightarrow$ RGE for physical coupling

some tricky

$\beta_g^{can} = \frac{d g_{can}^{-2}}{d\ln\mu} = \frac{b}{8\pi^2}\left(b + \frac{1}{2}\sum_i \gamma_i\, Tr(R_i)\right) f(g_{can})$

where $\gamma_i = -\frac{d\ln Z_i}{d\ln\mu}$ exact β-fcn result N

$\mathcal{N}=1$ $\beta_g \sim \frac{3}{2} T(adj) + \frac{1}{2}\sum_i (\gamma_i - 1)\, T(R_i)$

$\beta_\lambda \sim -3 + \sum_i r_i (1 + \frac{1}{2}\gamma_i)$ $\leftarrow$ superpotentia coupling for s

F. Leigh-Strassler $\to$ exactly marginal operators

Ex: $SU(N_c)$ with $2N_c$ Q^i in N_c, $2N_c$ $\tilde{Q}_i$ in $\overline{N_c}$

$1\ \phi$ in adjoint (all χSF)

$W = \lambda \sum_i tr(\tilde{Q}_i \phi Q^i)$

a.

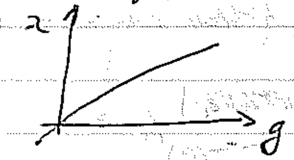
global symmetries of W: $U_i(2N_c)$ acting on ⓘ index ~~gauge it~~

$$\gamma_{Q_i} = \gamma_{\tilde{Q}_i} \equiv \gamma_Q$$

$$\beta_g \sim \frac{3}{2}\, 2N_c + \frac{1}{2} 2N_c (\gamma_\phi - 1) + \frac{1}{2} 4N_c (\gamma_Q - 1)$$

$$\sim \gamma_\phi + 2\gamma_Q \quad \leftarrow \text{gauged } SU(N_c)$$

$$\beta_\lambda \sim \gamma_\phi + 2\gamma_Q \quad \leftarrow \text{thanks to choice of } W$$



$$\beta_g = \beta_\lambda = 0 \qquad \lambda = 4\sqrt{2}\, g$$

1-complex parameter family of fixed point theories

Ex: $\mathcal{N}=1$ SQCD – let's start new part :-)

II. $\mathcal{N}=1$ SQCD, Seiberg duality

χSF: N_f Q^i in N_c, N_f $\tilde{Q}_i$ in $\overline{N}_c$ $\Rightarrow$ global symm. $SU(N_f) \times SU(N_f)$ $\Rightarrow \gamma_{Q_i} = \gamma_{\tilde{Q}_j} = \gamma$

no superpotential

$$\beta_g \propto 3N_c - N_f + N_f \gamma$$

$$N_f < 3N_c \quad \text{asympt. free}$$

$$D[Q^i] = \underset{\text{canonical}}{1} + \underset{\text{anomalous}}{\frac{1}{2}\gamma} = \frac{3}{2} R(Q)$$

scaling dimension

At a fixed point

$$\sum_i \left(R(Q_i) + R(\tilde{Q}^i)\right) = 2(N_f - N_c)$$

R-symmetry charges are anomaly free under following assignment

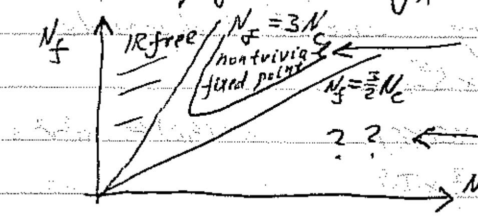
	R
Q_i	$1 - \frac{N_c}{N_f}$
$\tilde{Q}^i$	$1 - \frac{N_c}{N_f}$
W_α	1
Ω	0

Arguments for $\exists$ of nontrivial fix point:

1) consistent, non-anomalous assignment of R

2) $\beta_g \simeq (3N_c - N_f) - g^2\, \mathcal{O}(N_c^2) + \dots$ (for large N_f)

fixed point $\beta_g = 0 \Rightarrow g_*^2 = \frac{3N_c - N_f}{N_c^3}$



nontrivial fixed point means SCFT

$$M^i_j = Q^i_a \tilde{Q}^a_j \quad \text{"meson field"}$$

$$R(M) = \frac{2(N_f - N_c)}{N_f} = D(M) \Rightarrow D(M) < 1 \text{ when } N_f < \frac{3}{2} N_c$$

violates unitarity at fixed point

– something wrong for $N_f < \frac{3}{2} N_c$

Note: $N_f \to 0$ from lattice calculations seems to have confinement

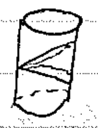
$\boxed{\text{SUSY \& real world 2}}$

Holography & Quantum foam 2

AdS / CFT : Basics repeated

$$ds^2 = -\sec^2\xi \, dt^2 + \ell^2 d\xi^2 + \ell^2 \tan^2\xi \, d\Omega^2$$

geodesics bounce back and forth, boundary $\sim S^{d-1} \times R$



Poincaré patch – suitable coords

$$ds^2 = \ell^2 \frac{-dt^2 + d\vec{x}^2 + d\xi^2}{\xi^2} \qquad \xi \to 0 \text{ boundary} \to R^{d-1,1}$$

In string theory suitable backgrounds

IIB $\quad AdS_5 \times S^5_M$

isometry $\quad SO(4,2) \times SO(6) \qquad SO(4,2) \Leftarrow -u^2 - v^2 + x_1^2 + \dots + x_4^2 = -\ell^2$

$\qquad AdS_3 \times S^3 \times T^4$

Duality – difficult to state in string theory

– in the field theory limit (in $D = 10$) bulk theory

AdS_{d+1} $= \dim M = d+1$ with isometries $SO(d,2)$

$$\int_{\phi \to J \text{ on } \partial M} \mathcal{D}\phi \, e^{iS(\phi)} = \left\langle e^{i\int_{\partial M} J\mathcal{O}} \right\rangle_{\text{vacuum}}$$

conformal field theory with conf. group $SO(d,2)$

boundary cond. play role of sources

Note : $SO(d,2)$ on both sides, in different context (isometry vs. conf.)

Note : CFT doesn't contain gravity at all.

References : hep-th/0309246, 9802150, 9902131, 990511
Witten

Where is duality coming from?

D3-branes

low-energy desription $\int d^{3+1}x \, Tr(F^2) + \dots$

$\mathcal{N} = 4$ SYM $U(N)$

Metric $\quad ds^2 = F^{-1/2}(-dt^2 + dx^2) + F^{1/2}(dr^2 + r^2 d\Omega_5^2) \qquad Q \propto$

$F = 1 + \frac{Q}{r^4} \qquad$ in IIB (class. soln. of SUGRA)

carries the same charges as D3-brane

horizon $r = 0$ near horizon limit accompanied by $Q(N)$

$$ds^2 = \frac{r^2}{\sqrt{Q}}(-dt^2 + dx^2) + \frac{\sqrt{Q}}{r^2}(dr^2 + r^2 d\Omega_5^2)$$

def. $\sqrt{Q} = \ell^2 \qquad \xi = \ell^2 / r$

$$ds^2 = \underbrace{\ell^2 d\Omega_5^2}_{\text{sphere } S^5 \text{ of radius } \ell} + \underbrace{\frac{\ell^2}{\xi^2}(-dt^2 + d\vec{x}^2 + d\xi^2)}_{\text{Poincaré patch of } AdS_5}$$

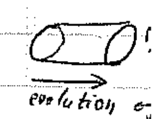
$\Rightarrow$ two diff. low energy description

$\mathcal{N} = 4$ SYM in 3+1 dim. $\qquad$ SUGRA $AdS_5 \times S^5$

both valid as $N \to \infty$ $\quad$ both valid $\Rightarrow$ should coinc

seems tobe valid even for finite N in near horizon limit

Can be considered an analog of duality between closed & open strings



Example : IIB: $AdS_5 \times S^5 \Longleftrightarrow \mathcal{N} = 4$ superconformal $SU(N)$ YM on boundary of AdS_5

scale $\ell \quad \ell$

$\ell = (g_s N)^{1/4} \ell_s$ $\qquad$ (32 fermionic SUSY generators)

$g_{YM}^2 = g_s$

$g_s N = $ 't Hooft $= \lambda \qquad \lambda$ large ($\Rightarrow$ no $1/N$ expansion)

SUGRA acts as "master field" summing ∞ sequence of Feynman diagrams

Group $SO(4,2) \times SO(6)$ gets interpretation in SCFT
conformal R-symmetry group

On string side we take semiclassical limit (usually)

$\ell/\ell_s \to \infty$ (or $\hbar \to 0$, $N \to \infty$)

$\Rightarrow \quad e^{iS_{cl}(J)} = e^{iW_{CFT}(J)}$ ← generating fcn of correlation fcns

$e^{iS_{cl}[\phi_0(\ )_{\text{on } \partial M}]} = e^{iW_{CFT}[\phi_0(\ )_{\text{on } \partial M}]}$

$S_{cl} = \int d^5x \sqrt{g} [R + \partial\phi^2 + \underbrace{\cdots}_{\text{incl. massive Kaluza-Klein}}]$

$g_{\mu\nu}$, scalars etc.

ϕ_i → modes from S^5 compactification

$S_{N=4} = \int d^4x \{ \mathrm{Tr} F^2 + \psi^2 \ldots + [D\phi_i]^2 + R(\phi^i)^2 + [\ ]\}$

to do QFT build composite operators from A_μ, ψ_i, ϕ_i

$\frac{1}{n!} \frac{\delta^n S_{cl}}{\delta \phi_0^n}\Big|_{\phi_0=0} = \underbrace{\langle \mathcal{O} \ldots \mathcal{O} \rangle}$

Lorentz & gauge inv. operators (i.e. all indices contracted)

hep-th/9802150

Fields	Composite operators $\mathcal{O}$
ϕ dilaton	$\mathrm{Tr}\, F^2$
a axion	$\mathrm{Tr}\, F\tilde{F}$
g_{ij} graviton	T_{ij} stress tensor
KK modes ϕ_i	$\mathrm{Tr}\,(F^2 X^k)$

map constructed so that it preserves charges, masses, symmetries

masses ↕ conf. dimension in CFT

Matching masses & conformal dimension

scalar field of mass m in AdS_{d+1}

$\Box\phi = m^2\phi$

Poincaré coords $ds^2 = \frac{\ell^2}{\xi^2}(-dt^2 + d\vec{x}^2 + d\xi^2)$

$\phi = e^{i\vec{k}\vec{x} - i\omega t} \chi(\xi) \Rightarrow$

$\xi^2 \partial_\xi^2 \chi + \xi \partial_\xi \chi - [(m^2 + \frac{d^2}{4}) + (\vec{k}^2 - \omega^2)\xi^2]\chi = 0$

Bessel eqn. ⇒ solution

$\phi = e^{i\vec{k}\vec{x} - i\omega t} \xi^{d/2} [A J_\nu(|q|\xi) + \phi_0 Y_\nu(|q|\xi)]$

as $\xi \to 0$

$\phi \sim e^{\cdots}(A\xi^{2h_+} + \phi_0 \xi^{2h_-})$

where $2h_\pm = \frac{d}{2} \pm \frac{\sqrt{d^2 + 4m^2\ell^2}}{2} = \frac{d}{2} \pm \nu$

for big mass $\boxed{2h_+ > 0,\ 2h_- < 0}$ for small mass needs not to be

$(\phi, \phi) = \int \sqrt{g}\, g^{tt} \phi \partial_t \phi < +\infty$ only for $\phi_0 = 0$

$\phi_0 = 0$ .. N mode $\quad A = 0$ .. NN mode

$\phi_N \leftrightarrow \mathcal{O}|0\rangle$

ϕ_{NN} as before turned on and off in computation of correlation fcns OR S_{CFT} deformed to $S_{CFT} + \int \phi_0 \mathcal{O}$

Conformal dimension Δ

2dCFT in Euclidean $ds^2 = dr^2 + r^2 d\vartheta^2$ $\quad \Delta$ = eigenvalue of $r\partial_r$

Lorentz $r = e^t \to ds^2 = \cancel{e^{2t}}_{\text{conform.}} [dt^2 + d\vartheta^2]$

$\xrightarrow[t \to it]{} (-dt^2 + d\vartheta^2)$, in higher dim. ~~cylinder~~ sphere replace

$\Rightarrow \Delta = r\partial_r \to \partial_t = H \Rightarrow$ conf. dimension = eigenvalue of H

$ds^2 = \frac{\ell^2}{\xi^2}(-dt^2 + d\vec{x}^2 + d\xi^2)$

- motion in ξ induces conf. transf.

fix $\xi = \epsilon \Rightarrow \phi_{NN} = \underset{\underset{\phi^0}{\shortparallel}}{J}\, \xi^{2h_-} [\quad]$ ϕ is regulated

Proposition: ϕ_{NN} corresp. to $S \to S + \int_{\partial M} dx \sqrt{h}\, [\phi_{NN}\, \mathcal{O}]$

$\sqrt{h}$ scales like ϵ^{-d}, $\phi_{NN} \sim \epsilon^{2h_-}$, $\mathcal{O} \sim \epsilon^{\alpha}$

What should be α?

scale invariance $-d + 2h_- + \alpha = 0 \Rightarrow \alpha = d - (\frac{d}{2} - \nu)$

i.e. $\alpha = \frac{d}{2} + \nu = 2h_+$

NN modes $\phi_{NN} + \Sigma \phi_N$ remains NN

- which one shall we pick?

$\phi \sim A J_\nu(q\xi) + J Y_\nu(q\xi) \sim J[\underbrace{\tfrac{A}{J}}_{B} J_\nu + Y_\nu]$

in general Y_ν singular after Wick rotation $t \to it$, for one special lin. comb. it is nonsingular — pick up this, Wick rotate back

$\longrightarrow$ bulk-boundary propagator $G_{B\partial}$

Exercise - repeat in global coords. on AdS

(solve for wavefcn, hypergeometric fcn, N vs NN modes)

QFT: vacuum $|0\rangle$ unique (annihilated by all symms. of AdS)

$G_F(b,\xi, b', \xi')$ (to compute Wick rotate back and forth)

$\lim_{\xi \to 0} \xi^{-2h_+} G_F(b,\xi,b',\xi') = G_{B\partial}$

$\langle \mathcal{O}(b_1) \dots \mathcal{O}(b_n) \rangle = [\dots$

2-point fcns

$\langle \mathcal{O}(b)\, \mathcal{O}(b') \rangle \sim \lim_{\substack{\xi \to 0 \\ \xi' \to 0}} G_F(b, b', \xi, \xi') \cdot \xi^{()} \xi'^{()}$

$\sim [\int \underset{\sim}{\mathcal{D}P}\, e^{i \frac{m}{\hbar} \ell(g)}]$

all paths of class. particle connecting (b,ξ), $(b',\dots)$

$\ell(g) = \int dt \sqrt{g_{\mu\nu} \partial^\mu x \partial^\nu x}$

semiclass. limit $\hbar \to 0$, i.e. $m \to \infty \Rightarrow$ effectively geodesics $\langle \mathcal{O}\mathcal{O} \rangle \sim \sum_{g \text{ geodesics}} e^{i \frac{m}{\hbar} \ell(g)}$

$\Rightarrow$ if $b \not\approx b'$ we have IR region which probes deeper into AdS

(cp.  UV vs.  IR)

Liu: $\boxed{\text{Exact solutions of SUGRA 1}}$

SUSY $\delta(\text{Fermion}) = \partial(\text{Boson})\, \epsilon$ $\delta(\text{Boson}) = \bar{\epsilon}/\text{Fermi}$

ϵ.. spinor parameter

$\boxed{\text{Holography \& Quantum foam 3}}$

Exercise in AdS_3 global show $\langle \mathcal{O}(\theta', t)\, \mathcal{O}(\theta, t) \rangle \propto \frac{1}{\sin(\theta - \theta')^2}$

$\mathcal{O}$ dual to scalar field ϕ of mass M

using geodesics approximation using gr-qc/9809087, 9906226

2-point fcn using action techniques

$S = \int d^5x \sqrt{g}\,(\partial_\mu\phi\partial^\mu\phi) + m^2\phi$

$\phi = \phi_0\, \xi^{2h_-}(\ldots)$ subst. into S

$\Rightarrow$ bulk term vanishes due to EoM

$S = \int_{\partial M} \phi\,\partial_\xi\phi = \int_{\partial M} (\phi_0)^2\,\xi^{4h_-}(\ldots) \xrightarrow[\xi\to 0]{} \infty$

i.e. $\left.\frac{\delta S^2}{\delta\phi_0^2}\right|_{\phi_0=0} = \infty$ Why? like in ordinary CFT $\langle\phi(x)\phi(y)\rangle \sim \frac{1}{|x-y|^{2\Delta}}$

Solution:

Method 1: Normalize $G_{B\partial} \xrightarrow[\xi\to\epsilon]{} 1$, calculate and finally set $\epsilon\to 0$ (hep-th/9804058)

Method 2: improved action - add boundary term

$\int_{\partial M}\sqrt{h}\,[a\phi^2 + b\,\phi\,\underbrace{\Box}_{\text{in tangent direction}}\phi\ldots]$ $\to$ doesn't change EoM

$\Rightarrow$ finite action, finite ~~...~~ correl. fcns

Holographic probes (hep-th/9808017)

"classical probe" in a classical state operators have expectation values $\langle S|\phi|S\rangle$

$e^{iS(\phi_0)} = \langle e^{\int J\mathcal{O}}\rangle$

$\left.\frac{\delta S}{\delta\phi_0}\right|_{\phi\to\phi_0 \text{ on boundary}} = \langle\mathcal{O}\rangle_{\mathcal{L}\to\mathcal{L}+\phi_0\mathcal{O}}$

$\left.\frac{\delta S}{\delta\phi_0}\right|_{\phi_0=0,\ \text{state } S} = \langle S|\mathcal{O}|S\rangle$

Ex: $S = \int d^{d+1}x\,\sqrt{g}\,(\partial_\mu\phi\partial^\mu\phi + m^2\phi^2)$

$\underset{\text{on-shell}}{=} \int d\Sigma^\mu\,(\partial_\mu\phi)\phi \sim \int \phi\,\partial_\xi\phi$ ξ: radial coordinate

Assume $\phi \xrightarrow[\xi\to 0]{} \phi_v(x)\,\xi^{2h_+}\ldots + \phi_0(x)\,\xi^{2h_-}\ldots$

term interesting for us $\sim \phi_v\phi_0$ (gives something in $\left.\frac{\delta S}{\delta\phi_0}\right|_{\phi_0=0}$, i.e. 1-point fcn.

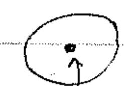
After some calculation

$\langle S_{\phi_V}|\mathcal{O}|S_{\phi_V}\rangle\big|_{\phi_0} = 2h_+\,\phi_v(x) + 2h_+\int d^4x'\,\frac{\phi_0(x')}{(x-x')^{4h_+}}$

$ds^2 = +\frac{\ell^2}{\xi^2}(-dt^2 + d\xi^2 + d\vec{x}^2)$ the rest skipped - said to be messy

D-instanton probe

$EAdS_5$ $ds^2 = \ell^2\frac{dt^2 + d\vec{x}^2 + d\xi^2}{\xi^2}$



$\xi_0, t_0, \vec{x}_0$

$(\xi_0, \vec{x}_0)$

D-instanton $e^\phi = g_{st}\,\frac{c\,\xi^4\xi_0^4}{|\xi_0^2 + |\vec{x}-\vec{x}_0|^2|^4}$

localized at $(\xi_0, \vec{x}_0)$

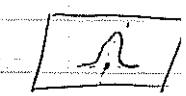
Axion $\chi = \chi_\infty + \left(e^{-\phi} - \frac{1}{g_s}\right)$

ϕ is massless $2h_+ = \Delta = d$, set $d=4$ (i.e. $EAdS_5$)

$\Rightarrow$ $\frac{\langle \mathrm{tr}F^2\rangle}{\# g_{YM}^2} = \frac{48}{g_{YM}^2}\left[\frac{\xi_0^4}{\xi_0^2 + |\vec{x}-\vec{x}_0|^2}\right]^2$

probably $\frac{1}{4\pi}$ $\to \#$

reminds of 't Hooft YM instanton



big $\xi_0 \to$ big instanton

String probe AdS_5 in Poincaré coords

string

$S_F = -\frac{1}{2\pi\alpha'}\int d^2\sigma\,\{\sqrt{-\det g_{\mu\nu}\partial X^\mu\partial X^\nu} - \frac{1}{2}\epsilon^{mn}B_{mn}$

(~~D3-brane action has term $\int d^4x\, b^{mn}\,\mathrm{Tr}\,F_{mn}$~~)

In Einstein frame $S_F = \int d\xi^2 \frac{\phi}{\xi_0^2} + \dots$

$\Rightarrow$ dilaton profile

$$\langle \mathrm{Tr}\, F^2 \rangle = \left(2 \frac{g_{YM}^3 \sqrt{N}}{\pi^2} \right) \frac{\xi_0^2}{[\xi_0^2 + |\vec{x}_\perp - \vec{x}_{0\perp}|^2]}$$

RGE

in CFT $\langle T^\mu{}_\mu \rangle = 0$

breaking of conf. $\langle T^\mu{}_\mu \rangle = \frac{\delta W}{\delta \log \underset{\text{scale}}{\lambda}} = \frac{\delta g_i}{\delta \log \lambda} \frac{\delta W}{\delta g_i}$

$$= \beta \langle \mathcal{O}_i \rangle$$

$\Rightarrow T_{\mu\nu}$ important

Boundary stress tensor hep-th/9902121, 9806087

graviton $g_{\mu\nu} \longleftrightarrow T_{\mu\nu}$

$$ds^2 = N_\xi^2 d\xi^2 + h_{\mu\nu} [dx^\mu + N^\mu d\xi][dx^\nu + N^\nu d\xi]$$

(any asymp. AdS can be written in this way)

$h_{\mu\nu}$ boundary metric

$$\langle T_{\mu\nu} \rangle = \frac{1}{\sqrt{h}} \frac{\delta S}{\delta h_{\mu\nu}}$$

$$S_{cl} = \frac{1}{16\pi G_N} \int_M d^{d+1}x \sqrt{g} \left[R - \frac{d(d-1)}{\ell^2} \right] + \frac{1}{8\pi G_N} \int_{\partial M} \sqrt{h}\, \underset{\text{extrinsic curv.}}{\Theta}$$

$\Theta_{\mu\nu} = \nabla_\mu n_\nu + \nabla_\nu n_\mu$ $\quad n$ inner point. normal $\quad \Theta = \Theta^\mu{}_\mu$

Classical mechanics $E = \frac{\partial S}{\partial t} \rightsquigarrow E \sim \frac{\delta S}{\delta h_{tt}}$

$\Rightarrow$ mass $M = \int_\Sigma d^{d-1}x \left(\sqrt{\sigma}\, N_\Sigma \right) T_{\mu\nu} u^\mu u^\nu$

$$\langle T_{\mu\nu} \rangle = \frac{1}{8\pi G_N} \left[\Theta^{\mu\nu} - \Theta h^{\mu\nu} \right]_{\xi = \epsilon} \xrightarrow[\epsilon \to 0]{} \infty$$

Why the divergence is occuring

bulk $\quad S_{cl} = \int (R - \Lambda) + \int \Theta \longrightarrow \infty$ large volume

CFT $\quad$ composite operator $\rightarrow$ divergence expected

$\Rightarrow$ need for finite action for gravity

$$S = \frac{1}{16\pi G_N} \int d^{d+1}x \sqrt{g} \left[R - \frac{d(d-1)}{\ell^2} \right] + \frac{1}{8\pi G_N} \int_{\partial M} \sqrt{h}\, \Theta$$

$$+ \frac{1}{8\pi G_N} \int_{\partial M} \sqrt{h} \left[a + b R_h + c R^2 + d R_{\mu\nu} R^{\mu\nu} + \dots \right]$$

AdS_3 counterterm $\mathcal{L}_{ct} = -\frac{1}{\ell} \Rightarrow T_{\mu\nu} = \frac{1}{8\pi G}\left[\Theta_{\mu\nu} - \Theta h_{\mu\nu} - \frac{1}{\ell} h_{\mu\nu} - \right.$

AdS_5 $\quad$ —— $\quad \mathcal{L}_{ct} = -\frac{3}{\ell}\left(1 - \frac{\ell^2 R}{12}\right) \Rightarrow T_{\mu\nu} = \frac{1}{8\pi G}\left[\dots - \frac{3}{\ell} h_{\mu\nu} - \frac{\ell}{2} G \right.$

logarithmic divergences remain, powerlike are removed

Ex: AdS_5 Schwarzschild in global AdS coords.

$$ds^2 = -\left[1 - \frac{r^2}{\ell^2} - \frac{r_0^2}{r^2}\right] dt^2 + [\quad]^{-1} dr^2 + r^2 d\Omega^2$$

$$M = \int d^3x \sqrt{h}\, N_\Sigma\, u^t u^t\, T_{tt} = \frac{3\pi r_0^2}{8G} + \frac{3\pi \ell^2}{32G}$$

even if $r_0 = 0$ we have ℓ^3 term, $\quad G_5 = \frac{G_{10}}{\ell^5}, \; G_{10} \sim g_s^2$

$\frac{\ell^3}{G} \sim \frac{\ell^8}{G_{10}} \sim \frac{3N^2}{\pi} \Rightarrow M_0 = \frac{3N^2}{16\ell} \qquad \ell \sim (g_s N)^{\frac{1}{4}}$

CFT $\Rightarrow$ Casimir energy $E_{cas} \sim \frac{3N^2}{16\ell}$ in $\mathcal{N}=4$ SYM on $S^3 \times \mathbb{R}$

$\boxed{\text{Holography \& Quantum foam 4}}$

Scales $\longleftrightarrow$ radial distance in AdS

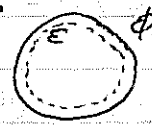
$\Rightarrow$ RG flow of deformed CFT $\overset{?}{\longleftrightarrow}$ radial flow of soln

RGE flow $\longleftrightarrow$ radial eqn. of motion

(see e.g. hep-th/9903190, 9912012)

Ex: "local" RG schemes for field th $\Leftrightarrow$ diffeomorphisms

Euclidean AdS



$$Z = \int_{\phi\to\phi_0} \mathcal{D}\phi\, e^{-S}$$

originally i in front of S, W, since Euclidean I assume should be -1 instead

$$\tilde{Z} = \int \mathcal{D}\phi_{in} \int \mathcal{D}\phi_{out}\, e^{-S} = \int_{\phi\to\phi_\varepsilon} \mathcal{D}\phi_{in}\, e^{-S_{eff}} = e^{-W_{eff}[\phi_\varepsilon]}$$

$\overset{?}{\phi_0 \leftrightarrow \phi_\varepsilon}$?

$$G_{B\partial} = \frac{\varepsilon^\Delta}{(\varepsilon^2+|b-b'|^2)^\Delta} \qquad \phi_\varepsilon(b') = \int_{\partial M} db \sqrt{h}\, G_{B\partial}(\varepsilon, \vec{b}', 0, \vec{b})\, \phi_0(\vec{b})$$

Def. $C_n(\varepsilon) \doteq \langle \theta_\varepsilon \dots \theta_\varepsilon \rangle = \dfrac{\delta \ln \tilde{Z}}{n!\, \delta \phi_\varepsilon^n}$

$S_{eff} \sim \int d(\text{inter})\, \mathcal{L} + \underbrace{S_{CL}^{out}[\phi_\varepsilon]}$ semi-classicaly

integrated out between "$\xi = 0$ & $\xi = \varepsilon$"

after computation $\delta S_{eff}/\delta\phi_\varepsilon, \dots$

$\langle \theta_\varepsilon \dots \theta_\varepsilon \rangle = \langle (\int G\theta)(\int G\theta) \dots \rangle$

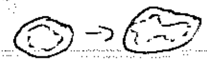
in Poincaré coords

in principle ε needs not to be in ξ coord. this would be e.g. in Poinc. patch

$$\left(\varepsilon\frac{\partial}{\partial\varepsilon} - n(\Delta - d)\right) C_n(\Sigma, k_1, \dots, k_n) + \sum \varepsilon \frac{\partial}{\partial\varepsilon} \ln(1+F(g_i^2\varepsilon^2)) C_n(\varepsilon, k_1, \dots, k_n) = 0$$

form of Callan-Symanzik eqn.

diff. choice of ε=konst. surfaces



i.e. diffeomorphisms $\longleftrightarrow$ scheme dependence of local RG

Area scaling in AdS_{d+1}

$$ds^2 = -\left(1+\frac{r^2}{\ell^2}\right)dt^2 + (\ )^{-1} dr^2 + r^2 d\Omega_{d-1}^2$$

$r = L$ cutoff $\qquad$ area $= L^{d-1} S_{d-1}$

$$\text{volume} = \int d\Omega \int_0^L dr \frac{r^{d-1}}{\sqrt{1+\frac{r^2}{\ell^2}}} = S_{d-1} L^{d-1}$$

$\Rightarrow$ area scales like volume

$AdS_5 \times S^5$, ℓ, ℓ

# of degrees of freedom $\sim A/G_5$

scale $\ell = (g_s N)^{1/4} \ell_s$, $\quad G_5 = \dfrac{G_{10}}{Vol(S^5)} = \dfrac{G_{10}}{\ell^5} \sim \dfrac{g}{(g_s}$

$L = \dfrac{\ell}{\delta}$

$$\#_{\text{degr. of fr.}} \sim S_{d-1} L^3 / G_5 = \frac{\ell^3}{\delta^3} \frac{\ell^5}{G_{10}} S_{d-1} \sim \frac{(g_s N)^2 \ell_s^8}{\delta^3 g_s^2 \ell_s^8} S_{d-1} \sim \frac{N^2}{\delta}$$

Recall $SU(N)$ $\quad \#_{\text{deg. of fr.}} \sim N^2$ $\qquad \delta \to 0 \Rightarrow$ bou

$\delta \to \infty \Rightarrow$ inter

rough counting: N^2 degrees of freedom in cells of dimension $\sim \ell_p$ on boundary

as $\varepsilon \to 0 \Rightarrow \delta \to 0$ # $\to \infty$, $\varepsilon \to \infty \Rightarrow \delta \to \infty$ # $\to$

Black holes in AdS_5 space

Schwarzschild $\quad ds^2 = -\left(1+\frac{r^2}{\ell^2} - \frac{r_0^2}{r^2}\right)dt^2 + (\ )^{-1} dr^2 + r^2 d\Omega_3^2$

mass $\quad M = \dfrac{3\pi r_0^2}{8 G_5} + \dfrac{3\pi \ell^2}{32 G_5} = \dfrac{3\pi r_0^2}{8 G_5} + \dfrac{3N^2}{16\ell}$

horizon $\quad r_h^2 = \dfrac{\ell^2}{2}\left[-1 + \sqrt{1+\frac{4 r_0^2}{\ell^2}}\right]$

$r_0 \ll \ell \qquad r_h^2 \sim r_0^2$

$r_0 \gg \ell \qquad r_h^2 \sim \ell r_0$

in Euclidean AdS

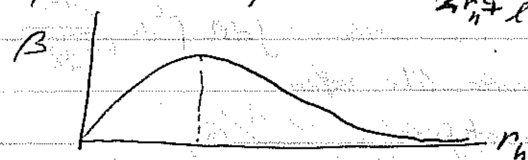
$$ds^2 = \left(1+\frac{r^2}{\ell^2} - \frac{r_0^2}{r^2}\right)d\chi^2 + (\ )^{-1} dr^2 + r^2 d\Omega_3^2$$

as $r \to r_h$ conical singularities

Exercise r-χ plane near $r=r_h$

$\Rightarrow$ to get rid of sing. $\chi \sim \chi + \frac{2\pi \ell^2 r_h}{2r_h^2+\ell^2}$

$\Rightarrow \beta$ = inverse temperature $= \frac{\ell^2 r_h}{2r_h^2+\ell^2}$



$\exists\ \beta_{max} = 1/T_{min}$ for AdS black holes

$r_h = \frac{\ell^2}{4\beta}\left[1 \pm \sqrt{\left(1-\frac{8\beta^2}{\ell^2}\right)}\right]$ r_h for given inv. temp. β

$\xrightarrow{\beta\to 0} \frac{\ell^2}{4\beta}\left(1 \pm \left(1-\frac{4\beta^2}{\ell^2}\right)\right) = \begin{cases} \frac{\ell^2}{4\beta} & \text{large BHs} \\ \beta & \text{small BHs} \end{cases}$

- For $\beta > \frac{\ell}{2\sqrt{2}}$ no black holes
- large BH positive specific heat $\left(\frac{\partial m}{\partial t} > 0\right)$
- small BH negative —"— $\left(\frac{\partial m}{\partial t} < 0\right)$

Why? SBH act like in flat space & can evaporate
LBH come into equilibrium with radiation

Subtlety: $S_{AdS} = \frac{A}{4G_5} \sim \frac{\ell^8}{G_{10}}\left(\frac{G_{10} M}{\ell^{\cdot}}\right)^{3/2}$

but BH live in $AdS_5 \times S^5$

$\Rightarrow$ small BH.. compactified S^5 cannot be neglected
is of scale $\ell > r_h$

$S_{10} \sim \frac{1}{G_{10}}(M G_{10})^{8/7} = \frac{\ell^8}{G_{10}}\left(\frac{G_{10} M}{\ell^{\cdot\cdot}}\right)^{\cdot\cdot} > S_{AdS}$

$\Rightarrow$ entropy instability

localization : 1) black circle $\to$ BH
2) Horowitz etc.
$\Big\}$ conjectures, no examples or proofs

Canonical ensemble

AdS $\sim$ box effectively

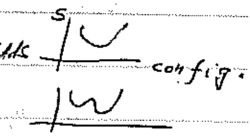
canonical ensemble of asymp. AdS spaces $\Longleftrightarrow$ thermal ensemble of $\mathcal{N}$...

in Euclidean section

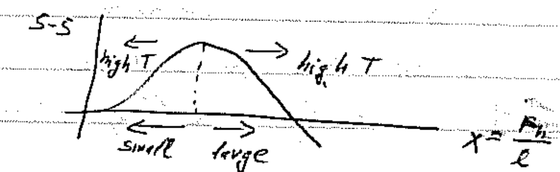
$$\sum_{\substack{\text{Euclidean}\\ \text{sol}^{\text{ns}}}} e^{-S(m)} = Z^{SYM}_{thermal}$$

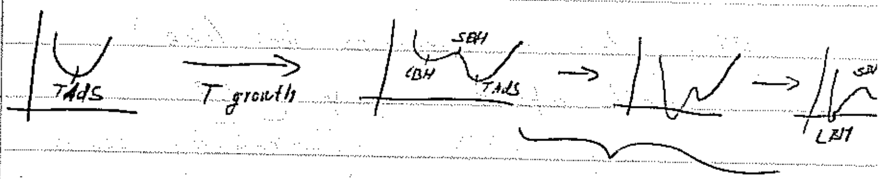
lowest action dominate

small $T \to$ only 1 saddle point thermal AdS
large $T \to$ 3 —"— thermal AdS, SBH, LBH



$S_{AdS_{th}} - S_{BH}$





$\Rightarrow$ expect phase transition (Hawking–Page)
$\sim$ deconfinement transition

LBH $r_0 \gg \ell$ $r_h \sim \sqrt{\ell r_0}$

$S = \frac{A}{4G_5} \sim N^2 \sim$ # of gluons

duality statement $|\text{black hole}\rangle_{AdS} \longleftrightarrow |\text{thermal state}\rangle_{CFT}$

Singularity resolution, information loss paradox

Perspective Thermodynamics = effective (meanfield) description of underlying pure quantum ...

$ds^2 = -\left(1+\frac{r^2}{\ell^2}-\frac{r_0^2}{r^2}\right)dt^2 + (\quad)^{-1}dr^2 + r^2 d\Omega^2$

$r_0 \gg l \qquad M \sim \frac{r_0^2}{G_5} \qquad Ml = \Delta$

LBH $\qquad \Delta \sim \frac{l^3}{G_5} = N^2$

r_H, r_0 comparable to l — independent of coupling

$|\text{black hole state}\rangle \longleftrightarrow \mathcal{O}|0\rangle \qquad \Delta(\mathcal{O}) \sim N^2$

Comparison

SUGRA states $\quad \Delta \sim \mathcal{O}(1)$

string $\quad \Delta \sim (g_s N)^{1/4} \sim \lambda^{1/4}$

D-branes $\quad \Delta \sim N$

$\Rightarrow$ BH like N D-branes

expecting e^{N^2} operators of $\Delta = N^2$

What do such operators look like?

fields of SYM $\quad A_\mu, \psi, \underbrace{X, Y, Z}_{\text{complex scalar}}$

gauge inv. ops. = "long polynomials in the fields with contracted indices" — Lorentz & gauge

e.g. $\mathcal{O} = \mathrm{Tr}[XXYYZ]\,\mathrm{Tr}[YYYXXZ]$ — $\sim \mathcal{O}(N^2)$ factors

temporarily pretend only one Tr

$\boxed{\text{SUSY \& real world 3}}$

Becker: $\boxed{\text{Flux compactification 1}}$

a.k.a. Flux backgrounds & dualities in string theory

Motivations + overview:

1) M-theory compactified on 8-manifolds

2) Flux compactifications of Type IIB

3) — " — and torsion of heterotic strings

11-dim SUGRA

vielbein $\quad e_M^A$ ← tangent space indices (A), ← curved indices (M)

gravitino $\quad \psi_M \quad \forall M$ this is 32-component Majorana spinor

$A_{MNP} \qquad A \to A + d\Lambda \qquad F_4 = dA_3$

HW: check matching of propagating degrees of freedom

Action

$$2\kappa_{11}^2 S = \int d^{11}x \sqrt{-G}\left(R - \tfrac{1}{2}|F_4|^2\right) - \tfrac{1}{6}\int A_3 \wedge F_4 \wedge F_4 + \text{fermio.}$$

$$|F_n| = \frac{1}{n!} G^{M_1 N_1} \cdots G^{M_n N_n} F_{M_1 \dots M_n} F_{N_1 \dots N_n}$$

SUSY

$$\delta e_M^A = \bar{\epsilon}\Gamma^A \psi_M \qquad \epsilon\text{ - susy param, 11d Major.}$$

$$\delta A_{MNP} = -3\bar{\epsilon}\,\Gamma_{[MN}\psi_{P]}$$

$$\delta \psi_M = \nabla_M \epsilon + \tfrac{1}{12}\left(\Gamma_M \mathcal{F}_4 - 3\mathcal{F}_M^{(4)}\right)\epsilon + \text{higher o. in } \psi$$

where $\mathcal{F}_4 = \frac{1}{4!} F_{MNPQ}\Gamma^{MNPQ}$

$\mathcal{F}_M^{(4)} = \frac{1}{3!} F_{MNPQ}\Gamma^{NPQ}$

$$\nabla_M \epsilon = \partial_M \epsilon + \tfrac{1}{4}\omega_{MAB}\Gamma^{AB} + \text{h. ord. in fermions}$$

higher order terms not important for class. solns

$\mathcal{M} + Y \qquad \mathcal{M}$ Minkowski, Y internal mfd

$ds^2 = \underbrace{\eta_{\mu\nu}dx^\mu dx^\nu}_{} + \underbrace{g_{mn}dy^m dy^n}_{}$

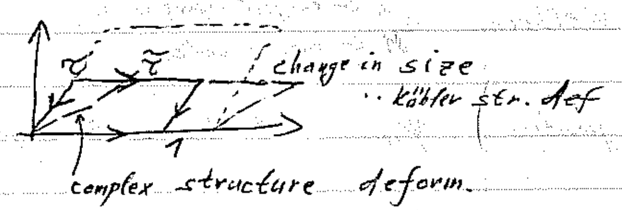
Calabi-Yau mfld   Kähler, vanishing 1st Chern class

$d J = 0$, J : (1,1)-form

holomorphic (4,0) form Ω

Variations of CY metric arise either from variation of complex or Kähler structure

Ex: Torus



The complex structure parametrized by $t_i \in \mathbb{C}$

$i = 1, .., h^{3,1}$ $\longrightarrow$ these terms give scalar fields in $d=3$

The Kähler structure parametrized by $K_i \in \mathbb{R}$

$i = 1, .., h_{1,1}$

No potential for t_i, K_i $\to$ flat directions, i.e. moduli fields (t_i, k_i) label many quasistable vacuum states.

But there are quantum corrections

$$\delta S = - \underset{\text{membrane action}}{T_2} \int A_3 \wedge X_8 \qquad X_8 = \frac{1}{192 (2\pi)^4} \left[\mathrm{tr} R^4 - \frac{1}{4} (\mathrm{tr} R^2)^2 \right]$$

~~Note: We put $F_{(4)} = 0$~~

$\Rightarrow$ modified EoM $d * F = \frac{1}{2} F \wedge F + 2 \kappa^2 T_2 X_8$

integrating over internal CY mfld

$$\frac{1}{4\alpha^2} \int_{CY_4} F \wedge F = - T_2 \int_{CY_4} X_8 \overset{HW}{\underset{c_1 = 0}{=}} T_2 \frac{\chi}{24} \leftarrow \text{Euler charact. of } CY_4$$

Generically $\chi \neq 0 \Rightarrow F$ is nonvanishing whereas classically we might put $F = 0$

We find SUSY soln of the EoM by solving

$$\delta_\epsilon \psi_M = 0$$

Classical background $\Rightarrow$ all fermionic fields = 0 (by definition of cl. back.)

Must be careful with quantum corrections to SUSY transf. - we will work in leading order

HW 1) $F_{(0,4)} = F_{(1,3)} = \text{c.c.} = 0$ (holom., antiholom.) indices

2) $F_{(2,2)} \wedge J = 0$ from $\delta_\epsilon \psi = 0$ (using cov. constant spinor ∇_η & $F_{mn} F^{mn} \epsilon = 0$ due to complex

the same conditions are obtained using superpotentials $W = \frac{1}{2\pi} \int \Omega \wedge F$
holomorphic (4,0) form

$W = 0 \Rightarrow F_{(4,0)} = F_{(0,4)} = 0$

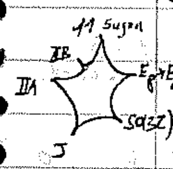
$D_\alpha W = 0 \Rightarrow F_{(3,1)} = F_{(1,3)} = 0$ since $D_\alpha \Omega = \chi_\alpha$ is basis of (3

Kähler derivative $\partial_\alpha W + (\partial_\alpha K) W$

$\tilde{W} = \int J \wedge J \wedge F$ $\tilde{W} = d\tilde{W} = 0 \Rightarrow F \wedge J = 0$

Also, W and $\tilde{W}$ follow from KK reduction. In 3d theory they give rise to a scalar potential for t_i, K_i.

Compactification of type IIB (IIB flux backgrounds)



IIB on $CY_3 \Rightarrow$ superpotential $W = \int G_{(3)} \wedge \Omega$
holom. (3,0) form

$G_{(3)} = F_{(3)} - \tau H_{(3)}$

$\tau = c_{(0)} + i e^{\phi}$ ($c_{(0)}$: axion, ϕ: dilaton)

Kähler potential

$$V = \frac{1}{2\kappa_{10}^2} e^{K} \left(G^{a\bar b} D_a W \overline{D_b W} - 3|W|^2 \right)$$

a labels all scalar fields

(complex structure + Kähler moduli, the size ρ)

$K(\rho) = -3\log(-i(\rho-\bar\rho))$

size

$K(\tau, z^\alpha) = -\log[-i(\tau-\bar\tau)] - \log(-i\int \Omega\wedge\bar\Omega)$

holomorphic (3,0) form

Note: $\mathcal{N}=2$ broken to $\mathcal{N}=1$

$G^{a\bar b}$ $\quad G_{\tau\bar\tau} = \frac{\partial K}{\partial\tau\partial\bar\tau}$

$$G^{\rho\bar\rho} \underbrace{D_\rho W}_{\partial_\rho W + \partial_\rho K\cdot W} \overline{D_\rho W} - 3|W|^2 = 0$$

$\Rightarrow$

$$V = \frac{1}{2\kappa_{10}^2} e^{K} G^{ij} D_i W \overline{D_j W}$$

all fields excluding ρ

Consequences

1) no potential for $\rho \Rightarrow$ no-scale SUGRA

2) V is minimized if $D_\alpha W = \partial_\alpha W + (\partial_\alpha K) W =$
$= \int G_{(3)} \wedge \chi_\alpha = 0 \quad (*)$

$\chi_\alpha = D_\alpha\Omega$ basis of (2,1) forms

& $D_\tau W = \partial_\tau W + (\partial_\tau K) W = \frac{1}{\bar\tau - \tau}\int \overline{G_{(3)}} \wedge \Omega = 0 \quad (**)$

(**) & (*) imply $G_{(3)} = i * G_{(3)}$, condition for unbroken SUSY

3) Generically we will have $D_\rho W = -\frac{3W}{\rho-\bar\rho} \neq 0$

4) In general, since $W\neq 0$, SUSY can be broken, $\Lambda = 0$ cosm. const.

SUSY can be unbroken if $D_i W = D_\rho W = W = 0$

$\underbrace{G_{(3)} \text{ is of type } (2,1),\ G_{(3,0)} = G_{(0,3)} = 0}_{\text{constraints on complex structure}}$

in Δ

Exact solns of SUGRA 2

Holography & Quantum foam 5

Caveats: 1) No SUSY $\rightarrow$ one may expect large renormali[zation]

but the conf. dimension of ops. $\Delta = 1$ doesn't change significantly

2) $U(N)$ gauge $\Rightarrow$ $Tr(X^{N+k})$ splits up into product of traces

3) Even in free field theory ops. with $\Delta > \sqrt{N}$ mix strongly

Ignore this, irrelevant for argument

for simplicity assume polynomial in X, Y, Z in one trace

$\mathcal{O} \sim Tr[XXY\bar X\bar Z Z\bar X \ldots]$

$\mathcal{O}|0\rangle = |\mathcal{O}\rangle$

Let $\mathcal{O}_p$ probe $\Rightarrow$

$\langle\mathcal{O}|\mathcal{O}_p|\mathcal{O}\rangle$ depends only on Δ and global charges of $\mathcal{O}, \mathcal{O}_p$ up to

$\rightarrow$ exponentially tiny corrections for almost all $\mathcal{O}$,

i.e. probe says almost nothing about BH.

Why?

1) almost all long polynomials belong to a so-called typical set of statistically random strings

prob[random letter $= X$] $= \frac{1}{|\text{alphabet}|}$ ... typical distr. of le[tters]

2) Fraction of strings that are atypical goes to 0 as length $\Delta \to \infty$

(Sanov's thm $\Pr[\text{Dist of letters} = g(x)] = e^{-\Delta D(p\|g)}$

$D(p\|g) = \sum_i -p(i) \ln \frac{p(i)}{g(i)}$

3) Correlation fcns are purely controlled by statistics

since $\langle 0 | \mathrm{Tr}(\quad)\mathrm{Tr}(\quad)\mathrm{Tr}(\quad)\mathrm{Tr}(\quad) | 0 \rangle$

all possible Wick rotations

$\Rightarrow$ a) leading individual term is pattern match in $\mathcal{O}$

b) universal contribution from each pattern in contraction

$\Rightarrow$ correlation fcn is independent of details of operator

exponential suppresion $\sim e^{-N^2} = e^{-\text{entropy}}$

What probes can detect the microstate

Δ(black hole) $\sim N^2$

gravity probes $\Delta \sim 1$ useless

strings $\Delta \sim (g_s N)^{1/4}$ —//—

D-branes $\Delta \sim N$ better

other BH probes $\Delta \sim N^2$ if you pick by chance the copy of examined BH you get huge response, generic BH will give small, universal response

$\Rightarrow$ to identify $\mathcal{O}_{BH}$ we need $\mathcal{O}(e^{N^2})$ measurements

$\frac{1}{2}$ BPS states

$\mathcal{N}=4$ SYM $\quad \overset{\text{susy}}{16} + \overset{\text{conf}}{16}$ supercharges

from superalgebra $SO(6)$ R-symmetry $\ni U(1)^3$

$\frac{1}{2}$ BPS in $(0,p,0)$ irrep of $SO(6)$

$J \sim U(1)$

highest weight states constructed out of single scalar X

- polynomials in X

$\Delta(X) = 1 \qquad J(X) = 1$

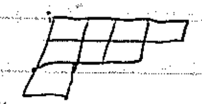
$\frac{1}{2}$ BPS states saturate $H = J$ (Witten, Olive)

e.g. $\mathcal{O} = \mathrm{Tr}(X^p) \Rightarrow \Delta(\mathcal{O}) = J(\mathcal{O}) = p$

Suitable OG basis of $\frac{1}{2}$ BPS states

$\sum_\sigma (-1) X^{i_1}_{\sigma(1)} X^{i_2}_{\sigma(2)} \dots X^{i_p}_{\sigma(p)}$

parametrizes Young tableaux with p-boxes (using isomorphism between repr. theory of $U(N)$ and S_p)

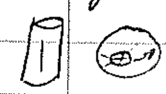


What are these states $N\{ \sim \det X \quad p<N \{ $ subdet pX

giant gravitons $S = \int \sqrt{\det(G-F)} + \int A_4$

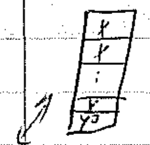
radius $= r = \sqrt{j}\, \ell \qquad j$ angular momentum

$E = \frac{j}{\ell}$



$AdS_5 \times S^5$

dual $J = j, \quad \Delta = E\ell = j \Rightarrow \Delta = j$ $\frac{1}{2}$ BPS state



$= \varepsilon_{i_1 \dots i_N} \varepsilon^{j_1 \dots j_N} Y^{i_1}_{j_1} \cdots X^{i_{N-1}} (Y^j)^{i_N}_{j_N}$

D-brane in S^5 moving at high speed

$$\langle O_G(x)\, O_G(y) \rangle \propto \frac{1}{(x-y)^{2\Delta}}$$

=> Δ turns out to be {D-brane energy + energy of excited string}

Columns of tableaux ~ D-branes in S^5

Typical $\Delta \sim N^2$ $\frac{1}{2}$BPS state

1) 1/2 BPS states $\Delta = J$

Fourier exp. $X(\vec{y}) = \underbrace{X(0)}_{\Delta=J} + \sum \underbrace{(\partial_{y_1 y_2 \dots} X)\, Y^{\mu_1} \cdots Y^{\mu_n}}_{\Delta > J}$

=> only keeps s-wave $X(0) \to$ matrix

2) S-wave reduction of YM $S = \int dt\, (D_t X)^2 + RX^2 + [X, \bar{X}]$

$X \sim$ complex scalar $\Delta = J \to 1$ real scalar

3) Matrix model of single scalar X

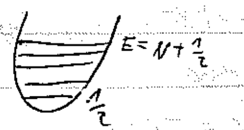
$$\int \mathcal{D}X\; e^{i\int (D_t X)^2 + R\, \mathrm{Tr}\, X^2}$$

using gauge field diagon. X in each point

↳ $\mathcal{D}X \sim N$ eigenvalues, gauge fixing → ghosts → ..

→ N fermions in a S. harm. oscillator

ground state



filled Fermi sea

should corresp to empty AdS

excited state $\delta_1 = E_1 - (N + \frac{1}{2})$ $\quad \delta_2 = E_2 - [(N-1) + \frac{1}{2}] \dots$

$\delta_1 \geq \delta_2 \geq \dots$

=> Young tableaux

wave fcn slater determinant

$$e^{-(\dots)} \begin{vmatrix} H_{n_1}(x_1) & H_{n_1}(x_2) & \dots \\ H_{n_2}(x_1) & H_{n_2}(x_2) & \dots \\ \dots & & \dots \end{vmatrix}$$

Given $\Delta = N^2$ what is the typical operator?

Using canonical ensemble description

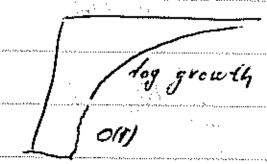
$\langle n_i \rangle = \langle \#$ of columns of length $i \rangle$

$= \langle \#$ of boxes in row i − # of boxes in row i

$= \dfrac{e^{-\beta i}}{1 - e^{-\beta i}}$

Fluctuations $\mathrm{var}(n_i) = \dfrac{e^{-\beta i}}{(1 - e^{-\beta i})^2}$

if one fixes $\beta \sim N^2$



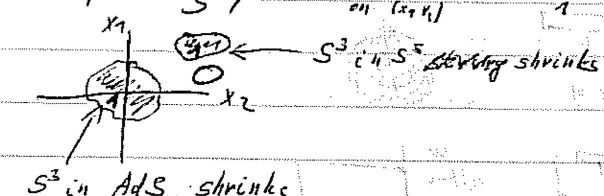
map to gravity

1/2 BPS states in gravity in semi-classical limit

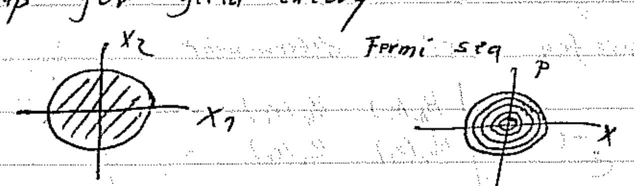
in 10d asympt. $AdS_5 \times S^5$

radial coord y, as $y \to 0$ $\exists$ special 2-plane (x_1, x_2) with a complicated topology

As $y \to 0$ $\xi(y) \to$ boundary values on (x_1, x_2) $\begin{matrix} 0 \\ 1 \end{matrix}$ to avoid singulari



Map for field theory



looks also like a blob

Wigner phase space distr

$\rho = |\psi(x_1 \cdots x_n)\rangle\langle\psi(y_1 \cdots y_n)|$

integrate out $y_2 \cdots y_n$, Fourier transf $\cdot i(x_1 - y_1)$

$\Rightarrow W(p,q) \quad q \equiv x_1 + y_1$

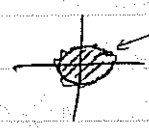
$\langle \mathcal{O}(F(p,q))_{\text{Weyl ordered}}\rangle = \int dp\,dq\; F(q,p)\, W(p,q)$

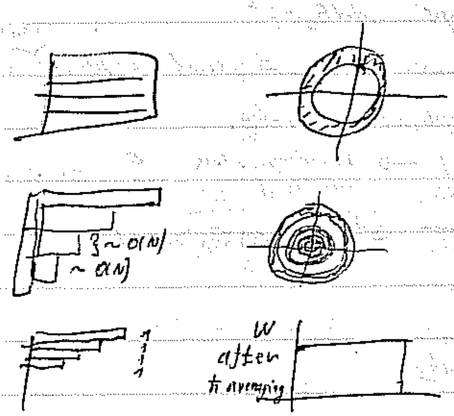
$|\psi_n\rangle \longrightarrow W_n(p,q) \sim$ Laguerre polynoms $\Rightarrow$

but $\sum_{n=0}^{\infty} W_n = 1$

$\Rightarrow$

$\hbar \to 0 \quad N\hbar$ fixed





$\langle \mathcal{O}_W\rangle = \int f(p,q)\, W(p,q)$

Hussini distribution - other possible distrib. $H(p,q)$

$\langle \mathcal{O}_{\text{normal}}\rangle = \int H(p,q)\, F(p,q)$

Summary : generic microstate of field theory is dual to quantum foam whose appearance depends on the kind of the probe, but generically it appears classicaly to be singular.

SUSY gauge theory 3

Classical moduli space

no F-terms, we need to classicaly solve the D-term $\Rightarrow$ gauge invariant combinations of $Q_a^i, \tilde{Q}_i^a$

mesons : $M^i_j = Q_a^i \tilde{Q}^a_j$

baryons : $B^{i_1 \cdots i_{N_c}} = \epsilon^{a_1 \cdots a_{N_c}} Q^{i_1}_{a_1} \cdots Q^{i_{N_c}}_{a_{N_c}}$ $\quad$ ($=0$ if $N_f <$

anti- -"- : $\tilde{B}_{i_1 \cdots i_{N_c}} = \epsilon_{a_1 \cdots a_{N_c}} \tilde{Q}^{a_1}_{i_1} \cdots \tilde{Q}^{a_{N_c}}_{i_{N_c}}$

$\Rightarrow$ moduli space is generated by these ($\Leftarrow \epsilon, \delta$ base of i tensors of

$\mathcal{M}_{\text{class.}} = \{M, B, \tilde{B}\} / \sim$

equivalence : $B^{i_1 \cdots i_{N_c}} \tilde{B}_{j_1 \cdots j_{N_c}} \cong M^{[i_1}_{j_1} \cdots M^{i_{N_c}]}_{j_{N_c}}$

$B^{[i_1 \cdots i_{N_c}} M^{i_{N_c+1}]}_j \cong 0 \cong \tilde{B}_{[i_1 \cdots i_{N_c}} \tilde{B}^j_{i_{N_c+1}]} = $

others fo $B=0$ or $\tilde{B}=0$

Simple cases

$N_f < N_c \Rightarrow B = \tilde{B} = 0$, no relations

$\mathcal{M}_{class} \simeq \mathbb{C}^{N_f^2}$

typically Higgs field

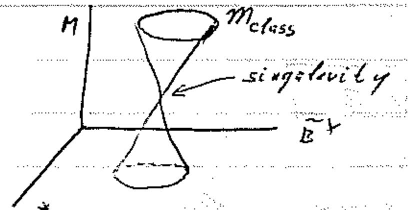
$\langle M \rangle : SU(N_c) \to SU(N_c - N_f)$

$N_f = N_c$ $\quad B^* = \epsilon_{i_1 \dots i_{N_f}} B^{i_1 \dots i_{N_f}}, \quad \tilde{B}^* = \dots$

$\mathcal{M}_{class} = \{ M^i_j, B^*, \tilde{B}^* \} \, / \, y = B^* \tilde{B}^* - \det M = 0$

singularity $\; y = dy = 0 \Leftrightarrow B^* = \tilde{B}^* = (\det M)(M^{-1})^i_j = 0 \quad \forall i,j$

(to have $y = dy = 0$ we need rank $M \leq N_c - 2$)



outside sing. typically $SU(N_c)$ totally broken

in sing. at least $SU(2)$ left

$N_f = N_c + 1$ $\quad B^*_i = \epsilon_{i i_1 \dots i_{N_f-1}} B^{i_1 \dots i_{N_f-1}}, \quad \tilde{B}^{*i} = \dots$

$\sim \, : \quad M^i_j B^*_i = \tilde{B}^{*j} M^i_j = 0, \quad B^*_i \tilde{B}^{*j} = (\det M)(M^{-1})^i_j$

again $SU(N_c)$ completely broken except singularities where symmetry is enhanced

Let us investigate $N_f = N_c + 1$ quantum mechanically

guess: $\{ M^i_j, B^*_i, \tilde{B}^{*i} \}$ are a good & complete set of low energy degrees of freedom

$W_{eff}(M, B, \tilde{B}, \Lambda)$

strong coupling scale

Global symmetries $SU(N_f)$ acting on $B^*_{(i)}, M^{(i)}_j$

another $SU(N_f)$ $\quad \tilde{B}^{*(i)}, M^j_{(i)}$

$\Rightarrow$ invariants $\det M$, $B^* \cdot M \cdot \tilde{B}^*$

	$U(1)_{axial}$	$U(1)_R$
Q	1	$(N_f - N_c)/N_f$
$\tilde{Q}$	1	$(N_f - N_c)/N_f$
W	0	1
Λ^b	$2N_f$	0
$\det M$	$2N_f$	$2(N_f - N_c)$
$B^* M \tilde{B}^*$	$2(N_c+1)$	$\frac{2(N_c+1)}{N_f}(N_f - N_c)$

$N_f = N_c + 1$, W_{eff} should have $U(1)_R$ charge 2

0 under $U(1)$

$$\Rightarrow W_{eff} = \frac{\alpha \det M + \beta B^* \cdot M \cdot \tilde{B}^*}{\Lambda^b}$$

problematic in weak coupling $\Lambda^b \to$

$\Rightarrow$ one would assume $\alpha = \beta = 0 \Rightarrow W_{eff} = 0$

loophole: constraints $\Rightarrow B^* \cdot M \cdot \tilde{B}^* = \det M$

$\Rightarrow$ if we set $-\alpha = \beta \Rightarrow W_{eff} = 0$ on $\mathcal{M}_{class}$

on the other hand

$0 = \partial_M W_{eff} = \partial_B W_{eff}$ gives back the classical constraints $\Rightarrow$ desription using W ~~are~~ is consistent with previous classical discussion

What's happening at singularity?

Perturbation $\delta W_{tree} = m^i_j \tilde{Q}^a_j Q^i_a \to \delta W_{eff} = m^i_j M^j_i$, i.e. add masses, then integrate flavours, i.e. massive fields, out start with ~~$N_f \leq N_c$~~ $N_f = N_c + 1$, integrate one flavour

$\Rightarrow N_f = N_c$ $\quad \Rightarrow W_{eff} = X \cdot (\det M - B^* \tilde{B}^* - \Lambda^b)$

submatrix of orig. M — comp. of orig. B

$\Rightarrow$ deformed surface, singularities smoothed out

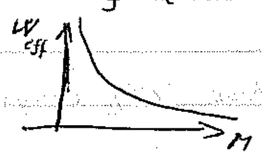
no vacuum in which symm. is restored

if we integrate more flavours $\Rightarrow N_f < N_c$

$$W_{eff} = (N_f - N_c)\left(\frac{\Lambda^b}{\det M}\right)^{\frac{1}{N_c - N_f}}$$

$N_f = N_c - 1 \Rightarrow$ one instanton

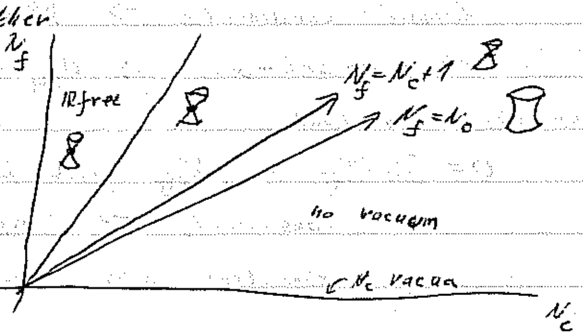
$N_f < N_c - 1 \Rightarrow$ fractional instantons



no vacuum ! (no minimum of W_{eff})

Finally $N_f = 0 \Rightarrow$ SYM $\quad W_{eff}$ gives N_c discrete vacua ~ matches Witten index calculation of # of vacua

$\Rightarrow$ All together



Back to singularity question?

for $N_f \geq N_c + 1$ there are new light degrees of freedom in singularity — Seiberg duality

Seiberg duality hep-th/9411149

Proposal: new light degrees of freedom are in UV described by $SU(\tilde{N}_c)$, $\tilde{N}_c = N_f - N_c$

"Seiberg dual theory" $\begin{cases} N_f \text{ fields } q_i, \tilde{q}^i \text{ in fund.-antifund. of } SU(\tilde{N}_c) \\ N_f^2 \text{ fields } n_i^j \quad \text{singlets} \quad \text{—} \not\text{—} \\ W = n_i^j\, q_i^a\, \tilde{q}_a^i \end{cases}$

Checks: 1) global symmetries are the same

2) 't Hooft anomaly matching works

3) $N_f < \frac{3}{2} N_c \Rightarrow N_f > 3\tilde{N}_c \Rightarrow$ dual is IR free

4) Moduli spaces are the same $\left(\begin{array}{l} b = B^*, \tilde{b} = \tilde{B}^*, \\ m = 0 \text{ from } F \\ \text{classical } \end{array}\right.$

5) Mass deformations

for $N_c < N_f < \frac{3}{2} N_c$ dual theory is IR free, direct th. no

for $\frac{3}{2} N_c < N_f < 3N_c$ 2 theories (direct & dual) RG flowing from UV to the same IR SCFT

Generalizations:

other gauge groups - lot of things can get wrong (no anomaly free R-symm. etc. / χ symm. breaking, ...)

$\boxed{\text{Flux compactification 2}}$

References: hep-th/9605053, 0107044 $\}$ M-theory

hep-th/9906070 relation

IIB: hep-th/0105097, 9908088 $\quad$ Comp. mflds, CY: Candelas Proc. of ICTP Spring

"String theory & M-theory: A modern primer"
Becker, Becker, Schwarz CUP 2005(?)

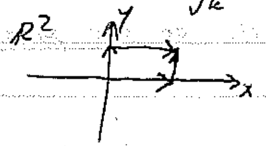
Fluxes, torsion & heterotic strings

there is only one real 3-form H
Once $\langle H\rangle \neq 0$ the spacetime acquires
1) nonconstant warp factor
not $M^{10} \to M^4 \times M^6$, instead metric looks like

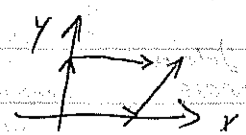
$$g_{MN}(x,y) = e^{2D(y)} \begin{pmatrix} g_{\mu\nu}(x) & 0 \\ 0 & g_{mn}(y) \end{pmatrix}$$

scalar "warp" fcn external spacetime internal mfld

SUSY will require $D(y)$ non-constant
2) torsion $T^{\ell}_{\ jk} = \Gamma^{\ell}_{kj} - \Gamma^{\ell}_{jk} \neq 0$ nonvanishing tensor
Ex: R^2



We can define parallel transport in the sense of trivial geometry
$\Rightarrow$ the parallelogram closes – we are using torsion free Christoffel connection
In polar coords $\omega^{\varphi}_{\ r} = d\varphi$
We can choose any connection compatible with the metric. This means that we can choose any connection $\omega^{\alpha}_{\ \beta} = h(\sigma_2)^{\alpha}_{\ \beta}$
compatible with metric any 1-form, choose $h = a(1-x^2)dX$
$\Rightarrow$ leads to nonvanishing torsion



~~Ex~~ infinitesimal parallelogram

Gravity theories with torsion considered already in 1922 by Cartan.

Pseudotensors

tensors, e.g. vectors $X_i \to X_i'(x) \Rightarrow A_i' = \dfrac{\partial x_i'}{\partial x_j} A_j$, $A^{i\prime} = \dfrac{\partial}{\partial x_j}$
a_{ij}

pseudoscalar $S' = |\alpha|\, S$
det a_{ij}

$\Rightarrow$ e.g. reflection in 3d gives extra sign

Unbroken SUSY and heterotic strings

$\delta\psi_M = \nabla_M \varepsilon - \frac{1}{4}\mathcal{H}_M \varepsilon = 0$
$= \frac{1}{2!} H_{MPQ}\Gamma^{PQ}$ H real 3-

$\delta\lambda = \not\nabla\phi \cdot \varepsilon - \frac{1}{2}\mathcal{H}\cdot\varepsilon = 0$ ϕ dilaton
$\delta\chi = \frac{1}{2} F^{(2)}\cdot\varepsilon = 0$ F YM field

Bianchi identity $dH = \frac{\alpha'}{4}(\mathrm{tr}\, R\wedge R - \mathrm{tr}\, F\wedge F)$
Poincaré invariance of soln (we impose it) implies
$H_{\mu\nu\rho} = F_{\mu\nu} = 0$
Any kind of even-dimensional internal mflds are to be con

One set of solns $H=0$ $\nabla_M\varepsilon = 0$ $\phi = $ const. $\Rightarrow$ CY comp

We are interested in $H \neq 0$
from $\delta\psi_M = \nabla_M \varepsilon - \frac{1}{4}\mathcal{H}_M\cdot\varepsilon = 0$
$\nabla_M\varepsilon = \partial_M\varepsilon + \frac{1}{4}\omega^a_{\ bM}\Gamma_a^{\ b}\varepsilon$
If torsion free $\omega^a_{\ bM} = -E^N_{\ b}(\partial_M e^a_{\ N} - \Gamma^L_{MN} e^a_{\ L})$

$\Rightarrow$ we are going to shift the spin connection

$$\tfrac{1}{2} H^a{}_{bM} = \tfrac{1}{2} e^a{}_L E_b^N H^L{}_{MN} \quad (E \text{ is inverse of } e)$$

We see that $H^L{}_{MN}$ acts as a torsion

$\Rightarrow \tilde{\nabla}_M \varepsilon = (\nabla_M - \tfrac{1}{4} \mathcal{H}_M) \varepsilon$

The new connection has nonvanishing torsion.

We have several components in SUSY transf.

$$\delta \psi_M = \nabla_M \varepsilon = 0 \Rightarrow \Gamma^{\mu\nu} \nabla_\mu \nabla_\nu \varepsilon = -\tfrac{1}{4} R \varepsilon = 0$$

$\Rightarrow$ either AdS or Minkowski (dS has no SUSY)

AdS we also don't like for some reason

$\Rightarrow$ external space is Minkowski

$$g_{MN} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & g_{mn}(y) \end{pmatrix}$$

what is this?

Warp factor absorbed in transf. from string to Einstein frame. This is in Ein. frame

Properties of the internal dimensions

1) $\tilde{\nabla}_m \eta_\pm = 0$ follows from $\delta \psi_M = 0$ ($\eta_\pm$ chiral spinors from decomp. of ε)

$\Rightarrow$ the scalar quantity $\eta_\pm^+ \eta_\pm$ ~~are~~ is const, let's scale $= 1$

Define a tensor

$$J_m{}^n = i \eta_+^+ \Gamma_m{}^n \eta_+$$

J has the property $J_m{}^n J_n{}^p = -\delta_m{}^p$ (can be showed using Fierz rearrangements)

$\Rightarrow$ internal mfld is almost complex

can be showed that metric is hermitean $g_{mn} = J_m{}^k J_n{}^l g_{kl}$

$\Rightarrow$ define $J_{mn} = J_m{}^k g_{kn}$, contracting with $J_k{}^n$

we get $J_{mn} = -J_{nm} \Rightarrow$ natural 2-form "fundamental form"

If $dJ = 0$ this is a Kähler mfld.

In the present case $dJ \neq 0 \Rightarrow$ non-Kähler mfld

$$\tilde{\nabla}_m J_n{}^p = \nabla_m J_n{}^p - H_{sm}{}^p J_n{}^s - H^s{}_{mn} J_s{}^p = 0$$

Can we show the manifold is complex?

$\Rightarrow$ Nijenhuis tensor $N_{mn}{}^l = J_m{}^q \nabla_{[q} J_{n]} - J_n{}^q \nabla_{[q} J_{n]}{}^p$

in our case vanishes due to $\tilde{\nabla}_m J_n{}^p = 0$ (and SUSY) after some calcu

$\Rightarrow$ Mfld. is complex

$$J = \tfrac{1}{2!} J_{mp}\, dx^m \wedge dx^p = \tfrac{1}{2} J_m{}^n g_{np}\, dx^m \wedge dx^p \overset{!}{=} i g_{a\bar{b}}\, dz^a \wedge$$

One finds that in complex coords.

$$H = i(\bar{\partial} - \partial) J \qquad \partial J = dz^a \wedge \frac{dJ}{dz^a}, \quad \bar{\partial} J = d\bar{z}^a \wedge \frac{\partial}{\partial \bar{z}^a} J$$

Unbroken SUSY implies $\exists \omega$ $\bar{\partial}\omega = 0$ $\quad \omega = e^{2\phi} \eta_-^T \Gamma_{a_1 \dots a_n} \eta_- dz^{a_1 \dots}$

& $\phi(y) = \tfrac{1}{2} \log \|\omega\|$ $\quad$ in our case $n = 3$ $\quad$ n co

From SUSY further follows

$$* d * J = i(\bar{\partial} - \partial) \log \|\omega\| \quad (\Leftrightarrow d(e^{-2\phi} J \wedge J) = 0)$$

F must satisfy $J_{a\bar{b}} F^{a\bar{b}} = F_{ab} = F_{\bar{a}\bar{b}} = 0$

Bianchi leads to $i \partial\bar{\partial} J = \frac{\alpha'}{8} (\mathrm{tr}\, R \wedge R - \mathrm{tr}\, F \wedge F)$

$\boxed{\text{SUSY \& real world 4}}$

$\boxed{\text{SUSY gauge theory 4}}$

$N = 2$ SQCD $\qquad$ Seiberg-Witten theory

$N = 2$ algebra

$\{Q_{\alpha i}, \bar{Q}_{\dot\alpha j}\} = \delta_{ij}\, \sigma^{\mu}_{\alpha\dot\alpha} P_\mu$

$\{Q_{\alpha i}, Q_{\beta j}\} = \varepsilon_{ij}\, \varepsilon_{\alpha\beta}\, Z \qquad [Z, Q_\alpha] = [Z, P_\mu] = 0$

Z ... central charge

bound on mass $m \geq |Z|$

we investigate particle in common eigenstate of P_μ^2

$P^\mu = (m, \vec{0})$ by Lorentz transf.

$\Rightarrow \{c_a, c_b^+\} = \delta_{ab}(m+|Z|) \qquad \{\tilde{c}_a, \tilde{c}_b^+\} = \delta_{ab}(m-|Z|)$

c arise from $Q, \tilde{Q}$, $a, b = 1, 2$

$a = b \Rightarrow \langle\psi|\{\ \}|\psi\rangle \geq 0 \Rightarrow m \pm |Z| \geq 0$ BPS bound

if $m = |Z|$ state is called BPS state

$\Rightarrow$ short multiplet $\{|\psi\rangle, c_a^+|\psi\rangle, c_b^+|\psi\rangle, c_a^+ c_b^+|\psi\rangle\}$

($\tilde{c}$ annihilate $|\psi\rangle \Rightarrow$ drop out)

decay $\underset{z_1}{1} \rightarrow \underset{z_2}{2} + \underset{z_3}{3} \Rightarrow Z_1 = Z_2 + Z_3, \ |Z_1| \leq |Z_2| + |Z_3|$

BPS states $m_1 \leq |Z_2| + |Z_3|$, if $<$

then since $|Z_2| \leq m_2, |Z_3| \leq m_3 \Rightarrow m_1 < m_2 + m_3 \Rightarrow$ no decay possible $\Rightarrow$ stability of BPS states

($\Rightarrow$ BPS states can decay only into BPS states of the same total mass)

if extended SUSY then phases of Z_i almost never match up $\Rightarrow$ BPS states are stable

IR effective action $\rightarrow \forall$ massive integrated out $\Rightarrow$ effectively $Z \rightarrow 0$

Massless particles must be neutral under Z

$(0 \geq |Z|$:-))

Multiplets $\qquad N=1$ SUSY $\qquad$ gauge representation

hypermultiplet $\quad H = Q + \tilde{Q} \qquad R + \bar{R}$

vector $\quad V = \underset{\text{chiral}}{\Phi} + \underset{\text{vector}}{W_\alpha} \qquad$ adjoint

<u>Classical SQCD</u>

$\mathcal{L}_{UV} = \int d^4\theta \left(Q e^V \bar{Q} + \tilde{Q} e^V \bar{\tilde{Q}} + \Phi e^V \bar{\Phi} \right) + \frac{i\tau}{16\pi} \int d^2\theta \left(\mathrm{tr}\, W^2 + \sqrt{2}\, Q \Phi \tilde{Q} + m^i_j\, Q \tilde{Q}^j \right) + \text{c.c}$

$[m, m^+] = 0 \qquad$ let's assume $m = 0$

F- & D-term eqns.

(1) Coulomb branch (CB) $\quad \langle\Phi\rangle \neq 0 \quad \langle Q \tilde{Q}\rangle = \langle Q\rangle = 0$

$\Rightarrow SU(N_c) \rightarrow U(1)^r, \ r = \text{rank } SU(N_c) = N_c - 1$

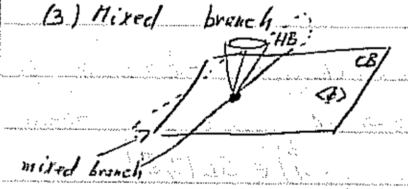
$\mathcal{M}_{CB} \simeq \mathbb{C}^r \quad$ basis - gauge invariants made from Q's $\quad \mathrm{tr}\,\Phi^2 = u_1, \dots, \mathrm{tr}\,\Phi^{N_c} = u_{N_c-1}$

(2) Higgs branch

$\langle Q, \tilde{Q}\rangle \neq 0, \quad \langle\Phi\rangle = 0 \qquad G \rightarrow \{\emptyset\}$

$\Rightarrow \mathcal{M}_{HB} \simeq \mathbb{C}^{2n}, \ n \in \mathbb{N}$

(3) Mixed branch



classical analysis can be quite believed in QM far from origin (large vevs)

Effective actions (IR)

hypers $(q, \tilde{q})$ $\qquad$ vectors (A, w_α)

$\mathcal{L}_{eff} = \int d^4\theta \left[\mathcal{K}_H(q, \tilde{q}, \bar{q}, \bar{\tilde{q}}) + \mathcal{K}_V(A^i, \bar{A}^i) \right] + \int d^2\theta\, \tau_{ij}(A)\, W^i W^j + \text{c.c}$

no other terms allowed under assumption $q, \tilde{q}$ neutral

coming from classical ...

No potential for $q, \tilde{q}$ → CB, Higgs branches are not lifted

Coupling Λ can appear only where A^i's are

⇒ Higgs branch gets no quantum correction

⇒ we should be interested in Coulomb branch only

Note: $\mathcal{K}_H$ gives hyper Kähler structure

in $\mathcal{L}_V$ there are terms $\mathrm{Im}\ \tau_{ij}(A)\,\partial_\mu A^i \partial^\mu \bar{A}^j$

$\partial_{[i}\tau_{j]k} = 0$ $\quad$ τ_{ij} can be taken symmetric,

$\mathrm{Im}\ \tau_{ij} > 0$

τ_{ij} understood as metric on Coulomb branch

– Rigid special Kähler geometry (RSK)

E-M duality redundancy in the decription of low energy U(1) actions

$\vec{E} \to \vec{B}$, $\vec{B} \to -\vec{E}$

$\tau \sim$ coupling to el. charges, Dirac: $-\frac{1}{\tau} \sim$ coupling to magn. charges

also symm. under $\tau \to \tau + 1$ doesn't change physics

⇒ invariance $\tau \to \frac{a\tau + b}{c\tau + d}$ $\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$

if $U(1)^n$ group $\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2r, \mathbb{Z})$

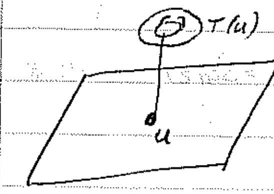
---

locally $\tau_{ij} = \frac{\partial B_i}{\partial A^j}$ $\qquad$ A^j, B_i .. special coordinates on Coulomb branch

RSK geometry (hep-th/9510101)

$\mathcal{L}_{eff} \sim \mathrm{Im}\,[\tau_{ij}\,(\partial A^i \cdot \partial A^j + \mathcal{F}^i \cdot \mathcal{F}^j)]$

$\mathcal{F}_{\mu\nu} = \frac{1}{2}(F_{\mu\nu} + i F^*_{\mu\nu})$



fibre bundle

$T(u)$ r-complex dim. torus

(2 directions for each U(1))

$\{x_i, y^i\}$ $\quad i = 1, .., r$ $\qquad x_i \simeq x_i + 1,\ y^i \simeq y$

let's put $\quad z_i = x_i + \tau_{ij}\, y^j$

can include arbitrary const.

$t = dx_i \wedge dy^i$ $\quad$ closed, integral ($\int t$ along cycles $\in \mathbb{Z}$)

positive, (1,1)-form

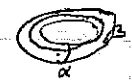
⇒ Hodge form / polarization on $T(u)$

⇒ $T(u)$ is "Abelian variety", can be written as solution set of polynomial eqns. in projective space

$\tau_{ij} = \mathcal{B}_{ik}\,(\mathcal{A}^{-1})^k{}_j$ $\qquad \mathcal{B}_{ik} = \oint_{\beta_i} \omega_k\,,\quad \mathcal{A}^j{}_k = \oint_{\alpha^j} \omega_k$

ω_k any basis of holom (1,0) forms on T

α, β 1-cycles (basis of homology)



$\int_{\alpha^i \wedge \beta_j} t = \delta_{ij}$ .. canonical basis of cycles

$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \to M \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ $\qquad M \in Sp(2r, \mathbb{Z})$

⇒ new canonical basis, τ doesn't change

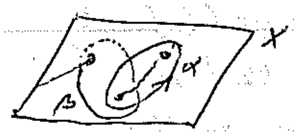
$\partial_{[i}\tau_{j]k} = 0 \iff \exists$ of a closed holomorphic (2,0) form Ω (in z-coords. $\Omega = dz_i \wedge dA^i$)

$d\Omega = 0 \Rightarrow$ locally $\Omega = d\lambda$ $\qquad \lambda|_{Torus}$ = Seiberg-Witten 1-f

$\exists \omega \Leftrightarrow \exists \{\omega_i(u)\}$ $\partial_{[j}\, \omega_{i]} = d_T\, g_{[jk]}$ (along fibre)
where $\partial_{[i}\, g_{jk]} = 0$

$r=1$ for simplicity
$\operatorname{rank} G = 1 \Rightarrow$ G abelian or $G = SU(2)$
Six $SU(2)$ theories

u $\quad y^2 = x^3 + f(u)\,x + g(u)$ describes a torus
2-cover of x-plane $\to$ 2-sheeted Riemann surface
3 branch points $y=0$ $(+\ \infty)$



$\omega = h(u)\,\dfrac{dx}{y}$ holom. 1-form

Let's look for scale invariant solns.
assign scaling dim $n = [u]$ to u as $u \to 0$
$\Rightarrow$ dominant powers $y^2 = x^3 + \alpha u^n x + \beta u^m$
to be single valued on u plane $n, m \in \mathbb{Z}$
$u \to 0$ we want scaling behaviour $\Rightarrow n, m \geq 0$
to get new physics at scale-invariant point
we need ~~bind~~ points colliding, to get interacting
theory of massless particles we need all 3 branch
points coming together
$2[y] = 3[x] = n[u] + [x] = m[u]$
using $\oint a_D = \oint \omega = \oint \frac{dx}{y}$ $\quad \overset{=1}{[a]} - [u] = [x] - [y]$
$\Rightarrow [u] = \frac{6}{6-m} = \frac{4}{4-n} > 0$

$\Rightarrow$ possibilities

$[u]$	$y^2 = x^3 + \ldots$	global symmetry
$6/5$	u	$U(1)$
$4/3$	ux	$U(2)$
$3/2$	u^2	$U(3)$
2	$u^2 x + \tau u^3$	$SO(8)$
3	u^4	E_6
4	$u^3 x$	E_7
6	u^5	E_8

$SU(2)$ with 4 doublets $\underline{2}$
mass deform. can be performed by adding subleading terms $\to$ global sym S

others are non-Lagrangian theories

Flux compactification 3

Examples:

1. $g_{mn} = \Omega^2(y)\, g^{K3}_{mn}$ $\quad \Omega$ scalar fcn.
Using $*d*J = i(\bar\partial - \partial)\log\|\omega\|$
$2i(n-1)\,\partial\log\Omega = in\,\partial\,|\log\Omega|$
n: complex dim $\Rightarrow \boxed{n=2}$
$(\cancel{\partial}\Phi + \frac{1}{2}\cancel{\partial}h)\,\epsilon = 0$ $\qquad dh = *H$
$\nabla_m \epsilon - \frac{1}{2}\partial h\, \Gamma_m{}^n \epsilon = 0 \Rightarrow \Phi = h,\ \Omega = e^{-\Phi}$
$d*d\,\Omega^2 = -\frac{\alpha'}{4}[\operatorname{tr} R\wedge R - \operatorname{tr} F\wedge F]$, using
$F_{ab} = F_{\bar a\bar b} = F_{a\bar b}\, g^{a\bar b} = 0 \Rightarrow F \Rightarrow \Omega = \ldots$
There is a bound on the volume of the internal dimensions (so that in metric no singularities arise)

T-duality

$$S=\int d^2\sigma\left(\tfrac{1}{2}V^\alpha V_\alpha-\varepsilon^{\alpha\beta}X\partial_\beta V_\alpha\right)$$

EoM for $X \Rightarrow \varepsilon^{\alpha\beta}\partial_\beta V_\alpha=0 \Rightarrow V_\alpha=\partial_\alpha\tilde{X}$

$\Rightarrow S=\frac{1}{2}\int d^2\sigma\left(\partial^\alpha\tilde{X}\partial_\alpha\tilde{X}\right)$

for $V \to V_\alpha=-\varepsilon_\alpha{}^\beta\partial_\beta X \Rightarrow S=\frac{1}{2}\int d^2\sigma\left(\partial^\alpha X\partial_\alpha X\right)$

in fact $\partial_\alpha X=-\varepsilon_\alpha{}^\beta\partial_\beta X$

$\Rightarrow$ change of sign of right movers

$X_R\to+\tilde{X}_R=-X_R\,,\quad X_L\to\tilde{X}_L=X_L$

fermions (type II strings)

$X_L^9\to X_L^9\,,\quad X_R^9\to-X_R^9$

$\Rightarrow \psi_L^9\to\psi_L^9\quad \psi_R^9\to-\psi_R^9$

$\Rightarrow$ T-duality changes the chirality of the right moving R sector ground state $\Rightarrow$ IIA and IIB are interchanged

Suppose $\partial_\sigma X^\mu\big|_{\sigma=0}=\partial_\sigma X^\mu\big|_{\sigma=\pi}=0\,,\quad \mu=0,\dots,p$

$X^i\big|_{\sigma=0}=d_1^i\,,\quad X^i\big|_{\sigma=\pi}=d_2^i\,,\quad i=p+1,\dots,9$

positions of D-branes

$$X^\mu=x^\mu+p^\mu\tau+i\sum_{n\neq0}\frac{1}{n}\alpha_n^\mu\cos n\sigma\, e^{-in\tau}$$

$$X^i=d_1^i+(d_2^i-d_1^i)\frac{\sigma}{\pi}+\sum_{n\neq0}\frac{1}{n}\alpha_n^i\sin n\sigma\, e^{-in\tau}$$

$\Rightarrow$ under T-duality Dirichlet & Neumann boundary conditions are interchanged

$\Rightarrow$ for type IIA we have Dp-branes (p even) $\Leftrightarrow$ type IIB Dp-branes where p odd

This is an example of a perturbative duality.

In 10 dim $\frac{1}{g_s^2}\int d^{10}x\, L_{NS}$

after compactification

IIA $\quad \frac{2\pi R}{g_s^2}\int d^9x\, L_{NS} \underset{R\tilde{R}=\alpha'}{\Longleftrightarrow} \frac{2\pi\tilde{R}}{\tilde{g}_s^2}\int d^9x\, L_{NS}\quad$ IIB

$\Rightarrow \tilde{g}_s=\frac{\sqrt{\alpha'}}{R}g_s$

in the presence of NS background fields

$S=S_g+S_B+S_\phi$

$S_g=\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{h}\,h^{\alpha\beta}g_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu$

$S_B=\frac{1}{4\pi\alpha'}\int d^2\sigma\,\varepsilon^{\alpha\beta}B_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu$

$S_\Phi=\frac{1}{4\pi}\int d^2\sigma\sqrt{h}\,\Phi R^{(2)}$

suppose nothing (g, B, ϕ) depends on X^9

$$S=\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{h}\,h^{\alpha\beta}\Big(g_{99}V_\alpha V_\beta+2g_{9\mu}V_\alpha\partial_\beta X^\mu+g_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu$$
$$+\varepsilon^{\alpha\beta}\left(B_{9\mu}V_\alpha\partial_\beta X^\mu+B_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu\right)+$$
$$\tilde{X}^9\varepsilon^{\alpha\beta}\partial_\alpha V_\beta+\alpha'\sqrt{h}R^{(2)}\phi\Big)$$

Eliminate V with EoM

$\tilde{g}_{99}=\frac{1}{g_{99}}\qquad \tilde{g}_{9\mu}=\frac{B_{9\mu}}{g_{99}}\qquad \tilde{g}_{\mu\nu}=g_{\mu\nu}-\frac{(g_{9\mu}g_{9\nu}-B_{9\mu}B_{9\nu})}{g_{99}}$

$\tilde{B}_{9\mu}=-\tilde{B}_{\mu9}=\frac{g_{9\mu}}{g_{99}}\qquad \tilde{B}_{\mu\nu}=B_{\mu\nu}+\frac{g_{9\mu}B_{9\nu}-B_{9\mu}g_{9\nu}}{g_{99}}$

$\tilde{g}_s=g_s\frac{\sqrt{\alpha'}}{R}\qquad g_{99}=\frac{R^2}{\alpha'}\qquad g_s=e^\Phi\quad\Rightarrow$

$$\boxed{\tilde{\Phi}=\Phi-\tfrac{1}{2}\ln g_{99}}$$

Ex: Six-dimensional example in the orbifold limit

Type IIB (without fluxes) compactified on

$T^2/\mathbb{Z}_2 \times K3$

$X_{8,9}$ $\quad \underbrace{\mathbb{Z}_2 = \Omega(-1)^{F_L} I_{89}}$

All are symmetries of action:

$X_{8,9} \to -X_{8,9}$

Ω world-sheet parity $\sigma \to 2\pi - \sigma$

- interchanges X_L, X_R

$(-1)^{F_L}$ changes sign of left-moving fermions

$K3 = T^4/\mathbb{Z}_2 = (T^2 \times T^2)/\mathbb{Z}^2 \leftarrow I_{4567}$ reflection of $X_4, \dots, X_7$

One can check that Betti # of $K3$ & $T^4/\mathbb{Z}_2$ are the same etc.

(References: Lectures on orientifolds & duality, hep-th/9804208)

$\mathbb{Z}_2 = \Omega(-1)^{F_L} I_{89} \xrightarrow[T_{8,9}]{} \Omega$

(One can check $\Omega(-1)^{F_L} I_{89} = T_{8,9}^{-1}\Omega T_{8,9}$)

After two T-dualities we have IIA background in which only Ω invariant states are kept

→ this is a background for type I.

To get background for Het we invert g

$g^{het}_{MN} = g_I g^{I}_{MN} \qquad g_{het} = g_I^{-1}$

metric, string coupling

without fluxes the metric on Het side

$$ds^2 = ds^2_{0123} + \underbrace{ds^2_{K3} + dz_3 d\bar{z}_3}_{K3\times T^2}$$

$b_i(T^2) = 1, 2, 1 \qquad b_i(K3) = 1, 0, 22, 0, 1$

$b_i(K3\times T^2) = 1, 2, 23, 44, 23, 2, 1 \qquad i = 0..6$

$\chi(K3\times T^2) = 0$

Include fluxes on IIB side

$* G_{(3)} = i G_{(3)} \qquad G_{(0,3)} = 0$

$A, D = \text{const.} \quad H_{(3)} = A\, d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_3 + D\, dz_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 + c.c$

$F_{(3)} = \tau(-A\, d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_3 - D\, dz_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3) + c$

set $\tau = i$, $\quad G_{(3)} = F_{(3)} - \tau H_{(3)}$

$(A, D) = (1+i, 0) \quad \text{or} \quad (A, D) = (1, 1)$

After T-duality on IIA side (on Het then proportional

$$ds^2 = 2g_{1\bar{1}}\, dz^1 d\bar{z}^1 + 2g_{2\bar{2}}\, dz^2 d\bar{z}^2 + \frac{1}{2g_{3\bar{3}}} |dz^3 + 2D\bar{z}^2 dz^1 - 2A\bar{z}^1 dz^2|^2 \; (+ds^2_{01\ldots}$$

$\Rightarrow (z_1, z_2, z_3) \simeq (z_1, z_2, z_3 + 1) \simeq (z_1, z_2, z_3 + i)$

$(z_1, z_2, z_3) \simeq (z_1, z_2 + 1, z_3 + 2Dz_1) \simeq (z_1, z_2 + i, z_3 - 2iD \ldots$

$(z_1, z_2, z_3) \simeq (z_1 + 1, z_2, z_3 - 2Az_2) \simeq (z_1 + i, z_2, z_3 + 2Ai\, z_2 \ldots$

⇒ changed topology

$b_i(K3 \times T^2_{\text{twisted}}) = 1, 0, 20, 42, 20, 0, 1$

B-field in I type

$$B^I = \frac{1}{g_{13}}(A\bar{z}^1 dz^2 \wedge d\bar{z}^3 - D\bar{z}^2 dz^1 \wedge d\bar{z}^3 + c.c)$$

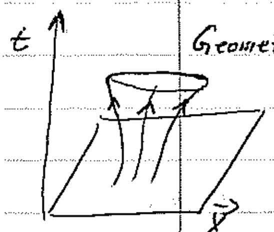
$g_I = g_{13}/2g_{3\bar{3}}$

Its possible to check $H = +i(\partial - \bar{\partial})J$

⇒ $\boxed{\text{Fluxes change topology, may introduce torsion, etc.}}$

Kofman

Geometry & dynamics of very early universe 1



Geometry: perturbed Friedmann model

$$ds^2 = a^2(\tau)\left[(\eta_{\mu\nu} + \delta g_{\mu\nu})dx^\mu dx^\nu\right] + ds_6^2$$

τ .. conformal time $\eta_{\mu\nu}dx^\mu dx^\nu = d\tau^2 - d\vec{x}^2$

physical time $ds^2 = dt^2 - a^2(t)d\vec{x}^2$, $\tau = \int^t \frac{dt'}{a(t')}$

in physical time $\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\sum_a \varepsilon_a$ (density) $\Leftrightarrow$ (0,0) comp. of Eins. eqn.

$\frac{\ddot a}{a} + \frac{\dot a^2}{a^2} = \frac{4\pi G}{3}\sum_a (\varepsilon - 3p)_a$ (pressure) $\Leftrightarrow$ trace of —— // ——

Eqn. of state = relation between ε & p

e.g. CDM $p \approx 0$ more or less nothing else needed for cosmology

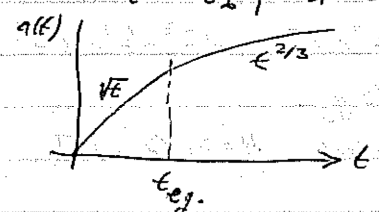
matter $\varepsilon_m = \frac{\varepsilon_{m0}}{a^3}$, $p_m = 0$

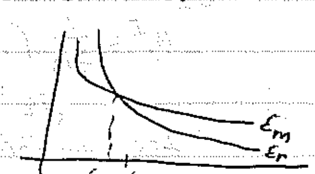
radiation $\varepsilon_r = \frac{\varepsilon_{r0}}{a^4}$, $p_r = \frac{1}{3}\varepsilon_r$

in conformal time $a'' = 4\pi G\,\varepsilon_m \Rightarrow a(\tau) = a_0\tau(\tau_0 + \tau)$

$\tau \ll \tau_0$, $a \sim \tau$ $t = \int d\tau\, a(\tau) \sim \tau^2 \Rightarrow a \sim \sqrt{t}$

$\tau \gg \tau_0$, $a \sim \tau^2$ $t \sim \tau^3$ $a \sim t^{2/3}$





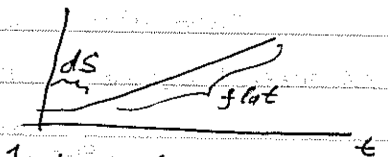
One needs ε_r dominating in early time $\rightarrow$ nucleosynthesis

not many massive states coming from string theory (or any other HE theory) are allowed, otherwise $t_{eq.}$

Cosmological const. $p_\Lambda \approx -\varepsilon_\Lambda$

$a = a_0 e^{Ht}$ $-\infty < t < \infty$ $a(\tau) = \frac{1}{H(-\tau)}$ $-\infty < \tau < 0$

Horizons

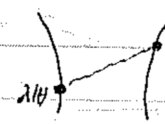


$H = \frac{\dot a}{a} \sim \frac{1}{t}$

$\frac{1}{H} \sim t$ in flat

$\frac{1}{H} \sim$ const. in deSitter

photon emitted at $a(t_0)$ $\lambda_0 \rightarrow$ detected at $a(t)$ $\lambda = \lambda_0 \frac{a(t)}{a(t_0)}$



$\lambda(t+\Delta t) = \lambda(t)(1 + \frac{v}{c}) = \lambda(t)(1 + H\Delta t)$

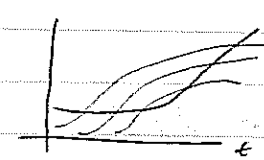
$\|$ $V = H\Delta l = H\Delta t \cdot c$

$\lambda(t) + \dot\lambda(t)\Delta t \Rightarrow$ $\boxed{\frac{\dot\lambda(t)}{\lambda(t)} = H = \frac{\dot a}{a}}$

red-shift applies to all physical phenomena

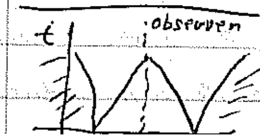
$\varphi(t,x) = \int d^3k\, \varphi(t,k)\, e^{ikx}$ $\lambda = \frac{2\pi}{k}a(t)$

e.g. fluctuations in gravity & matter



only heuristic meaning, should be considered more carefully

Particle horizon



particle horizon – points reached by light from past light cone of observer

$\Delta x = \Delta\tau = \int_0^t \frac{dt'}{a(t')}$, $\Delta r = a(t)\Delta x$

$\rightarrow$ $r_H = a(t)\int_0^t \frac{dt'}{a(t')} \underset{today}{=\!=\!=} 1{,}456$ Mpc

$1\,Mpc = 3{,}09 \cdot 10^{24}$ cm

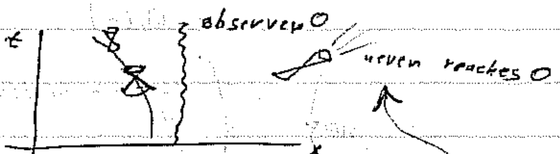
inflation (de Sitter) stage $r_H \simeq \frac{1}{H}(e^{Ht}-1)$

Event horizon $r_e = a(t)\int_{t_{now}}^{t_{fut}} \frac{dt'}{a(t')}$ dS $r_e = \frac{1}{H}$

- area from which we can see events happening now in future till t_{fut}

makes not much sense in Friedman

Apparent horizon



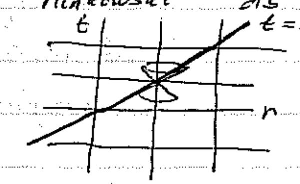
distance from which light may reach observer O, i.e. light cones are not tilted to much like

dS: $\frac{dr}{dt} = Hr > 0$ $r = \frac{1}{H}$ like event horizon

Friedman: $r \sim t^{1/3}$ different from —"—

Penrose diagrams

Minkowski $ds^2 = dt^2 - dr^2 - r^2 d\Omega_2^2$

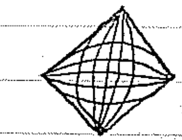


conf. transf. $t+r = f(t'+r')$, $t-r = g(t'-r')$

$dt^2 - dr^2 = f'g'(dt'^2 - dr'^2)$

usual choice $f = \tan\left(\frac{t'+r'}{2}\right)$, $g = \tan\left(\frac{t'-r'}{2}\right)$

conf. transf $\Rightarrow$ light still propagates along diagonal $\left(\frac{\pi}{4}\right)$ lines, we rescale locally metric

$\rightarrow$  $-\pi < t'+r' < \pi$
$-\pi < t'-r' < \pi$

de Sitter geometry

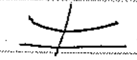
in $\mathbb{R}^5$: $ds^2 = -dv^2 + dw^2 + dx^2 + dy^2 + dz^2$

hyperboloid $-v^2 + w^2 + x^2 + y^2 + z^2 = \frac{1}{H^2}$

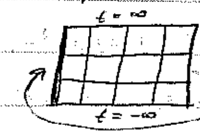
(if we Wick rotate $v \to iv \Rightarrow S^5 \Rightarrow$ dS still has many

in suitable coords

$ds_4^2 = dt^2 - \frac{1}{H^2}\cosh^2 Ht \big[\underbrace{d\chi^2 + \sin^2(\chi)\underbrace{(d\theta^2 + \sin^2\theta\, d\varphi^2)}_{S^2\ metric}}_{S^3\ metric}$

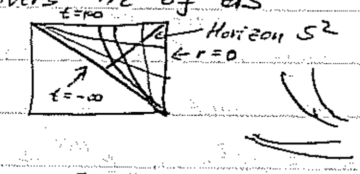
$a(t)$ ..  $\Rightarrow$ shrinks then expands

Penr. diag



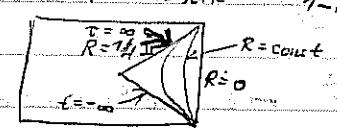
flat slicing - covers 1/2 of dS

$ds^2 = dt^2 - e^{2Ht} d\vec{x}^2$



static patch

$ds^2 = (1 - H^2R^2)dR^2 - \frac{dR^2}{1 - H^2R^2} - R^2 d\Omega_S^2$



$\Rightarrow$ QFT in each patch may look very differently, hard to translate

Friedmann in Penrose diagram

GR arguments for inflation

$$R^{\mu}{}_{\nu} - \delta^{\mu}{}_{\nu} R = 8\pi G\, T^{\mu}{}_{\nu}$$

generic solns anisotropic, inhomogeneous,

dynamical (Friedmann's soln have "measure 0")

near $t=0$: $ds^2 = dt^2 - t\, d\vec{x}^2$ Friedmann

$ds^2 = dt^2 - (t\,\alpha_{\alpha\beta} + t^2 \beta_{\alpha\beta} + \ldots)\, dx^\alpha dx^\beta$ inhomogen.

$\Rightarrow$ dynamics for $T_{\mu\nu}$ defined by $\alpha_{\alpha\beta}(\vec{x})$

$\Rightarrow$ we have 3 fcns, probability to have them const. is 0

or anisotropic soln

$ds^2 = dt^2 - a(t)\,dx^2 - b(t)\,dy^2 - c(t)\,dz^2 \Rightarrow$

doesn't need T to | again hardly $a=b=c=t$

Once we introduce effective Λ, i.e. E.e. $+ 8\pi G\, \delta^{\mu}{}_{\nu}\, \varepsilon_{\Lambda}$

$\rightarrow$ smoothes out anisotropies & inhomogeneities

$$ds^2 = dt^2 - (a_{\alpha\beta}\, e^{2Ht} + b_{\alpha\beta} + c_{\alpha\beta}\, e^{-Ht} + \ldots)\, dx^\alpha dx^\beta$$

e.g. anisotropy $c_{\alpha\beta\gamma\sigma} c^{\alpha\beta\gamma\sigma} \sim e^{-6Ht}$ dies out fast

Ex: $ds^2 = dt^2 - \cosh^2 Ht\, dx^2 - b_0\, (d\theta^2 + \sin^2\theta\, d\varphi^2)^2$

anisotropic

E.e. $R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu} R = 8\pi G\, \varepsilon_{\Lambda}$

belongs to a class of metric $ds^2 = dt^2 - a^2(t)dx^2 - b(t)\,d\Omega_2^2$

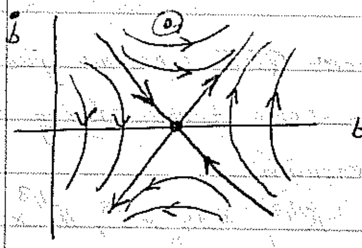
subst. into E.e. $\Rightarrow \frac{2\ddot{b}}{b} + \left(\frac{\dot b}{b}\right)^2 + \frac{1}{b^2} = \Lambda, \quad \frac{\dot a}{a} = \frac{\ddot b}{\dot b}$

Solution using <u>method of phase portraits</u>

$\frac{dy}{dt} = Q(y,x), \quad \frac{dx}{dt} = P(y,x)$

we solve $P(y,x) = Q(y,x) = 0 \Rightarrow$ critical points (x_*, y_*)

then look at near critical points (saddle points, ...)



asymptotically

$a(t) = \sinh Ht$

$b(t) = \cosh Ht \quad H = \sqrt{\frac{\Lambda}{3}}$

$\Rightarrow$ from our original soln

which is unstable we get after perturbation

& time evolution asymp. dS

$\boxed{\text{Exact solns of SUGRA 3}}$

Denef: $\boxed{\text{Landscape studies 1}}$

Mysteries of nature

* Standard model $SU(3)\times SU(2)\times U(1)$, its parameters
* $m_{ew}^2 \sim 10^{-32}\, m_P^2$ fine tuning ??
* $\Lambda \sim 10^{-120}\, m_P^4$ ridiculous fine tuning ????

If SUSY at $\sim 1\,\mathrm{TeV} \Rightarrow m_{ew}^2 = 10^{-2}\, m_{SUSY}^2 \Rightarrow$ reasonable

but why $m_{SUSY}^2 \sim 10^{-30}\, m_P^2$??

(Experimentally SUSY at 1 TeV looks suspicious, it should already show some indications)

* Weinberg ~~bound~~ bound for galaxy formation

$-10^{-120}\, m_P^4 < \Lambda < 10^{-118}\, m_P^4$

- We are "lucky bastards"

* m_{ew} 1 order of magnitude different $\Rightarrow$ no complex atom

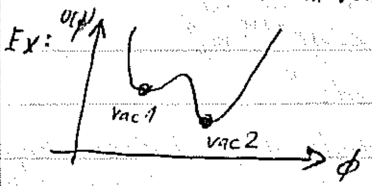
Why? Answer not from QFT – params just input – but may come from string theory.

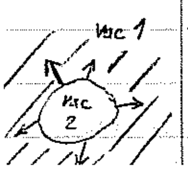
- may be only one vacuum
  - definitely no $\mathbb{R}^{1,9}$, $AdS_5 \times S^5$, $\mathbb{R}^{1,3} \times CY_3$, ...
- may be only 1 vacuum with large 4D part and no massless moduli
  - no, generic $\mathcal{N}=1$ compactifications have potential for moduli → massive
    (KKLT: scenario for construction of such vacua)
    for fixed topology like 10^{300} vacua
- may be only 1 vacuum with $\Lambda > 0$ as observed (⇒ broken SUSY)
  KKLT also proposal for $\Lambda > 0$ but may not be stable (⇐ broken SUSY) ⇒ again like
  ⇒ might be only 1 vacuum but unlikely

⇒ Working hypothesis: landscape
i.e. complicated effective potential with many local minima

Selection principle?
nothing known close to selecting 1 or few vacua

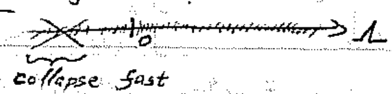
Opposite possibility: enormous # of vacua realised in universe (multiverse)

Ex: 
originally vac 1 but inside by tunnel effect vac 2 may arise



realistically
⇒ 10^{300} vacua ⇒ mess 

discretum of Λ



may be statistical predictions possible
- technically rather unfeasible, not clear how to dec

easier questions
- discretum sufficiently fine?
  similarly for other constants
  i.e. distribution of vacua over parameter space
  - count number of vacua with certain property
  → "statistics of vacua"

observation parameters – can they fix vacuum?
continuous $\xrightarrow{?}$ string vacuum (discrete pa

- hard problem – viz complexity theory NP-hard

  NP = nondeterministic polynomials – takes polynomial time to check a candidate polynomial but not to find it – like prime factorization

  NP-hard – any NP can be reduced to NP-hard in polynomial t

No one is sure that $N \neq NP$.
In practice NP-hard problems tend to require exponential time. It seems to be better to study statistics and assume that probability for certain params is suff. large, a suitable vacuum exists.

Basics of statistics   Bousso – Polchinski

$$\Lambda = \underset{\uparrow\ \text{bare}}{-\lambda_0} + \frac{1}{2}\int \underset{\uparrow\ \text{fluxes}}{G \wedge * G} \qquad G = \underset{\uparrow\ \text{integers}}{N^\alpha}\, \Sigma_\alpha \ \nwarrow \text{harmonic forms}$$

$$= -\lambda_0 + \frac{1}{2} M^4 g_{\alpha\beta} N^\alpha N^\beta$$

$g_{\alpha\beta}$ ← depend on moduli but temporarily forget about it

$$N_{\text{vacua}}(\Lambda \le \Lambda_*) = \#\text{ fluxes with } \frac{1}{2} g_{\alpha\beta} N^\alpha N^\beta \le \frac{\lambda_0 + \Lambda^*}{M^4}$$

$= \#$ —— " —— in sphere of radius $\sqrt{2L}$ $\quad L = \sqrt{\frac{\lambda_0 + \Lambda^*}{M^4}}$

$\sim$ volume of the sphere of radius $\sqrt{2L}$

↗ for large Λ^*

$$\text{Vol}\, S^k = \int d^k N\ \underset{\swarrow\text{Heaviside}}{\theta}\Big(L - \frac{1}{2}\overbrace{g_{\alpha\beta} N^\alpha N^\beta}^{N^2}\Big) = \int d^k N \frac{1}{2\pi i}\int_C \frac{dz}{z} e^{(L - \frac{1}{2}N^2) z}$$

$$= \frac{1}{2\pi i}\int \frac{dz}{z} e^{zL} \underbrace{\int d^k N e^{-\frac{1}{2} g_{\alpha\beta} N^\alpha N^\beta z}}_{= (2\pi)^{K/2} \big/ \sqrt{\det g}\ z^{-K/2}}$$

$$= \frac{(2\pi)^{K/2}}{\sqrt{\det g}} \frac{L^{K/2}}{(K/2)!} \qquad \text{(for } K \text{ odd express using } \Gamma \text{ fcn)}$$

$$N_{\text{vacua}}(0 \le \Lambda \le \varepsilon) \approx \frac{dN}{d\Lambda}\Big|_{\Lambda = 0 \text{ i.e. } L = \frac{\sqrt{\lambda_0}}{M^2}} \varepsilon = \frac{1}{\sqrt{\det g}} \frac{(2\pi)^{K/2} (\lambda_0/M^4)^{\frac{K}{2}-1}}{(K/2 - 1)!} \frac{\varepsilon}{M^4}$$

$\Rightarrow$ if $\lambda_0 \gg M^4$, $K \gg 1$ the number can be huge even for $\varepsilon \sim 10^{-120} M_p^4$ → exists moduli such that $\lambda_0 \gg M^4$

if $\lambda_0 \sim M^4$ then doesn't work

Main shortcoming: in reality we have to find moduli which are critical points of V_{eff} $\quad V_N'(z) = 0,\ V_N''(z) > 0$ and then investigate Λ – next time

Landscape studies 2

Moduli stabilization (KKLT)

setting IIB on CY 3-orientifold ($\Rightarrow \mathcal{N} = 1$ SUSY in 4d)

→ O3 and O7 space filling planes

→ D3 and D7 branes to cancel tadpoles coming from O3, O7

moduli: $h^{2,1}(X)$ $\mathbb{C}$-structure moduli

$h^{1,1}(X)$ Kähler moduli (volumes of 4-cycles)

axion-dilaton $\quad \tau = C_0 + \frac{i}{g_s}$

to give them masses turn on fluxes

RR: F_3 $\qquad$ NSNS: H_3

→ superpotential $\quad W = \int_X (F_3 - \tau H_3) \wedge \Omega^{(3,0)}$

$\{\Sigma_\alpha\}$ basis of $H^3(X, \mathbb{Z})$, $\quad F = \overset{\in\mathbb{Z}}{N^\alpha}\Sigma_\alpha$, $H = \overset{\in\mathbb{Z}}{M^\alpha}\Sigma_\alpha$

$$\Rightarrow W = (N^\alpha - \tau M^\alpha) \underbrace{\Pi_\alpha(z)}_{\text{depend on } \mathbb{C}\text{-str. moduli}}$$

$$\Rightarrow \text{potential} \quad V = \underset{\uparrow\ \text{Kähler potential}}{e^K} \big(g^{ab} D_a W \overline{D_b W} - 3|W|^2\big) \qquad D_a = (\partial_a + \partial_a K)$$

for $\mathbb{C}$-str. & ax-dil. moduli

tadpole constraint: total D3 charge = 0

$$0 = N_{D3} + \underbrace{\int_X F \wedge H}_{N^\alpha M^\beta \eta_{\alpha\beta}} - \text{contrib. from } \underset{\downarrow\ -\frac{1}{4}}{\text{O3}},\ \underset{\downarrow\ -\frac{\chi(D7)}{24}}{\text{curvature on D7}}$$

Kähler moduli classically not fixed but

* D3 instantons wrapping 4-cycles $\quad \Delta W = B e^{-(V + i\int C_4)}$

* gaugino condensation in gauge theory on 7-branes

$$\Delta W = B e^{-\frac{1}{N}(V + i\int C_4)} \quad \leftarrow \text{vol. of (wrapped) 4-cycle}$$

condition 4cycles can contribute if there are exactly 2 fermion zeromodes ($\Leftarrow \partial_i \partial_j W \psi^i \psi^j$)

In principle these contributions sufficient to stabilize Kähler moduli

$$W = W_{flux}(z,\tau) + \sum B_i e^{\underbrace{-V_i + i\int C_4}_{S_i}}$$

SUSY vacuum

$D_a W = 0 \qquad D_a = \partial_a + \partial_a K$

$K = -2\ln V(g) - \ln i\int \Omega \wedge \bar{\Omega}$

to trust approximations $\quad e^{-S_i} \ll 1,\ |W_{flux}| \ll 1$

generically $\quad |W_{flux}| > 1$

$\qquad\qquad\quad \overset{\shortparallel}{|(N^\alpha + \tau M^\alpha)\Pi_\alpha|}$ but sometimes cancellations

can occur $\rightarrow$ for fraction $\sim \lambda_*$ is $|W_{flux}|^2 < \lambda_*$

Statistics

SUSY vacua $\quad$ flux $(\overset{F_3}{N^\alpha}, \overset{H_3}{M^\alpha})$ + crit. point of W

$$D_i W_{(M,N)}(\tau, z) = 0$$

approx. how many vacua are inside region S in moduli space

simplest case: rigid CY $\rightsquigarrow$ only $\tau = C_0 + \frac{i}{g_s}$

$$W = (f_1 + i f_2) - \tau(h_1 + i h_2)$$

tadpole cancellation $\quad \underbrace{f_1 h_2 - f_2 h_1}_{\le L_*} + \overset{\ge 0}{N_{D3}} = L_*$

$K = -\ln(-i(\tau - \bar\tau))$

$D_\tau W = 0 \Rightarrow \tau = -\frac{f_1 - i f_2}{h_1 - i h_2}$

duality $\Rightarrow \tau$ in fundam. domain $\Rightarrow$ finite # of τ

e.g. $L_* = 150 \Rightarrow$ # of τs $= 37208$

$$N_{vac}(\tau \in S) = \sum_{f,h} \int_S d^2\tau\, \delta^2(D_\tau W) \left|\det\begin{pmatrix} D_\tau D_\tau W & D_\tau \bar D_{\bar\tau} \bar W \\ \bar D_{\bar\tau} D_\tau W & \bar D_{\bar\tau} \bar D_{\bar\tau} \bar W \end{pmatrix}\right|$$

for $\int \delta^2(\ )\,|\det(\ )|$ compare $\int dx\, \delta(x) = \int dx\, \delta(f(x))\,|f'(x)|$

in our case $\qquad$ metric from Kähler potential

$D_\tau D_\tau W = 0 \qquad D_\tau \bar D_{\bar\tau} \bar W = [D_\tau, \bar D_{\bar\tau}]\bar W = g_{\tau\bar\tau} \bar W$

$\Rightarrow N_{vac}(\tau \in S) = \sum \int_S d^2\tau\, g_{\tau\bar\tau}\, \delta^2(DW)\,|W|^2 \approx \int_S d^2\tau \int d^2f\, d^2h\, \delta(\ldots)$ $\quad f_1 h_2 - f_2 h_1 \le L_*$

Linear change of variables $(f_1, f_2, h_1, h_2) \to W = W(\tau), F = DW(\tau)$, Jacobian $= g_{\tau\bar\tau}$

$f_1 h_2 - h_1 f_2 = |W|^2 - |F|^2 \Rightarrow N_{vac} \approx \int_S d^2\tau\, g_{\tau\bar\tau} \int_{|W|^2 - |F|^2 \le L_*} d^2W\, d^2F\, \delta^2(F)\,|W|^2 =$

$= \int d^2\tau\, g_{\tau\bar\tau} \int_{|W|^2 \le L_*} d^2W\, |W|^2 = 2\pi L_*^2 \int_S d^2\tau\, g_{\tau\bar\tau} = 2\pi L_*^2\, \mathrm{vol}(S)$

$N_{tot} = 2\pi L_*^2$ area (fund. domain) $= \frac{\pi^2 L_*^2}{6}$ $\quad$ where $g_{\tau\bar\tau} = \frac{1}{4(\mathrm{Im}\,\tau)^2}$

$L_* = 150 \Rightarrow N_{tot} = 37011$ $\quad$ i.e. 0,5% error

$N(g_s < g_*) = N(\mathrm{Im}\,\tau > \frac{1}{g_*}) \sim$ area (fund. domain above $1/g_*$) $\sim \int_{1/g_*}^{\infty} \frac{d(\mathrm{Im}\,\tau)}{(\mathrm{Im}\,\tau)^2} = g_*$

generalizations: distrib. of SUSY vacua in general IIB case

— " — cosmol. const. $\Lambda = -3e^K |W|^2$

— " — scalar masses

— " — non SUSY vacua, distrib of SUSY breaking s

$$dN \sim M_{susy}^{10}\, dM_{susy}^2$$

In general case $\qquad W = N^\alpha \Pi_\alpha(z) \qquad K = -\ln \bar\Pi_\alpha \eta^{\alpha\beta} \Pi_\beta$

(e.g. compactification of M-theory) $\frac{1}{2}N^2 = \frac{1}{2} N^\alpha \eta_{\alpha\beta} N^\beta \le L_* = \frac{\chi_4}{24}$

$N_{vacua} = \sum_N \theta(L_* - \frac{1}{2}N^2) \int d^{2n}z\, \delta^{2n}(DW)\, |\det D^2 W|$

$\mathcal{I} = \quad$ — " — $\quad \delta^{2n}(DW)\ \det D^2 W$ rough approx. for

we change $\sum_N$ into $\int$, $\theta() = \frac{1}{2\pi i}\int_p \frac{ds}{s} e^{s(\ldots)}$, $\quad \delta(\ ) = \int d^{2n}\lambda\, e^{i\pi(\lambda^i D_i \Pi_\alpha \ldots}$

$\det D^2 W = \int d^n\psi\, d^n\bar\psi\, d^n\theta\, d^n\bar\theta\, \exp(\psi^i \theta^j D_i D_j \Pi_\alpha N^\alpha + \bar\psi^i \bar\theta^j \bar D_i \bar D_j \bar\Pi_\alpha N^\alpha + c$

integrate out $N \to \mathcal{I} = \int \frac{d\psi d\theta}{2\pi i} \int \frac{ds}{s} e^{sL_*} Z(s)$, $\quad Z(s) = \left(\frac{2\pi}{s}\right)^{K/2} \exp[\frac{1}{2s}(i\pi\lambda^i D_i \Pi_\alpha - c.c.$
$+ \psi^i \theta^j D_i D_j \Pi_\alpha + \bar\psi \bar\theta^j \bar D_i \bar D_j \bar\Pi$

after manipulation

$Z(s) = \int d\psi\, d\theta \exp[\frac{1}{2s}(g_{i\bar j} \lambda^i \bar\lambda^j + R_{i\bar j k \bar l} \psi^i \bar\psi^j \theta^k \bar\theta^l + g..g..\theta..)$ $\quad$ w.r. like

$$\Rightarrow \boxed{\mathcal{I} \cong \frac{(2\pi L_*)^{K/2}}{(K/2)!} \int_{\text{moduli space}} \frac{1}{\pi^n} \det(R + \omega \mathbb{1})}$$

R.. curvature form, ω.. Kähler form

Geometry & dynamics of very early universe 2

to get effect. cosmological const. (i.e. with $p \approx -\varepsilon$)
one needs scalar field (otherwise anisotropy would arise)

$\mathcal{L} = \frac{1}{2}\varphi_{,\mu}\varphi^{,\mu} - V(\varphi) \quad (+\frac{1}{3}\xi R\varphi^2)$ ← GR term, forget it temporarily

$T^\mu{}_\nu = \varphi^{,\mu}\varphi_{,\nu} - \delta^\mu{}_\nu\left(\frac{1}{2}\varphi_{,\sigma}\varphi^{,\sigma} - V(\varphi)\right)$

$T^0{}_0 = \frac{1}{2}\dot\varphi^2 + V(\varphi), \quad p = -T^1{}_1 = -T^2{}_2 = -T^3{}_3 = \frac{1}{2}\dot\varphi^2 - V(\varphi)$

$\frac{1}{2}\dot\varphi^2 \ll V(\varphi) \Rightarrow p \approx -\varepsilon$

if $ds^2 = dt^2 - a^2(t)\,d\vec{x}^2$, $\Box = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\mu}\sqrt{g}\,g^{\mu\nu}\frac{\partial}{\partial x^\nu}\varphi$

$\boxed{M_p^2 = G}$

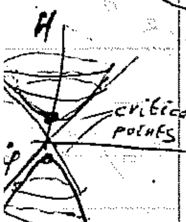
$\Rightarrow \ddot\varphi + 3H\dot\varphi + V_{,\varphi} = 0 \qquad 3H^2 = \frac{8\pi}{M_p^2}\left(\frac{1}{2}\dot\varphi^2 + V\right)$

$\Rightarrow \dot H = -\frac{4\pi}{M_p^2}\dot\varphi^2 \qquad H = \frac{\dot a}{a}$

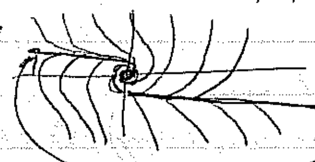
$\frac{d\varphi}{dt} = \dot\varphi \qquad \frac{d\dot\varphi}{dt} = -3\dot\varphi H - V' \qquad \frac{dH}{dt} = -\frac{1}{2}\dot\varphi^2 + \frac{1}{3}V - H^2$

using method of phase portrait, let's assume $V = \frac{m^2\varphi^2}{2}$

2dim hyperboloid in 3D (for $\varphi, \dot\varphi, H$)



in $\varphi, \dot\varphi$ plane



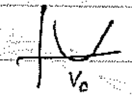
→ any soln. approaches separatrix and moves along it to asymptotic inflationary state

– since phase portrait is topological, the general behaviour will not change if V is perturbed

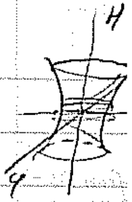
Initially $\ddot\varphi, 3H\dot\varphi \gg V \quad \varphi \sim \log t \quad p \approx \varepsilon, \ \varepsilon \sim \frac{1}{a^6}, \ a \sim t^{1/3}$

later ~~finally~~ $\ddot\varphi \ll 3H\dot\varphi, \ p \approx -\varepsilon, \ \dot H \sim H^2 \qquad a = e^{\int H dt}$ ∴ inflation

finally, along separatrix pressure oscillates $p \approx -\frac{m^2\varphi_0^2}{a^3}\cos 2mt \Rightarrow \langle p\rangle = 0$

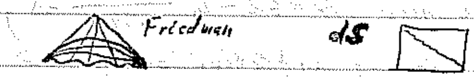


if $V(\varphi)$ is slightly negative $\Rightarrow$ hyperboloid in $\varphi, \dot\varphi, H$

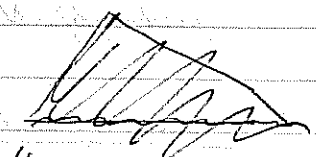


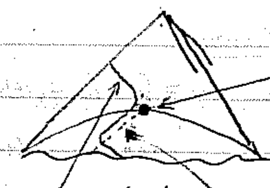
$\Rightarrow$ originally expanding universe reaches $H<0$ and recollapses

Penrose diagrams



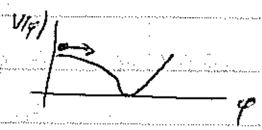
Friedman now, but with inflation





structure of horizons locally like in dS

Inflation potentials 1) SU(5) GUT



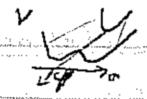
but param $\lambda \approx 0.1$ which observationally should be $\lambda \approx 10^{-\text{big}}$

2) chaotic inflation $\quad V(\varphi) = \frac{m^2\varphi^2}{2} \quad m = 10^{-5}M_p$

3) $\quad V(\varphi) = \frac{\lambda\varphi^4}{4} \quad \lambda \sim 10^{-13}$

4) power-law inflation $\quad V(\varphi) = e^{-\varkappa\varphi} \quad \varkappa = \sqrt{\frac{16\pi}{p}}, \quad p \gg 1$

5) hybrid inflation $\quad$ 2 fields



can arise from D3-D̄ in string theory

Slow-roll inflation $\quad \ddot\varphi \ll 3H\dot\varphi, \ a \approx a_0 e^{\int dt H}, \ N = \int dt H = \#$ of e-folds

$N = \int\frac{d\varphi}{\dot\varphi}H = \int\frac{d\varphi\,H^2}{V_{,\varphi}} \sim \frac{8\pi}{M_p}\int\frac{d\varphi\,V(\varphi)}{V'(\varphi)} \sim 8\pi\frac{\varphi_0^2}{M_p^2}$ ← initial value of φ

$\Rightarrow$ duration of inflation doesn't depend much on m, λ, $N \approx 65$ for $\varphi_0 \sim$ fi

In string cosmology ξ in $\xi R\varphi^2$ appears to be generically $\xi \sim \frac{1}{6}$

if $V=0$ $\quad \Box\varphi + \frac{1}{6}R\varphi = 0$ we may conf. rescale $g \to \Omega^2\tilde g_{\mu\nu} = a_0^2\eta_{\mu\nu}, \ \varphi \to \varphi/\Omega \Rightarrow \tilde\Box\tilde\varphi = 0$,

$\ddot\varphi + 3H\dot\varphi + V_{,\varphi} + 2H^2\varphi = 0 \Rightarrow \ddot\varphi \sim H\varphi$ not any more slow roll

$\varphi \sim \frac{const}{a}$ $\quad N = \int dt H \sim \int \frac{d\varphi}{\varphi} \sim \log \frac{\varphi}{\varphi_0} \sim \Delta\varphi =$

from D3-$\overline{D3}$ collision we get $\varphi \sim e^{-\gamma/R}$ $=37$

is $N=37$ enough to describe homogeneity of our universe

We need $N = 62 - \ln \frac{10^{16} GeV}{V^{1/4}} - \Delta$, $\quad V$.. inflation scale

$\Rightarrow$ if inflation occurs at TeV scale we

need only $N \sim 34 \Rightarrow N=37$ might be o.k.

Fluctuations generated by inflation

$\varphi = \varphi(t) + \delta\varphi \qquad \Box\varphi + m^2\varphi = 0, \; \delta\varphi = \int d^3k (a_k f_k(t) e^{i\vec{k}\vec{r}} + h.c.)$

guantum fluctuation

in Friedman $\langle\varphi^2\rangle = 0 = \langle T^\mu{}_\nu\rangle$

in dS $\langle\varphi^2\rangle = \frac{3H^2}{8\pi^2 m^2}$ if $m \to 0$ diff. calculation,

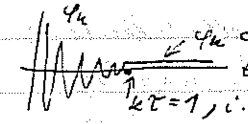
in diff slice of dS $\Rightarrow$ modes $\varphi_k = \frac{\sqrt{\pi}}{2} H(\tau)^{3/2} H_\nu^{(1)}(k|\tau|)$

Hankel fcn

selected so that if $\tau \to -\infty \quad \varphi_k \sim \frac{1}{\sqrt{2k}} e^{ik|\tau|}$

$\Rightarrow \tau \to 0 \quad \varphi_k \sim \frac{H}{\sqrt{2} k^{3/2}} \Rightarrow \langle\varphi_k^2\rangle \simeq \int d^3k |\varphi_k|^2 = \frac{H^3}{4\pi} t$

$\Rightarrow$



$\Rightarrow$ the modes freeze out in differen times

if $m > 0$ then the tail goes gradually to 0

Fluctuations of the metric

metric related to φ

fluctuations small – linear version of E.r., in

longitudinal gauge $\quad ds^2 = (1+2\phi)dt^2 - (1-2\phi)d\vec{x}^2$

introduce $u = a\delta\varphi \Rightarrow u'' + (k^2 - \frac{z''}{z})u = 0, \; z = \frac{a\dot{\varphi}}{H}$

$\Rightarrow$ oscillator behaviour of ~~[illegible]~~ fluctuations

originally starts to exponentially grow (at least part of it)

after certain t, $\quad |\phi_k|^2 = \left|\frac{u_k}{z}\right|^2_{k=aH}$

$\Rightarrow$ it is possible to calculate fluctuations of metric.

Quantum fluctuations may become strong and overcome global background

Aharony: $\boxed{\text{Non AdS/non CFT correspondence 1}}$

(1) Introduction $\qquad$ (2) Deformations

(3) Compactifications $\qquad$ (4) Cascades $\qquad$ $\boxed{\text{hep-th/02121}}$

(1) String theory $\Leftrightarrow$ field theory

$g_s \sim \frac{1}{N} \quad \Leftrightarrow$ Large N limit of $SU(N)$ (fixed $\lambda = g_{YM}^2 N$)

or guantum theory $\xleftrightarrow{holography}$ FT

1st example of such correspondence: AdS/CFT

IIB on $AdS_5 \times S^5 \Leftrightarrow \mathcal{N}=4$ SYM $SU(N)$

$\frac{R_{AdS}^4}{\alpha'^2} = \lambda = 4\pi g_s N \quad \Leftrightarrow \quad \lambda = g_{YM}^2 N$ non-running continuous;

$\lambda \to 0$ strongly curved ~~coupled~~ $\Leftrightarrow$ weakly coupled

$\lambda \to \infty$ weakly curved ~~coupled~~ ($\Rightarrow$ SUGRA) $\Leftrightarrow$ strongly coupled

Phenomenologically more interesting $SU(3)$ YM, i.e. QCD at strong

weakly coupled string theory (may be even SUGRA) NOT

via $U(N)$, $N \to \infty$ firstly, then understanding N expansion enough to be able to extrapolate to $N=3$ ($\Rightarrow$? confinement, mass gap,

– in SUGRA gap between scale of SUGRA modes and string excitations, not in QCD

To get SUGRA limit one may modify QCD at scale $\Lambda_{new} \gg \Lambda$

and then find properties invariant under change of Λ_{new}/Λ

study $\Lambda_{new}/\Lambda_{QCD} \to 1 \Rightarrow$ SUGRA description possib

(2) Deformation of CFT, namely $\mathcal{N}=4$ SYM in $d=4$

$A_\mu, \lambda^a, \varphi^i \quad a = 1,..,4, \quad i = 1,..,6$, to get QCD we add masses

$\mathcal{L} + m^2 tr(\varphi^i\varphi^i) + n\, tr(\lambda^a\lambda^a) + c.c.$

at $\lambda \to 0 \qquad \Delta = 2 \qquad\qquad \Delta = 3 \quad \to$ relevant deforms.

$\lambda \to \infty \qquad ? \qquad\qquad ? \quad ?$

Note: chiral primary ops $\mathrm{tr}(\varphi^{i_1}\dots\varphi^{i_k})$ ← symm. traceless $SO(6)$, also $Q\,\mathrm{tr}(\varphi^{i_1}\dots\varphi^{i_k})$

$\Delta = k$ is protected by SUSY, i.e. not running

(stringy excitations, not SUGRA state) ⟷ not-chiral e.g. $\mathrm{tr}(\varphi^i\varphi^i)$ $\Delta = 2 + (.)\lambda + (.)\lambda^2 \dots$

$\Rightarrow$ $n\,\mathrm{tr}(\lambda^a\lambda^a)$ has $\Delta = 3$ also for $\lambda \to \infty$

$m\,\mathrm{tr}(\varphi^i\varphi^i)$ $\Delta \gtrsim \lambda^{1/4}$ as $\lambda \to \infty \Rightarrow$ not relevant op.

$\Rightarrow$ we add $\mathcal{L}_{CFT} + (n_1 \mathrm{tr}\,\lambda^1\lambda^1 + n_2 \mathrm{tr}\,\lambda^2\lambda^2 + n_3 \mathrm{tr}\,\lambda^3\lambda^3 + n_4 \mathrm{tr}\,\lambda^4\lambda^4) + c.c.$)

only, mass of scalars is generated in higher loops.

Simplest case: $\mathcal{N}=1$ SUSY preserved $\Rightarrow n_4 = 0,\ n_1, n_2, n_3 \neq 0$

$$W = \frac{2\sqrt{2}}{g_{YM}^2}\,\mathrm{tr}\left(\Phi_1[\Phi_2,\Phi_3]\right) + \frac{1}{g_{YM}^2}\left(n_1 \mathrm{tr}(\Phi_1)^2 + n_2 \mathrm{tr}(\Phi_2)^2 + n_3 \mathrm{tr}(\Phi_3)^2\right)$$

$\Rightarrow$ scalars have same masses as fermions $\Rightarrow$ no tachyons arise

Naively $\varepsilon \ll n_i \rightsquigarrow \mathcal{N}=1$ SYM but what is

relation between dynamically generated Λ_{SYM} and n_i

RG flow $\Rightarrow \Lambda_{SYM}^3 = n_1 n_2 n_3\, e^{-\frac{8\pi^2}{g_{YM}^2 N}}$

$\lambda = g_{YM}^2 N \ll 1 \Rightarrow \Lambda_{SYM} \ll n_i$ naive concl. o.k. but no SUGRA description

$\lambda = g_{YM}^2 N \gg 1 \Rightarrow \Lambda_{SYM} \sim n_i \Rightarrow$ problem

Many discrete SUSY $\overbrace{\text{vacua}}^{\text{classical}}$ $\frac{\partial W}{\partial \varphi^i} = 0$ / complexified gauge group $SL(N,\mathbb{C})$

put $n_i = n$: $\frac{\partial W}{\partial \Phi_i} = 0 \Leftrightarrow [\Phi_i, \Phi_j] = -\frac{n}{\sqrt{2}}\,\epsilon_{ijk}\Phi_k \Rightarrow$ def. of $SU(2)$ representation, N-dim.

unitary transf. $\Rightarrow \Phi$ block diagonal $\left(\begin{smallmatrix} \delta^{n_1} & 0 \\ 0 & \ddots \end{smallmatrix}\right)$ $\sum n_i = N$

$n_1 = N$ $SU(N)$ completely broken "Higgs vacuum"

in all other vacua some part of $SU(N)$ preserved, if $n_i = 1 \to \varphi_i = 0$

$\to$ classically $SU(N)$ preserved, QM N diff. vacua (for $n_i = 1$)

("confining vacua") distinguished by $\langle \mathrm{tr}(\lambda\lambda)\rangle$ (gluino condensate)

i.e. QM even more vacua

other interesting $n_i = q$, $p \cdot q = N$ $SU(N) \xrightarrow{\text{Higgs}} SU(p)$

Note: $\cancel{SU(N)} = SU(N)$ breaking
$\cancel{SUSY} = SUSY$ breaking etc.

$\xrightarrow{QM}$ p different vacua

other vacua – at least $U(1)$ preserved $\to$ no mass gap

If we break SUSY, i.e. $n_4 \neq 0$, we get one vacuum

but may have multiple metastable vacua.

String dual $\to$ specific boundary conditions

$ds^2_{AdS} = \frac{dx_\mu dx^\mu + dz^2}{z^2}$ $\quad \mathcal{O}_i(x) \Leftrightarrow \chi_i(x,z)$

$\left\langle e^{i\int \lambda_i(x)\mathcal{O}_i(x)}\right\rangle_{\substack{\mathcal{N}=4\\ SYM}} \Longleftrightarrow Z_{IIB}\left(\chi_i(x,z) \xrightarrow{z\to 0} \lambda_i(x)\, z^{4-\Delta_i}\right)$.

$\mathcal{L} + n_a \underbrace{\mathrm{tr}(\lambda^a\lambda^a)}_{\mathcal{O}_i} + \dots$ (coming from SUSY) $\quad \mathcal{O}_i \Leftrightarrow \chi_i = \dots \begin{cases} B_{ab} \in S^5 \\ C_{ab} \end{cases}$ some spherical harm. corresp. to

$\Rightarrow$ dual to is SUGRA ($\mathcal{N}=1$ SUSY) solution with $\chi_i(x,z) \xrightarrow{z\to 0} n z$

unfortunately this leads to singular SUGRA solutions

$\Rightarrow$ ~~SUGRA~~

Mathur: $\boxed{\text{Quantum structure of black holes 1}}$

Puzzles: entropy, information $\boxed{\text{hep-th/05020}}$

Entropy puzzle $\frac{dS_m}{dt} < 0$?? after throwing gas into

$S_{BH} = \frac{Ac^3}{4G\hbar} \to \frac{A}{4G}$ $\quad [c = 1, \hbar = 1]$

$\Rightarrow \frac{dS_m}{dt} + \frac{dS_{BH}}{dt} \geq 0$ $\quad$ 2nd law of thermodynamics save

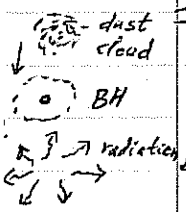
but $S_{BH} = \ln(\#\text{states}) \to$ BH should have $e^{S_{BH}}$ states

for given M, J, Q (e.g. solar mass $10^{10^{77}}$ states)

But "Black holes have no hair" $\Rightarrow S = \ln[1] = 0$???

Puzzle: Where are the states of BH?
- at the horizon?, at the singularity?, everywhere inside?

Information puzzle $\quad dS = \frac{1}{T} dE$



Schwarzschild in 3+1 $\quad S = \frac{4\pi (2GM)^2}{4G} \Rightarrow \frac{1}{T} = 8\pi GM$

Hawking: Black holes radiate with temperature T

BUT radiation has no information about initial matter so we lose the reversibility of quantum mechanics

Hawking "theorem" is very robust
if (a) all quantum gravity effects are confined to within a given distance, e.g. $\ell < l_p, l_s$
(b) vacuum is unique (in bounded volume finite # of states with $E_{min} < E < E_{max}$, $E_{min} \to E$ as $E \to \infty$)
then black hole radiation will cause information loss.

String theory $\to$ indications that (a) may be false
since to make BH requires lot of quanta $N \gg 1$
might be QGravity effects $\sim N^\alpha l_{Planck} \sim$ radius of horizon

BH in string theory (Strominger, Vafa)
concentrated mass using only objects present in string theory
e.g. IIA on $M_{9,1} \longrightarrow M_{8,1} \times S^1$ ($M_{p,q}$ = Minkowski)
n_1 NS1 branes (fundamental strings) $\longrightarrow$ Bound state "long string"
in $M_{8,1}$ looks like $\quad$ mass = (winding) charge

$$ds^2_{string} = H_1^{-1}[-dt^2 + dy^2] + \sum_{i=1}^{8} dx_i dx_i \quad \text{in stringy frame}$$
$$e^{2\phi} = H_1^{-1}, \quad H_1 = 1 + \frac{Q_1}{r^6} \quad (n_1 \approx Q_1)$$

Entropy $\quad S_{Bek} = \frac{A_9}{4G_9} = \frac{A_{10}}{4G_{10}} = \frac{A_{11}}{4G_{11}}$
all areas in Einstein frame, $\quad A_{10} = A_9 L, \; G_{10} = G_9 L$

$\Rightarrow S_{Bek} = 0, \quad S_{micro} = \ln[256] \sim 0$ (doesn't grow wit[h]

Why was $A = 0$ in M-theory (11d) $\quad \times x^{11}$

Branes: shrink directions along brane, expand directions $\perp$ brane
$x^{11} \to 0 \Rightarrow \phi \to -\infty \quad y = x^9 \to 0 \Rightarrow A_{11} \to 0, S_{Bek} \to 0$
to prevent x^{11} from shrinking (and $\phi \to -\infty$) put M5-brane $\perp$ to $x^{11} \Rightarrow$
also NS5 in IIA $\quad M_{9,1} \to M_{4,1} \times \underbrace{T^4 \times S^1}_{NS5}$ (S^1: NS1)

$\Rightarrow \; e^{2\phi} = H_5 / H_1$
$H_1 = 1 + \frac{Q_1}{r^2} \qquad H_5 = 1 + \frac{Q_5}{r^2}$
$$ds^2 = H_1^{-1}[-dt^2 + dy^2] + H_5 \sum_{i=1}^{4} dx_i dx_i + \sum_{a=1}^{4} dz_a dz_a$$

but still y shrinks $\to$ add momentum P along Y, $E_p = \frac{M}{R_y}$
$K = \frac{Q_p}{r^2} \quad ds^2$ changed only inside $[\;] \to [-dt^2 + dy^2 + K(dt + d$…

$\Rightarrow$ stable black hole configuration with 3 charges

near $r = 0$ transverse space $\quad H_5 \sum_{i=1}^{4} dx^i dx^i \approx Q_5 \left(\frac{dr^2}{r^2} + d\Omega_3^2\right)$
$\Rightarrow A^{S^3}_{string} = 2\pi^2 Q_5^{3/2}$
$A^E_{10} = 2\pi^2 (2\pi)^4 V \cdots (Q_1 Q_5 Q_p)^{\#}$ $\leftarrow$ probably 1/2

$Q_1 = \frac{g^2 \alpha'^3}{V} n_1, \; Q_5 = \alpha' n_5, \; Q_p = \frac{g^2 \alpha'^4}{V R^2} n_p \Rightarrow \boxed{S_{Bek} = 2\pi \sqrt{n_1 n_5 n_p}}$

all moduli V, R, g had cancelled out - entropy of BPS holes
(without cancellation we would not be able to explain S_{Bek} from microscopic description)

Dualities e.g. $\quad$ NS1-NS5-P (IIA) $\xrightarrow{T_6}$ NS1-NS5-P (IIB) $\xrightarrow{S}$ D1-D5-P (IIB)

by composition of many T, S e.g. $\quad$ D1-D5 (IIB) $\rightarrow$ P-NS1 (IIB)
- all 2charge BH can be connected in this way

Microscopic count of states
NS1 (n_1 turns, L) $\quad L_T = n_1 L \quad$ one long string $\quad S_{micro} = 0$
2 charges NS1-P $\quad$ boundstate $\quad$ travelling wave on str[ing] with momentum P

momentum partitioned among different Fourier modes

$e_k = p_k = \frac{2\pi k}{L_P}$, total momentum $p = \frac{2\pi n_p}{L} = \frac{2\pi n_1 n_p}{L_T}$

$\Rightarrow \sum_i m_i k_i = n_1 n_p$ (m_i units of harmonic k_i)

partitions of (large) $n_1 n_p$ $\quad e^{2\pi\sqrt{\frac{n_1 n_p}{6}}}$

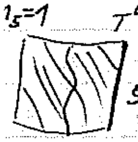
BUT 8 transverse directions, SUSY $\Rightarrow$ 8 fermionic degrees of freedom, behave like 4 bosonic d. of freedom

$\Rightarrow$ effectively 12 d. of freedom $\Rightarrow$

$[e^{2\pi\sqrt{\frac{n_1 n_p}{6 \cdot 12}}}]^{12} = e^{2\pi\sqrt{2}\sqrt{n_1 n_p}}$

$S_{micro} = \ln[\#] = 2\pi\sqrt{2}\sqrt{n_1 n_p}$

$l_s = 1$ $\quad T^4$



S^1

3charges NS1-P-NS5

NS1 can vibrate only in T^4, i.e. stuck to NS5

$\Rightarrow$ 6 eff. bosons $\Rightarrow S_{micro} = 2\pi\sqrt{n_1 n_p} \Longrightarrow 2\pi\sqrt{n_1 n_5 n_p}$ duality implies symm. between n's

$\Rightarrow \boxed{S_{Bek} = 2\pi\sqrt{n_1 n_5 n_p} = S_{micro} \;\; \boxed{\text{o.k.}}}$

Note: in 2 charge case there are R^2 corrections changing effect. A

$\Rightarrow$ then also $S_{Bek} = S_{micro}$, whereas without corrections $S_{Bek} = 0$

What do the microstates look like?

e.g. NS1-P fundamental strings have no longitudinal vibration modes

$\Rightarrow$ transverse vibr. $\rightarrow$ microstate not concentrated at $r=0$, instead

classical limit for $n_1, n_p \to \infty$ microstates are "spread out" over $r \leq r_0$

"Naive" 2-charge NS1-P geometry is singular at $r=0$ $\Rightarrow$ may $\not\exists$ in full string theory

$\vec{F}(v)$

$\vec{F}(t-y)$

$\rightarrow$ correction, for a single strand carrying a travelling wave

$ds^2 = H\,[-du\,dv + K du^2 + A_i\, dx^i dy]$

A_i, K, H involve $\vec{F}(v)$, for many strands $\sum \frac{-\dot{F}^{(s)}(0)}{\ldots}$

many strands, end of one beginning of next

instead of $\sum \rightarrow \frac{1}{L_p}\int_0^{L_p} \frac{F(v)}{\ldots}\, dv$

Dualities, e.g. NS1-P $\rightarrow$ D1-D5 — since dualities are in compact directions, the noncompact directions remain the same.

Shenker

$\boxed{\text{Behaviour of horizons 1}}$

Maldacena hep-th/ 0106112 $\quad$ FHKS hep-th/ 0306170

Description of behind horizon metric from outside horizon measurements via analyticity assumption - should be reasonable due to symmetries of usual solns.

In string theory - use D-branes as probes of geometry - suitable because heavy as $g_s \to 0$, small

Ex: Eternal AdS-Schw. BH in $D=3$ (BTZ, i.e. orbifold of AdS)

$ds^2 = R_{AdS}^2\left(-f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\theta^2\right)$, $f(r) = (r^2 - r_+^2)$

put $R_{AdS} = 1$ $\Rightarrow$ by scaling $r_+ = 1$, $f(r) = r^2 - 1$

Penrose diagram

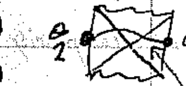
singularity: BH

white hole

horizons

CFT$_1$ and CFT$_2$ are entangled together $|\psi\rangle = \sum_n e^{-\beta E_n/2}|n\rangle_1 |n\rangle_2$

$\langle\psi|\theta_1 \tilde{\theta}_1|\psi\rangle = \# \sum_{n,\tilde{n}} e^{-\beta(E_n \tilde{E})} \langle\tilde{n}|\theta_1 \tilde{\theta}_1|n\rangle_1$



$\langle\psi|\theta_1 \theta_2|\psi\rangle = ?$

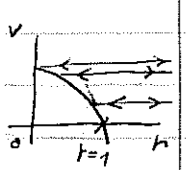
, but the result shouldn't due to

Geodesics in AdS-Schw. – extremize $S=\int d\tau \frac{\partial x^\mu}{\partial\tau}\frac{\partial x_\mu}{\partial\tau}$

radial geodesics $r(\tau), t(\tau)$ $S=\int d\tau\left(-f(r)\dot t^2+\frac{1}{f(r)}\dot r^2\right)$

$\Rightarrow E=f(r)\dot t$ is const. along geodesics, in unit speed parametrization

$\dot y^\mu \dot y_\mu = 1 = -f(r)\dot t^2+\frac{1}{f}\dot r^2 = \frac{-1}{f}E^2+\frac{1}{f}\dot r^2 \Rightarrow E^2=-f(r)+\dot r^2 \Rightarrow$ class mechanics



$\Rightarrow$  $\Rightarrow$ geodesics arbitrarily closed to singularity

$\Rightarrow \mathcal{L}=2\ln\cosh(t)+\text{const.}$ – distance along geodesics

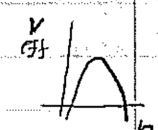
let's assume $t_1=t_2=t$ $\langle\phi\phi\rangle(t)=e^{-m\mathcal{L}}=\frac{1}{(\cosh t)^{2m}}$

Compare directly to CFT $r_+\to\infty, T\to\infty, \theta\to X$

in Euclidean $\langle\phi\phi\rangle_\beta = \frac{1}{(\cosh^2 t)^\Delta}$ $\Delta=m+\mathcal{O}(1/m)\ldots$ ok, match

$\Rightarrow$ constraints on geometries infalling observer can see.

For general geodesics another conservation $L=r^2r_+^2\dot\theta$



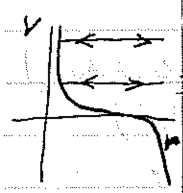
$\Rightarrow V_{eff}(r)=f(r)-\frac{L^2}{r_+^2r^2}$ $\Rightarrow$ complications

e.g. for slowly shrinking circles tachyon may arise

AdS-Schwarzschild in D=5

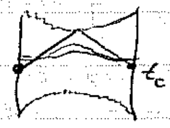
very large BH $T\to\infty, r_+\to+\infty$ $ds^2=-f(r)dt^2+\frac{1}{f(r)}dr^2+r^2r_+^2d\Omega^2$

$f(r)=r^2-\frac{1}{r^2}$, consider radial geodesics ($\Omega=$const.)



$V=-f$ if $r_{turn}\to 0 \Rightarrow E\to\infty, \dot r\to\infty, \Delta\tau\to 0$

– almost null geodesics $\Rightarrow 45°$ in Penrose diag.



$\Rightarrow$ singularity is no longer straight line in PD

$\langle\varphi\varphi\rangle_{t\sim t_c}\sim e^{-m\mathcal{L}}\sim\frac{1}{(t-t_c)^{2m}}$

BUT the almost null geodesics is no longer dominant in semiclassical analysis $\Rightarrow$ (for $m\to\infty, g_s\to 0$) $\langle\varphi\varphi\rangle$ has branch cut, singularity is on second Riemann sheet

– pole on 2nd sheet can be viewed as signature of BH singularity

Mateos

Black rings & supertubes 1 see lecture notes

Non AdS / non CFT correspondence 2

Hints: 1) H_3, F_3 grow in interior $\to$ may resolve singularity – need source for them D5, NS5

2) Higgs vacuum if we give masses to ϕ_i

$\langle\phi_i\rangle\neq 0 \to$ D3-brane in bulk arises via AdS/

$\sum\langle\phi_i\rangle^2\simeq n^2N^2\cdot\mathbb{1}$

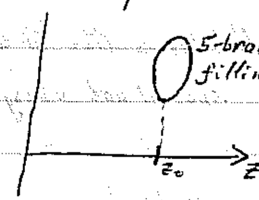
$\to$ eigenvalues of ϕ_i sit on a 2-sphere

(more exactly "fuzzy sphere" $z_0\sim\frac{1}{\alpha' nN}$)

$\Rightarrow$ possible to replace N Dp-branes by D(p+2)-brane on S^2

$AdS_5\times S^5$

$\int_{S^5}F_5=N$



$\Rightarrow$ correct limit should be SUGRA + D-b

qualitative arguments known for existence, exact form not in regime $g_s\ll 1, g_sN\gg 1$

Similarly for intermediate vacua $\Rightarrow$ p D5-branes sources SUGRA re

– should be feasible for $g_sp\ll 1, g_sN\gg 1$

Confining vacua – we naively need $g_sN\ll 1$ $\wedge g_sN\gg 1$

BUT

$\tau=\frac{4\pi i}{g_{YM}^2}+\frac{\theta}{2\pi}$ $\tau\to-\frac{1}{\tau}$ E-M duality

Higgs vacuum: electric condensate (electric screened, magnetic confi

$\updownarrow \tau\to-\frac{1}{\tau}$

magnetic condensate $\Rightarrow$

$\Rightarrow$ Higgs vacuum $\xleftrightarrow{\tau\leftrightarrow-\frac{1}{\tau}}$ Confining vacuum

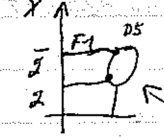
We know how to realize duality also in string theory

S-duality $D5 \leftrightarrow NS5 \Rightarrow$ confining vacuum contains one NS5-brane

<u>Mass gap</u>

look for $e^{ik_\mu x^\mu} f(z,\Omega)$ solns in D-brane background (metric, F, …)

$M^2 = -k^2 \Rightarrow$ discrete spectrum of masses



<u>Confinement, screening</u> ? (in dual, stringy NS5 description)

Higgs vacuum $\Rightarrow$ screening

$E \sim T_S R \Rightarrow$ confinement

lowest energy configuration for given $q, \bar{q}$ positions

Instanton, baryons, condensates – all behaves like in QCD, qualitative properties should not change as $\lambda \to \infty$

<u>Advantages, disadvantages of this method</u>

⊕ 1) Simple field theory interpretation

⊕ 2) Simple non-SUSY backgrounds generalization

⊖ 3) At large λ not pure SUGRA, we had to add branes, NS5-branes not described by perturbative string theory (solitons)

(3) <u>Compactifications</u>

e.g. – N D4-branes on S^1, anti-periodic fermions

or – N NS5 branes on $S^2 \Rightarrow \mathcal{N}=1$ SYM

↳ weakly coupled + curved background → promising

<u>NS5 in flat space (in IIB)</u> S-duality to D5-branes

D5 weak coupling limit g_{YM} fixed $l_p \to 0$

$g_{YM}^2 \sim g_s (\alpha')^{\frac{p-3}{2}}$ $l_p^8 \sim g_s^2 \alpha'^4 \Rightarrow$ for $p=5$: $g_s \to \infty, \alpha' \to 0$

$\xrightarrow{\text{S-duality}}$ NS5 $g_s \to 0$ α' fixed $\to SU(N)$, $g_{YM}^2 = \alpha'$ "$\underline{\text{little string th}}$…"

(since α' fixed it is not ordinary SU(N) YM, e.g. continuous spectrum above gap, T-duality etc., …

Holographic dual of little ST (LST)

$ds^2 = dx_{\mu}^2 + d\rho^2 + N d\Omega_3^2$ $g_s = e^{-\rho/\sqrt{N}}$, $\int_{S^3} H_3 = N$

LST compactified on S^2, by choice of spin structure $\Rightarrow \mathcal{N}=1$ SY… + KK modes on S^2 + LST modes

$g_{YM,4d}^2 \Big|_{\text{scale } 1/R_{S^2}} = \frac{\alpha'}{R_{S^2}^2}$ $\Lambda_{SYM} \simeq \frac{1}{R_{S^2}} e^{-\frac{8\pi^2}{3 N \alpha'/R_{S^2}^2}}$

we want $\Lambda_{SYM} \ll R_{S^2} \Rightarrow \boxed{R_{S^2} \gg \sqrt{N\alpha'}} \Rightarrow$ weakly c…

Maldacena + Nuñez: $\rho \to \infty$ similar to before $\mathbb{R}^6 \to \mathbb{R}^4 \times S^2$, radius of $S^2 \sim \rho$

$\rho \to 0$ radius of $S^2 \to 0$ (like $d\rho^2 + \rho^2 d\Omega_2^2$)

$\mathbb{R}^7 \times S^3$ at $\rho = 0$ $\longrightarrow \rho$ $\sim \mathbb{R}^4 \times S^2 \times$ linear dilaton $\times S^3$

max. $\boxed{g_s = e^{\phi_0} \text{ at } \rho = …}$

$\forall g_s$ $\exists$ appropriate soln.

at $R^2 \gg N\alpha'$: $R^2/N\alpha' \sim \phi_0 \Rightarrow$ we are interested (to decouple SYM) in $\phi_0 \to \infty$

other limit $\phi_0 \to -\infty \Rightarrow$ very good SUGRA approximation, $G_{\mu\nu}$ … only, no RR-fields – good string description

<u>Mass gap</u> discrete spectrum from SUGRA

<u>Confinement</u> NS5 $\rho = \rho_0$ — D1 — $\rho = \infty$ the force between D1 branes $E \sim T_S R = \frac{1}{g_s^{\rho=\rho_0} \alpha'}$…

<u>Chiral SUSY breaking</u>

$\mathcal{N}=1$ SYM $U(1) \xrightarrow[\text{anomaly}]{} \mathbb{Z}_{2N} \xrightarrow[\text{vacuum}]{} \mathbb{Z}_2$

in SUGRA one can see $U(1)$ broken to $\mathbb{Z}_{2N}$ by looking on solns. in asymptotic … by investigating full solns. also $\mathbb{Z}_{2N} \to \mathbb{Z}_2$ turns out in S…

whereas at $\lambda \to 0$

<u>(Dis)advantages</u>: ⊖ high energy theory = LST mysterious, not well understoo…

⊕ …

Seiberg:

$\boxed{\text{Low-dimensional string theories 1}}$

Worldsheet perspective in 2 dimension

$$\mathcal{L}_{WS} = \frac{1}{4\pi\alpha'}\left[(\partial\varphi)^2 + (\partial X)^2 + 2\sqrt{\alpha'}\,\varphi R\right]$$ "enhancement" — X.. time, Euclid. Lorentz

$$\Rightarrow T = -\frac{1}{\alpha'}\left[(\partial\varphi)^2 + (\partial X)^2 + Q\,\partial^2\varphi\right]$$ dilaton $Q\varphi$

$\boxed{\alpha' = \frac{1}{2}}$

set $c = 26 = 1 + 1 + \alpha' Q^2 \cdot 6 \Rightarrow Q = \frac{2}{\sqrt{\alpha'}} = 2\sqrt{2}$ $\quad g_s = e^{\frac{Q}{2}\varphi}$

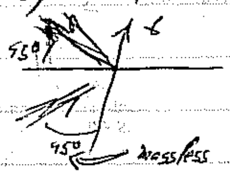
$\Delta(e^{ipx}) = \frac{1}{2}p^2 \quad \Delta(e^{\alpha\varphi}) = -\frac{1}{2}\alpha(\alpha - Q)$

particles:	graviton	$B_{\mu\nu}$	Dilaton	
degrees of freedom	-1	0	1	(graviton cancels against graviton)

"tachyon" massless scalar

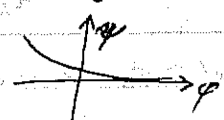
vertex op. $\underbrace{e^{-iE(t-\varphi)}}_{\text{right-moving (incoming)}}\ \underbrace{e^{\frac{Q}{2}\varphi}}_{g_s}$, $E>0$, $t = iX$

$e^{-iE(t+\varphi)}\,g_s$ left-moving (outgoing)



euclidean version

$e^{ipx + (\frac{Q}{2} - |p|)\varphi} \quad p \gtrless 0$

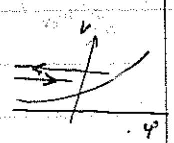


in spacetime $\mathcal{L}_{st} = \frac{1}{g_s^2(\varphi)}\left((\partial T)^2 - 2T^2 + ..\right) + ...$

define $\tilde{T} = e^{-\sqrt{2}\phi}T \Rightarrow \mathcal{L}_{st} = (\partial\tilde{T})^2 +$ total derivative $+ ...$

to get reflection off the strong coupling region

one turns on background $\langle T\rangle = \mu e^{\varphi}$, $V = \mu\varphi e^{\varphi} \Rightarrow g_s \sim \frac{1}{\mu}$ at $\varphi \sim -\log\mu$



Type 0 (fermionic) string

critical dimension $d = 10$, GSO $f_L(f_R)$ left (right) world-sheet fermions

NS-NS $(-1)^{f_L + f_R} = 1$, R-R $(-1)^{f_L + f_R} = \begin{cases} 1 & 0B \\ -1 & 0A \end{cases}$

0B NS "tachyon" $e^{-\varphi - \tilde{\varphi} + ipx + (1-|p|)\varphi}$ vertex op.

ghosts

c RR scalar $\partial_+ c \sim e^{-\frac{\varphi}{2} - \frac{\tilde{\varphi}}{2} + \frac{i}{2}(H + \bar{H}) + ipx + (1-|p|)\varphi}$ $\quad p>$

$\partial_- c \sim e^{-\frac{\varphi}{2} - \frac{\tilde{\varphi}}{2} - \frac{i}{2}(H + \bar{H}) + ipx + (1-|p|)\varphi}$ $\quad p<$

zero modes $c = \nu\varphi + \tilde{\nu}t = \overbrace{\nu_{in}(t-\varphi)}^{c_-} + \overbrace{\nu_{out}(t+\varphi)}^{c_+}$ created

by vertex ops $\sim \partial_+ c, \partial_- c$ for $p = 0$, $\nu, \tilde{\nu}$ specify

$\Rightarrow \mathcal{L}_{st} = e^{-2\varphi}\left((\partial T)^2 - T^2 + ..\right) + (\partial c)^2 + ... \quad \Rightarrow \tilde{T} = e^{-\phi}T$

Symmetries of the system 1) left, right fermion numbers

2) trivial $\mathbb{Z}_2\ (-)^E$, $E_L(F_R) = \#$ left (right) moving spacetime fermion in ... $T \to -T$, $c \to$

($\sim$ charge conjugation in spacetime) NS-NS $\to$ NS-NS, R-R $\to$ -

3) another $\mathbb{Z}_2 \quad (-1)^{S_L} \quad T \to -T,\ c_+ \to c_+,\ c_- \to -c_-$

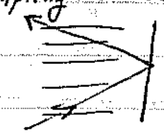
S-duality (duality in spacetime) $\Rightarrow \nu \leftrightarrow \tilde{\nu}$

4) $\mathcal{L}_{st} \ni (\partial c)^2 \ \to$ symmetry $c \to c +$ const.

S-duality $\Rightarrow$ c is compact $c \cong c + 2\pi R_0$ self-dual radi

5) At weak coupling $SU(2)_{incoming} \times SU(2)_{outgoing}$

toy model



comes from higher order c

S-duality in ST: $(\partial\tilde{T})^2 + e^{2T}(\partial c)^2 \leftrightarrow (\partial\tilde{T})^2 +$

exact symmetry even in interacting theory

Turning on background $\langle T\rangle = \mu e^{\phi}$

EoM $\partial(e^{2T}\partial c) = 0 \Rightarrow c = \nu\int^{\varphi} e^{-2T(\varphi)}d\varphi + \tilde{\nu}t$

$\|\partial c\|^2 = \int d\varphi\, e^{2T(\varphi)}\,\partial c^2$

if $\mu > 0 \Rightarrow \nu\int^{\varphi}..$ normalizable as $\phi \to +\infty$, $\tilde{\nu}t$ not suppressed $\|\nu\|$ as $\phi \to$

divergence indicates a presence of source - an object at $\varphi \Rightarrow$

$\mu < 0 \quad \tilde{\nu}t$ has finite norm as $\phi \to +\infty \Rightarrow$ not sourced by an object.

$\nu\int^{\varphi}$ has infinite norm as $\phi \to +\infty \Rightarrow$ sourced by — " —

$\Rightarrow$ small change of μ which is not significant at weakly coupled region is dramatical at strongly coupled region

Imagine $\tilde{\nu}=0, \nu\neq 0 \Rightarrow \mathcal{L} \supset (\partial c)^2 \sim \nu^2 \Rightarrow$

energy $= \log\Lambda\, \nu^2$ $\quad \log\Lambda$ .. volume of space = volume of φ

$\tilde{\nu}\neq 0, \nu=0$ $\quad$ energy $= \log\Lambda \cdot \tilde{\nu}^2$

both situations time independent

if both $\tilde{\nu}\neq 0\neq\nu$ $\quad T_{++}\sim \nu_{in}^2,\ T_{--}\sim \nu_{out}^2$

$\Rightarrow \int T_{++} \neq \int T_{--}$ $\quad$ ingoing flux is not the same as outgoing

to fix it, add more quanta

total energy $\frac{E}{\log\Lambda} \sim \max(\nu_{in}^2, \nu_{out}^2) \sim (|\nu|+|\tilde{\nu}|)^2$

0A system

NS tachyon $e^{\varphi-\tilde{\varphi}+ipx+(1-|p|)\varphi}$, in R-R sector only 0-momentum

fields $e^{-\varphi/2-\tilde{\varphi}/2 \pm i/2 (H-\bar{H})+\varphi}$ - vertex ops for 2 field strengths $F, \tilde{F}$

$\mathbb{Z}_2$ symmetries $\Rightarrow$ $\mathcal{L}_{ST} = e^{2T}F^2 + e^{-2T}\tilde{F}^2 +$ higher order in derivatives $\quad$ (S-duality $T\leftrightarrow -T, F\leftrightarrow \tilde{F}$)

$\Rightarrow F = q e^{2T}, \tilde{F} = \tilde{q} e^{-2T}$, turn on background as before

$\mu>0$ $\quad q$ is not sourced, $\tilde{q}$ is sourced by a brane

$\mu<0$ $\quad \tilde{q}$ - " - , q - " -

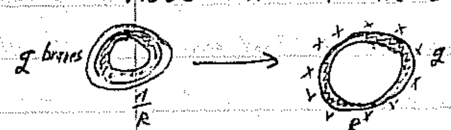
in this case $q, \tilde{q}$ quantised $\in \mathbb{Z}$ (Dirac charge quantisation)

$\frac{E}{\log\Lambda} \sim q^2\ (\tilde{q}=0)$ or $\sim \tilde{q}^2\ (q=0)$

when $q\cdot\tilde{q}\neq 0$ we need to add (to have constant bckgr.)

$q\tilde{q}$ fundamental strings along φ $\quad \frac{E}{\log\Lambda} \sim q^2+\tilde{q}^2+2|q\tilde{q}| = (|q|+|\tilde{q}|)^2$

Note: 0A is T-dual to 0B if we compactify Euclidean time



as $R\to\infty$ we must also send

$q\to\infty$ to have macroscopic effect

keeping $\nu = \frac{q}{R}$ fixed $\Rightarrow q$ can be quantised and ν not quantised

Quantum structure of black holes 2

Properties of these D1-D5 geometries (defined by $\vec{F}(v)$)

A) no horizons $\quad$ B) no singularities (only coordinate $\sim \frac{1}{|\vec{x}|}$, resemble KK monop

Example: particular profile $\vec{F}(v)$

- uniform helix with a single turn in covering space

$F_1 = a\cos\omega v, F_2 = a\sin\omega v$ $\quad \omega = \frac{2\pi}{L_T}$ $\quad F_3=F_4=0$

x_i $\quad i=1,..,4$ transverse space $\to r,\theta,\varphi,\psi \Rightarrow$

$H^{-1} = 1+\frac{Q_1}{r^2+a^2\cos^2\theta}$ $\quad$ (axial symmetry of $F \Rightarrow$ no ψ,φ,

$K = ..., A_i = ...$

$\Rightarrow$ explicit $ds^2_{string} = ...$ (5 lines :-)

$r\to\infty \to$ flat spacetime

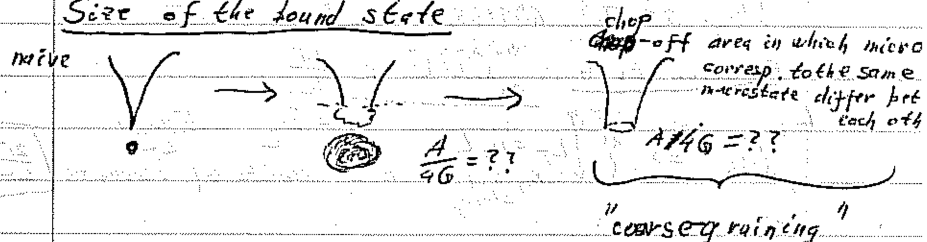
$r \ll \sqrt{Q_1Q_5}$ $\quad Q_1$ .. D1-charge, Q_5 .. D5-charge

after coordinate change

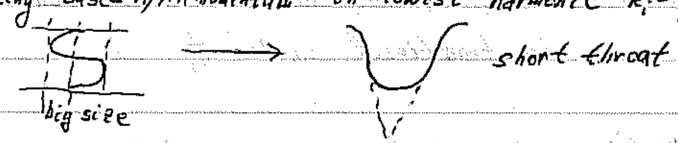
$ds^2_{string} = \sqrt{Q_1Q_5}\,[AdS_3$ global$] + \sqrt{Q_1Q_5}\,[S^3] + \sqrt{\frac{Q_1}{Q_5}}$

$\to$ no horizon, no singularity, nicely smooth

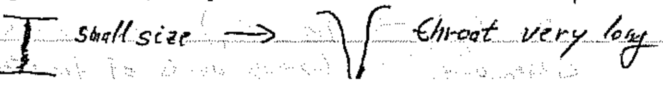
Size of the bound state



Limiting cases: A) All momentum in lowest harmonic $k_i=1, m_i=n_1n_p$



B) 1 quantum of highest harmonic $k_i = n_1n_p$, $m_i = 1 \Rightarrow$ 1 state of ea



generic state $\sum_i m_i k_i = n_1 n_p$

$k_i \sim \sqrt{n_1 n_p}$

$m_i \sim \sqrt{n_1 n_p}$ $\qquad \frac{A^E}{4G} \sim \frac{\sqrt{Q_1 Q_p}}{g^2 \alpha'^4} L V \alpha'^{1/2} \sim \sqrt{n_1 n_p} \sim S_{micro}$

All moduli cancel out.

$\frac{A^E}{G}$ is duality invariant $\Rightarrow$ the result holds for any 2-charge BH configuration

(Other effects are duality dependent, e.g.
NS1-P cut at $R \sim \frac{1}{\alpha'}$ $\qquad$ D1-D5 cut at $R \sim R_{AdS}$)

Other example: Similarly 2-charge BH with rotation (angular momentum J)

$$S_{micro} = 2\pi\sqrt{2}\sqrt{n_1 n_5 - J}$$

* $J = J_{max} = n_1 n_5$ $\quad$ 1 turn of uniform helix
* $J = J_{max}$ $\quad$ some extra energy allows "wiggling" around circular profile

coarse graining $\rightarrow$ toroidal "horizon"

$$\Rightarrow \frac{A^E}{4G} \sim \sqrt{n_1 n_5 - J} \sim S_{micro}$$

D1-D5 CFT

$AdS_3 \times S^3 \times T^4$ (D1+D5) $\longrightarrow$ D=2 CFT

Fractionation $\qquad p = \frac{2\pi}{L}$

$\Rightarrow$ $L_T = n_1 L$ (length of string) $\qquad P = \frac{2\pi n_p}{L} = \frac{2\pi n_p n_1}{L_T}$ $\quad$ $n_1 n_p$ units of strength $\boxed{\frac{2\pi}{L_T} = \frac{1}{n_1}\left(\frac{2\pi}{L}\right)}$

$\Rightarrow$

When you bind n_1 NS1-branes together, the P charges comes in fractional units $\frac{1}{n_1}$.

Duality $\quad$ NS1 $(n_1) \longrightarrow$ D5 (n_5)

P $(n_p) \longrightarrow$ D1 (n_1)

$n_5 = 1$ $\qquad\qquad$ $n_5 > 1$ $\quad$ $n_1 n_5$ fractional components

n_1 D1 branes $\qquad\qquad$ of an "effective string"

First application

$n_1 n_5$ $\longrightarrow$ $L_T = n_1 n_5 L$

2-charge $\quad S = 2\pi\sqrt{\frac{c}{6} n_1 n_p}$ $\quad$ $L_T = n_1 L$ $\quad$ momentum $\qquad c = $ # of eff. degrees of freedom

3-charge $\quad S = 2\pi\sqrt{\frac{c}{6}(n_1 n_5) n_p} \xrightarrow{c=6} 2\pi\sqrt{n_1 n_5 n_p}$ $\quad$ agrees

Second application $\qquad$ D1-D5 [no P] $\qquad n_1 n_5$ $\qquad n_1 n_5$ $\quad$ component strings

states = partitions of $n_1 n_5$
(4 bos., 4 ferm. degr. of freedom)

$= e^{2\pi\sqrt{2}\sqrt{n_1 n_5}} \Rightarrow \boxed{S = 2\pi\sqrt{2}\sqrt{n_1 n_5}}$

Geometry for a given D1-D5 microstate

profile $\vec{F}(v)$ in NS1-P

$\sum_i m_i k_i = n_1 n_p$ $\qquad\qquad$ $\sum m_i k_i = n_1 n_5$

m_i units of harmonic k_i $\qquad\qquad$ m_i component strings with winding # k_i

NS1-P geometry $\xleftrightarrow{\text{duality}}$ D1-D5 geometry

AdS/CFT correspondence : microstate physics (in the bulk SUGRA, or CFT boundary CFT) $\leftrightarrow$ shape of geometry in the bulk

"Blackness" of stringy BH arises from disorder, just as in ordinary physical systems (destructive interference of wavepackets reflected by different modes, e.g. different (with diff $L_i = k_i \cdot L$) component strings)

"No hair thm" - microstates don't have horizons (instead caps)

$\Rightarrow$ No hair thm doesn't hold

horizon - microstates break spherical symmetry ($\rightarrow$ horizon is not

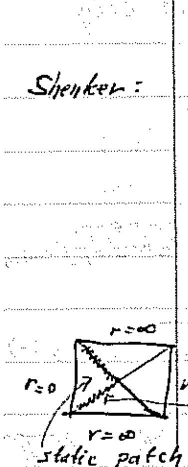
Shenker: $\boxed{\text{Inflation in AdS/CFT}}$ work in progress

inflation + string theory = difficult

de Sitter space – hard to understand

attempts to do analog of AdS/CFT

$\Rightarrow$ dS / ? — no one knows what to put instead of CFT on horizons



or dS/CFT on $r=\infty$ regions – also strange

also in dS S-matrix cannot be constructed – no suitable spacial, flat asymptotic regions

Current attempts – joining AdS (with one spatial infinity untouched – AdS/CFT possible) with dS ($\to$ inflation) separated by thin domain wall

$\to$ lot of open problems (entropy etc.)

$\boxed{\text{Non AdS/non CFT correspondence 3}}$

(4) Duality cascade

D3-branes at a conifold, take near horizon limit

$\mathbb{R}^4 \times CY \quad ds^2_{CY} = dr^2 + r^2 dS^2_{M_5}$

D3-branes at $r=0 \to AdS_5 \times M_5$

Conifold $\quad z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0, \quad z_i = 0 \to$ conifold singularity

$$M = T^{1,1} = \frac{SU(2)\times SU(2)}{U(1)}$$

$$ds^2_{T^{1,1}} = \frac{1}{9}\left(d\psi + \cos\theta_1 d\varphi_1 + \cos\theta_2 d\varphi_2\right)^2 + \frac{1}{6}\left(d\theta_1^2 + \sin^2\theta_1 d\varphi_1^2\right) + \frac{1}{6}\left(d\theta_2^2 + \sin^2\theta_2 d\varphi_2\right)$$

N D3-branes $\quad \mathcal{N}=1$ SYM $\quad \overset{g_1}{SU(N)} \times \overset{g_2}{SU(N)}$

$A_1, A_2 \quad \overset{SU(N)\times SU(N)}{\square \quad \overline{\square}} \qquad B_1, B_2 \quad \overline{\square} \quad \square$

$N_f = 2N$ symmetries $SU(2)\times SU(2)\times U(1)_B \times U(1)_R \Rightarrow$

one anomalous dimension γ, RG flow $1 + \gamma_1 \overset{!}{\underset{CFT}{=}} 0$

SCFT fixed point $\gamma(g_1, g_2) = -\frac{1}{4}$

add $W = h\,\mathrm{tr}(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1) \Rightarrow \gamma(g_1, g_2, h) = -\frac{1}{4}$

gives 2-dim. surface of SCFT fixed points

dual – strings IIB on $AdS_5 \times T^{1,1}$ – two complex params $g_s = e^{\phi}, \int_{S^2} B_2, \int_{S^2} C_2$

Moduli space = conifold$^N / S_N$

$$e^{-\phi} \sim \frac{1}{g_1^2} + \frac{1}{g_2^2}, \quad 2e^{-\phi}\left[\int_{S^2} B - \frac{1}{2}\right] \sim \frac{1}{g_1^2}$$

Modification to get non CFT – take $SU(N+M)\times SU(N)$

– N D3-branes $\quad M$ D5-branes on S^2 ("fractional D

$\Rightarrow$ no SCFT fixed point, not asymptotically free – potential trouble
o.k. that's what we wanted

$\frac{1}{g_1^2} + \frac{1}{g_2^2} \sim$ const. $\to$ we start from $g_2 \ll 1, h \ll 1$

$\to g_1$ large, effect. $SU(N+M) \quad N_f = 2N \quad$ SQCD

$\downarrow IR$

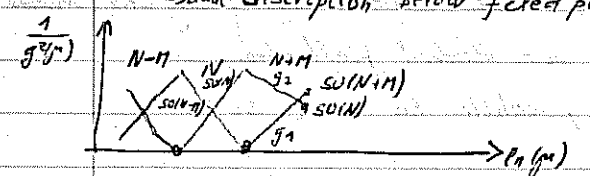
SCFT $\underset{IR}{\Leftarrow} SU(N-M), N_f = 2N$ magnetic description

adding superpotential $W \to$ relevant deform. of IR SCFT fixed

in magnetic description $W = Mq\tilde{q} + hM^2 \xrightarrow{IR} W - \frac{1}{h} q\tilde{q}q\tilde{q}$

$\Rightarrow SU(N-M) \quad N_f = 2N$ at SCFT fixed point

$\Rightarrow$ dual description below fixed point $SU(N)\times SU(N-M)$
g_2 grows $\quad g_1$ decreases



finally as $\mu \to 0$

$\to SU(2M)\times SU(M) \to SU(M)$

$\Rightarrow$ pure $SU(M)$ SYM as we wanted

$g_s M \gg 1 \Rightarrow$ no separation of scales

$g_s M \ll 1 \Rightarrow \Lambda_{...} \ll \Lambda \quad \Rightarrow$ good low-energy SYM high energy limit $N \to \infty$, $M \to \infty$ in a controlled way

in dual, gravity description $AdS_5 \times T^{1,1}$, $\int_{S^3} F_3 = M$

EoM $\Rightarrow$ $\int B_2 \sim 3 g_s M \ln\left(\frac{r}{r_0}\right) \longleftrightarrow \frac{1}{g_1^2} - \frac{1}{g_2^2}$

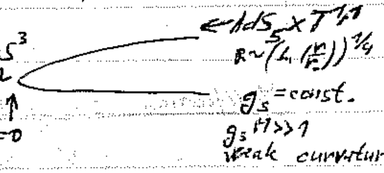
$\int_{T^{1,1}} F_5 \sim N_c + \# g_s M^2 \ln\left(\frac{r}{r_0}\right)$ (valid for $g_s M \gg 1$)

$\uparrow$ original value

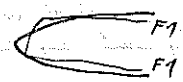
$r \to \infty$ $ds^2 \sim \frac{r^2}{R^2 \sqrt{\ln(\frac{r}{r_0})}} dx_\mu dx^\mu + \frac{R^2 \sqrt{\ln(\frac{r}{r_0})}}{r^2} dr^2 + R^2 \sqrt{\ln(\frac{r}{r_0})}\, ds^2_{T^{1,1}}$

$r \to 0$ to get proper limit with spontan. broken symmetries one needs to deform conifold $z_1^2 + \ldots + z_4^2 = \epsilon \Rightarrow U(1)_R \to \mathbb{Z}_2$

topologically $T^{1,1} \cong S^2 \times S^3$ $\quad \mathbb{R}^3_2 \times S^3$



Confinement



$V_{q\bar{q}} \sim T_s R$ o.k.

Mass gap, spectrum $\longrightarrow$ o.k. $\quad M^2 \ll T_s$

$\searrow$ no mass gap $\Leftarrow$ ($\cancel{U(1)}_{Baryon}$ $\to$ 1 massless complex scalar)

Instantons, baryons, condensates qualitatively o.k.

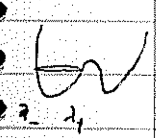
(Dis) advantages

$\oplus$ weak coupling & curvatures (even as $\lambda \to 0$ coupling weak, curvature not)

$\ominus$ UV theory is not standard field theory

$\ominus$ no mass gap, subtle to break SUSY

Summary: $\overset{SYM\,or\,YM}{0} \leftarrow \lambda \rightarrow \overset{SUGRA}{\infty}$, 3 examples of construction of SUGRA backgrounds then extending qualitative (& RR fields)



Low-dimensional field theory 2

Matrix models

1d matrix model

$\Phi(t)$ $N \times N$ hermitean matrix, $U(N)$ gauge symmetry $\Phi \to V^\dagger(t) \ldots$

$\mathcal{L} = \mathrm{tr}\left[\frac{1}{2}\left(\partial_t \Phi + [A, \Phi]\right)^2 - U(\Phi)\right]$ $\quad A(t) \to V^\dagger \partial_t V + V^\dagger A V$

gauge choices (then global $U(N)$ symmetry)

"axial gauge" 1. $A = 0$ $\quad$ Gauss law $\left.\frac{\delta S}{\delta A(t)}\right|_{A=0} = 0 = -\partial_t \Phi \Phi + \Phi \partial_t \Phi$

"unitary gauge" 2. diagonalize $\Phi \ldots \Phi(t) = \begin{pmatrix} \lambda_1(t) & & \\ & \ddots & \\ & & \lambda_N(t) \end{pmatrix} \Rightarrow$ remnant of gauge symmetry

$\mathcal{L}_{eff} = \sum_i \left(\frac{1}{2} \dot\lambda_i^2 - U(\lambda_i)\right)$, $\quad H \text{"="} \sum\left(\frac{1}{2}\partial_{\lambda_i}^2 + U(\lambda_i)\right)$ - wrong

in QM measure $D\Phi(t) = D\Omega(t) \prod_i d\lambda_i\, J(\lambda)$

$J(\lambda) = \Delta^2(\lambda)$ $\quad \Delta(\lambda) = \prod_{i>j} (\lambda_i - \lambda_j)$ (superficially comp. measure in polar coordinates)

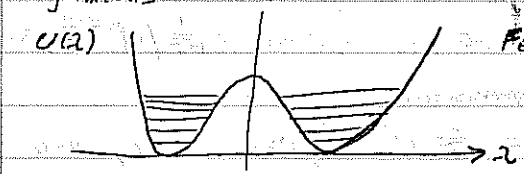
$\Rightarrow -\sum_i \frac{1}{2} \partial_{\lambda_i}^2 \to -\frac{1}{2} \sum_i \frac{1}{\Delta^2} \partial_{\lambda_i} \Delta^2 \partial_{\lambda_i} = -\frac{1}{2} \frac{1}{\Delta} \sum_i \partial_{\lambda_i}^2 \Delta$

Wave fcn of the form $\frac{1}{\Delta} \psi(\lambda_i) \Rightarrow$

$H \frac{1}{\Delta} \psi(\vec\lambda_i) = -\frac{1}{2} \frac{1}{\Delta}\left(\sum_i \partial_{\lambda_i}^2 \psi(\vec\lambda_i) - U(\lambda)\psi(\vec\lambda_i)\right)$

$\Rightarrow H\psi = E\psi$ with $H = -\frac{1}{2} \sum_i \partial_{\lambda_i}^2 + U(\lambda)$

but $\psi(\lambda_i, \lambda_j) = -\psi(\lambda_j, \lambda_i) \Rightarrow$ wave fcn of N-independent fermions



Semiclassical quantization $\oint p\, d\lambda = 2\pi n$

$\int_{\lambda_-}^{\lambda_+} \sqrt{2(E - U(\lambda))}\, d\lambda = \pi n$

$\rho(E) = \sum_n \delta(E - E_n)$ $\quad N = \int_{\mu_0}^{\mu} \rho(E)\, dE$

$\Rightarrow E = \int_{\mu_0}^{\mu} E\, \rho(E)\, dE \qquad \rho(\mu) = \frac{\Delta N}{\Delta \mu} = \frac{1}{\pi}\int_{\lambda_-(\mu)}^{\lambda_+(\mu)} d\lambda \Big/ \sqrt{2(\mu - V(\lambda))}$

$\mu \approx 0 \qquad \rho(\mu) \approx \frac{1}{\pi}\log\mu + \dots$ (approx. $\underset{\text{near maximum}}{U(\lambda) \sim -\frac{1}{2}\lambda^2}$)

$E = \frac{1}{\pi}\int^{\mu} \log E \; E\, dE = \frac{1}{2\pi}\mu^2 \log\mu$

effectively $H = -\frac{1}{2}\partial_\lambda^2 - \frac{1}{2}\lambda^2$ How to diagonalize it?

$s = \frac{1}{\sqrt{2}}(\lambda + P) \qquad U = \frac{1}{\sqrt{2}}(\lambda - P)$ analogs of a, a^+

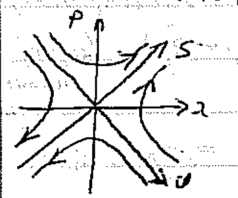
but $s = s^+,\ U = U^+,\ P = -i\frac{\partial}{\partial\lambda} \Rightarrow [U, s] = i \Rightarrow$ "$U = i\frac{\partial}{\partial s}$"

$\hat{S}|s\rangle = s|s\rangle,\ \hat{U}|U\rangle = U|U\rangle \quad \langle\lambda| : \langle\lambda|H = \lambda\langle\lambda|$

$\Rightarrow \langle\lambda|s\rangle = \frac{2^{1/4}}{\sqrt{2\pi}} e^{i(-\frac{\lambda^2}{2} + \sqrt{2}\lambda s - s^2/2)}$

$\langle\lambda|U\rangle = \frac{2^{1/4}}{\sqrt{2\pi}} e^{i(\frac{\lambda^2}{2} - \sqrt{2}\lambda U + U^2/2)}$

$\langle s|U\rangle = \sqrt{\frac{i}{2\pi}}\, e^{-isU}$



classical trajectories, correspond to

$\mathcal{L} = \frac{1}{2}(\dot{\lambda}^2 + \lambda^2) \qquad H = \frac{1}{2}(p^2 - \lambda^2) = -\frac{1}{2}(sU + Us) = -is\frac{\partial}{\partial s} - i/2$

$\Rightarrow s^{iE - \frac{1}{2}}$ is eigenfcn with energy E, singularity at 0

def. $|E, out, +\rangle,\ |E, out, -\rangle$

$\langle s|E, out, +\rangle = \frac{1}{\sqrt{2\pi}} s^{iE - \frac{1}{2}}, \ s > 0 \qquad = 0,\ s < 0$

$\langle s|E, out, -\rangle = 0, \ s > 0 \qquad = \frac{1}{\sqrt{2\pi}}|s|^{iE - \frac{1}{2}}, s < 0$

similarly in U representation

$\langle U|E, in, +\rangle = \frac{1}{\sqrt{2\pi}} U^{-iE - \frac{1}{2}}, \ U > 0 \qquad = 0,\ U < 0$

$\langle U|E, in, -\rangle = 0, \ U > 0 \qquad = \frac{1}{\sqrt{2\pi}}|U|^{-iE - \frac{1}{2}}, \ U < 0$

but they must be related

$$\begin{pmatrix} |E, in, +\rangle \\ |E, in, -\rangle \end{pmatrix} = S \begin{pmatrix} |E, out, +\rangle \\ |E, out, -\rangle \end{pmatrix}$$

$\langle E, out, + | E', in, +\rangle = \int_0^\infty ds \int_0^\infty dU \frac{i}{2\pi} \frac{e^{-isU}}{2\pi} s^{-iE - \frac{1}{2}} U^{-iE' - \frac{1}{2}} \qquad \underset{s = re^{\varphi}}{U = re^{-\varphi}}$

$= \delta(E - E') \int_0^\infty dr\, e^{-r^2(i + 0^+)} r^{-2iE} \sqrt{\frac{i}{2\pi}}$

$= \frac{1}{\sqrt{2\pi}}\delta(E - E')\, \Gamma\left(\frac{1}{2} - iE\right) e^{-\frac{\pi E}{2}}$, similarly others

$\Rightarrow S = \frac{\Gamma(\frac{1}{2} - iE)}{\sqrt{2\pi}} \begin{pmatrix} e^{-\pi E/2} & i e^{\pi E/2} \\ i e^{\pi E/2} & e^{-\pi E/2} \end{pmatrix} = e^{i\phi(E)} \begin{pmatrix} \frac{1}{\sqrt{1 + e^{2\pi E}}} & \frac{i}{\sqrt{1 + e^{-2\pi E}}} \\ \frac{i}{\sqrt{1 + e^{-2\pi E}}} & \frac{1}{\sqrt{1 + e^{2\pi E}}} \end{pmatrix}$

Physics

incoming states $|U| \gg 1,\ S \approx 0$ $\quad +, -$ supported on the right or

outgoing states $|s| \gg 1,\ U \approx 0$ $\quad$ — " —

check: $E \to +\infty \quad S \to fcn \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad |E, in, \pm\rangle \approx |E, out, \mp\rangle$

$E \to -\infty \quad S \to fcn \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad |E, in, \pm\rangle \approx |E, out, \pm\rangle$

$\Rightarrow$ really reasonable S-matrix S

Second quantisation for a bunch of fermions

$\mathcal{L} = \int dt\, \psi^+(\lambda, t)\left(i\partial_t + \frac{1}{2}\partial_\lambda^2 + \frac{1}{2}\lambda^2 - \mu\right)\psi(\lambda, t)$

$\psi_s(s, t) = \frac{(2i)^{1/4}}{\sqrt{2\pi}} \int d\lambda\, e^{i(\frac{\lambda^2}{2} - \sqrt{2}s\lambda + \frac{s^2}{2})} \psi(\lambda, t)$

$\psi_U(U, t) = \frac{(2i)^{1/4}}{\sqrt{2\pi}} \int d\lambda\, e^{i(-\frac{\lambda^2}{2} + \sqrt{2}U\lambda - \frac{U^2}{2})} \psi(\lambda, t)$

$\psi_s(s, t) = \int \frac{dU}{\sqrt{2\pi}} e^{-isU} \psi_U(U, t)$

$\Rightarrow \mathcal{L} = \int_{-\infty}^{\infty} ds\, \psi_s^+\left(i\partial_t + \frac{i}{2}(s\partial_s + \partial_s s) - \mu\right)\psi_s = \int_{-\infty}^{\infty} dU\, \psi_U^+\left(i\partial_t + \frac{i}{2}(U\partial_U + \partial_U U)\right)\dots$

define $\psi_\pm^{in}(r, t) = e^{r/2}\psi_s(s = \pm e^r, t)$, $\psi_\pm^{out}(r, t) = e^{r/2}\psi_U(U = \pm e^r, \dots$

$\Rightarrow \mathcal{L} = \underbrace{\int dr \sum_{i=\pm} \psi_i^{in}(i\partial_t - i\partial_r - \mu)\psi_i^{in}}_{\text{free relativistic fermions}} = \int dr \sum_{i=\pm} \psi_i^{out}(i\partial_t + i\partial_r - \mu)\psi_i^{o}$

free relativistic fermions, where is the nontrivi[al]

physics hidden – in transform between ψ_s & ψ_U

– nonlocal

Symmetries $SU(2)_{in}$ fermionic transf. (similarly $SU(2)_{out}$)

– arising from 2 species of fermions – left & right coming

but it is not a symmetry of $\psi_s = \int \dots \psi_U$

Bosonisation of fermions — in previous lecture

$J^{in} \sim (\partial_t + \partial_r)\phi^{in} = \text{"}\sim\text{"}\, \partial_+ c$

$J^{out} \sim (\partial_t - \partial_r)\phi^{out} \text{"}\sim\text{"}\, \partial_- c$

also other state arises, nonvisible in worldsheet

perturbative description — massless nonperturbative solitons

Symmetries:

$(-)^{F_L}$ charge conjugation $\to (\lambda, V, S) \to (-\lambda, -V, -S)$

$(-)^{F_L}$ S-duality $\quad \psi^{in}_{\pm} \to \psi^{in\,+}_{\mp},\ \psi^{out}_{\pm} \to \psi^{+\,out}_{\pm}$

Quantum structure of black holes 3

Conjecture: Bound states in a consistent theory of quantum gravity "swell up" to a size depending on their degeneracy.

$$\frac{A}{4G} \sim S = \ln[\#\text{states}]$$

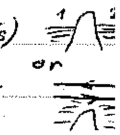
$\Rightarrow$ May be there are no BH horizons.

Black rings & supertubes 2

Low-dimensional string theory 3

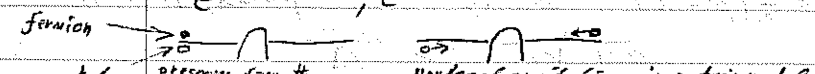
$\mathbb{Z}_2$ symmetry $\quad \lambda \to -\lambda,\ p \to -p,\ s \to -s,\ u \to -u$

$\to (-)^{F_L}$ charge conjugation $c \to -c,\ T \to T$

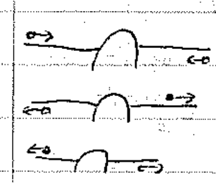
quanta of $T \leftrightarrow$ even perturbations of the Fermi sea(s)  or

quanta of $c \leftrightarrow$ odd —"—

nonperturbative massless excitations: $e^{\pm i\sqrt{2}c^{in}},\ e^{\pm i\sqrt{2}c^{out}}$

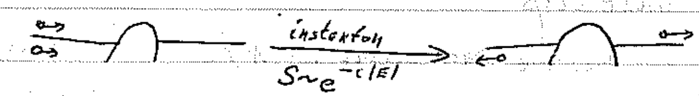


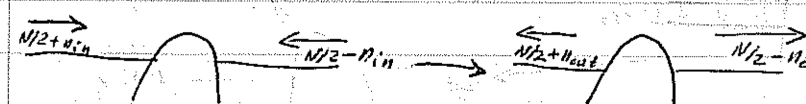
$\partial c \leftrightarrow$ # of fermions with $\lambda > 0$ − # of ferm. with

 $\sim e^{-i\sqrt{2}c_{in}}$

$e^{i\sqrt{2}c_{out}}$

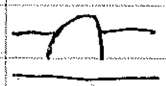
$e^{-i\sqrt{2}c_{out}}$

E.g. 



n_{in} eigenvalue of $J_3^{in} \subset SU(2)_{in}$

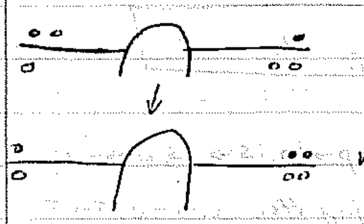
n_{out} —"— of $J_3^{out} \subset SU(2)_{out}$

 $\mu < 0 \quad S$ large for $n_{in} \approx n_{out}$

 $\mu > 0 \quad S$ large for $n_{in} \approx -n_{out}$

$\Rightarrow \mu < 0 \quad J_3^{in} \approx J_3^{out}$

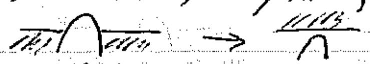
$\mu > 0 \quad J_3^{in} \approx -J_3^{out}$

Ex:  $n_{in} = 1$, $n_{out} = 0$

$(-)^{F_L}$ $\quad T \to -T, \quad c_{in} \to c_{in}, \quad c_{out} \to -c_{out} \Rightarrow c \to \tilde{c}$

$\Rightarrow \mu \to -\mu$

$H = \frac{1}{2}(p^2 - \lambda^2)$ duality symm. $\lambda \to -p$, $p \to \lambda$, $H \to -H$



we also must $\psi \to \psi^+$ (fermions → holes) (to get $(\text{transf.})^2 = \mathbb{1}$) $\Rightarrow$

$\psi(\lambda, t) \to \frac{1}{\sqrt{2\pi}} \int dp \, e^{ip\lambda} \psi^+(p, t)$

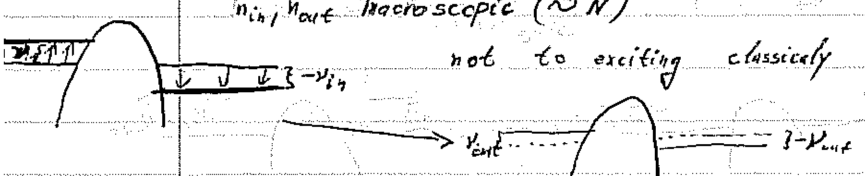
i.e. $\psi_S(s,t) \to \psi_S^+(s,t)$ $\quad \psi_U(u,t) \to \psi_U^+(-u,t)$

$\psi_\pm^{in} \to \psi_\mp^{in+}$ $\quad \psi_\pm^{out} \to \psi_\pm^{out+}$

Next step:

n_{in}, n_{out} macroscopic $(\sim N)$

not to exciting classicaly



$\nu_{in} \sim \nu_{out}$ but $\neq$ $\Rightarrow$ tunneling, exponentially suppresed by $e^{-N\#} e^{(\nu_{in} - \nu_{out})\#}$

also $E_{in} = (\mu + \nu_{in})^2 + (\mu - \nu_{in})^2$

$E_{out} = (\mu + \nu_{out})^2 + (\mu - \nu_{out})^2$

$\hookrightarrow E_{in} - E_{out} \neq 0 \quad \sim 2(\nu_{in}^2 - \nu_{out}^2) = 2\nu\tilde{\nu}$

in the picture out-state less energetic than in

$\Rightarrow$ "ripples" in out-state carrying excess energy

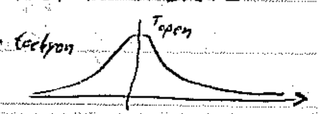
$\Rightarrow E = f(\max(|\nu_{in}|, |\nu_{out}|)) = \frac{1}{2}\left(E(\mu+\nu) + E(\mu-\nu)\right)$

$E = \int_{-\infty}^{\mu} dE\, \rho(E)(E-\mu) = \int_{-\infty}^{\mu} dE\, \phi(E)(E-\mu) \overset{\text{per partes}}{=} -\int_{-\infty}^{\mu} dE\, \phi(E)$

$e^{-iET} = \langle 0 | e^{-iHT} | 0 \rangle$ traditional interpretation

$= \langle 0, out | 0, in \rangle = e^{iL\int dE\, \phi(E)}$ $\quad$ L.. length of the system $= T$

Why do matrix models work?

Semi-classical system has unstable D-branes with open string potential for tachyon 

– looks like

N such D-branes $\Rightarrow T_{open}$ $N \times N$ matrix with $U(N)$, potential $V(T_{open}) \approx -\mathrm{Tr}(T_{open}^2)$

– not well defined in full QFT

– in $\mu > 0$ the image changes to unstable hole

OA theory

unstable branes $D\bar{D}$, N of them

$\phi = T_{open}$ complex $N \times N$ matrix

symmetry $U(N) \times U(N)$

$\mathcal{L} = \mathrm{Tr}(D\Phi)^2 - V(\phi^+\phi)$

gauge fixing $\phi = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}$ $\quad \lambda_i \to e^{i\alpha_i}\lambda_i$ residual $U(1)^N$

$\lambda_i \in \mathbb{C} \Rightarrow N$ fermions moving in $\mathbb{C}$

Gauss law: angular momentum of each fermion $= 0$

$\Rightarrow$ can gauge $\lambda_i \geq 0$ .. radial direction

$H = -\frac{1}{\lambda_j} \partial_{\lambda_j} \lambda_j \partial_{\lambda_j} + V(\lambda)$

$\psi = \frac{1}{\sqrt{\lambda}} \tilde{\psi} \rightarrow \tilde{H} = -\partial_\lambda^2 + \frac{1}{4\lambda^2} + V(\lambda^2)$

better not to gauge λ and use 2 superimposed harm. oscillators

$(s_i, u_i) \quad i=1,2$

$s = \sqrt{s_1^2+s_2^2} \geq 0,\ u = \sqrt{u_1^2+u_2^2} \geq 0 \Rightarrow s, u$ arent conjugate to P_s, P_u

$H = \frac{1}{2}\{s, P_s\} = -\frac{1}{2}\{u, P_u\}$

$\langle s | E, out\rangle = \frac{1}{\sqrt{2\pi}} s^{iE-\frac{1}{2}} \qquad \langle u | E, in\rangle = \frac{1}{\sqrt{2\pi}} u^{-iE-\frac{1}{2}}$

but only 1 state $\forall E$ ($\Leftarrow s, u \geq 0$)

Ooguri: $\boxed{\text{Topological string theory 1}}$

tool for understanding of string compactification

$CY_3 \times \mathbb{R}^{3,1}$

1. Compactification of $\mathbb{I}$ string on CY_3

$\Rightarrow \mathcal{N}=2$ supersym. in $\mathbb{R}^{3,1}$

massless super multiplet ?

SUGRA $\ni$ graviton, $U(1)$ gauge field $A^0_{\mu=0,1,2,3}$

+ superpartners

matter multiplets:

n_V vectors $A_\mu^{i=1,..,n_V}$ gauge field, z^i complex scalars

expectation values $\langle z^i\rangle$ characterize the theories – moduli space $\mathcal{M}_V$

$\mathcal{M}_V = \begin{cases} \mathbb{I}B & \text{complex structure of } CY \\ \mathbb{I}A & \text{complexified Kähler structure} \end{cases}$

$\mathbb{I}B \quad n_V = h^{2,1} \quad \dim H^{2,1}$

in 10dim $\mathbb{I}B$ massless fields $A_{MN}, A_{MNPQ}, A_{MNPQRS}, A_{MNP\ldots}$

for generic CY $h^{1,0}=0$ we canot get rid off one index in A_{MN} $\rightarrow$ no gauge field from A_{MN}

$H^3 \ni \omega \quad \omega \wedge A_{4} \longrightarrow A_\mu$

but 4-form is self-dual $F_5 = dA_4 = * dA_4$

$H^3 = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}, \ \dim H^{3,0} = 1$

– unique $\Omega_{ijk}\, dz^i \wedge dz^j \wedge dz^k$ holomorphic 3-form

$\dim_{\mathbb{R}} H^3 = 2 + 2h^{2,1}, \ H^3 = \mathrm{span}\{a_I, b^I\} \quad \int_{CY_3} a_I \wedge b^J = \delta$

symplectic basis

self-duality of $A \Rightarrow A_I \wedge a^I, B^I \wedge b_I$ are dependent

$\Rightarrow \boxed{h^{2,1}+1}$ gauge fields in $d=4$ coming from A_4 ind

in fact unique source of gauge fields

– too many by 1 ($h^{2,1}+1 \neq h^{2,1}$:-)

for vector multiplet, the additional one goes into SUGRA multiplet – "grav. photon"

Convention: $I = 0,1,..,n_V, \quad i = 1,..,n_V$

X^I with gauge symmetry $X^I \to cX^I$, c arbitrary

X's projective coordinates of $\mathcal{M}_V$, e.g. fix gauge $X^0=1 \Rightarrow z^i = \frac{X^i}{X^0}$

hypermultiplets $i=1,..,n_H$ 2 complex scalars

neutral with respect to A_μ^I (true for generic CY compactification)

$\mathcal{M}_H$ quaternionic Kähler

no neutral coupling between hyper and vector

(true for any $\mathcal{N}=2$ theory)

$\Rightarrow$ generic moduli space $\mathcal{M}_H \times \mathcal{M}_V$, can break down at specific CY_3, e.g. conifold singularity (then hyper becomes charged w.r.t. gauge field in $d=4$)

origin of hypermultiplet

e.g. in IIA $\quad n_H = h^{2,1}+1 \quad n_V = h^{1,1}$ (Kähler moduli)

$$h^{2,1} = \dim H^{2,1} \quad \text{complex moduli}$$

RR field A_3 ~~[illegible]~~

can be integrated $A_3 \wedge \omega \to h^{2,1}$ $\mathbb{C}$-scalars

$+ \; 1 \; \underbrace{\text{dilaton} \quad B^{NS}_{\rho\nu}}_{\text{1 complex scalar}} \quad * dB^{NS} = d\phi$ axion

$\Rightarrow \underbrace{\text{1 complex scalar} + RR \to \text{komplex scalar}}_{\text{universal multiplet}}$

in IIB using mirror symmetry

IIB $\quad \mathcal{M}_V \quad z^i$ complex structure moduli $i = 1, \dots, n_V$

$$X^I \quad I = 0, 1, \dots, n_V = h^{2,1}$$

$$\Omega = \Omega_{ijk}\, dz^i \wedge dz^j \wedge dz^k \quad \text{(diff. index range)}$$

holomorphic 3-form

$$T\mathcal{M}_V \Longleftrightarrow \text{deformations of } \Omega \to \Omega + \delta z^i \eta_i$$
(modulo $\Omega \to c\Omega$) $\quad \eta \in H^{2,1}$

basis of $H_3 \quad \{A_I, B^I\}_{I=0,1,\dots,h^{2,1}}$

$$A_I \cap A_J = 0 \quad B^I \cap B^J = 0 \quad A_I \cap B^J = \delta_I^{\ J}$$

$X^I = \int_{A_I} \Omega \quad$ additional $+1$ corresponds to $\Omega \to c\Omega$

flat coords on moduli space $\frac{X^i}{X^0}$

$\int_{B^I} \Omega = F_I(X)$ homogeneous of degree 1

even $\quad F_I(X) = \frac{\partial}{\partial X^I} F(X) \quad$ F - prepotential, homog. fcn. of degree 2

$$\tau_{IJ} = \partial_I \partial_J F \quad \text{period matrix}$$

$$C_{IJK} = \partial_I \partial_J \partial_K F \quad \text{Yukawa coupling}$$

The low-energy effective action for the vector multiplet (at the string tree level) is completely determined by $F(X)$

e.g. $\quad G_{i\bar{j}}(z,\bar{z})\, \partial_\mu z^i \partial_\mu z^{\bar{j}}$ where $G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$

$$e^{-K} = \int_{CY3} \Omega \wedge \bar{\Omega} \sim X^I \bar{F}_I - \bar{X}^I F_I$$

$F(X)$ is computable

- IIB $\quad X^I = \int_{A_I} \Omega, \quad \frac{\partial F}{\partial X^I} = \int_{B^I} \Omega$
- IIA more complicated

complex str. $\to \mathcal{M}_H \times \mathcal{M}_V$ Kähler $\quad$ (in IIB reversed $\Rightarrow$ for $\mathcal{M}_V$ size, i.e. Kähler structure was not important $\Rightarrow$ eff. one can make strings arbitrarily small $\to$ like point particles)

$\Rightarrow$ use of

Gromov-Witten invariant

mirror symmetry

2. Formal theory of topological string

$F_g(z) \quad$ g-loop topological string amplitude

$$F_{g=0} = F \quad \text{prepotential}$$

in type II in RNS formalism $\quad CY_3 \times \mathbb{R}^{3,1}$

$$\underbrace{\mathcal{N}=2 \;\; SCFT\, (\Sigma \to CY_3)}_{\text{we shall focus on this part}} \otimes (X^\mu, \psi^\mu, \mu = 0,\dots,3) \otimes (\text{ghosts})$$

$\underset{(2,0)}{T_L}, \underset{(\frac{3}{2},0)}{G_L^\pm}, \underset{(1,0)}{J_L} \; : \; T_R, G_R^\pm, J_R \qquad \hat{c} = \frac{c}{3} = 3$

Topological twist $\quad T_L \to T_L \pm \frac{1}{2} \partial J_L = T_L'$

equivalently new interaction on the worldsheet $\pm\frac{1}{2}\int_\Sigma \omega_{\bar z} J_L$

$\omega_{\bar z}$ spin connection

(equivalence follows from $\frac{\delta S}{\sqrt{g}\,\delta g_{\bar z\bar z}} = T_L$)

After topological twist:

T_L: (2,0) but G_L^+ (1,0), G_L^- (2,0) or vice versa dep. on $\pm$

$\hat c = 3 \to 0$ $\Rightarrow$ string theory without ghosts etc.

similarly for right movers $\Rightarrow$

4 types of twisting A, B, A^*, B^*

let's choose convention

$G_L^+ (1,0)$ $G_L^- (2,0)$ $G_R^+ (0,1)$ $G_R^- (0,2)$

$Q = \oint G_L^+ + \oint G_R^+$, $Q^2 = 0$ $\Rightarrow$ BRST charge $Q_{BRST} = Q$

$\{Q_{BRST}, G_{\substack{L\\R}}^-(z)\} = T'_{\substack{L\\R}}(z)$

in the following we shall forget prime in $T_L'(z)$

define $F_g = \int_{\mathcal{M}_g} \langle \prod_{i=1}^{3g-3} (\eta_i, G_L^-)(\bar\eta_i, \bar G_R^-) \rangle$

η_i Beltrami differential, $(\eta_i, G_L^-) = \int_\Sigma \eta_i G_L^-$

makes sense only if $\hat c = 3$ to cancel anomaly

F_g naively homogeneous of degree $2-2g$ $F_g(cX) = c^{2-2g} F_g(X)$

F_g depends only on $\mathcal{M}_V$

$F_g(X^0, X^i) = \left(\frac{1}{X^0}\right)^{2g-2} F_g\left(\frac{X^i}{X^0}\right)$ overall scale of X $\sim$ 1/coupling const

forget about this

$\exp\left(\sum_g F_g(X) \lambda^{2g-2}\right) = \psi$ partition fcn ψ is naturally associated to quantisation of H^3 in B-model

Yi: $\boxed{\text{Moduli dynamics of } \mathcal{N}=2 \text{ SU(2) pure YM BPS states}}$

Sen: $\boxed{\text{Black holes, attractors, elementary strings etc. 1}}$

hep-th/0506177 hep-th/0505122

full effective action contains not only 2-derivative terms but also higher derivatives

$\int d^Dx \sqrt{-\det G}\, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$

Extremal black holes - Hawking temperature is 0

$\Rightarrow$ don't radiate

With higher derivatives SUSY difficult

$\to$ we don't assume SUSY.

<u>How to recognize</u> extremal black holes (without SUSY, i.e. BPS condition)?

We shall assume spherical symmetry of BH.

Ex. $D=4$ Reissner-Nordstrøm soln $ds^2 = -(1-\frac{\rho_+}{\rho})(1-\frac{\rho_-}{\rho})dt^2 + \frac{1}{(\cdot)(\cdot)} d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta\, d\phi^2)$

Extremal limit $\rho_+ = \rho_-$

define $\tau = t/\rho_+^2$ $r = \rho - \rho_+$

$\Rightarrow ds^2 \simeq \rho_+^2\left(-r^2 d\tau^2 + \frac{dr^2}{r^2}\right) + \rho_+^2 (d\theta^2 + \sin^2 d\phi^2)$ near horizon

$\to$ in near horizon limit $AdS_2 \times S^2$

all known extremal BHs in $d=4$ with nonsingular near horizon geometry have near horizon $AdS_2 \times S^2$ geometry.

General gauge & general coordinate inv. theory with $g_{\mu\nu}$, $\{A_\mu^{(i)}\}$ & neutral scalars $\{\phi_s\}$

Def: Extremal BHs in such theory have
1. near horizon geometry $AdS_2 \times S^2$
2. $SO(2,1)\times SO(3)$ symmetry of all other background fields

$$\Rightarrow ds^2 = v_1\left(-r^2dt^2 + \frac{dr^2}{r^2}\right) + v_2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

$\phi_s = u_s$ scalar values at horizon

$F_{rt}^{(i)} = e_i$ radial electric fields at the horizon

$F_{\theta\phi}^{(i)} = \frac{p_i}{4\pi}\sin\theta$, $\frac{p_i}{4\pi}$ radial magnetic —"—

$R_{mnpq} = -v_2^{-1}(g_{mp}g_{nq} - g_{mq}g_{np})$

$\alpha, .., \delta = r,t \qquad m,..,q = \theta,\phi \quad \Rightarrow R_{\alpha\beta\gamma\delta} = -v_1^{-1}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma})$

$F_{rt}^{(i)} \propto$ volume form on AdS_2 $\qquad F_{\theta\phi}^{(i)} \propto$ volume form on S^2

$D_\mu R_{\nu\rho\sigma\tau} = 0, \quad D_\mu F_{\nu\rho}^{(i)} = 0, \quad D_\mu \phi_s = 0$

$\Rightarrow$ only terms in $\mathcal{L}$ not involving explicit covariant derivatives affect the background

def. $f(\vec{u}, \vec{v}, \vec{e}, \vec{p}) \equiv \int d\theta\, d\phi \sqrt{-\det g}\,\mathcal{L}$ $\qquad \sqrt{-\det g}\,\mathcal{L}$ Lagrangian density

indep. of r,t due to symmetry

$q_i \equiv \frac{\partial f}{\partial e_i} \qquad F(\vec{u},\vec{v},\vec{q},\vec{p}) \equiv 2\pi\left(e_i q_i - f(\vec{u},\vec{v},\vec{e},\vec{p})\right)$ (Legendre transf. $\cdot 2\pi$)

For an extremal BH of electric charge $\vec{q}$ and magnetic charge $\vec{p}$

1. the values of $\{u_s\}$, v_1, v_2 are obtained by extremizing $F(\vec{u},\vec{v},\vec{q},\vec{p})$ w.r.t. to them

$$\frac{\partial F}{\partial u_s} = 0 \qquad \frac{\partial F}{\partial v_1} = 0 \qquad \frac{\partial F}{\partial v_2} = 0$$

2. the near horizon electric field variables e_i given by $e_i = \frac{1}{2\pi}\frac{\partial F}{\partial q_i}$

3. entropy S_{BH} is given by $\boxed{S_{BH} = F(\vec{u},\vec{v},\vec{p},\vec{q})}$ at the extremum given by $\vec{u}, v_1, v_2$

Significance

1. F without flat directions $\rightarrow$ complete near horizon soln determined from these eqns.
   $\rightarrow$ independent of asymptotic values of moduli fields also S_{BH} indep.
2. F with some flat directions $\rightarrow$ some params not fixed from F, may depend on asympt. values of moduli fields, but F is flat along those directions $\Rightarrow S_{BH} = F$ still independent of asymptotic moduli $\rightarrow$ generalized attractor mechanism

Similar results hold in higher dimensions for near horizon geometry AdS_2.

BH entropy for spherically symm. BHs with higher derivative terms

$$S_{BH} = -8\pi \int_H d\theta\, d\phi \frac{\delta S}{\delta R_{rtrt}} \sqrt{-g_{rr}g_{tt}} \qquad \text{(Wald et al.)}$$

express in terms of symm. cov. derivatives & treat $R_{\mu\nu\rho\sigma}$ as independent variables

in our case we can neglect all covariant derivatives in S $\Rightarrow$

$S_{BH} = -8\pi \int_H d\theta\, d\phi \sqrt{-\det g}\,\frac{\partial \mathcal{L}}{\partial R_{rtrt}}\sqrt{-g_{rr}g_{tt}}$ and this can be proofed to be $= F$ (see video)

Ex: Reissner-Nordström $\qquad \boxed{S_{BH} = F = \frac{1}{4}(q^2 + p^2)}$ o.k.

Topological string theory 2

hep-th/9112056, 9309140

BH entropy hep-th/0007195, 0405146

$$F_g = \int_{\mathcal{M}_g} \langle \prod_{a=1}^{3g-3} (\eta_a, G_L^-)(\bar\eta_a, G_R^-) \rangle$$

$\mathcal{M}_g$ ← Riemann surface

A-model $G_L^+ = \psi_i^L \partial X^i,\ G_R^+ = \psi_j^R \bar\partial X^{\bar j}$ $X^{i=1,2,3}$

$G_L^- = \psi_{\bar j}^L \partial X^{\bar j},\ G_R^- = \psi_i^R \bar\partial X^i$

B-model $G_L^\pm \to G_R^\mp$

F_g depend on Kähler moduli of CY_3 in A-model

complex structure of CY_3 in B-model

In fact F_0 prepotential for type $\frac{IIA}{IIB}$ on CY_3

IIA $\mathcal{M}_V$ ← Kähler IIB $\mathcal{M}_V$ ← complex

$\theta_\alpha^1, \theta_\alpha^2$ α = spinor index = 1,2

$Z^i(\theta) = z^i + \dots$ scalar in vector multiplet

$g=0$ $\int d^4\theta\, F_0(z(\theta))$

$g \geq 1$ $\int d^4\theta\, F_g(z(\theta))(W(\theta)^2)^{2g-2}$

where $W_{\alpha\beta}(\theta) = F_{\alpha\beta} + \dots + R_{\alpha\beta\gamma\delta}\theta_1^\gamma\theta_2^\delta$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ graviphoton

can be decomposed selfdual $\to F_{\alpha\beta}$ $F_{\dot\alpha\dot\beta}$ ← antiselfdual spinor

$g=1$ $F_{g=1}(\overset{\text{scalar}}{z}) \cdot R^2$

$g > 1$ $F_g(z)\, R^2 (F^2)^{2g-2} + \dots$

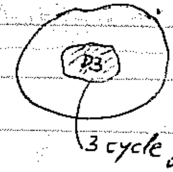
$\Rightarrow$ $F^2 \sim$ topological string coupling

$$Z_{top.} = \exp\left(\sum_g F_g(z) \cdot \lambda^{2g-2}\right)$$

Application of topological string theory:

BPS black holes from D-branes

IIB



$CY_3 \times \mathbb{R}^{3,1}$

D3 on a cycle γ

$M \geq |e^{\frac{1}{2}K} \int_\gamma \Omega|$ Note: invariant under rescaling of Ω

where

$$e^{-K} = \int_{CY_3} \Omega \wedge \bar\Omega$$

Mass bound depends only on cohomology,

branes tend to shrink to minimal configuration

(in their cohomology class) - special Lagrangian cycle (subm...

saturates the bound

$\{A_I, B^I\}$ homology 3-cycles of CY_3 $I = 0,1,\dots,b^{2,1}$

$\gamma \sim q_I$ times A_I, p^I times B^I

$\int_\gamma \Omega = q_I \int_{A_I} \Omega + p^I \int_{B^I} \Omega = q_I X^I + p^I F_I$, $F_I = \frac{\partial F}{\partial X^I}$

define

$W_{p,q}(X) = q_I X^I + p^I F_I(X)$ $M_{BPS} = |e^{\frac{1}{2}K} W_{p,q}(X)|$

M_{BPS} is a function on $\mathcal{M}_V$, does it make phys. sense to minimize i...

Attractor mechanism minimizes M_{BPS}

When $p,q \gg 1$ then $R_{\text{schwarzschild}} \gg R_{\text{compton}}$

$\Rightarrow$ gravity description is good.

$$ds^2 = -e^{2u(\rho)} e^{2\rho} dt^2 + e^{-2u(\rho)} d\rho^2 + e^{-2u(\rho)} d\Omega_2^2 + dS_{CY_3}^2$$

Some SUSY preserved $\Rightarrow$ $\exists$ Killing spinor $\Rightarrow$ BPS equation

$$\frac{du}{d\varsigma} = -1 + e^{u} M_{BPS}(\overset{\text{CY moduli}}{z})$$

$$\frac{dz^i}{d\varsigma} = 2 e^{u} G^{i\bar{j}} \frac{\partial}{\partial \bar{z}^{\bar{j}}} M_{BPS} \qquad G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$$

if z^i is const $\Rightarrow \frac{\partial}{\partial \bar{z}^{\bar{j}}} M_{BPS} = 0$

similarly for $\bar{z}^i \Rightarrow \underline{M_{BPS} \text{ is minimized}}$

– attractor mechanism

if $u = \text{const} \Rightarrow e^{-u} = M_{BPS}$

Attractor eqn. $\quad dM_{BPS} = 0 \qquad M_{BPS}^2 = e^{K} W \bar{W}, \; W = q_I X^I + p^I F_I$

$\Leftrightarrow \; e^{-K} d(e^{K} W) = DW = 0 \quad (\Leftarrow \bar{\partial} W = 0, \partial \bar{W} = 0, d = \partial + \bar{\partial})$

"covariant derivative w.r.t. connection on line bundle above moduli space of CY_3s of $W = 0$"

define $a^I, b_I \quad \int_{B^J} a^I = \delta^I_J$ etc. basis of H^3

$\Rightarrow F_3 = q_I a^I + p^I b_I \qquad W = \int_{CY_3} F_3 \wedge \Omega$

$\Rightarrow D_i W = \int_{CY_3} F_3 \wedge \underbrace{D_i \Omega}_{\eta_i \text{ basis of } H^{2,1}}$

$= \int_{CY_3} F_3 \wedge \eta_i \overset{!}{=} 0 \Rightarrow F_3 \in H^{3,0} + H^{0,3}$

$\Rightarrow$

$F_3 = Re(\overset{\text{const.}}{C}\,\Omega)$

(F_3 real $\Rightarrow$ if there would be $H^{2,1}$ they also $H^{1,2}$ $\Rightarrow$ contradiction with $\int F_3 \wedge \eta_i = 0$)

$$\Rightarrow \begin{array}{l} \int_{A^I} F_3 = \\ \int_{B^I} F_3 = \end{array} \boxed{\begin{array}{l} p^I = Re(c X^I) \\ q_I = Re(c F_I(X)) \end{array}} \quad \underline{\text{attractor equation}}$$

def: $\Omega(p,q)$ # of ground states of BH w/ charges p, q

If $p, q \gg 1 \Rightarrow$ classical description is good

$$\ln \Omega(p,q) \sim \frac{1}{4} A_{horizon} = \frac{1}{4} \pi M_{BPS}^2 \Big|_{attractor}$$

When p, q finite the entropy receives string loop corrections

$$\mathcal{F} = \sum_g F_g + \sum_g \bar{F}_g \qquad X^I = p^I + \frac{i}{\pi} \phi^I$$

entropy:

$S(p,q) = \mathcal{F}(p,\phi) + q_I \phi^I$, where $q_I = -\frac{\partial}{\partial \phi^I} \mathcal{F}(p,\phi)$

correction (conjecture)

$$\sum_q \Omega(p,q)\, e^{-q_I \phi^I} = \left| \exp\left(\sum_g F_g\right)\right|^2$$

to all orders in perturbation theory

IIA $\quad CY_3 =$

fibre bundle $\mathcal{L}_{-m} \oplus \mathcal{L}_{2g-2+m} \overset{\text{sum of}}{\longleftarrow}$ line bundles above Σ_g
$\downarrow$
Σ_g

F_g computed, $\quad \Omega(p,q) \quad$ D4 wrapping $\mathcal{L}_{-m} \to \Sigma_g$

($\Rightarrow$ noncompact CY_3)

since $(M_p^{(4d)})^2 = (\text{Vol}\, CY_3) \times (M_p^{(10d)})^8 \Rightarrow$ problem, can be considered only as local model and consider limits

4d entropy $\sim \frac{M_{BH}^2}{(M_p^{(4d)})^2} \to$ finite

Counting $\Omega(p,q)$ – using SYM theory on cocycle & index theory

e.g. $g = 1 \qquad Z_{D4} = tr\left(q^{-mH} e^{i\theta P}\right)$

$$q = e^{-\lambda} \qquad P_i = \pm\frac{1}{2}, \pm\frac{3}{2}, \dots, \; H = \frac{1}{2}\sum_{i=1}^{N} P_i^2, \; N \# D$$

$\Rightarrow$ one confirmation of the conjecture

Low-dimensional string theory 4

Review:

0A $\quad \mathbb{C}$ $N\times N$ matrices with $U(N)\times U(N)$

$$\phi \to U^+ \phi V \qquad \overset{\psi}{U} \quad \overset{\psi}{V}$$

N complex eigenvalues λ_i angular momentum $=0$

$$H = -\partial_\lambda^2 - \frac{1}{4\lambda^2} + \lambda^2$$

2 harm. osc. (S_i, U_i) $\quad i=1,2 \qquad S=\sqrt{S_1^2+S_2^2},\ U=\sqrt{U_1^2+U_2^2}$

and corresp. angle $\theta_S \quad \theta_U$

we want soln. indep. of θ_S, θ_U

$\Rightarrow$ angular momentum $=0$ after some computation

$$H = \tfrac{1}{2}\{S, P_S\} = -\tfrac{1}{2}\{U, P_U\}$$

$\langle S | U\rangle = \sqrt{SU}\ I_0(SU)$ $\leftarrow$ Bessel fcn.

$$\langle S | E\ out\rangle = \frac{1}{\sqrt{2\pi}} S^{iE-\frac{1}{2}}$$

$$\langle U | E\ in\rangle = \frac{1}{\sqrt{2\pi}} U^{-iE-\frac{1}{2}}$$

$$S = \langle E\ out | E'\ in\rangle = \delta(E-E')\, i\, 2^{-iE}\underbrace{\frac{\Gamma(\frac{1}{2}-iE)}{\Gamma(\frac{1}{2}+iE)}}_{\text{phase shift}}$$

2nd quantisation

$$\mathcal{L} = \int_0^\infty d\lambda\, \psi^+(\lambda,t)\left(i\partial_t + \tfrac{1}{2}\left(-\partial^2 - \lambda^2 - \tfrac{1}{4\lambda^2}\right) - \mu\right)\psi$$

def:

$$\psi_S(S,t) = \int_0^\infty d\lambda \sqrt{2\lambda S}\ e^{i\left(\frac{\lambda^2}{2}+\frac{S^2}{2}\right)} J_0(\sqrt{2}S\lambda)\,\psi(\lambda t)$$

$$\psi_U(U,t) = \int_0^\infty d\lambda \sqrt{2\lambda U}\ e^{-i\left(\frac{\lambda^2}{2}+\frac{U^2}{2}\right)} J_0(\sqrt{2}U\lambda)\,\psi(\lambda t)$$

$$\psi^{in} = e^{r/2}\,\psi(U=e^r) \qquad \psi^{out} = e^{r/2}\,\psi_S(S=e^r)$$

($\Rightarrow$ $\langle S|E out\rangle, \langle U| E in\rangle$ are plane waves)

$$\mathcal{L} = \int_{-\infty}^{+\infty} dr\ \psi^{in}\left(i(\partial_t - \partial_r) - \mu\right)\psi^{in}$$
$$= \int_{-\infty}^{+\infty} dr\ \psi^{out}\left(i(\partial_t + \partial_r) - \mu\right)\psi^{out}$$

$$\psi^{out}(r,t) = \int dr'\, e^{r+r'} J_0(e^{r+r'})\,\psi^{in}(r',t)$$

not a Fourier transf. as before

Symmetries: $U(1)$ rotating both ψ^{in} & ψ^{out}

$(-)^{F_L}$ trivial

S-duality $(-)^{F_L}$ $\quad \psi^{in}, \psi^{out} \to (\psi^{in}, \psi^{out})^+$

How to include RR fluxes?

start with unstable D-branes $(q+N)D + N\bar{D}$ $\Rightarrow$ N will decay, q will rema[in]

What to do in matrix model?

$N\{(\overbrace{\phi_2}^{N+q})$ $\qquad U(N)\times U(N+q)$ symmetry

$\rightarrow$ "quiver models"

by picking gauge $\left(\overset{N}{\phi_0} \,\middle|\, \overset{q}{0}\right)$ $\Rightarrow$ matrix model for ϕ_0 with modified measure - a factor of $(\det \phi_0\phi_0^+)^q$

Now the complex eigenvalues λ correspond to fermions moving in a plane with angular momentum q

$\Rightarrow$ $H = \partial^2 - \lambda^2 - \frac{q^2-\frac{1}{4}}{\lambda^2}$ $\Rightarrow$ $J_0(\cdot) \to J_q(\cdot)$ in all formulae

phase shift from in to out

$$e^{i\phi(E)} = i^{q+1}\, 2^{-iE}\,\frac{\Gamma(\frac{1}{2}(1+q-iE))}{\Gamma(\frac{1}{2}(1+q+iE))}$$

effect of RR flux $\tilde{q}$ on D-branes

$\phi \quad \underset{A}{U(N)} \times \underset{\tilde{A}}{U(N)}$ $\Rightarrow$ additional $\overbrace{\tilde{q}\int dt\,(\mathrm{Tr}A - \mathrm{Tr}\tilde{A})}^{\text{Chern-Simons term}}$ in Lagrang.

in our gauge (ϕ diagonal) $\rightarrow \tilde{q}\int dt \sum_i (a_i - \tilde{a}_i)$

$\Rightarrow$ angular momentum $=\tilde{q}$ - seems the same as for q

but the physics is different

both q & $\tilde{q}$ $\Rightarrow$ rectangular ϕ + Chern-Simons term $\Rightarrow$

answers depend only on $\mu(1+\frac{\eta}{2})$ — holds for any $\mu \gtreqless 0$

- great simplification but interpretation in spacetime not clean — should be something simple

New topic — new theories — type II (at the moment corresponding matrix models not known)

type IIB 0B modulo $T \to -T$, $C \to \tilde{C}$ (s-duality)

spectrum NS–NS $\cancel{T}$

R–R C_{in} ~~C_{out}~~ C_{out}

vertex operator $e^{-\varphi/2 - \bar{\varphi}/2 + \frac{i}{2}(H + \bar{H}) + ipx + (1-p)\phi}$ $p \geq 0$

NS–R $e^{-\varphi/2 - \bar{\varphi} - i\frac{H}{2} + ipx + (1-|p|)\phi}$ $p < 0$

R–NS $e^{-\varphi - \bar{\varphi}/2 - i\bar{H}/2 + ipx + (1-|p|)\phi}$ $p \leq 0$

$C_{out} \Longrightarrow$ c_{out} selfdual form

$\varphi \longleftarrow$ non-chiral

$\tilde{\varphi} \longleftarrow$

On world sheet no unitary S-matrix

- solution — presence of massless solitons
  - fermions made out of φ

Similarly IIA 0A mod s-duality

R–R F $\cancel{F}$

NS–NS $\cancel{T}$

NS–R $e^{-\varphi/2 - \bar{\varphi} - iH/2 + ipx + (1-|p|)\phi}$ $p \leq 0$

R–NS $e^{-\varphi - \bar{\varphi}/2 + iH/2 + ipx + (1-|p|)\phi}$ $p \geq 0$

$\rightleftarrows$ fermions → chiral on worldsheet, not in spacetime