Symmetries of differential equations

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Abstract

The purpose of this short course is to introduce the concept of point symmetries of differential equations. Next, we shall use point symmetries to solve a given ordinary differential equation. The method is based on finding a suitable transformation of independent and dependent variables after which we can reduce the order trivially. We shall also briefly indicate other applications.

Contents

1 Definition of Lie group and its Lie algebra

Let us consider a real smooth manifold G (of finite dimension). If the manifold G is also a group, i.e. equipped with an associative product such that a multiplicative unit e and an inverse g^{-1} exist, we may contemplate the compatibility of these two structures on G. When both the product[1]

$$\cdot \ : G \times G \to G$$

and the inverse

$$(\ )^{-1} \ : G \to G$$

are smooth (i.e. differentiable) maps, we call G a *Lie group*. One may also consider *complex Lie groups* which are complex manifolds such that the group operations are holomorphic but we shall not use them here.

Lie groups form a class of manifolds with rather special properties. Let us define two particular sets of diffeomorphisms of G, the left and right translations

$$L_g \ : G \to G, \quad L_g(h) = gh$$

and

$$R_g \ : G \to G, \quad R_g(h) = hg$$

defined for any chosen $g \in G$. Since these maps are diffeomorphisms their tangent maps $(L_g)_*, (R_g)_*$ define isomorphisms of the infinite–dimensional Lie algebra $\mathfrak{X}(G)$ of vector fields on G. A vector field $X \in \mathfrak{X}(G)$ is called *left–invariant* if

$$(L_g)_* X = X$$

for all $g \in G$. (Similarly for right–invariant fields.) The definition of a left–invariant vector field can be phrased also in a different way. Let us view both $X \in \mathfrak{X}(G)$ and the pullback $(L_g)^*$ as endomorphisms of the vector space $\mathcal{F}(G)$ of all smooth functions on G. Then X is left–invariant if and only if

$$X \circ (L_g)^* = (L_g)^* \circ X, \qquad \forall g \in G. \tag{1}$$

The formulation (1) makes evident a crucial property of left–invariant vector fields: they form not only a subspace but a subalgebra of $\mathfrak{X}(G)$ because

$$\begin{aligned}[X,Y] \circ (L_g)^* &= X \circ Y \circ (L_g)^* - Y \circ X \circ (L_g)^* \\ &= (L_g)^* \circ X \circ Y - (L_g)^* \circ Y \circ X = (L_g)^* \circ [X,Y]\end{aligned}$$

[1]often written without an explicit product sign $\cdot$

for any left–invariant vector fields X, Y. The algebra of left–invariant vector fields is called the *Lie algebra of the Lie group G and denoted by* $\mathfrak{g}$.[2]

Elements of $\mathfrak{g}$ are uniquely specified by their value at any chosen point $g \in G$. Conventionally, this identification is performed at the group unit, i.e. we identify

$$\mathfrak{g} \simeq T_e G.$$

Therefore, the dimension of $\mathfrak{g}$ is the same as dimension of the Lie group G.

One of the properties of left–invariant vector fields is that they are complete, i.e. any integral curve $\gamma(t)$

$$\dot{\gamma}(t) = X(\gamma(t))$$

of $X \in \mathfrak{g}$ can be extended to all real values of the curve parameter $t \in \mathbb{R}$. This property allows us to define the *exponential map* from the Lie algebra to the Lie group

$$\exp : \mathfrak{g} \to G : \quad X \to \gamma_X(1) \qquad \text{where } \dot{\gamma}_X(t) = X(\gamma_X(t)), \quad \gamma_X(0) = e. \quad (2)$$

The exponential map is a local diffeomorphism of $\mathfrak{g}$ into G, i.e. is smooth and is a diffeomorphism of some open neighborhood U of $0 \in \mathfrak{g}$ onto the open neighborhood $\exp(U)$ of $e \in G$.

Using the exponential map one may relate properties of Lie groups and their Lie algebras. In essence any local property of Lie groups has its counterpart in the properties of Lie algebras. Therefore, one may say that locally, i.e. up to topological issues, a Lie group and its Lie algebra encode the same information. Because Lie algebras are vector spaces, most computations in the theory of Lie algebras reduce to problems of linear algebra and consequently are much easier to handle than the corresponding computation

[2]More generally, an abstract *Lie algebra* $\mathfrak{g}$ is a vector space over a field $\mathbb{F}$ equipped with a multiplication (also called a bracket), i.e. a bilinear map $[\ ,\ ] \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, such that

$$\begin{aligned} [y, x] &= -[x, y] && \text{(antisymmetry)} \\ 0 &= \big[x, [y, z]\big] + \big[y, [z, x]\big] + \big[z, [x, y]\big] && \text{(Jacobi identity)} \end{aligned}$$

for all elements $x, y, z \in \mathfrak{g}$. In what follows we shall consider the fields $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ and finite–dimensional Lie algebras only.

The structure of the Lie algebra $\mathfrak{g}$ can be represented in any chosen basis $(e_j)_{j=1}^{\dim \mathfrak{g}}$ by the corresponding *structure constants* ${c_{jk}}^l$ in the basis $(e_j)_{j=1}^{\dim \mathfrak{g}}$

$$[e_j, e_k] = \sum_{l=1}^{\dim \mathfrak{g}} {c_{jk}}^l e_l.$$

in Lie groups. Therefore, using the local diffeomorphism exp (2) one may solve many problems on Lie groups which would be intractable on a general smooth manifold (or on a general, e.g. discrete, group).

2 Actions of Lie groups

For applications in both mathematics and physics we need a formalism allowing us to view Lie groups as sets of certain transformations of some objects. This leads us to the notion of an action of the group.

A (left) *action* of the Lie group G on a manifold M is a smooth map

$$\triangleright : \ G \times M \to M : \ (g, m) \to g \triangleright m$$

such that $g_1 \triangleright (g_2 \triangleright m) = (g_1 g_2) \triangleright m$ and $e \triangleright m = m$ for all $g_1, g_2 \in G$, $m \in M$.

Similarly one may consider also *right actions* $\triangleleft : M \times G \to G$ which satisfy $(m \triangleleft g_1) \triangleleft g_2 = m \triangleleft (g_1 g_2)$ and $m \triangleleft e = m$. Any left action $\triangleright$ defines a right action $\triangleleft$ through $m \triangleleft g = g^{-1} \triangleright m$ and vice versa.

An action $\triangleright$ of G on M is called *effective* if for every $g \in G$ different from the group unit e an element $m \in M$ exists such that $g \triangleright m \neq m$. Consequently, we can reconstruct the group multiplication on the group G from the knowledge of its effective action.

Examples of left actions of the group G on itself are

$$g \triangleright h = gh, \qquad g \triangleright h = h \cdot g^{-1}$$

and the *adjoint action*

$$Ad : \ G \times G \to G : \ Ad_g(h) \equiv Ad(g, h) = g \cdot h \cdot g^{-1}.$$

When the manifold M is a vector space and the action of G on M is *linear*

$$g \triangleright (av + w) = a(g \triangleright v) + g \triangleright w, \qquad \forall g \in G, v, w \in M, a \in \mathbb{R}$$

it is equivalent to a *representation* of the group G on the vector space M. A representation of the Lie group G on a vector space V is any (smooth) map

$$\rho : G \to \mathrm{End}(V)$$

which satisfies

$$\rho(e) = \mathbf{1}, \qquad \rho(g_1 g_2) = \rho(g_1) \circ \rho(g_2), \quad \forall g_1, g_2 \in G.$$

A representation can be associated to any linear action by the prescription

$$\rho : G \to \mathrm{End}(M) : \ \rho(g)v = g \triangleright v.$$

Whether we speak about a linear action or a representation is just a matter of convenience in the problem at hand.

A particular representation of the Lie group G on its algebra $\mathfrak{g}$ is defined by the derivation of the adjoint action

$$\mathrm{Ad} : \ G \to \mathfrak{gl}(\mathfrak{g}) : \ \mathrm{Ad}(g) = (Ad_g)_*.$$

This representation is called the *adjoint* representation of G.

Further differentiating we get the adjoint representation of the Lie algebra $\mathfrak{g}$ on itself

$$\mathrm{ad} : \ \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) : \ \mathrm{ad} = \mathrm{Ad}_*.$$

It can be shown that the adjoint representation of the Lie algebra satisfies

$$\mathrm{ad}(x)\, y = [x, y] \tag{3}$$

for any pair x, y of elements of $\mathfrak{g}$.

Sometimes we may encounter actions which are not well–defined for all pairs (g, m). Formally, one defines a *local (left) action* of a Lie group G on a manifold M to be a smooth map $\triangleright : U \to M$ where U is some open neighborhood in $G \times M$ which contains the whole subset $\{e\} \times M$ and satisfies the properties

$$e \triangleright m = m, \qquad \forall m \in M$$

and

$$g_1 \triangleright (g_2 \triangleright m) = (g_1 g_2) \triangleright m$$

whenever (g_2, m) and $(g_1, g_2 \triangleright m) \in U$.

When we consider an abstract Lie group G together with its prescribed (local) effective action on some manifold M we often speak about a (local) *group of transformations* or *group of motions* of M. In fact, this notion was what Sophus Lie had in mind in his pioneering works [1, 2, 3, 4] on what we now call Lie groups and Lie algebras.

An *infinitesimal action* of the Lie algebra $\mathfrak{g}$ on M is a homomorphism $\mu : \mathfrak{g} \to \mathfrak{X}(M)$. We often write the image of $x \in \mathfrak{g}$ in capital letters, $\mu(x) \equiv X$. A Lie algebra equipped with an injective infinitesimal action on some manifold M is called an *algebra of infinitesimal transformations*.

Any local action of G on M gives rise to an infinitesimal action of the Lie algebra $\mathfrak{g}$ on M through the prescription

$$(\mu(x)f)(m) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f\left(\exp(tx) \triangleright m\right), \qquad \forall f \in \mathcal{F}(M), m \in M. \tag{4}$$

3 Symmetries of algebraic equations

Now we shall introduce the notion of a symmetry of a given equation. Next, we apply it in particular to differential equations. Again we present only the essential notions and ideas. For proofs see [5, 6].

Let

$$f(x) = 0, \qquad f : \mathrm{Dom}(f) \subset \mathbb{F}^N \to \mathbb{F}^{\tilde{N}} \tag{5}$$

be a system of algebraic equations (or just one equation when $\tilde{N} = 1$) and $\mathcal{S}_f$ be its *solution set*

$$\mathcal{S}_f = \{x \in \mathrm{Dom}(f) \ | f(x) = 0\}.$$

A *symmetry* of the equation (5) is any transformation

$$T : \mathrm{Dom}(f) \to \mathrm{Dom}(f)$$

such that it preserves the solution set

$$T(\mathcal{S}_f) = \mathcal{S}_f. \tag{6}$$

Usually, we restrict our attention to transformations T which are diffeomorphisms, $T \in \mathrm{Diff}(\mathrm{Dom}(f))$.

It follows from the definition of a symmetry that symmetries of a given equation form a group, i.e. a subgroup of $\mathrm{Diff}(\mathrm{Dom}(f))$. Let us denote this *group of symmetries* of the equation (5) by $Sym(f = 0)$.

The group of all diffeomorphisms $\mathrm{Diff}(\mathrm{Dom}(f))$ is infinite–dimensional. While the use of the theory of Lie algebras as introduced above is not completely rigorous in this case, we may in a certain sense view the algebra $\mathfrak{X}(\mathrm{Dom}(f))$ of vector fields on $\mathrm{Dom}(f)$ as a Lie algebra of $\mathrm{Diff}(\mathrm{Dom}(f))$. When $Sym(f = 0)$ happens to be a a Lie group (more precisely, a Lie group of transformations), the corresponding algebra $\mathfrak{sym}(f = 0)$ of infinitesimal transformations defines a subalgebra of $\mathfrak{X}(\mathrm{Dom}(f))$. Its relation to the function f is derived using the notion of a 1–parameter subgroup.

A *1–parameter subgroup* σ of a group G is a homomorphism of the additive group $(\mathbb{R}, +)$ into the group G. While G may not necessarily be a Lie group (cf. $\mathrm{Diff}(\mathrm{Dom}(f))$), the image $\sigma(\mathbb{R})$ has a natural structure of a 1–dimensional Lie group (or 0–dimensional if $\sigma(t) = e$ for all $t \in \mathbb{R}$). Consequently, one may consider its Lie algebra. When G is a group of transformations of M and σ its 1–parameter subgroup we have a 1–dimensional algebra of infinitesimal transformations spanned by its generator $X_\sigma \in \mathfrak{X}(M)$:

$$X_\sigma j(m) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} j\left(\sigma(t) \triangleright m\right), \qquad \forall j \in \mathcal{F}(M).$$

Let $Sym(f=0)$ be the group of symmetries of the equation (5). We shall call the vector subspace of $\mathfrak{X}(\mathrm{Dom}(f))$ spanned by all generators X_σ of 1–parametric subgroups of the group $Sym(f=0)$ the *algebra of infinitesimal symmetries* of the equation $f=0$ and denote it by $\mathfrak{sym}(f=0)$. It turns out that $\mathfrak{sym}(f=0)$ is a subalgebra of $\mathfrak{X}(\mathrm{Dom}(f))$. The algebra $\mathfrak{sym}(f=0)$ coincides with the algebra of infinitesimal transformations arising from the group of transformation $Sym(f=0)$ when $Sym(f=0)$ is a Lie group.

Let us take $m \in \mathcal{S}_f$ and $X_\sigma \in \mathfrak{sym}(f=0)$. Because $\sigma(t)$ lies in the symmetry group $Sym(f=0)$ for all $t \in \mathbb{R}$ we have $f(\sigma(t) \triangleright m) = 0$ and consequently

$$X_\sigma f(m) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f\left(\sigma(t) \triangleright m\right) = 0.$$

That means that the vector fields X in the algebra of infinitesimal symmetries $\mathfrak{sym}(f=0)$ of the equation $f=0$ satisfy

$$X f\big|_{f=0} = 0, \qquad \text{i.e. } X f(m) = 0, \quad \forall m \in \mathcal{S}_f. \tag{7}$$

Let us consider the converse problem. We recall that the *flow of the vector field* X is the map

$$\Phi_X : U \to M : \Phi_X(0,m) = m, \quad \frac{\mathrm{d}}{\mathrm{d}t}\Phi_X(t,m) = X(\Phi_X(t,m)), \quad \forall (t,m) \in U, \tag{8}$$

where U is some open neighborhood $U \subset \mathbb{R} \times M$ such that $(0, M) \subset U$.

Let $X \in \mathfrak{X}(\mathrm{Dom}(f))$ satisfy the condition (7). Is it true that the flow Φ_X defines a 1–parameter group of symmetries of the equation (5)?

In general, the answer is negative for two reasons.

Firstly, the flow may not be defined on the whole $\mathbb{R} \times \mathrm{Dom}(f)$, i.e. the vector field may not be *complete*. That is why we introduced the notion of a local action of a group: the flow of a vector field defines in general a local action of a 1–parameter group.

Secondly, even locally the flow may not define symmetries of the given equation (5).

Example 1 *Let us consider a system of equations*

$$x_1 - x_2^2 = 0, \qquad x_1 = 0. \tag{9}$$

Its set of solutions is $\mathcal{S} = \{(0,0)\}$. On the other hand, the condition (7) is satisfied by the vector field

$$X = \partial_{x_2}$$

whose flow is

$$\Phi_X : \mathbb{R} \times (\mathbb{R} \times \mathbb{R}) \to (\mathbb{R} \times \mathbb{R}) : \Phi_X(t, x_1, x_2) = (x_1, x_2 + t).$$

Now the action of the group element $t \neq 0$, $\Phi_X(t, \cdot)$, *takes the solution* $(0, 0)$ *to a point* $(0, t)$ *which is not a solution of the equation* (9).

It turns out that the condition on the function f which prevents such pathological behaviour is the maximality of the rank of the Jacobian, rank $\frac{\partial f_j}{\partial x_k}\Big|_{\mathcal{S}_f} = \tilde{N}$. These results are the content of

Theorem 1 (On infinitesimal generators of symmetries) *Let* $f : \mathrm{Dom}(f) \subset \mathbb{R}^N \to \mathbb{R}^{\tilde{N}}$ *define a system of equations*

$$f(x) = 0 \tag{10}$$

such that

$$\operatorname{rank} \frac{\partial f_j}{\partial x_k}(x) = \tilde{N}, \qquad \forall x \in \mathcal{S}_f. \tag{11}$$

Then a vector field $X \in \mathfrak{X}(\mathrm{Dom}(f))$ *generates a local 1–parameter group of symmetries of the equation* (10) *if and only if*

$$(Xf)(m) = 0, \quad \forall m \in \mathcal{S}_f. \tag{12}$$

We see that under the assumption of regularity of the function f (11) we can determine the algebra of infinitesimal symmetries $\mathfrak{sym}(f = 0)$ of the given equation $f = 0$ through solution of a linear system of equations (7) for the coefficient functions $X^i \in \mathcal{F}(\mathrm{Dom}(f))$ of the vector field

$$X : \; X(x) = \sum_{i=1}^{N} X^i(x) \frac{\partial}{\partial x^i}\bigg|_x .$$

Infinitesimal symmetries can be converted into actual symmetries through computation of the corresponding flows; composing the flows one may construct a local group of symmetries of the given equation $f = 0$. In this way, the description of infinitesimal symmetries in terms of the condition (7) significantly simplifies the search for symmetries of the given equation.

Detection of symmetries which cannot be connected to identity transformation by flows of infinitesimal symmetries, e.g. belonging to different connected components of the symmetry group, is a much harder problem and we shall not discuss it here.

4 Symmetries of differential equations

Let us now shift our attention to differential equations.

Let us for simplicity start with one ordinary differential equation

$$F(x, u(x), u'(x), \ldots, u^{(p)}(x)) = 0 \tag{13}$$

on some domain $M \subset \mathbb{R}$.

The concept of symmetry remains the same: symmetries are transformations leaving the set of solutions invariant. The question is what kind of transformations do we admit?

In principle, we may allow any transformation on the infinite–dimensional space of all functions on M differentiable up to order p. Such a broad definition would, however, entail numerous computational difficulties. Therefore, one a priori restricts the class of allowed transformations.

The most restrictive and most often used class of allowed transformations is the following one: we allow any invertible transformation on the space of dependent and independent variables, i.e. u and x,

$$\hat{x} = g_1(x, u), \qquad \hat{u} = g_2(x, u). \tag{14}$$

Such transformations are called *point transformation.* The effect of such a transformation on any function $f : M \to \mathbb{R}$ is defined using the transformation of the graph of the function $f(x)$.

Let f be a function on the domain $M \subset \mathbb{R}$. Its *graph* is the following subset of $M \times \mathbb{R}$

$$\Gamma_f = \{(x, f(x)) \mid x \in M\}. \tag{15}$$

$\Gamma \subset M \times \mathbb{R}$ defines a function f on some subset of M such that $\Gamma = \Gamma_f$ if and only if for every pair of points $(x_1, u_1), (x_2, u_2) \in \Gamma$ the relation $x_1 = x_2$ implies $u_1 = u_2$.

When f is at least k–times differentiable we define also the k^{th}*–prolonged graph* of the function f

$$\Gamma_f^{(k)} = \{\big(x, f(x), f'(x), \ldots, f^{(k)}(x)\big) \mid x \in M\} \subset M \times \mathbb{R}^{1+k}. \tag{16}$$

We denote the coordinates on $M \times \mathbb{R}^{1+k}$ by $x, u, u', \ldots, u^{(k)}$ for obvious reasons.

Let us assume that a (local) group G of transformations of the form (14) is given. We define the action of $g \in G$ on the graph Γ_f in a natural way

$$g \triangleright \Gamma_f = \{g \triangleright (x, f(x)) \mid x \in M\}.$$

In this way we obtain a new subset $g \triangleright \Gamma_f$ of $M \times \mathbb{R}$. When $g \triangleright \Gamma_f$ is a graph of some function $\hat{f}$

$$g \triangleright \Gamma_f = \Gamma_{\hat{f}}$$

we call $\hat{f} \equiv g \triangleright f$ the transformation of the function f under the point transformation g of $M \times \mathbb{R}$.

Such construction of the transformation $f \to \hat{f}$ introduces another source of locality into our transformation groups. In particular, even if the action of G on the space of dependent and independent coordinates $M \times \mathbb{R}$ is globally defined, its induced action on functions is not: $g \triangleright \Gamma_f$ may fail to define a graph of a new function; there may be two different points (x, u_1) and (x, u_2) in $g \triangleright \Gamma_f$. Therefore, the induced action of G on the space of functions $\mathcal{F}(M)$ is only a local action.

A local 1–parameter group of point transformations

$$(\hat{x}, \hat{u}) = t \triangleright (x, u) : \qquad \hat{x} = g_1(x, u; t), \quad \hat{u} = g_2(x, u; t) \tag{17}$$

of $M \times \mathbb{R}$ is a *1–parameter symmetry group* of the differential equation (13) if for every solution $u : M \to \mathbb{R}$ of equation (13) and every $t \in \mathbb{R}$ such that $\hat{u} = t \triangleright u$ is defined we have

$$F(x, \hat{u}(x), \hat{u}'(x), \ldots, \hat{u}^{(n)}(x)) = 0.$$

In order to establish a symmetry criterion in terms of a vector field generating the 1–parameter group of transformations we have to analyze how do the derivatives transform. Let us assume that a function $u = f(x)$ is given. We have its graph Γ_f and its prolonged graph $\Gamma_f^{(1)}$. We transform Γ_f by a 1–parameter group of point transformations $\phi : \mathbb{R} \to \mathrm{Diff}(M \times \mathbb{R})$ and consequently we also obtain $\hat{f}_t = t \triangleright f$ whenever it is defined. What is the relation between the derivatives of the function f and of the functions $\hat{f}_t$? In other words, how are the prolonged graphs of these functions related?

The points of the graph Γ_f transform under the action (17) into the points of the graph $\Gamma_{\hat{f}}$ as

$$\hat{x} = g_1(x, f(x); t), \qquad \hat{f}(\hat{x}) = g_2(x, f(x); t).$$

We obtain by differentiation and use of the chain rule an expression for the derivative of $\hat{f}$,

$$\hat{f}'(\hat{x}) \equiv \frac{\mathrm{d}\hat{f}}{\mathrm{d}\hat{x}}(\hat{x}) = \frac{\frac{\mathrm{d}}{\mathrm{d}x} g_2(x, f(x); t)}{\frac{\mathrm{d}}{\mathrm{d}x} g_1(x, f(x); t)} = \left. \frac{\frac{\partial g_2}{\partial x} + f'(x) \frac{\partial g_2}{\partial u}}{\frac{\partial g_1}{\partial x} + f'(x) \frac{\partial g_1}{\partial y}} \right|_{(x, f(x); t)}.$$

We see that the transformation (17) induces a unique point transformation of $U \times \mathbb{R}^2$

$$\hat{x} = g_1(x, u; t), \quad \hat{u} = g_2(x, u; t), \quad \hat{u}' = \frac{\frac{\partial g_2}{\partial x}(x, u; t) + u' \frac{\partial g_2}{\partial u}(x, u; t)}{\frac{\partial g_1}{\partial x}(x, u; t) + u' \frac{\partial g_1}{\partial u}(x, u; t)} \tag{18}$$

such that the prolonged graph $\Gamma_f^{(1)}$ of any function $f : M \to \mathbb{R}$ is transformed by the transformation (18) into the prolonged graph $\Gamma_{\hat{f}_t}^{(1)}$ of the transformed function $\hat{f}_t = t \triangleright f$ whenever $\hat{f}_t$ exists. By induction, this concept can be readily generalized to k^{th}–prolonged graphs.

Let us now convert these ideas to the infinitesimal language. Let us assume that the 1–parameter group of transformations (17) is generated by the vector field X on $M \times \mathbb{R}$,

$$X \in \mathfrak{X}(M \times \mathbb{R}), \qquad X = \xi(x, u) \frac{\partial}{\partial x} + \eta(x, u) \frac{\partial}{\partial u}. \tag{19}$$

What is the corresponding vector field $\tilde{X} \in \mathfrak{X}(M \times \mathbb{R} \times \mathbb{R})$ generating the action on the prolonged graphs?

We differentiate equation (18) with respect to t and set $t = 0$. We notice that by definition of the generator X of the 1–parameter group (17) we have

$$g_1(x, u; 0) = x, \; g_2(x, u; 0) = u, \quad \frac{\partial g_1}{\partial t}(x, u; 0) = \xi(x, u), \; \frac{\partial g_2}{\partial t}(x, u; 0) = \eta(x, u).$$

Altogether, we find that

$$\tilde{X} = \xi(x, u) \frac{\partial}{\partial x} + \eta(x, u) \frac{\partial}{\partial u} + (\mathcal{D}_x \eta(x, u, u') - u' \mathcal{D}_x \xi(x, u, u')) \frac{\partial}{\partial u'} \tag{20}$$

where $\mathcal{D}_x = \frac{\partial}{\partial x} + u' \frac{\partial}{\partial u}$ is called the *operator of total derivative* on $\mathcal{F}(M \times \mathbb{R})$. We call the vector field (20) the first *prolongation* of the vector field X and denote it by $\mathrm{pr}^{(1)} X$. Repeating the same procedure for higher derivatives we find that the action of the 1–parameter group (17) on k^{th}–prolonged graphs is generated by the vector field $\mathrm{pr}^{(k)} X \in \mathfrak{X}(M \times \mathbb{R}^{1+k})$

$$\mathrm{pr}^{(k)} X = \xi(x, u) \frac{\partial}{\partial x} + \eta(x, u) \frac{\partial}{\partial u} + \sum_{j=1}^{k} \eta^{(j)}(x, u, u', \ldots, u^{(j)}) \frac{\partial}{\partial u^{(j)}} \tag{21}$$

where the components $\eta^{(j)}(x, u, u', \ldots, u^{(j)})$ are constructed recursively

$$\eta^{(j)}(x, u, u', \ldots, u^{(j)}) = \mathcal{D}_x \eta^{(j-1)} - u^{(j)} \mathcal{D}_x \xi \tag{22}$$

using the operator of total derivative

$$\mathcal{D}_x = \frac{\partial}{\partial x} + u' \frac{\partial}{\partial u} + \sum_{j=1}^{k-1} u^{(j+1)} \frac{\partial}{\partial u^{(j)}}.$$

That means that the vector field (21) encodes in itself the fact that the derivatives $u'(x), \ldots, u^{(n)}(x)$ in the differential equation (13) transform in a unique way once the point transformation (14) is chosen. Provided that we work only with generators of the form (21), we may now for our purposes view the differential equation (13) as an algebraic equation for a set of unknowns $x, u, u', \ldots, u^{(p)}$. This determines certain solution hypersurface Σ_F in $M \times \mathbb{R}^{1+p}$,

$$\Sigma_F = \{(x, u, u', \ldots, u^{(p)}) \in M \times \mathbb{R}^{1+p} | F(x, u, u', \ldots, u^{(p)}) = 0\}.$$

Any p–times differentiable function $f : M \to \mathbb{R}$ whose p^{th}–prolonged graph $\Gamma_f^{(p)}$ lies in the hypersurface Σ_F is a solution of the differential equation (13).

Combining the results on symmetries of algebraic equations and the prolongation of vector fields, we can formulate a criterion on generators of point symmetries of differential equations.

Theorem 2 (On generators of symmetries of ODEs) *Let $M \subset \mathbb{R}$ and let $F : M \times \mathbb{R}^{1+p} \to \mathbb{R}$ define a differential equation*

$$F(x, u(x), u'(x), \ldots, u^{(p)}(x)) = 0. \tag{23}$$

Let

$$\Sigma_F = \{(x, u, u', \ldots, u^{(p)}) \in M \times \mathbb{R}^{1+p} | F(x, u, u', \ldots, u^{(p)}) = 0\}$$

and

$$\mathrm{d}F(v) \neq 0, \qquad \forall v \in \Sigma_F. \tag{24}$$

Then a vector field $X \in \mathfrak{X}(M \times \mathbb{R})$ generates a local 1–parameter group of point symmetries of the differential equation (23) *if and only if*

$$\mathrm{pr}^{(p)} F(v) = 0, \quad \forall v \in \Sigma_F. \tag{25}$$

We notice that the regularity condition (24) is satisfied e.g. for any differential equation solved with respect to the highest derivative.

Let us mention that point transformations are not the only class of transformations one may consider in the context of symmetry analysis of differential equations. Another, less restrictive choice is defined by transformations on $\mathbb{R}^3$ (with coordinates x, u, u') of the form

$$\hat{x} = g_1(x, u, u'), \qquad \hat{u} = g_2(x, u, u'), \qquad \hat{u}' = g_3(x, u, u') \tag{26}$$

subject to a consistency condition

$$\frac{\partial g_2(x,u,u')}{\partial u'} = g_3(x,u,u')\frac{\partial g_1(x,u,u')}{\partial u'}. \tag{27}$$

This condition comes from the requirement that first derivatives of the function $u = f(x)$ should transform independently of second and higher derivatives of $f(x)$.

Transformations (26) are called *contact transformations*. While for certain differential equations the group of contact symmetries is larger than the group of point symmetries, in most cases both groups are isomorphic.

We shall restrict ourselves to point transformations in the following.

Theorem 2 can be readily generalized to systems of ordinary differential equations and also to partial differential equations.

Let us consider a system of q partial differential equations of order at most p

$$F_\nu(x^i, u^\alpha, \ldots, u^\alpha_J) = 0, \qquad \nu = 1, \ldots, q,\ |J| \leq p. \tag{28}$$

where $(x^i)_{i=1}^m$ are independent variables, $(u^\alpha)_{\alpha=1}^n$ are dependent variables. (Collectively, we denote them x and u, respectively.) We define the multi–index $J = (j_1, \ldots, j_m)$, where $j_i \in \mathbb{N} \cup \{0\}$, $|J| = j_1 + \ldots + j_m$ and

$$u^\alpha_J = \frac{\partial^{|J|} u^\alpha}{\partial^{j_1} x_1 \partial^{j_2} x_2 \ldots \partial^{j_m} x_m}.$$

We suppose that solutions $u(x)$ of PDE (28) are defined on a domain $M \subset \mathbb{R}^m$ and take values in $N \subset \mathbb{R}^n$ where M and N are some open subsets.

As before, the coordinates x^i, u^α on $M \times N$ are formally extended to the so–called k^{th} jet bundle

$$\mathcal{J}_k = \{(x^i, u^\alpha, u^\alpha_J) \,|\, |J| \leq k\} \tag{29}$$

which includes both coordinates on $M \times N$ and all derivatives of the dependent variables u^α of order less or equal to k (we identify $\mathcal{J}_0 \equiv M \times N$). On the jet bundle, we define the total derivatives

$$\mathcal{D}_i = \frac{\partial}{\partial x^i} + \sum_{\alpha, J} u^\alpha_{J_i} \frac{\partial}{\partial u^\alpha_J}, \tag{30}$$

where

$$J_i = (j_1, \ldots, j_{i-1}, j_i + 1, j_{i+1}, \ldots, j_m).$$

More generally, for $J = (j_1, j_2, \ldots, j_m)$, we define

$$\mathcal{D}_J = \underbrace{\mathcal{D}_1 \mathcal{D}_1 \cdots \mathcal{D}_1}_{j_1} \cdots \underbrace{\mathcal{D}_n \mathcal{D}_n \cdots \mathcal{D}_n}_{j_m}. \tag{31}$$

The prolongation of a 1–parameter group action to the jet bundle $\mathcal{J}_k$ as before induces a prolongation of the generating vector field. For the vector field X given by

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}, \tag{32}$$

the k^{th} order prolongation of X is

$$\mathrm{pr}^{(k)}(X) = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha} + \sum_{\alpha,|J|\neq 0} \eta^\alpha_J(x,u,\ldots,u_{(|J|)})\frac{\partial}{\partial u^\alpha_J}, \tag{33}$$

where $\eta^\alpha_J(x,u,\ldots,u_{(|J|)})$ are functions on the $|J|$–th jet bundle and are given by the recursive formula

$$\eta^\alpha_{J_j} = \mathcal{D}_j \eta^\alpha_J - \sum_i (\mathcal{D}_j \xi^i) u^\alpha_{J_i} \tag{34}$$

or, equivalently, by the formula

$$\eta^\alpha_J = \mathcal{D}_J\left(\eta^\alpha - \xi^i \frac{\partial u^\alpha}{\partial x^i}\right) + \xi^i u^\alpha_{J_i}. \tag{35}$$

An analogue of the symmetry criterion 2 can now be stated as follows

Theorem 3 (On generators of symmetries of PDEs) *Let*

$$F_\nu(x^i, u^\alpha, \ldots, u^\alpha_J) = 0, \qquad \nu = 1,\ldots,q,\ |J| \leq p.$$

be a non–degenerate system of partial differential equations (meaning that the system is locally solvable with respect to highest derivatives and is of maximal rank at every point $p \in \mathcal{J}_k$ such that $F_\nu(p) = 0$, $\nu = 1,\ldots,q$) and G be a connected Lie group (locally) acting on $\mathcal{J}_0 = M \times N$ through the transformations

$$\tilde{x}_i = A^i(x,u,g), \qquad \tilde{u}^\alpha = B^\alpha(x,u,g).$$

Let the Lie algebra $\mathfrak{g}$ of the Lie group G together with its induced infinitesimal action (4) *be the corresponding algebra of infinitesimal transformations. Then G is a group of point symmetries of the PDE system $F = 0$ if and only if*

$$\left[\mathrm{pr}^{(p)}(X)\right](F_\nu) = 0, \quad \nu = 1,\ldots,q, \qquad \text{whenever } F = 0 \tag{36}$$

for every infinitesimal generator X representing the infinitesimal action of $x \in \mathfrak{g}$.

Theorem 3 applies also to ODEs and systems of ODEs (when $m = 1$).

A practical determination of the symmetry algebra of a given p^{th} order system (28) of differential equations

$$F_\nu(x^i, u^\alpha, \dots, u^\alpha_J) = 0, \qquad \nu = 1, \dots, q$$

involves several steps:

1. we have to compute p^{th} prolongation of an arbitrary vector field X (32) on $\mathcal{J}_0$,
2. evaluate $\mathrm{pr}^{(p)}(X)F_\nu$,
3. substitute into it all equations $F_\nu = 0$ and their differential consequences (if necessary); preferably, we eliminate the highest order derivatives using $F_\nu = 0$.

   These three steps can be rather lengthy and tedious, but are algorithmic and can be efficiently and reliably performed using computer algebra systems.
4. Now that $F = 0$ was imposed, the resulting equations

   $$\mathrm{pr}^{(p)}(X)F_\nu|_{F=0} = 0$$

   are to be viewed as equations for the unknown components ξ^i, η^α of the vector field X which must hold for any values of the remaining jet space coordinates $u^\alpha_J, |J| \geq 1$. After we separate independent terms in u^α_J, we obtain a highly overdetermined[3] system of linear partial differential equations for the functions $\xi^i(x,u), \eta^\alpha(x,u)$. Its solution provides us with all generators X which satisfy equation (36) of Theorem 3.

   Although this step is often also entrusted to computers, it does sometimes happen that computer programs miss some of the solutions and the resulting symmetry algebra is incomplete.

After the symmetry generators are found, it is sensible to check their consistency by verifying that the symmetry algebra is closed under commutators. Next, one may integrate the generators to 1–parameter subgroups and compose them to obtain the connected component of the symmetry group.

Other possible components of the symmetry group cannot be deduced directly from the infinitesimal approach. Although some methods for their determination exist (see e.g. [5, 6]) we shall not consider them here.

Let us now iluminate the presented abstract concepts by concrete examples. We use an abbreviated notation, $\partial_a \equiv \frac{\partial}{\partial a}$.

[3]in almost all cases

Example 2 *Let us consider the ODE*

$$y''(x) = \frac{(y'(x))^2}{y(x)} - y^2(x) \tag{37}$$

The second prolongation of an arbitrary vector field $X = \xi(x,u)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u}$ *is computed*

$$\mathrm{pr}^{(2)}(X) = \xi\partial_x + \eta\partial_u + (\mathcal{D}_x\eta - u'\mathcal{D}_x\xi)\partial_{u'} + (\mathcal{D}_x(\mathcal{D}_x\eta - u'\mathcal{D}_x\xi) - u''\mathcal{D}_x\xi)\partial_{u''} = \ldots$$

We apply $\mathrm{pr}^{(p)}(X)$ *on the function*

$$F = u'' - \frac{(u')^2}{u} + u^2$$

and substitute into it $u'' = \frac{(u')^2}{u} - u^2$, *i.e. we restrict ourselves to points laying on the solution hypersurface* Σ_F. *We obtain an equation*

$$\begin{aligned} \partial_{xx}\eta + 2u'\partial_{xu}\eta + u'^2\partial_{uu}\eta + \frac{u'^2}{u}\partial_u\eta - u^2\partial_u\eta - u'\partial_{xx}\xi - 2u'^2\partial_{xu}\xi + \\ +2u^2\partial_x\xi - u'^3\partial_{uu}\xi - 3\frac{u'^3}{u}\partial_u\xi + 3u'u^2\partial_u\xi - 2\frac{u'}{u}\partial_x\eta - 2\frac{u'^2}{u}\partial_u\eta + \\ +2\frac{u'^3}{u}\partial_u\xi + \frac{u'^2}{u^2}\eta + 2u\eta &= 0 \end{aligned}$$

Isolating different powers of u' *and setting to zero each of the coefficients we obtain four linear partial differential equations*

$$-\partial_{uu}\xi - \frac{1}{u}\partial_u\xi = 0, \tag{38}$$

$$\partial_{uu}\eta - 2\partial_{xu}\xi - \frac{1}{u}\partial_u\eta + \frac{1}{u^2}\eta = 0, \tag{39}$$

$$2\partial_{xu}\eta - \partial_{xx}\xi + 3u^2\partial_u\xi - 2\frac{1}{u}\partial_x\eta = 0, \tag{40}$$

$$\partial_{xx}\eta - u^2\partial_u\eta + 2u^2\partial_x\xi + 2u\eta = 0. \tag{41}$$

for the unknown functions $\xi(x,u)$ *and* $\eta(x,u)$. *Solving the system of equations (38–41) by simple manipulations we obtain the general solution of the system (38–41) involving two arbitrary constants of integration*

$$\xi(x,u) = Cx + b, \qquad \eta(x,u) = -2Cu. \tag{42}$$

Correspondingly, there are two linearly independent infinitesimal symmetries of the ordinary differential equation (37), *namely the translation*

$$X_1 = \frac{\partial}{\partial x} \tag{43}$$

with the flow $\Phi_1(x,u;t) = (x+t,u)$*, and a scaling symmetry*

$$X_2 = -x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u} \tag{44}$$

with the flow $\Phi_2(x,u;t) = (e^{-t}x, e^{2t}u)$*. The action of these point transformations on functions* $y(x)$ *is*

$$(t \triangleright_1 y)(x) = y(x-t)$$

and

$$(t \triangleright_2 y)(x) = e^{2t}y(e^t x),$$

respectively.

In the case of PDEs the full derivation and intermediate calculations are very long. Therefore, we shall only review and interpret the results.

Example 3 *The heat equation*

$$\partial_t u - \partial_{xx} u = 0 \tag{45}$$

has an infinite dimensional algebra of infinitesimal point symmetries. It consists of the six vector fields

$$\begin{aligned}
X_1 &= 4xt\partial_x + 4t^2\partial_t - (2t+x^2)u\partial_u, \\
X_2 &= 2x\partial_x + 4t\partial_t - u\partial_u, \\
X_3 &= \partial_t, \\
X_4 &= -2t\partial_x + xu\partial_u, \\
X_5 &= u\partial_u, \\
X_6 &= \partial_x
\end{aligned}$$

together with an infinite set of generators

$$X_V = V(x,t)\partial_u$$

where $V(x,t)$ *is an arbitrary solution of the heat equation (45).*

It is instructive to interpret these vector fields in terms of the corresponding finite transformations. The vector fields X_3, X_6 *generate translations in* t *and* x*. These symmetries are obvious from the onset – they just represent the fact that the heat equation* (45) *is autonomous, i.e. does not involve* t *and* x *explicitly.*

The vector fields X_2, X_5 *represent invariance of the heat equation under two independent scalings* $u \to \lambda u$ *and* $x \to \lambda x, t \to \lambda^2 t$*.*

The vector field X_4 indicates invariance under the Galilei transformation $x \to x - \lambda t$ accompanied by a suitable redefinition of $u(x,t)$.

Finally, X_V generates the invariance under the transformation $u \to u + \lambda V$ where V is another arbitrary solution of the heat equation (45), i.e. represents its linearity.

Altogether, all the symmetry generators $X_2, \dots, X_6, X_V$ can be guessed without any calculations. They close into a subalgebra of the full symmetry algebra $\mathfrak{sym}(\partial_t u - \partial_{xx} u = 0) = \mathrm{span}\{X_1, \dots, X_6, X_V\}_{\partial_t V - \partial_{xx} V = 0}$. Without explicit computation of the symmetry algebra one would probably miss the generator X_1 which does not possess any obvious physical interpretation.

As far as the algebraic structure of the Lie algebra $\mathfrak{sym}(\partial_t u - \partial_{xx} u = 0)$ is considered, we notice that it splits into a semidirect sum,

$$\mathfrak{sym}(\partial_t u - \partial_{xx} u = 0) = \mathrm{span}\{X_V\}_{\partial_t V - \partial_{xx} V = 0} \ni\!\!\!\supset \mathrm{span}\{X_1, \dots, X_6\}$$

where $\mathrm{span}\{X_V\}_{\partial_t V - \partial_{xx} V = 0}$ is an infinite–dimensional Abelian Lie algebra and $\mathrm{span}\{X_1, \dots, X_6\}$ is a finite dimensional Levi decomposable algebra. It has a simple factor $\mathrm{span}\{X_1, X_2, X_3\}$ isomorphic to $\mathfrak{sl}(2)$ and a nilpotent radical $\mathrm{span}\{X_4, X_5, X_6\}$ isomorphic to the Heisenberg algebra $\mathfrak{h}(1)$ which is a nilpotent Lie algebras spanned by three vectors e_1, e_2, e_3 with the only nonvanishing Lie bracket

$$[e_2, e_3] = e_1.$$

We observe that the infinite dimensional algebra $\mathrm{span}\{X_V\}_{\partial_t V - \partial_{xx} V = 0}$ is often truncated to a finite dimensional subalgebra when the symmetries are computed using algorithms implemented in computer algebra systems (e.g. procedure Infinitesimals *in Maple 13).*

We have noticed in this example that often most, if not all, infinitesimal symmetries of the given differential equation can be found by inspection, without any computation. Unfortunately, there is no easy way of establishing the completeness of the symmetry algebra guessed in this way, e.g. there is no method of independent determination of dimension of the symmetry algebra. The only reliable method is to perform the full computation of symmetries and check whether anything unexpected arises.

5 Applications: Reduction of the order of a given ODE and others

Once the symmetry algebra of the given equation(s) is determined, one can use it in several different ways such as:

1. Exponentiate infinitesimal symmetries to 1–parameter subgroups and use the resulting transformations to generate new solutions from the known ones.

2. Use the symmetry algebra as a necessary criterion for equivalence of two differential equations. If any pair of differential equations can be transformed one into the other by a point transformation then necessarily their symmetry algebras must be isomorphic. Thus we have a necessary (though far from sufficient) condition for equivalence. In addition, when an explicit transformation between two equations is sought, it is often convenient to construct point transformations taking one symmetry algebra into the other and only then look for transformations taking one equation into the other inside this class.

   In particular, when a given PDE has an infinite dimensional Abelian subalgebra of infinitesimal symmetries involving an arbitrary solution of some linear PDE we may interpret it as a strong indication that our prescribed equation may be linearizable by some point transformation.

3. Reduce the order of an ODE. This method is based on a simple observation that an ODE
$$F(x, y, \dots, y^{(p)}) = 0$$
   which possesses an infinitesimal symmetry ∂_y must be independent of the dependent variable y, i.e. in the form
$$F(x, y', \dots, y^{(p)}) = 0 \tag{46}$$
   (possibly up to a multiplication by a common nonvanishing y–dependent prefactor which does not affect its solutions). Obviously, we may lower its order by one through the substitution $z = y'$, then attempt to solve the new ODE
$$F(x, z, \dots, z^{(p-1)}) = 0$$
   and once its solution $z(x)$ is known, we may write the solution of the original equation (46) in quadrature
$$y(x) = \int z(x)\mathrm{d}x.$$
   Hence, the substance of the method is the following: starting from an arbitrary nonvanishing infinitesimal symmetry $X = \xi\partial_x + \eta\partial_y$ we look for a point transformation, i.e. a change of coordinates on $M \times N$, such that in the new coordinates $\tilde{x}, \tilde{y}$ our vector field X takes the form

$X = \partial_{\tilde{y}}$. According to the rules for transformation of the components of a vector field these new coordinates must satisfy equations

$$X(\tilde{x}) = 0, \qquad X(\tilde{y}) = 1.$$

These equations are solved using the method of characteristics. Their solution is in general not unique, but any particular solution with non-constant $\tilde{x}$ can be used.

Once $\tilde{x}, \tilde{y}$ are found, we lower the order of our equation in the new coordinates, solve it (if possible), and at the end transform the solution to the original coordinates.

This approach generalizes many particular methods used in solution of ODEs.

Example 4 *Let*

$$F(y, \ldots, y^{(p)}) = 0$$

be an autonomous ODE, i.e. not depending explicitly on x*. It is invariant under translations in the independent variable* x*, generated by* $X = \partial_x$*. Therefore, if we interchange the roles of independent and dependent variable* $\hat{x} = y$*,* $\hat{y} = x$*, the vector field becomes* $X = \partial_{\tilde{y}}$ *and we may lower the order of the differential equation for the inverse function* $x(y)$ *by one.*

Example 5 *Let*

$$F(x, y, \ldots, y^{(p)}) = 0 \tag{47}$$

be invariant under the scaling $x \to \lambda x$*,* $y \to \lambda^\alpha y$*. Such scaling is obtained as the 1–parameter group of transformations generated by the vector field*

$$X = x\partial_x + \alpha y \partial_y.$$

The new coordinates $\tilde{x}, \tilde{y}$ *can be chosen as*

$$\tilde{x} = \frac{y}{x^\alpha}, \qquad \tilde{y} = \ln x.$$

Once we rewrite the original ODE (47) *in these coordinates we may again lower its order by one.*

We remark that the reduced equation may have a group of symmetries rather distinct from the original one. In particular, other symmetries of the original equation may not survive the reduction. Only the symmetries generated by such vector fields $Y \in \mathfrak{X}(M \times N)$ that a constant $\alpha \in \mathbb{F}$ exists satisfying

$$[Y, X] = \alpha X$$

are guaranteed to survive the reduction.

By induction, a k–dimensional algebra of infinitesimal symmetries of a given ODE with a complete flag of ideals as in Lie's theorem[4] allows us to reduce the order by k provided we can find suitable coordinates in each step, of course. That was the original motivation for the definition of a solvable algebra – although, as we have seen in Lie's theorem, it is in the current terminology well justified only if we consider complex Lie algebras and complex (holomorphic) ODEs.

Example 6 *Let us use the symmetries computed in Example 2 to solve the ordinary differential equation* (37). *Since we have* $[X_1, X_2] = -X_1$ *we shall use the vector field* X_1 *first. The suitable new coordinates in which we have* $X_1 = \partial_{\tilde{u}}$ *are obviously*

$$\tilde{x} = u, \qquad \tilde{u} = x,$$

i.e. we use a so–called hodograph transformation. *The equation* (37) *when expressed in these new coordinates becomes*

$$\tilde{y}''(\tilde{x}) = -\frac{\tilde{y}'(\tilde{x})}{\tilde{x}} + \tilde{x}^2(\tilde{y}'(\tilde{x}))^3$$

and we can lower its degree using the substitution

$$\tilde{z}(\tilde{x}) = \tilde{y}'(\tilde{x}).$$

---

[4]

Theorem 4 (Theorem of Lie) *Any representation* ρ *of a solvable Lie algebra* $\mathfrak{g}$ *on a complex finite–dimensional vector space* V *contains a common eigenvector* $v \in V$, $v \neq 0$, *i.e.*

$$\rho(x)v = \lambda(x) \cdot v, \qquad x \in \mathfrak{g} \tag{48}$$

for some linear functional λ *on* $\mathfrak{g}$.

For any complex solvable Lie algebra $\mathfrak{g}$ *there exists a filtration by codimension 1 ad–invariant subspaces, i.e.*

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq \ldots \subsetneq V_{\dim \mathfrak{g}} = \mathfrak{g}, \quad \dim V_k / \dim V_{k-1} = 1, \quad [\mathfrak{g}, V_k] \subseteq V_k. \tag{49}$$

Lie's theorem implies that any complex solvable Lie algebra $\mathfrak{g}$ has only one–dimensional irreducible representations and that the adjoint representation of any complex non–Abelian solvable Lie algebra $\mathfrak{g}$ is not fully reducible.

We obtain an equation

$$\tilde{z}'(\tilde{x}) = -\frac{\tilde{z}(\tilde{x})}{\tilde{x}} + \tilde{x}^2(\tilde{z}(\tilde{x}))^3 \tag{50}$$

The vector field X_2 in the new coordinates becomes

$$X_2 = 2\tilde{x}\frac{\partial}{\partial \tilde{x}} - \tilde{u}\frac{\partial}{\partial \tilde{u}}.$$

Its first prolongation is

$$\mathrm{pr}^{(1)}X_2 = 2\tilde{x}\frac{\partial}{\partial \tilde{x}} - \tilde{u}\frac{\partial}{\partial \tilde{u}} - 3\tilde{z}\frac{\partial}{\partial \tilde{z}}.$$

We see that dropping the $\frac{\partial}{\partial \tilde{u}}$ term we obtain a well defined vector field

$$\tilde{X}_2 = 2\tilde{x}\frac{\partial}{\partial \tilde{x}} - 3\tilde{z}\frac{\partial}{\partial \tilde{z}}$$

on a two–dimensional space with coordinates $\tilde{x}, \tilde{z}$. The vector field $\tilde{X}_2$ is by construction an infinitesimal symmetry of the equation (50)*. Now, we use it to further lower the order of the equation* (50)*, i.e. to convert it into an algebraic equation. We find suitable new coordinates*

$$\hat{x} = \tilde{x}^3\tilde{z}^2, \qquad \hat{z} = -\frac{1}{2}\ln \tilde{x}$$

in which our equation (50) *becomes*

$$\hat{z}'(\hat{x}) = -\frac{1}{2(\hat{x} + 2\hat{x}^2)}. \tag{51}$$

Integrating the equation (51) *we find*

$$\hat{z}(\hat{x}) = \frac{1}{2}\ln\left|\frac{1+2\hat{x}}{\hat{x}}\right| - \frac{1}{2}\ln C$$

where C is a constant of integration. Going back to the coordinates $\tilde{x}, \tilde{z}$ we get an expression for the function $\tilde{z}(\tilde{x})$,

$$\tilde{z}(\tilde{x}) = \frac{1}{\tilde{x}\sqrt{C - 2\tilde{x}}}.$$

Now we can further integrate

$$\tilde{y}(\tilde{x}) = \int \tilde{z}(\tilde{x})\mathrm{d}\tilde{x} = -\frac{2}{\sqrt{C}}\mathrm{arctanh}\sqrt{\frac{C - 2\tilde{x}}{C}} + D$$

where D is a second constant of integration. Finally, we transform the function $\tilde{y}(\tilde{x})$ to the original coordinates and find a general solution of the ODE (37) *in the form*

$$y(x) = \frac{C}{2}\left(1 - \tanh^2\left(\frac{\sqrt{C}}{2}(D-x)\right)\right). \tag{52}$$

This reduction method can be immediately generalized to systems of ODEs but not to PDEs. For PDEs, another method is available.

4. Construction of group–invariant solutions of PDEs. As already mentioned, the method described above does not work for PDEs since the fact that a PDE does not involve the dependent variable explicitly does not in general provide any help in its solution. Nevertheless, we may employ the symmetries in construction of particular solutions of a given PDE.

   The essential observation is as simple as above. Let us suppose that a given PDE
   $$F(x^i, u^\alpha, \dots, u^\alpha_J) = 0$$
   has a symmetry generator
   $$X = \partial_{x^1}. \tag{53}$$
   That means that F is invariant with respect to translations in x^1, i.e. does not depend on it explicitly. Consequently, we may suppose that our solution u^α depends only on the remaining independent variables x^i, $i = 2, \dots, m$ and in this way we obtain a well–defined PDE with one less independent variables. Any solution of this PDE is also a solution of the original equation which in addition is invariant with respect to the 1–parameter group of symmetries generated by the vector field X; hence its name *group–invariant solution.*

   Similarly as before, the method boils down to the construction of suitable coordinates $\tilde{x}^i, \tilde{u}^\alpha$ on $M \times N$ in which a given symmetry generator X takes the form (53). Again, the method of characteristics is used. In fact, it turns out that we need to compute only the invariant coordinates
   $$\tilde{x}^i : X(\tilde{x}^i) = 0,\ i = 2, \dots, m, \qquad \tilde{u}^\alpha : X(\tilde{u}^\alpha) = 0,\ \alpha = 1, \dots, n$$
   in the process, as the following example will demonstrate.

Example 7 *Let us consider the heat equation of Example 3 and the vector field*

$$X_4 = -2t\partial_x + xu\partial_u.$$

This vector field has the following invariants

$$\tau = t, \qquad I = ue^{\frac{x^2}{4t}}.$$

Therefore, we substitute $u(x,t) = I(t)e^{-\frac{x^2}{4t}}$ *into the heat equation* (45) *and obtain a reduced equation for* $I(t)$

$$2tI'(t) + I(t) = 0.$$

Its general solution is $I(t) = \frac{C}{\sqrt{t}}$. *Altogether, we have recovered the fundamental solution (when* $C = \frac{1}{\sqrt{4\pi}}$*) of the heat equation*

$$u(x,t) = \frac{C}{\sqrt{t}}e^{-\frac{x^2}{4t}}$$

as the solution invariant with respect to Galilei transformations generated by the vector field X_4.

As before, the reduced equation may have symmetries which are of no direct relation to the original ones. If we want to be able to further reduce the number of independent variables we again need a solvable symmetry algebra and an appropriate choice of generators of 1–parameter subgroups (i.e. a basis respecting the flag of codimension 1 ideals, starting from the smallest one).

We notice that solutions invariant with respect to vector fields X and $\tilde{X} = Ad_g X$ are related: we may obtain a solution $\tilde{u}(x)$ invariant with respect to $\tilde{X}$ from $u(x)$ simply by setting $\tilde{u}(x) = g \triangleright u(x)$. Therefore, one shall first classify 1–dimensional subalgebras of the symmetry algebra under conjugation by $g \in G$ (or higher–dimensional subalgebras if reduction with respect to more independent variables is intended) and only then perform the reduction with respect to nonequivalent generators.

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