Introduction to Group Theory and Some Applications in Particle Physics

(Lecture Notes)

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INTRODUCTION TO GROUP THEORY AND SOME APPLICATIONS IN PARTICLE PHYSICS

(Lecture Notes)

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General Outline of the Course

I. Groups, Topological Groups, Lie Groups and Lie Algebras

II. Linear Representations of Lie Groups

III. The Poincare' Group and its Representations

IV. Application of the Representation Theory of the Poincare' Group and its Little Groups to Particle Scattering.

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Other references will be given in the process of the course.

Lecture 1

I. Introduction

The symmetry properties of the World in which we live in and their reflection in scientific theories have played an outstanding role in the development of natural sciences. The history of applications of symmetries, invariance principles, conservation laws etc. in physics can be traced back to Aristotle and Ptolemy, later to Galilei, Tycho de Brahe, Kepler and Newton, not to mention the crucial role these concepts played in the development of analytic mechanics by Lie, Cartan, Hamilton and others. A qualitatively new stage in the use of essentially group theoretical concepts started with the development of the special theory of relativity, with its emphasis on a precise and profound understanding of geometrical space-time concepts. Indeed, an explicit consideration of the Galilei group, consisting of space rotations and translations, time translations and special Galilei transformations connecting inertial frames of reference, moving with respect to each other with rectilinear uniform velocity, entered into the game of classical mechanics at a relatively late stage. The Lorentz transformations and the corresponding Lorentz group, on the other hand, played a crucial role right from the very first steps in the development of a relativistic theory.

Group theory has turned into a real working tool for physicists specially in the field of quantum physics. Indeed, the linear character of the Hilbert space of wave functions (or state vectors) makes this space particularly suitable for realizing representations of symmetry groups.

Let us just briefly mention some of the different aspects of the applications of symmetry principles and of group theory in physics. Conceptually the simplest application is typical for classical physics - the mechanics of continuous media, heat flow, etc. and also for atomic physics.

Namely, when the basical physical laws and equations are already known, but their solution presents a difficult and complicated problem, many simple, but crucial specific results can be obtained directly from the symmetries of the equations, without obtaining explicit solutions. The fact that essentially all basic results of atomic spectroscopy follow from the symmetry of the problem, mainly from the properties of the three-dimensional rotation group O(3) has been sufficiently stressed in E. Wigner's famous book "Group Theory and its Application to the Quantum Mechanics of Atomic Spectra". Group theory also helps to obtain exact solutions of known equations, relate general solutions to particular ones, etc.

A second aspect of group theory in physics, which receives great attention, whenever a new field is developed, flourishes at present in nuclear physics, elementary particle physics, quantum field theory and other fields. Namely, once certain fundamental symmetry principles are established from our experimental knowledge or are shown to follow, say, from some basical properties of space-time, then these principles can be imposed as "superlaws of nature". We can then demand that all (unknown) dynamical laws are compatible with these symmetry or invariance principles, which thus serve as part of the criteria for the acceptability of a suggested theory.

A third aspect has played and does play a significant role in high energy physics, namely group theoretical methods furnish convenient tools in terms of which it is possible to make dynamical assumptions (or educated guesses) and formulate hypotheses, the consequences of which can then be tested against experimental data.

In these lectures, after some general mathematical introduction, we shall mainly concentrate on well established symmetry groups, which we can intuitively call "geometrical symmetries", in that they mainly reflect the symmetries of the space-time continuum in which the studied processes occur and thus represent "kinematics" rather than specific "dynamics". We shall, however, also be interested in "dynamic" symmetries, specific for particular interactions, rather than broad classes of interactions. Thus, in nonrelativistic quantum mechanics the group O(3) represents a geometric symmetry group for an arbitrary spherically symmetric potential, V(r), the four-dimensional rotation group O(4), on the other hand, represents a dynamic symmetry, typical for the nonrelativistic Coulomb potential 1/r. Both types of symmetries play very significant roles in particle physics.

II. Elements of Abstract Group Theory

1. Definition of a Group

An abstract set of elements G is called a group if

(1) For each pair of elements $g_1 \varepsilon G$ and $g_2 \varepsilon G$ there exists a third element $g_1 g_2 \varepsilon G$, called the product of g_1 and g_2. This product is associative, but in general not commutative, i.e.:

$$g_1 g_2 g_3 = (g_1 g_2) g_3 = g_1 (g_2 g_3)$$

but, in general,

$$g_1 g_2 \neq g_2 g_1$$

2. An identity element e exists, such that

$$eg = ge = g$$

3. For each element $g \varepsilon G$ there exists an inverse element of g^{-1} such that

$$gg^{-1} = g^{-1} g = e$$

Problem: Discuss the possibility of having left and right identity operators

$$e_1 g = g \qquad g e_2 = g \qquad e_1 \neq e_2$$

and left and right inverse operators

$$g_1^{-1} g = e \qquad g g_2^{-1} = e$$

Prove that a "left identity" is simultaneously a "right identity" and that the same is true for an inverse element.

Examples: 1. Group of permutations of two elements:

$$e = \begin{pmatrix} 1 & 2 \\ \downarrow & \downarrow \\ 1 & 2 \end{pmatrix} \qquad a = \begin{pmatrix} 1 & 2 \\ \downarrow & \downarrow \\ 2 & 1 \end{pmatrix}$$

$$ea = \begin{pmatrix} 1 & 2 \\ \downarrow & \downarrow \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ \downarrow & \downarrow \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ \downarrow & \downarrow \\ 2 & 1 \end{pmatrix} = a \qquad aa = \begin{pmatrix} 1 & 2 \\ \downarrow & \downarrow \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ \downarrow & \downarrow \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ \downarrow & \downarrow \\ 1 & 2 \end{pmatrix} = e$$

$$ae = \begin{pmatrix} 1 & 2 \\ \downarrow & \downarrow \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ \downarrow & \downarrow \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ \downarrow & \downarrow \\ 2 & 1 \end{pmatrix} = a \qquad \Rightarrow \quad a = a^{-1}$$

2. Group SO(2): Rotations on a circle: Each rotation is given by one parameter - an angle α, $0 \leq \alpha < 2\pi$

$$g_\alpha g_\beta = g_{\alpha+\beta}$$
$$g_0 = e$$
$$g_{-\alpha} = (g_\alpha)^{-1}$$

3. Group SO(3): Relations on a two-dimensional sphere in a three-dimensional Euclidean space.

4. Group T_1: Translations on a straight line.

Let us discuss example 1 and finite (or discrete) groups in general.

Discrete groups. The order of the group = number of elements.

Order of an element a u is k if $a^k = e$ a no $\xi' < k$ exists, such that $a^{\xi'} = e$.

Isomorphism of discrete groups: The finite groups and ' are isomorphic if their elements can be put into one-to-one correspondence, such that if $g_1 \sim g_1'$, $g_2 \sim g_2'$, then $g_1 g_2 \sim g_1' g_2'$ (identity elements correspond to identities, inverse elements to inverse elements).

Isomorphic groups have the same structure - they are the same abstract group.

Example: All groups of order two are isomorphic:

Proof: Consider two elements a and e: $a \cdot e = a$ (By definition, since e is the identi

Try: $a \cdot a = a^2 = a \Longrightarrow a = e$ (this is impossible, since we would have one element only.)

Thus: $a \cdot a = e$

Isomorphic to the above group of permutations of two elements are e.g. the following groups α) a = reflection of coordinates $(x,y,z) \to (-x,-y,-z)$

β) a = rotation through π about fixed axis

γ) $e = 1$, $a = -1$ the group operation is ordinary multiplicat

Group of order 3: a, b, e

$$\left.\begin{array}{l} ab = a \Longrightarrow b = e \\ ab = b \Longrightarrow a = e \end{array}\right\} \begin{array}{l} \text{impossible since we would have only two elements} \\ \text{thus } ab = e \end{array}$$

$$\left.\begin{array}{l} a^2 = a \Longrightarrow a = e \quad \text{impossible} \\ a^2 = e \Longrightarrow b = a^{-1} = a \quad \text{impossible} \end{array}\right\} \Longrightarrow \begin{array}{l} a^2 = b \\ b^2 = a \end{array}$$

A convenient way of representing a finite group with its multiplication law is in the form of a group table:

Order two:

	e	a
e	e	a
a	a	e

Order three:

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

Problem: Consider all groups of order 4.

The other groups in our examples are continuous: SO(2) and T_1 are commutative or abelian, SO(3) is noncommutative

Abelian group: $g_1 \in G$, $g_2 \in G \Rightarrow g_1 g_2 = g_2 g_1$

In these lectures we shall concentrate mainly on continuous groups.

Remark: a group is a homogeneous manifold, i.e. we can introduce left and right translations

$$g \to g_0 g \qquad\qquad g \to g g_0$$

and for any two elements $g_1 \varepsilon G$, $g_2 \varepsilon G$ we can find a $g_0 \varepsilon G$, such that $g_2 = g_0 g_1$ and $g_0' \varepsilon G$ such that $g_2 = g_1 g_0$ and $g_2 = g_0' g_1$.

For continuous groups this provides a method by which local properties (volumes, derivatives etc.) can be established in the vicinity of a chosen point g_1 for instance $g = e$ and then transferred to arbitrary points (group elements).

Definition: Topological Space : a set of elements R is a topological space, if to each subset $M \subset R$ there corresponds a set $\bar{M}$, called the closure of M, such that

a) If M contains only one element X M, then $\bar{M} = M$ (i.e. $\bar{x} = x$)

b) If M and N are any two subsets of R, then $\overline{M \cup N} = \bar{M} \cup \bar{N}$ (the closure of the union is the union of the closure).

c) $\bar{\bar{M}} = \bar{M}$ (applying the operation of closure twice gives the same result as applying it once).

Basically: A topological space is a space in which we have introduced concepts like neighbourhood , closeness, etc.

Remark: The union $M \cup N$ of two spaces M and N is the set of all points lying in M or in N (or both)

Definition: A group G in a Topological Group if

1) G is a group

2) G is a topological space

3) The group operations in G are continuous in the topological space G .

Roughly speaking a group G is a topological group, if g_1g_2 depends continuously on g_1 and g_2 and g^{-1} depends continuously on g.

More precisely:

1) $g_1 \varepsilon G$, $g_2 \varepsilon G \Longrightarrow$ for every neighbourhood W of g_1g_2 there exist neighbourhoods U and V of g_1 and g_2 such that $UV \subset W$

2) $g \varepsilon G \Rightarrow$ for every neighbourhood V of a^{-1} there exists a neighbourhood U of a such that $U^{-1} \subset V$.

To introduce the concept of neighbourhood, let us made a few remarks.

Let M be an arbitrary subset of R.

A point of M is any element $a \varepsilon M$.

A limit point of M is a point a contained in the closure of the difference between M and a: $a \varepsilon \overline{M \setminus a}$ (the symbol $M \setminus N$ denotes the difference between the sets M and N, i.e. all points contained in M but not in N).

Definition: A set F in the topological space R is closed if $\bar{F} = F$.

A set G in the topological space R is open if $R \setminus G$ is closed: $\overline{R \setminus G} = R \setminus G$

Thus open and closed sets are in some sense dual to each other. To each assertion about an open set there corresponds an assertion about a closed one.

Definition: A collection Σ of open sets of a topological space R is a base for R, if every open set X of R is the union of some open sets belonging to Σ.

$$X = \Sigma_a \cup \Sigma_b \cup \Sigma_c \ldots$$

A base = a complete system of neighborhoods for R. Each open set Σ_ε is a neighborhood of every point of that open set.

There are many such bases - one of them with minimal cardinality this minimal cardinal number is the weight of R. Countable weight $\Rightarrow$ separable space.

Remark: Cardinality: two sets have the same cardinality if there exists a one-to-one mapping from one onto the other:

1) Finite cardinality
2) Countable - $alef_o$ - mapping onto integers
3) $Alef_1$ - mapping onto real numbers
4) $Alef_2$ - mapping onto of all real valued functions of real numbers

etc.

Compactness: Generalization of the properties of a closed interval of the real axi to general topological spaces.

Covering: A collection of sets Σ in a space R is a covering of a set $M \subset R$ if the union of all sets in Σ contains M:

$$M \subset \Sigma_a \cup \Sigma_b \cup \ldots \cup \Sigma_n$$

Definition: A topological space R is compact if from every covering of R by open sets it is possible to select a finite covering.

A topological space R is locally compact if every one of the points possesses a neighborhood, whose closure is compact.

Lecture 2

After this superficial excursion into topological spaces, let us return to some further relevent concepts of group theory

Consider a group G:

Notation: subsets A, B

Then AB set of elements ab with $a \in A$, $b \in B$

A^{-1} set of elements a^{-1} where $a \in A$

A^m defined by induction

$A^{-m} = (A^{-1})^m$

$A^0 = \{e\}$

We have: $AG = GA = G$

$G^{-1} = G$

$Ae = eA = A$

Definition: Subgroup of G: a subset H of G, which in itself is a group with respect to the same operation of composition, as defined in G.

Problem: Prove: the necessary and sufficient conditions for H to be a subgroup of G are that either one of the following conditions is satisfied:

(1) $a \in H,\ b \in H \Rightarrow ab^{-1} \in H$

i.e. $HH^{-1} \subset H$

(2) $a \in H,\ b \in H \Rightarrow ab \in H,\ b^{-1} \in H$

i.e. $H^2 \subset H,\ H^{-1} \subset H$

Equivalence relation: $a \sim b$

An equivalence relation must satisfy:

(a) Reflexivity $a \sim a$

(b) Symmetry $a \sim b \Rightarrow b \sim a$

(c) Transitivity $a \sim b,\ b \sim c \Rightarrow a \sim c$

<u>Equivalence classes of elements:</u>

Let H be a subgroup of G :

$a\varepsilon G$, $b\varepsilon G$: $a \sim b$ if $ab^{-1} = h\varepsilon H$, i.e. $a = hb$

Each equivalence class A is called a "right coset" of the subgroup H: $a\varepsilon A \rightarrow A = Ha$

Similarly: $a^{-1}b = h\varepsilon H$, i.e. $b = ah$ defines a "left coset" $A = aH$

<u>Definition:</u> A subgroup $N\subset G$ is an invariant (or normal) subgroup if $a^{-1}na\varepsilon N$, for every $n\varepsilon N$ and $a\varepsilon G$, i.e. $a^{-1}Na\subset N$ for every $a\varepsilon G$.

Obviously: If H is an invariant subgroup, then $aH = Ha$

i.e. the left and right cosets coincide.

<u>Example:</u> E_2 - the group of motions of an Euclidean plane

$$\begin{pmatrix} \cos\phi & \sin\phi & a \\ -\sin\phi & \cos\phi & b \\ 0 & 0 & 1 \end{pmatrix} \quad \text{acting on} \quad \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

<u>Problem:</u> show that the translations

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

constitute an invariant subgroup.

<u>Definition:</u> Let N be a normal subgroup of G and let A and B be cosets of N: $A = Na$, $B = Nb$. Introduce the produce: $AB = NaNb = NNab = Nab$ (since $Na = aN$), so that AB is again a coset of N. The group of cosets is called the <u>factor group</u> of the Group G by the normal subgroup N and is denoted G/N.

Proof that G/N is a group:

1) Associativity

$(AB)C = A(BC)$

$(NaNb)Nc = Na(NbNc)$

2) The identity is $E = N$

Indeed $NA = N(Na) = Na = A$

3) $A = Na \to A^{-1} = a^{-1}N$

Indeed $AA^{-1} = Na\ a^{-1}N = NN = N = E$

Every group has at least two invariant subgroups:

$$G \quad \text{and } \{e\}$$

(the entire group and the identity element alone)

Definition: a simple group has no other invariant subgroup

Isomorphism: Two groups G and G' are isomorphic if a one-to-one mapping $f(x)$ from one onto the other exists, preserving the group operation:

$$f(xy) = f(x)f(x) \qquad x, y \varepsilon G \qquad f(x), f(y) \varepsilon G'$$

Homomorphism: A mapping of G into G' is a homomorphism if it preserves the group operation: $g(x)g(y) = g(xy)$. The set $g^{-1}(e')$ of all elements $g \varepsilon G$ mapped into the identity $e' \varepsilon G'$ is the kernel of the homorphism.

Remark: For topological groups some of the above concepts should be further specified, but we shall not go into that.

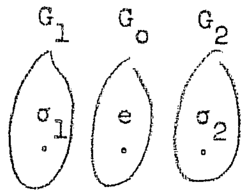
Automorphism : isomorphism of group G onto itself. Inner automorphism of group G: Take a fixed element $g \varepsilon G$ and put

$$f_g(x) = gxg^{-1}$$

for every $x \varepsilon G$.

Further important concepts for topological groups, to which we shall address ourselves are discreteness and connectivity. A topolog. space is connected if any two of its points can be connected by a continuous curve, belonging to the space.

Consider a topological group G: if it is not connected it decomposes into individual connected parts (sheets) $G_o, G_1, \ldots G_n$ (n can be infinite) where G_o is by definition that sheet of G_o, connected to the identity (thus, G_o is a subgroup of G)



If σ_1 is not connected to e, i.e. no $g\varepsilon$ exists such that $\sigma_1 g = e$, then all elements $\sigma_1 g_o$ with $g_o \varepsilon {}_o$ constitute the sheet g_1, "similar" to g_o. G_1 is not a group, since it does not contain the identity.

Remark: the number of sheets can be not only infiite, but continuously infinite.

An element σ_1, characterizing a sheet, can be called a "reflection". The product $\sigma_1\sigma_2$ is again a reflection - in general on a new sheet.

Discrete groups can be considered to be topological groups, in which each sheet consists of one point.

Examples:

(1) Group O(3): all linear transformations leaving the quadratic form

$$x^2 = x_1^2 + x_2^2 + x_3^2$$

invariant

$$g\varepsilon O(3) \quad \rightarrow \quad g^T g = 3$$ T denotes the transposed matrix

$(\det g)^2 = 1 \quad \rightarrow \quad \det g = \pm 1$

Thus: O(3) consists of two sheets:

$$O(3) = O^+ + O^-$$

O^+...has $\det g = 1$, O^- has $\det g = -1$

(2) Consider the group O(2,1), preserving the quadratic form

$$x^2 = x_o^2 - x_1^2 - x_2^2$$

invariant

$$x'_i = g_{ik} X_k \qquad g^T I g = I \qquad I = \begin{pmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

We have: $\det g^T I g = + (\det g)^2 = +1 \rightarrow \det g = \pm 1$

$$(g^T I g)_{00} = g^T_{0k} I_{kk} G_{k0} = 1$$

$$g_{00}g_{00} - g_{10}g_{10} - g_{20}g_{20} = 1 \rightarrow g_{00} \geq 1 \text{ or } g_{00} \leq -1 \,.$$

Thus

$$
\begin{array}{lll}
SO(2,1)_0 = L_+^{\uparrow} & \text{with } \det\Lambda = 1, & \Lambda_{00} \geq 1 \\
L_-^{\uparrow} & \text{with } \det\Lambda = -1, & \Lambda_{00} \geq 1 \\
L_+^{\downarrow} & \text{with } \det\Lambda = 1, & \Lambda_{00} \leq -1 \\
L_-^{\downarrow} & \text{with } \det\Lambda = -1\,, & \Lambda_{00} \leq -1 \,.
\end{array}
$$

Thus the three-dimensional Lorentz groups $O(2,1)$ consists of four connected parts, of which only $L_+^{\uparrow}$ - the proper ortrochronous Lorentz group - is a subgroup. Exactly the same is true for the four-dimensional Lorentz group $O(3,1)$.

We shall be mainly, but not exclusively, interested in connected groups (groups consisting of one connected sheet).

Parametrical groups:

A group G is parametrical or locally Euclidean if its elements can be parametrized by a finite number of real parameters:

$$g = g\,(t_1,\ldots,t_n)$$

The parametrization can be local: i.e. different coordinates can be used in different regions of the group.

The multiplication law is:

$$g_3 = g_1 g_2$$

i.e.

$$t_i(g_3) = F_i(t_j(g_1),\ t_k(g_2)) \qquad i,j,k = 1,\ldots,n.$$

and we have

$$t_i(g^{-1}) = \phi_i(t_j(g))$$

If is a topological group then $F = \{ F_i \}$

and $\{\emptyset_i\}$ are continuous functions.

Definition: A topological parametric group is a Lie group if the functions F_i and $\emptyset_i$, determining the multiplication are analytic functions.

Theorem (Hilberts problem V): Every parametric group is a Lie group

We shall not give the proof. (Continuity implies analyticity in this case). However, a simple analogy is that equation

$$f(x+y) = f(x)\, f(y)$$

with $f(x)$ analytic has $f(x) = e^{\alpha x}$ as only continuous solutions; these are analytic functions.

Lie Groups and Lie Algebras

Having the concept of analyticity , we can introduce differentiation, etc. and consider the tangential space in the unit point e (or in any arbitrary point, since the group is a homogeneous manifold).

Let us define a one-parameter subgroup of a Lie group (corresponding to a "direction" in an Euclidean space):

Assume: There exists a neighborhood Ω of e in which the equation

$$x^2 = g$$

has a solution $g^{1/2} \in \Omega$ for $g \in \Omega$.

Using this extraction of square roots and the group multiplication we can construct elements

$$g^r \quad \text{with } r = k(\tfrac{1}{2})^n \; \ldots k \text{ and } n \text{ integer}$$

satisfying

$$g^{r_1+r_2} = g^{r_1} g^{r_2}$$

Using a limiting procedure, we construct

$$g^{\lambda} \qquad 0 \leq \lambda < \infty \;,\; g^{0} = e$$

for arbitrary real non-negative λ.

If Ω contain g, it also contains g^{-1} = >

$$\rightarrow \quad g^{\lambda+\mu} = g^{\lambda} g^{\mu} \qquad -\infty < \lambda < \infty$$

We shall call such a family g^{λ} a <u>one-parameter subgroup</u> of G.

Thus: for each point $g \varepsilon G$ we define a one-parameter subgroup of g^{λ}, containing e $(g^{0} = e)$

Loosely speaking: we can uniquely draw a line g^{λ} in any direction δ_{g}.

<u>Example:</u> $GL(n,R)$: the group of all real nondegenerate nxn matrices:

Identity: $e = I = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$

Define the "norm" of matrix g as:

$$||g|| = \sum_{i,j} g_{ij}^{2}$$

and the neighborhood Ω of e by the condition

$$||g-I|| < 1$$

One can show that any $g \varepsilon \Omega$ can be written as

$$g = e^{a} = 1 + a + \frac{a^{2}}{2} + \ldots \frac{a^{n}}{n!} + \ldots$$

where the matrix a is $a = \ell n g$ and can be determined from the expansion of

$$\ell n(1+x), \; X = g - 1$$

$(||x|| < 1 \quad \rightarrow$ convergence).

In this form it is easy to exponentiate g:

$$g^{1/2} = e^{1/2 a} \qquad g^{\lambda} = e^{\lambda a}$$

Expanding

$$e^{\lambda a} = 1 + \lambda a + \dots$$

we see that a represents a "tangential vector" to the 'line" g^{λ}. Choosing all different one-parameter subgroup g^{λ} we obtain all tangential vectors and thus a tangential space A.

Let us see how group multiplication reflects itself in the tangential space

$$(I + \lambda a + \dots)(I + \lambda b + \dots) = I + \lambda(a+b) + \dots$$

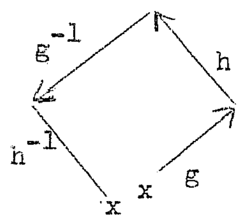
$$(1 + \lambda c + \dots)^m = I + \lambda m a + \dots$$

(we are in the neighborhood of $e \Rightarrow |\lambda| << 1$)

Introduce the "commutator" of two elements $h \varepsilon G$, $g \varepsilon G$ in the group:

$$k = g^{-1} h^{-1} g h$$

The geometrical meaning of k is that it is an operation necessary to "close" the quadrangle formed by the transformations



We have: $g = e^{\lambda a}$ $h^{\lambda b}$

$$k = e^{-\lambda a} e^{-\lambda b} e^{\lambda a} e^{\lambda b} = (I - \lambda a + \frac{\lambda^2 a^2}{2})(I - \lambda b + \frac{\lambda^2 b^2}{2})(I + \lambda a + \frac{\lambda^2 a^2}{2})$$

$$(+I + \lambda b + \frac{\lambda^2 b^2}{2}) = I + \lambda^2 (ab - ba)$$

Taking $2\lambda^2$ as a new parameter, we find that

$$[a,b] = ab - ba$$

is the vector tangential to the commutator k.

This matrix commutator obviously staisifes

$$[a,b] = -[b,a] \qquad \text{antisymmetry}$$

$$[a[bc]] + [b[ca]] + [c[ab]] = 0 \ldots \text{ the Jacobi identity}$$

We shall call the linear space A with the commutation operation $[a,b]$ the Lie algebra of the group G.

This concept can be generalized to an arbitrary Lie group. Actually, we shall only be interested in "linear Lie groups" - those that can be represented by linear transformations in a linear vector space.

Thus: a Lie algebra is the differential of a Lie group. Given a Lie group we can always construct a Lie algebra with commutators satisfying the antisymmetry condition and the Jacobi identity.

<u>Theorem:</u> (Lie): A Lie algebra can always be integrated to give the multiplication law for a Lie group in the neighborhood of the identity. The multiplication law for the whole group is a more complicated problem, to be treated below.

Let us consider the Lie algebra A and introduce a basis $e_1, e_2, \ldots e_n \in A$ (it is a finite algebra by definition). It is sufficient to know the commutators for the basis elements.

$$[e_i e_j] = C^k_{ij} \, e_k \qquad \text{(summed over k)}$$

The new constants C^k_{ij}, which are basis dependent, are called <u>structure constants.</u>

Thus - a complicated object - a Lie group, is to a large degree determined by the structure constants C^k_{ij}. It is easy to prove that the structure constants of a Lie algebra satisfy

$$C^k_{ij} = -C^k_{ji}$$

$$C^m_{in}C^n_{jk} + C^m_{kn}C^n_{ij} + C^m_{jn}C^n_{ki} = 0$$

Problem: 1) Prove the above assertion

2) Consider all possible two and three-dimensional Lie algebras,

Lecture 3

We have given the definition of a Lie algebra, starting from a Lie group. Let us now look at some purely algebraic concepts and properties. Most of todays lecture is contained in the book "Lie Algebras" by N. Jacobson,(Interscience Publishers, N.Y. 1962). We shall talk of algebras over fields.

<u>Definition</u>: A field $\emptyset$ is a set of elements which is a commutative group with respect to a group operation which we shall call addition $a + b$ and in which we introduce a further operation, which we call multiplication $a.b$, satisfying:

(1) Associativity $abc = a(bc) = (ab).c$

(2) Distributivity $(a+b)c = ac + bc$

$$a(b+c) = ab+ac$$

(3) The elements of $\emptyset$, different from zero in the additive group $(a + 0 = a)$ form a group with respect to multiplication

(4) Multiplication is commutative.

We shall only need two fields: the field of real numbers and the field of complex numbers.

<u>Definition:</u> A <u>linear vector space</u> L over a field $\emptyset$ is a set of elements for which we introduce the concept of addition of vectors (elements) in L and multiplication of vectors in L by "numbers" from $\emptyset$. These operations must satisfy:

(1) $x,y\varepsilon R \Rightarrow x+y\varepsilon R$

(2) $x+y = y+x$

(3) $(x+y)+z = x+(y+z)$

(4) There exists $0\varepsilon L$: $x + 0 = x$ for all $x\varepsilon L$

(5) $x\varepsilon L$, $\alpha\varepsilon\emptyset \Rightarrow \alpha x\varepsilon L$

(6) $x\varepsilon L$, $\alpha\ \beta\varepsilon\emptyset \Rightarrow \alpha(\beta x) = (\alpha\beta)x$

(7) $1\cdot x = x$

(8) $0 \circ x = 0$

(9) $\alpha(x+y) = \alpha x+\beta y$

(10) $(\alpha+\beta)x = \alpha x+\beta x$

Definition: An algebra A (not necessarily associative) is a vector space over a field $\emptyset$ in which a bilinear composition (multiplication) a.b for $a,b\varepsilon A$ is defined, satisfying:

(1) $(a_1+a_2)b = a_1 b + a_2 b$

$a(b_1+b_2) = a\, b_1 + a\, b_2$

(2) $\alpha(ab) = (\alpha a)b = a(\alpha b)$ $\qquad \alpha\varepsilon\emptyset$

Definition: An associative algebra is an algebra in which the multiplication satisfies

$$(ab)c = a(bc)$$

Definition: A Lie algebra is an algebra in which the multiplication satisfies:

$$ab = -ba \qquad \ldots \qquad \text{Antisymmetry}$$

$$(ab)c + (ca)b + (bc)a = 0 \qquad \ldots \qquad \text{Jacobi identity}$$

Thus: the differential of a Lie group, as introduced previously, is indeed a Lie algebra where the multiplication a.b is actually the commutators [a,b].

Question: When is an associative algebra a Lie algebra?

Example:

The associative algebra of linear transformations of a finite dimensional linear vector space into itself.

We can always use an associative algebra to construct a Lie algebra: Let A be an associative algebra. If $x,y\varepsilon A$ then we define a Lie product (or a commutator) of x,y as

$$[x,y] = xy - yx$$

Obviously the product $[x,y]$ satisfies all the conditions for a product in a Lie algebra. This can be used to show that every Lie algebra is isomorphic to a Lie algebra of linear transformations so that each n-dimensional Lie algebra can be considered as a subalgebra of the general Lie algebra of linear transformations of an n-dimensional vector space.

The dimension of a Lie algebra (or of an associative algebra, or of any linear vector space) is the maximal number of linearly independent vectors in the algebra (space). As usual e_i are independent if

$$\Sigma\, c_i e_i = 0 \quad \text{implies } c_i = 0 \quad \text{for all } i$$

where c_i belong to the field $\emptyset$.

Any set of linearly independent elements $\{e_i\}$ form a basis for the linear space (algebra) and an arbitrary element can be written as

$$a = \sum_{i=1}^{n} \alpha_i e_i .$$

An important concept for Lie algebras is the derived algebra. Let L be a Lie algebra with the basis $\{e_i\}$, $i = 1\ldots n$

Consider the set L^2 of all elements

$$d_k = [e_i, e_k]$$

L^2 is again a Lie algebra, called the derived algebra.

Problem: Prove that the derived algebra is indeed a Lie algebra.

Definition: A subalgebra B of L is a set of elements $b_i \in L$ which themselves form an algebra.

Definition: $N \subset L$ is an ideal (invariant subalgebra) if

$$[na] \in N \text{ for all } n \in L,\ a \in L$$

$$(\text{i.e. } [N\ L] \subset N)$$

Theorem: The derived algebra of a Lie algebra is an ideal

Proof: Left as a problem.

Definition: The centre C of a Lie algebra L is the set of all elements $c \in L$ such that $[ca] = 0$ for all $a \in L$.

Lie Algebras of Low Dimensions over the Field of Real Numbers

(1) dim L = 1 One element e : $L = \emptyset e$

$[e,e] = 0$ ... denote $\underline{L_1}$

(2) dim L = 2 Consider the derived algebra $L^2 = L'$

(a) $\dim L' = 0$, i.e. $L' = 0$

$[e_1 e_2] = 0$ ... an Abelian (commutative algebra)

(b) $\dim L' \neq 0$

$L = \emptyset e_1 + \emptyset e_2$, i.e. $e \in L \Rightarrow e = \alpha_1 e_1 + \alpha_2 e_2$

$\alpha_1, \alpha_2 \in \emptyset$

$$L' = \emptyset\,[e_1 e_2] \Rightarrow \dim L' = 1$$

Choose: e such that $L' = \emptyset e$

Let $\tilde{f} \in L$ $\tilde{f} \neq \alpha e$

$[e\tilde{f}] = \kappa e$ $\kappa \in \emptyset$

Put $f = \kappa^{-1}\tilde{f}$, then

$$[ef] = e$$

Thus, there only only two two-dimensional Lie algebras

(a) Abelian $[e_1 e_2] = 0$ ... $L_1 + L_1$

(b) Non-abelian $[e_1 e_2] = e_1$ ... L_2

(3) dim L = 3

Introduce the basis e_1, e_2, e_3 and consider the derived algebra $L^2 = L'$

(a) $L' = 0$ $\quad [e_i, e_k] = 0$ .. an abelian algebra

Thus $\boxed{[ef] = 0 \quad [fg] = 0 \quad]g,e] = 0}$ $\quad L_1+L_1+L_1$

(b_α) $\dim L' = 1$, $L' = \emptyset e$, $L' \subseteq C$ (the derived algebra is contained in the centre)

Put: $\left.\begin{array}{l} L = \emptyset e + \emptyset f + \emptyset g \\ L' = \emptyset e \end{array}\right\} \Longrightarrow L' = \emptyset . [fg]$

$\Longrightarrow$ we can put $[fg] = e$

Further: $[ef] = [eg] = 0$

Thus:

$\boxed{[fg] = e \quad [ef] = 0 \quad [eg] = 0}$ ... $L_{3,1}$

(b_β) $\dim L' = 1$, $L' = \emptyset e$ $\quad L' \not\subseteq C$

$e \notin C \Longrightarrow$ there exists an f such that $[ef] \neq 0$ and we can put: $[ef] = e$ and the algebra $\{e,f\}$ is an ideal.

We have: $[ef] = e$

$[e\tilde{g}] = \alpha e$

$[f\tilde{g}] = \beta e$

Put $g = \tilde{g} - \alpha f + \beta e$: $\quad [ef] = \alpha e - \alpha e = 0$

$[fg] = \beta e - \beta e = 0$

Thus:

$\boxed{[ef] = e \quad [e,g] = 0 \quad [f,g] = 0}$ ... $L_2 + L_1$

(c_α) dim $L' = 2$, L' abelian

Put: $L = \emptyset\tilde{e}_1 + \emptyset\tilde{e}_2 + \emptyset f$

$L' = \emptyset\tilde{e}_1 + \emptyset\tilde{e}_2$

Thus: $[\tilde{e}_1\tilde{e}_2] = 0$

$[\tilde{e}_k f] = \alpha_{ki}\tilde{e}_i \qquad i,k = 1,2 \qquad \det\alpha \neq 0$

Introduce: $e_i = \rho_{is}\tilde{e}_s \qquad \det\rho \neq 0$

$\tilde{e}_s = (\rho^{-1})_{s\,i}e_i$

We have: $[e_1e_2] = 0$

$$[e_if] = \rho_{is}[\tilde{e}_sf] = \rho_{is}\alpha_{sr}\tilde{e}_r = \rho_{is}\alpha_{sr}\rho^{-1}{}_{rk}e_k = $$
$$= (\rho\alpha\rho^{-1})_{ik}e_k$$

A real 2 x 2 matrix α can always be brought to one of the following standard forms by a similarity transformation $\rho\alpha\rho^{-1}$ where ρ is a real matrix:

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix} \quad -1 \leq h < 1, \quad h \neq 0$$

or

$$\begin{pmatrix} p & -1 \\ 1 & p \end{pmatrix} \quad 0 \leq p < \infty .$$

The first corresponds to dim $L' = 1$, the rest give new algebras, namely:

$\boxed{[ef] = 0 \qquad [eg] = e \qquad [fg] = f}$ $\qquad L_{3,2}$

$\boxed{[ef] = 0 \qquad [eg] = e \qquad [fg] = e+f}$ $\qquad L_{3,3}$

$\boxed{[ef] = 0 \qquad [eg] = e \qquad [fg] = hf \quad -1 \leq h \leq 1 \quad h\neq 0}$ $\qquad L_{3,4}$

$\boxed{[ef] = 0 \qquad [eg] = pe - f \quad [fg] = e+pf \quad 0 \leq p < \infty}$ $\qquad L_{3,5}$

Above, different values of h and p correspond to different (non isomorphic) algebras.

The last algebra corresponds to the algebra of the Euclidean group E_2 if $p = 0$.

(C_β) dim $L' = 2$, L' non abelian:

$$[ef] = e$$

$$[eg] = \alpha e + \beta f$$

$$[fg] = \gamma e + \delta f$$

The Jacobi identity:

$$0 = [[ef]g] + [[ge]f] + [[fg]e] =$$

$$= \alpha e + \beta f - \alpha e - \delta e \Rightarrow \beta = \delta = 0$$

This is impossible, since then dim $L' = 1$

(d) <u>dim $L' = 3$</u> e_1, e_2, e_3

$$[e_i e_k] = \varepsilon_{ik\ell} f_\ell = \varepsilon_{ik\ell}\alpha_{\ell m} e_m$$

$\varepsilon_{ik\ell}$ = totally antisymmetric tensor with $\varepsilon_{123} = 1$

dim $L' = 3 \Rightarrow \det \alpha \neq 0$

Jacobi identity $\Rightarrow \alpha_{\ell m} = \alpha_{m\ell}$

Choose a new basis: $\tilde{e}_i = \rho_{ir} e_r$ $\det\rho \neq 0$ $i, r = 1, \ldots, 3$

This induces a similarity transformation $\rho\alpha\rho^{-1}$. However a symmetric matrix α can be diagonalized.

Thus: $[e_1 e_2] = e_3$ $[e_2 e_3] = \alpha e_1$ $[e_3 e_1] = \beta e_2$

This can easily be reduced to two possibilities:

$[e_1 e_2] = e_3$	$[e_2 e_3] = e_1$	$[e_3 e_1] = e_2$	$L_{3,6}$
$[e_1 e_2] = -e_3$	$[e_2 e_3] = e_1$	$[e_3 e_1] = e_2$	$L_{3,7}$

The first is the Lie algebra of O(3), the second of O(2,1).

Similarly one can classify algebras of somewhat higher dimensions (classification of Lie algebras of dim L = 4,5 and 6 exist in the literature).

Definition: The direct sum L of two Lie algebras M and N: $L = M \oplus N$ if $L = M \cup N$ and $[M,M] \subset M$, $[NN \subseteq N$ and $[M,N] = 0$ (in obvious notations).

Example from above: $L_2 + L_1$: $L_2 = \{ef\}$, $L_1 = g$

Definition: Representation of a Lie algebra L: A homomorphism of the algebra L into the Lie algebra of linear transformations of a vector space M over $\emptyset$. The conditions for a homomorphism are:

$$\text{if} \quad \ell_1 \to L_1, \quad \ell_2 \to L_2, \text{ then}$$

$$\ell_1 + \ell_2 \to L_1 + L_2, \; \alpha\ell_1 \to \alpha\ell_2$$

$$[\ell_1, \ell_2] \to [L_1, L_2] = L_1 L_2 - L_2 L_1$$

If the homomorphism is an isomorphism (one-to-one, onto), then the representation it called faithful. The opposite extreme case is when all elements of the algebra get mapped onto a single element. Then the representation is trivial.

Definition: The adjoint representation of a Lie algebra L: Represent each element $\ell \varepsilon L$ by ad ℓ and for all $x \varepsilon L$ put x ad $\ell = [x\ell]$.

<u>Examples:</u> 1) <u>Algebra:</u> $e, f \quad [e,f] = e$

In the adjoint rep. we have: $\ell \to x \text{ ad } \ell = [x\ell]$

Find the operators $\text{ad}\ell$ in the basis e, f:

$$\begin{array}{lll} \text{ad } e & x = e & \to e \text{ ade} = [ee] = 0 \\ & x = f & \to f \text{ ade} = [fe] = -e \\ \text{ad } f & x = e & \to e \text{ adf} = [ef] = e \\ & x = f & \to f \text{ adf} = [ff] = 0 \end{array} \qquad (*)$$

Thus:

$$e \to \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \qquad f \to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Remark: in general we could have, say

$$e \to \text{ade} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

in an e, f basis. We can put

$$e = (1,0) \qquad f = (0,1).$$

With usual matrix multiplication, we have:

$$(1\ 0) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = (\alpha\beta), \text{ but } e \text{ ade} = 0, \text{ so that } \alpha=\beta=0$$

$$(0\ 1) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = (\gamma\ \ ), \text{ but } f \text{ ade} = e, \text{ so that } \gamma = -1,\ \delta = 0$$

2) The algebra of SO(3): $e_1, e_2, e_3 \quad [e_i e_k] = \varepsilon_{ik\ell} e_\ell$

$$\begin{array}{llll} \text{ade}_1 : & x = e_1 & e_1 \text{ade}_1 = [e_1 e_1] = 0 & \\ & x = e_2 & e_2 \text{ade}_1 = [e_2 e_1] = -e_3 & \text{ade}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \\ & x = e_3 & e_3 \text{ade}_1 = [e_3 e_1] = e_2 & \end{array}$$

$$\begin{array}{lll} \text{ade}_2 : & e_1 \text{ada}_2 = [e_1 e_2] = e_3 & \\ & e_3 \text{ade}_2 = [e_3 e_2] = -e_1 & \text{ade}_2 = \begin{pmatrix} 0 & 0 & +1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \end{array}$$

ade_3

$$e_1\mathrm{ade}_3 = [e_1e_3] = -e_2$$
$$e_2\mathrm{ade}_3 = [e_2e_3] = e_1 \qquad \mathrm{ade}_3 = \begin{pmatrix} 0 & -1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$e_3\mathrm{ade}_3 = [e_3e_3] = 0$$

Problem: 1) Find the adjoint representations of all three-dimensional Lie algebras.

2) When is the adjoint representation faithful?
When is it trivial?

3) What is the kernel of the adjoint representation
(when it is a homomorphism)?

Lecture 4

We have already defined the concept of an ideal, namely a subalgebra $N \subset L$ is an ideal if $[NL] \subseteq N$.

Obviously, we have:

(1) The intersection of ideals is an ideal

(2) The sum of ideals is an ideal

(3) The Lie product of ideals is an ideal.

The derived series of a Lie algebra:

$$L \supseteq L' = [L,L] \supseteq L'' = [L',L'] \supseteq \quad \dots\dots \quad \supseteq L^{(k)} = [L^{(k-1)},L^{(k-1)}] \supseteq \quad \dots\dots$$

The lower central series

$$\begin{aligned} L \supseteq L^2 &= L' = [L,L] \\ \supseteq L^3 &= [L^2,L] \supseteq \dots \\ \supseteq L^k &= [L^{(k-1)},L] \supseteq \dots \end{aligned}$$

All terms in both series are ideals.

Definition: A Lie algebra is solvable if $L^{(h)} = 0$ for some positive integer h.

Example: An abelian algebra is solvable.

Problem: Which two and three-dimensional algebras are solvable and which are nilpotent?

Definition: A Lie algebra is nilpotent if $L^k = 0$ for some positive integer k. Obviously, every nilpotent algebra is solvable, but not vice versa.

Example: $[ef] = e$ .. solvable, but not nilpotent.

Definition: The Radical of a Lie algebra L is the maximal solvable ideal of L, i.e. a solvable ideal, which contains all solvable ideals of L.

Definition: Algebra L is simple if it has no ideals except 0 and L and if $L' \neq 0$.

Definition: Algebra L is semisimple if it has no non-zero solvable ideal (i.e. if the radical is equal to zero).

Theorem: An algebra L is semisimple if it has no non-zero abelian ideals.

Proof: We must show that if L has a solvable ideal it has an abelian ideal (the converse is obvious). Let Σ be a solvable ideal: $[\Sigma L] \subseteq L$

$$\left.\begin{array}{l}\Sigma^{(h)} = [\Sigma^{(h-1)}, \Sigma^{(h-1)}] = 0 \\ \Sigma^{h-1} \neq 0\end{array}\right\} \Rightarrow \Sigma^{(h-1)} \text{ is a non-zero abelian ideal.}$$

We have thus introduced two distinct classes of algebras - semisimple and solvable - and the investigation of these classes is very important, in view of the existence of the following theorem:

Levi-Maltsev Theorem: Every Lie algebra L as a linear space can be considered to be the direct sum of two subspaces A and B. A is a semisimple subalgebra of L and B is a solvable subalgebra of L. B is the radical of L.

$$L = A \oplus B \qquad [L,B] \subseteq B\,.$$

Remark: $L = A \oplus B$ means that every element $\ell \in L$ can be written as $\ell = a + b$ with $a \in A$, $b \in B$.

Thus, the structure of general Lie algebras can, to a large degree, be understood in terms of semisimple and solvable ones.

The corresponding statement for Lie groups is:

Theorem: Every connected Lie group is locally isomorphic to the semidirect product

$$G = R \circ T$$

where R is a semisimple connected group and T is a solvable connected group. Further - T is an invariant subgroup of G, so that

$$G\,T\,G^{-1} \subseteq T, \text{ in particular } R\,T\,R^{-1} \subseteq T.$$

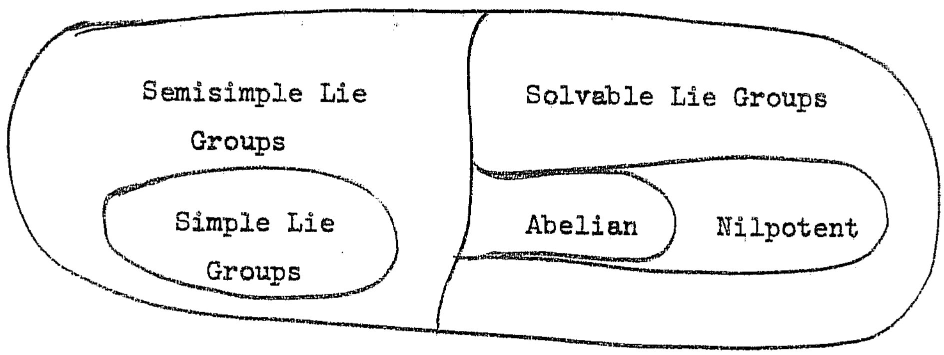
Example: An Euclidean group:

$$G = R.T$$

R - rotations, T - translations. Here R.T is a semidirect product, i.e. $r_1, r_2 \varepsilon R$, $t_1, t_2 \varepsilon T$ $rtr^{-1} \varepsilon T$.

Definition: A group G is solvable if the sequence of subgroups $Q_1, Q_2, \ldots Q_n \ldots$ contains the trivial subgroups $\{e\}$. Here Q_1 is the commutator group of G, i.e. the group consisting of all elements of the type $aba^{-1}b^{-1}$, where $a \varepsilon G$, $b \varepsilon G$. Q_n is the commutator group of Q_{n-1}.

Schematically we have the following picture for Lie groups (and Lie Algebras):



In a moment we shall return to semisimple Lie algebras, but in order to be able to move more freely between algebras and groups, let us discuss some further properties of groups.

We have given a definition of a compact topological space. In a metric space we can give a simpler definition: A set K in a metric space is compact if it can be covered by a finite number of spheres with equal radii $\varepsilon > 0$ where ε can be arbitrarily small.

If the space K lies in a Euclidean space, then it is a compact space if it is bounded and closed.

A metric space R is a set of elements in which we associate a distance $\rho(x,y)$ to each pair of points $x,y\epsilon R$. The distance $\rho(x,y)$ is a real number, satisfying

(1) $\rho(x,y) \geq 0$, $\rho(x,y) = 0$ iff $x = y$

(2) $\rho(x,y) = \rho(y,x)$

(3) $\rho(x,y) + \rho(y,z) \geq \rho(x,z)$ (The triangle inequality)

A locally compact space: each point of the space has a compact neighborhood. Otherwise: the space is essentially noncompact.

Theorem: Lie groups are either compact, or locally compact.

Proof: Each element is parametrized by a finite number of parameters and can thus be covered by a finite-dimensional sphere.

Example: Circle: compact

Straight line: noncompact

Properties of compact Lie groups:

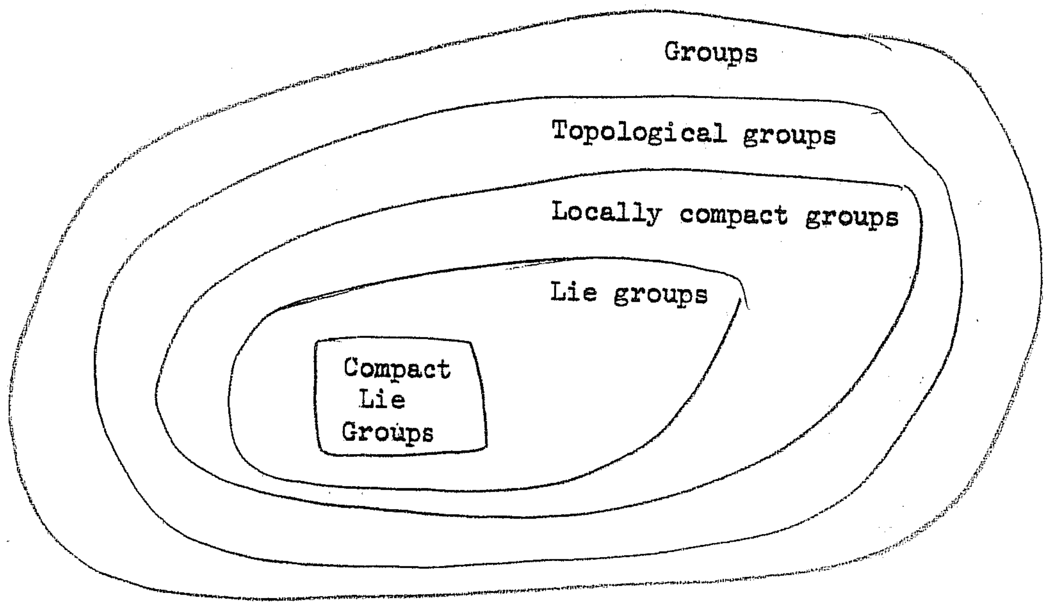
1. Every compact Lie group of dimension 1 is a circle. Thus any one-parameter subgroup of a Lie group is isomorphic to a circle.

2. A compact group is either a connected group, or it consists of a finite number of connected sheets.

3. If X is the Lie algebra of a compact group G, then there exists only a finite number of non-isomorphic groups having the same Lie algebra. These are called locally isomorphic groups.

From the point of view of physical applications the most important distinct properties of compact groups manifest themselves in the representation theory of these groups. This will be treated in detail in the second part of this course - here let us just note that all irreducible representations of compact groups are finite-dimensional and unitary (or at least equivalent to unitary ones), that expansions of functions, defined over compact groups lead to sums, rather than integrals, etc. An example is Fourier analysis: functions defined on a circle can be expanded into series, on a line - into integrals.

The general picture for the classification of groups that emerges is the following:



In the next lectures we shall give a classification of semisimple Lie groups and Lie algebras and investigate some of their properties. However, we still need some further preliminaries.

The adjoint representation of a Lie algebra: defined above.

Let G be a real connected Lie group and L its Lie algebra and let us again consider the adjoint representation:

$$\ell \varepsilon L \quad \to \quad \mathrm{ad}\ell . y \equiv \hat{\ell} y = [\ell , y]$$

Notation ℓ , y: elements of L

$\hat{\ell}$... an operator which can be written in the form of a matrix

$$\hat{\ell} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

The matrix is of order nxn where n is the dimension of the Lie algebra. The form of the matrix does of course depend on the choice of the basis.

Remark: This form of the adjoint representation differs by a sign from the one given in Lecture 3.

Properties:

(1) $\hat{\ell}[xy] = [\hat{\ell}x,y] + [x,\hat{\ell}y]$

(a differentiation formula)

(2) $\widehat{[\ell,x]} = [\hat{\ell},\hat{x}]$

Both properties follow from the Jacobi identity.

So: the adjoint representation is a representation, the linear operators of which act in the algebra itself.

If we can exponentiate such a representation we obtain the adjoint representation of the group G: $\rho(g) = e^{\hat{x}}$

Let G be a matrix group. Then:

$$\rho(g)y = gyg^{-1}$$

for any $y\varepsilon L$. Again the dimension of the adjoint representation is equal to the dimension of the group.

Thus: $g\varepsilon G$ is represented by $\rho(g)$ acting on $y\varepsilon L$ according to the above formula.

This finite-dimensional representation should not be confused with the regular representation of a group, which is in general infinite-dimensional and quite different (and will be treated in detail below).

Examples: (1) O(3) .. adjoint representation is 3 dimensional

(2) O(4) adjoint representation is 6 dimensional

(3) GL(n,R) group of nxn nonsingular real matrices. The basis for the adjoint representations of the algebra can be taken to be the operators E_{ij}.

$$E_{ij}y = [e_{ij}, y]$$

e_{ij} is a matrix consisting of zeros everywhere, except for a one at the intersection of the i'th row and j'th column.

Remark: In general the adjoint representation is not faithful. In particular, if G is abelian then $\rho(g) = 1$ for all $g\varepsilon G$ and the representation is trivial.

We have given a definition of an ideal in an algebra. An equivalent definition: Any subspace of an algebra X, invariant with respect to the adjoint representation, is an ideal.

Indeed $Y = \text{ideal}, Y \subset X \Rightarrow [x,y] \varepsilon Y \quad x\varepsilon Y,\ y \varepsilon Y$

To each ideal Y in the algebra, there corresponds an invariant subgroup H in the group:

$$ghg^{-1} \subset H \quad \text{for all } h\varepsilon H,\ g\varepsilon G\ .$$

The correspondence between ideals and invariant subgroups is not one-to-one: the group can have discrete invariant subgroups, not reflected in the algebra.

Centre X of an algebra: $x \varepsilon X,\ \ell \varepsilon L \quad [x\ell] = 0$

...a centre is a commutative ideal.

Centre C in the group: $c\varepsilon C,\ g\varepsilon G \Rightarrow g^{-1}cg = c$

– an abelian invariant subgroup.

Now we can return to the faithfulness of the adjoint representation:

(1) If $\hat{x}_1 = \hat{x}_2$ then x_1 and x_2 can only differ by an element from the centre of L:

$$x_1 = x_2 + Z$$

(2) If $\rho(g_1) = \rho(g_2)$ the g_1 and g_2 (in the group) can only differ by a factor from the centre of G.

$$g_1 = cg_2$$

Lecture 5

We ended last lecture with some remarks on the adjoint representation. We came to the conclusion that the adjoint representation of an algebra is faithful if the algebra has no centre (no abelian invariant subalgebra). Referring to the definitions of a semisimple algebra, we obtain:

Theorem: The derived algebra of a semisimple algebra is identical with the algebra itself: $L' = L$.

Further remarks on the relation between Lie Groups and Lie Algebras. We already know that given a Lie group, we can uniquely "differentiate" it to obtain a Lie algebra. We have also quoted one of Lie's theorems, telling us that a given Lie algebra can always be integrated, at least in the neighborhood of the identity, to give a Lie group. Let us consider the amount of arbitrariness in the integration.

Let D be a discrete invariant subgroup of the connected Lie group G. Then D must be contained in the centre of G.

Proof: $d_o \varepsilon D,\ g \varepsilon G \qquad g d_o g^{-1} \varepsilon D$

$$g d_o g^{-1} \text{ is } \left.\begin{array}{l}\text{(a) Connected}\\ \text{(b) Discrete}\end{array}\right\} \Rightarrow \quad g d_o g^{-1} = d_o$$

Thus : D is a commutative invariant subgroup

$\Rightarrow$ it is, by definition, in the centre.

Introduce the Factor Group G_o of the group G by the discrete invariant subgroup D:

$$G_o = \frac{G}{D}$$

We identify all elements belonging to the same conjuacy class, i.e. $g_1 \sim g_2$ if $g_1 = g_2 d$ (or dg_2). Each set of elements (each coset) is an element of the group G_o. The fact that G_o is a group is obvious.

1) $(g_1D\cdot g_2D)g_2D \;=\; g_1g_2g_3D \;=\; g_1D(g_2D\; g_3D)$

2) $gD \cdot D = gD$ D is the identity

3) $gD\; Dg^{-1} = D$ Dg^{-1} is the inverse of gD

Examples: Take the group of complex, nxn, unimodular matrices SL(n,C) and the subgroup of matrices λe. We have

$$\text{de}\ \lambda e = \lambda^n = 1$$

The group λe is an abelian invariant subgroup consisting of n elements. Consider n = 2: SL(2,C). Then $D = \{e,-e\}$. The Group $G_o = G/D$ is obtained by identifying g and -g. Thus:

Thus, we are "glueing" together individual "parts" of the group. A certain neighborhood of e is preserved: thus G and G_o have the same Lie algebra.

Let us now sketch the answer to the question: Given a Lie algebra L, how do you find all Lie groups having this algebra?

1) Find one such group G_1

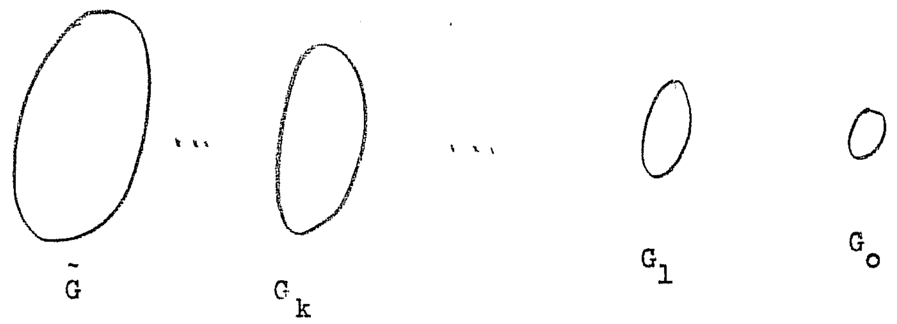
2) If discrete invariant subgroups D exist, then take G_1/D, this group has the same algebra.

3) If G_1 is not simply connected (i.e. if there exists at least one closed cycle in G_1, which cannot be contracted to a point), then there exists a larger group G_2, such that $G_1 = G_2/\tilde{D}$.

4) Among all Lie groups with a given algebra L there exists one unique simply connected group $\tilde{G}$ which cannot be further extended.

5) Take the maximal discrete invariant subgroup D_o. Then $G_o = \tilde{G}/D_o$ is a uniquely determined "minimal group". G_o has no further discrete invariant subgroups.

Thus, we obtain the following picture:



We have a whole series of locally: isomorphic groups:

$$G_i = \tilde{G}/D_i$$

where D_i are discrete invariant subgroups. The largest group $\tilde{G}$ is called the universal covering group. All these groups have the same algebra.

Example: $G = SU(n)$ is simply connected. The maximal discrete invariant subgroup is

$$D_o = \{\varepsilon_k; \ \varepsilon_k^n = 1\}$$

Find all subgroups $D_i \subset D_o$ and the series of factor groups

$$G_i = \tilde{G}/D_i$$

E.g.: For $n = 4$ we have

$$D_o = \{1, i, -1, -i\}$$
$$D_1 = \{1, -1\}$$
$$\tilde{D} = \{1\}$$

and

$$\tilde{G} = SU(n); \ G_1 = SU(n)/D_1 \qquad G_2 = SU(n)/D_o$$

Semisimple Lie Groups and Algebras

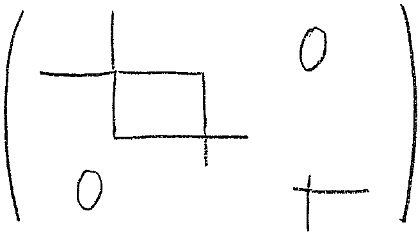
We have already given the definitions of simple and semisimple groups and algebras. Note: A simple Lie algebra has no centre, a simple Lie group may have a discrete centre.

Properties of a semisimple group:

(1) The adjoint representation of a semisimple group has no one-dimensional invariant subspaces (they would correspond to an abelian invariant subgroup).

(2) The adjoint representation of a semisimple Lie group is completely reducible, i.e. if the representation does leave a subspace invariant, then all matrices of the representation can be brought to a block-diagonal form:

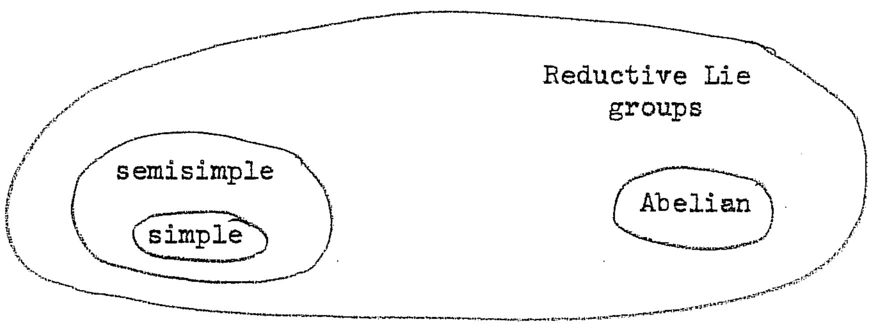
$\rho(g) =$



This however means that a semisimple group is locally isomorphic to the direct product of simple groups. (This will be shown below).

Definition: A group G is reductive, if its adjoint representation is completely reducible.

Thus we have:



Remark: Every compact group is reductive.

<u>Definition</u>: The <u>Cartan-Killing</u> form is an essential concept in the study of semisimple groups.

Let L be a Lie algebra and put

$$B(x,y) = \mathrm{Tr}\,\hat{x}\hat{y} = \mathrm{Tr}(\mathrm{ad}x)(\mathrm{ad}y)$$

for $x,y \in L$.

<u>Theorem</u> (The Cartan Criterion): Algebra L is semisimple iff $B(x,y)$ is not degenerate (i.e. iff $B(x,y_o) = 0$ for all $x \in L$ implies $y_o = 0$).

Sketch of Proof:

(1) Let $B(x,y) = 0$ for all $x \in L$, $y \in N \subset L$. Then N is an ideal, since $B(x,[y,x]) = \mathrm{Tr}\,\hat{x}\,\widehat{[yx]} = \mathrm{Tr}\,\hat{x}[\hat{y},\hat{x}] = \mathrm{Tr}(\hat{x}\hat{y}\hat{x} - \hat{x}\hat{x}\hat{y}) = 0$ i.e. $[yx] \in N$.

It can be shown (not trivially), that N has a one-dimensional subalgebra N_o (also an ideal). This is an abelian ideal $\Rightarrow$ L is not semisimple.

2). Let L not be semisimple $\Rightarrow$ there exists an abelian ideal N. Choose a basis for L such, that the first vectors form a basis for N. Then

$$\mathrm{ad}n = \left(\begin{array}{c|c} 0 & \vdots \\ \hline 0 & 0 \end{array}\right) \qquad \mathrm{ad}\ell = \left(\begin{array}{c|c} \vdots & 0 \\ \hline 0 & 0 \end{array}\right) \qquad \text{for } n \in N,\ \ell \in L$$

so that $\mathrm{Tr}\,\{\mathrm{ad}n . \mathrm{ad}\ell\} = 0$

For details of proof: See Jacobson.

<u>Example</u>: $[ef] = 0 \quad [e,g] = pe-f \quad [fg] = e+pf$

Put: $e = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad f = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad g = \begin{pmatrix}0\\0\\1\end{pmatrix}$

Then:

$$\operatorname{ad}e = \begin{pmatrix} 0 & 0 & p \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \operatorname{ad}f = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & p \\ 0 & 0 & 0 \end{pmatrix} \qquad \operatorname{ad}g = \begin{pmatrix} -p & -1 & 0 \\ 1 & -p & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Tr ade adf = 0 $\qquad$ Tr adg adg = $2(p^2-1)$

Tr ade ade = 0 $\qquad$ Tr adg adf = 0

Tr adf adf = 0 $\qquad$ Tr adf adf = 0

$$\left.\begin{array}{l} \text{Thus ade} \perp \text{adL} \\ \text{adf} \perp \text{adL} \end{array}\right\} \Longrightarrow \underline{\text{not}} \text{ semisimple}$$

<u>Remark:</u> 1) Consider algebra L in a basis $\{e_i\}$ $\quad i = 1,\ldots n$

We have: $\quad [e_i e_k] = C_{ik}^{m}\, e_m$

$$\operatorname{ad}e_k = \begin{pmatrix} C_{k1}^{1} & C_{k2}^{1} & \ldots C_{kn}^{1} \\ & & \\ C_{k1}^{n} & C_{k2}^{n} & \ldots C_{kn}^{n} \end{pmatrix}$$

the matrix $\operatorname{ad}e_k$ consists of the structure constants C_{ks}^{r}.

We can characterize the Cartan-Killing form by a symmetric tensor

$$b_{ik} = B(e_i, e_k) = \mathrm{Tr}(\operatorname{ad}e_i)(\operatorname{ad}e_k) = (\operatorname{ad}e_i)_{rs}(\operatorname{ad}e_k)_{sr} = C_{is}^{r} C_{kr}^{s} .$$

From here: $B(x,y)$ nondegenerate $\Leftrightarrow$ $\det(b_{ik}) \neq 0$.

2) $B(\hat{x}y, z) + B(y, \hat{x}z) = 0$

i.e. - it follows from the Jacobi identity that $\hat{x}$ is an antisymmetric operator with respect to the form $B(y,z)$.

Theorem: A semisimple group G is compact if and only if the bilinear Cartan-Killing form has a definite sign (is positive or negative definite).

We shall not give the proof. Its essence is that we use $B(x,y)$ to introduce a positive definite scalar product on the algebra, make use of the antisymmetry of operator $\hat{x}$ with respect to this scalar product. This can then be used to show that the matrices of the regular representation are orthogonal matrices. Thus: we obtain a finite dimensional unitary representation. This is only possible for compact groups, as we shall show when considering group representations.

Thus: we have a simple algebraic criterion for a basically topologic concept of compactness.

Remark: Even when the sign of $B(x,y)$ is not definite we can use the Cartan-Killing form to introduce an indefinite scalar product (indefinite metric). Then of course $B(x,x) = 0$ does not imply $x = 0$.

Let us note some further properties of semisimple Lie groups.

Definition: A linear operator D acting in the algebra L is called a differentiation, if

$$D[x,y] = [Dx,y] + [x,Dy]$$

Example: Action of the adjoint representation: $\hat{\ell}\cdot[x,y] = [\ell[x,y]] = -[y[\ell x]] - x[y\ell]] = [\hat{\ell}x,y] + [x,\hat{\ell}y]$.

Theorem: If L is semisimple, then every differention can be represented as an operator $\hat{\ell}$ in the adjoint representation (for some $\ell\varepsilon L$).

Thus: if $$D[x,y] = [Dx,y] + [x,Dy] \text{ then}$$
there exists $\ell\varepsilon L$, such that $Dx = \mathrm{ad}\ell\cdot x$ for all x.

(Every differentiation for a semisimple Lie Algebra is an "inner differentiation").

A similar property holds for Lie groups. An automorphism of a group is an isomorphism of a group onto itself, (i.e. : $g \to \tilde{g}$, $g_1g_2 \to \tilde{g}_1\tilde{g}_2$, $g^{-1} \to \tilde{g}^{-1}$). Automorphisms of a group G themselves form a group Aut G. Consider the subgroups of this group connected to the identity and call it $\mathrm{Aut}_o G$.

Theorem: If G is semisimple, then every connected automorphism is an inner automorphism, i.e.: every automorphism $g \to \tilde{g}$ contained in Aut_o G can be written as

$$g \to g_o g g_o^{-1}$$

where $g_o \varepsilon G$.

In other words: any connected automorphism in a semisimple group is just a transformation to a new basis.

Classification of Semisimple Complex Lie Groups and Lie Algebras

We shall consider all Lie algebras over a complex field for which the Cartan-Killing form

$$B(x,y) = \mathrm{Tr}\ \hat{x}.\hat{y}$$

is non-degenerate.

Below we shall also make some comments on Lie algebras over a real field, the theory of which is somewhat more complicated.

Formulation of the Problem: Consider the algebra L, satisfying the usual conditions on $x+y$, λx and $[x,y]$ and the condition that $B(x,y)$ is non-degenerate. We wish to find a "canonical" basis for the algebra and write down all commutation relations for this basis.

Remembering that it was quite complicated to do this for all two and three-dimensional algebras it is remarkable, that it can be done at all for so general a class as all semisimple algebras of arbitrary finite dimensions. One of the reasons why this is possible is that for semisimple Lie algebras the adjoint representation is faithful (the algebra has no centre), so that we can always just consider a finite dimensional matrix algebra.

We shall solve the problem in several steps.

I. The Maximal Commutative Subalgebra

1) Take an element $\ell_o \epsilon L$ and consider the equation

$$\mathrm{ad}\ell_o . x = 0$$

for all x. This is an eigenvalue problem - namely we are looking for the eigenvectors of the matrix $\mathrm{ad}\ell_o$ corresponding to the eigenvalue zero. For each matrix $\mathrm{ad}\ell_o$ the eigenvalue 0 has a definite multiplicity and clearly a minimal multiplicity of zero must exist.

Definition: An element $\ell_o \epsilon L$ is regular if $\mathrm{ad}\ell_o$ has the minimal possible multiplicity for zero as an eigenvalue.
(Many regular elements can exist, but in general not every element is regular).

2) Take all elements of L, commuting with ℓ_o - they form a Cartan subalgebra $H \subset L$.

Remarks: 1) The Cartan subalgebra depends on the choice of ℓ_o. However, we shall show that all different Cartan subalgebras are isomorphic to each other.

2) The fact that the Cartan subalgebra is not simply a maximal commutative subalgebra, but one that contains a regular element, is crucial. An example of the importance of this fact is the conformal group of space-time, the Cartan subalgebra of which has only three elements, although the maximal commutative subgroup has four-the translations in the Poincare' subgroup. (These remarks will become

clear to the uninitiated towards the end of these lectures.)

3) Consider all elements $h_i \varepsilon H$ in the adjoint representation (as matrices). Any set of commuting complex matrices can be simultaneously brought to the Jordan canonical form ("block-triangular"):

$$\hat{h} = \begin{pmatrix} \begin{array}{cc|cc|cc} \alpha(h) & 0 & & & & \\ * & \alpha(h) & & & & O \\ \hline & & \beta(h) & 0 & & \\ & & * & \beta(h) & & \\ \hline & & & & \mu(h) & 0 \\ & O & & & * & \mu(h) \end{array} \end{pmatrix}$$

where all matrices have the same structure(same dimension of each block).

We can now see that the eigenvalues are linear functions on the algebra H:

$$\begin{aligned} \alpha(h_1+h_2) &= \alpha(h_1) + \alpha(h_2) \\ \alpha(\lambda h) &= \lambda\alpha(h) \end{aligned}$$

(since adding up matrices $\hat{h}_1+\hat{h}_2$ gives a matrix with elements $\alpha(h_1) + \alpha(h_2)$ on the diagonal of the first block, etc.)

Lecture 6

We have defined the Cartan subalgebra $H \subset L$ where L is a semi-simple algebra and shown that all elements $h_i \epsilon H$ can be represented, in the adjoint representation, by matrices of the type

$$\hat{h}_i = \left(\begin{array}{cc|cc|cc} \alpha_1(h_i) & 0 & & & & \\ * & \alpha_1(h_i) & & & & \\ \hline & & \alpha_2(h_i) & 0 & & \\ & & * & \alpha_2(h_i) & & \\ \hline & & & & \alpha_n(h_i) & 0 \\ & & & & * & \alpha_n(h_i) \end{array}\right) \qquad (1)$$

By definition h_1+h_2 is represented by $\hat{h}_1+\hat{h}_2$ so that

$$\begin{aligned} \alpha_k(h_1+h_2) &= \alpha_k(h_1) + \alpha_k(h_2) \\ \alpha_k(\lambda h) &= \lambda\alpha_k(h) . \end{aligned} \qquad (2)$$

Again, by definition of the adjoint representation, the matrices $\hat{\ell}$, in particular $\hat{h}_i$, act on the algebra itself, considered as a vector space. Let us use the subalgebra H to split the space L into subspaces, namely subspaces invariant under the action of all $\hat{h}_i \epsilon H$. Let us denote the subspaces $X_{\alpha i}$ and we have

$$H\, X_{\alpha_i} \subset X_{\alpha_i} \qquad (3)$$

From the form of $\hat{h}$ we see that:

1. All the $\alpha_r(h)$ are eigenvalues of $\hat{h}$,

2. Each invariant subspace X_{α_i} contains at least one eigenvector, corresponding to α_i, but does not necessarily consist of eigenvectors alone.

3. One of the eigenvalues must be equal to zero. Indeed: $\widehat{h_i}x = [h_i x]$; take $x \epsilon H$, then $[h_i, h] = 0$ i.e. $\hat{h}_i h = 0$

We can now write the whole space L as a direct sum of invariant subspaces:

$$L = X_o + \sum_i X_o \tag{4}$$

and by definition $\alpha_i \neq 0$, since we have separated out the eigenvalue 0 explicitly.

<u>Example</u>: 3 x 3 matrices:

Put: $\hat{h} = \begin{pmatrix} \alpha & 0 & 0 \\ x & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}$

Find the eigenspaces: X_α and X_β :

$$\begin{pmatrix} \alpha & 0 & 0 \\ x & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} \alpha a \\ xa + \alpha b \\ \beta c \end{pmatrix}$$

Thus: $X_\alpha = \begin{Bmatrix} a \\ b \\ 0 \end{Bmatrix} \qquad X_\beta = \begin{Bmatrix} 0 \\ 0 \\ c \end{Bmatrix}$

The two dimensional invariant subspace X contains only one (independent) eigenvector $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Remark: This only examplifies the splitting of a linear space into invariant subspaces with respect to a given operator h. The picture for several commuting operators h_i is exactly the same. So far, this does not examplify the algebra of a semisimple Lie group - it is just an intermediary step.

Terminology: 1) Call all non-zero eigenvalues $\alpha(h)$ roots

2) Call the sum 4) - the canonical decomposition of the space L with respect to the Cartan subalgebra H.

3) The invariant subspaces X_α are root spaces.

We shall actually show, that everything is much simpler, namely that the matrices $\hat{h}$ are actually diagonal (or at least simultaneously diagonalizable.) Notice that each root space X_α is the maximal invariant subspace in which each operator $\hat{h}$ has just one eigenvalue α. Thus, in X_α (a space of lower dimension than L) we can write

$$\hat{h}\Big|_{X_\alpha} = \begin{pmatrix} \alpha(h) & & \\ & \alpha(h) & 0 \\ & * & \alpha(h) \end{pmatrix} \tag{5}$$

This implies: 1) For each $x \in X_\alpha$ (and only for such x) one can find a positive integer $k \leq n_\alpha$, where n_α is the dimension of X_α, such that

$$(\hat{h} - \alpha)^k \, x = 0$$

(in other words, X_α is a generalized eigenspace and all $x \in X_\alpha$ are generalized eigenvectors).

Example:

$$\hat{h} - \alpha = \begin{pmatrix} \alpha & 0 & 0 \\ x & \alpha & 0 \\ y & z & \alpha \end{pmatrix} - \alpha I = \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & z & 0 \end{pmatrix}$$

$$(\hat{h}-\alpha)\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ xa \\ yb + cz \end{pmatrix} \quad (\hat{h}-\alpha)^2\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ xza \end{pmatrix} \quad (\hat{h}-\alpha)^3\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

2) There exists at least one (eigenvector) $x \varepsilon X_\alpha$ for which k=1:

$$\hat{h}x = \alpha x$$

This is an eigenvector for all matrices $\hat{h} \varepsilon H$.

3) We have already stressed that $\hat{h}x = 0$ for all $\hat{h}$ if $x \varepsilon H$. Thus:

$$H \subset X_o \qquad (6)$$

II. Some properties of the Canonical Decomposition.

We have decomposed the space L into subspaces, not however subalgebras. We now wish to establish commutation relations between elements from various subspaces X_α (including X_o).

Basic Lemma:

$$[X_\alpha, X_\beta] \subset X_{\alpha+\beta} \qquad (7)$$

where $X_{\alpha+\beta} = \{0\}$ if $\alpha+\beta$ is not a root (then elements from the two spaces commute). In other words: $\hat{X}_\alpha X_\beta \subset X_{\alpha+\beta}$ i.e X_α takes elements of X_β into $X_{\alpha+\beta}$.

Proof: Put $z = [x_\alpha, x_\beta]$, $x_\alpha \varepsilon X_\alpha$, $x_\beta \varepsilon X_\beta$ (8)

(a) Let x_α and x_β satisfy

$$\hat{h}x_\alpha = \alpha x_\alpha \qquad \hat{h}x_\beta = \beta x_\beta$$

Then $\hat{h}z = \hat{h}[x_\alpha, x_\beta] = [\hat{h}x_\alpha, x_\beta] + [x_\alpha, \hat{h}x_\beta] = (\alpha+\beta)[x_\alpha y_\beta] = (\alpha+\beta)z$

$\Longrightarrow$ $z = 0$ or the eigenvalue of z is $\alpha+\beta$

(b) Let x_α and x_β not be eigenvectors. Then we can prove by induction that

$$\{\hat{h}-(\alpha+\beta)\}^m z = \sum_{k=0}^{m} C_m^k [(\hat{h}-\alpha)^k x_\alpha, (\hat{h}-\beta)^{mk} x_\beta] \quad (9)$$

Since $(\hat{h}-\alpha)^r x_\alpha = 0$ and $(\hat{h}-\beta)^s x_\beta = 0$ for some r and s, the r.h.s of (9) is equal to zero for large enough m. Thus in this case we again have $z = [x_\alpha, x_\beta] \in X_{\alpha+\beta}$ (or $z = 0$). QED

In particular we have:

$$[X_o, X_\alpha] \subset X_\alpha \quad (10)$$

...each subspace X_α is invariant with respect to X_o (and also with respect to H).

Putting $\alpha = 0$ in (10) we have

$$[X_o, X_o] \subset X_o \quad (11)$$

so that X_o is a subalgebra.

We already know that we have

$$x_o \varepsilon H \subset X_o,$$

where x_o is a regular element. More important, we have:

Theorem 1: $H = X_o$ - the Cartan algebra coincides with the root space X_o, corresponding to the root 0.

Proof: We shall show that

$$B([x_1, x_2]y) = 0 \quad (12)$$

for all $x_1, x_2 \varepsilon X_o$, $y \varepsilon L$. Since L is semisimple (12) implies that $[x_1 x_2] = 0$, that is $[X_o, X_o] = 0$. Thus: X_o, being a commutative subalgebra, is contained in the maximal commutative subalgebra. Thus:

$$X_0 \subset H \qquad (13)$$

Formulas (6) and (13) imply that $X_0 = H$.

Now let us prove assertion (12).

a) $y \varepsilon X_\alpha$, $\alpha \neq 0$. In this case we actually have a stronger statement, namely $B(x,y) = 0$ for all $x \varepsilon X_0$. Indeed: $\hat{x}\hat{y}\, X_\beta \subset X_{\alpha+\beta}$ i.e.: $\hat{x}\hat{y} \cdot x_\beta = x_{\alpha+\beta}$. It follows that the diagonal blocks of $\hat{x}\hat{y}$ must be zero, so as to take one subspace into another (without mixing them). Thus

$$\hat{x}_\alpha \hat{x}_\beta = \begin{pmatrix} 0 & & & * \\ & 0 & & \\ & & 0 & \\ * & & & 0 \end{pmatrix}$$

and

$$\mathrm{Tr}\, \hat{x}\hat{y} = 0 \quad \text{i.e.} \quad B(x,y) = 0$$

for all $x \varepsilon X_0$, $y \varepsilon X_\alpha$.

This does not mean that $B(x,y)$ is degenerate, since $y \varepsilon X_\alpha$, not $y \varepsilon L$ (x is not orthogonal to L)

b) $y \varepsilon X_0$ This case is more complicated and we shall only indicate the proof.

Lemma: Every inner differentiation in X_o is nilpotent:

$$\hat{x}^{n_o} = 0 \tag{14}$$

where n_o = dimension of X_o and (14) holds for all $x \varepsilon X_o$.

Proof of lemma: Write $\hat{x}$ in triangular form:

$$\hat{x}\Big|_{X_o} = \begin{pmatrix} v_1(x) & & \\ & v_2(x) & \\ * & & v_{n_o}(x) \end{pmatrix} \quad \text{(in the subalgebra } X_o\text{)}$$

and show that actually $v_1 = \ldots = v_{n_o} = 0$. Then apply the Engel theorem: If X is a linear algebra and all its elements are nilpotent: $x^n = 0$, then the matrices of X in any irreducible representation can be simultaneously brought to triangular form:

$$\hat{x} = \begin{pmatrix} \lambda_1(x) & 0 \\ * & \lambda_n(x) \end{pmatrix} \quad \text{in the whole algebra X.}$$

Thus:

$$B([x_1 x_2], y) = \sum_{i=1}^{n} [\lambda_i(x_1), \lambda_i(x_2)]_{iy} = 0 \qquad \text{QED}$$

Finally we obtain the canonical decomposition as

$$X = H + \sum_\alpha X_\alpha \tag{15}$$

III. <u>Orthogonality properties</u> (with respect to the Cartan-Killing form).

1) $\beta \neq -\alpha \qquad X_\alpha \perp X_\beta$

2) For any $x_\alpha \epsilon X_\alpha$, $x_\alpha \neq \{0\}$ we can find an $x_{-\alpha} \epsilon X_{-\alpha}$, which is not orthogonal to X_α.

<u>Proof:</u>

1) $\hat{x}_\alpha \hat{x}_\beta X_\gamma \subset X_{\gamma+\alpha+\beta} \qquad \alpha+\beta \neq 0$

$\Rightarrow$ The diagonal elements of $\hat{x}_\alpha \hat{x}_\beta$ are zero

$$\mathrm{Tr}\ \hat{x}_\alpha \hat{x}_\beta = B(x_\alpha, x_\beta) = 0$$

2) $B(x,y)$ is not degenerate $\Rightarrow$ there must exist an element y, not orthogonal to x_α. It follows from assertion 1 that y can only be contained in $X_{-\alpha}$. QED

Thus: $B(x,y)$ is non-degenerate on the pair $(X_\alpha, X_{-\alpha})$... we shall call two such spaces <u>dual</u> to each other.

<u>Corollaries</u>: 1) $\dim X_\alpha = \dim X_{-\alpha}$. (16)

A root $(-\alpha)$ exists for every root α.

2) Taking $\alpha = 0$ we find: The Cartan subalgebra H is orthogonal to all root spaces X_α. The form $B(h_1,h_2)$ is not degenerate on H.

Since $\hat{h}_i$ are triangular matrices, we have

$$B(h_1,h_2) = \mathrm{Tr}\ \hat{h}_1\hat{h}_2 = \sum_\alpha n_\alpha\ \alpha(h_1)\alpha(h_2)$$

where n_α is the dimension of X_α (only the diagonal matrix elements of $\hat{h}_1$ and $\hat{h}_2$ figure in the trace of $\hat{h}_1 \cdot \hat{h}_2$ for triangular matrices).

3) If $h_o \varepsilon H$ has only zero roots, then $h_o = 0$ (since $B(h_o,h) \equiv 0$ for all $h\varepsilon H$).

Let us make use of the dual properties of roots and separate all roots into "positive" and "negative" ones. Let us introduce a basis in H, so that eac hεH has coordinates $h = (\xi_1,\ldots,\xi_r)$.

Then

$$\alpha(h) = \alpha\,\xi_1 + \ldots + \alpha_r\xi_r \tag{17}$$

so that each root is given by a set of real numbers

$$\alpha = (\mathrm{Re}\alpha_1, \mathrm{Im}\alpha_1 \ldots, \mathrm{Re}\alpha_r, \mathrm{Im}\alpha_r) \tag{18}$$

(we are considering an algebra over the field of complex numbers).

We introduce an ordering of the roots, saying $\alpha > \beta$ if $\mathrm{Re}\alpha_1 \geqslant \mathrm{Re}\beta_1$; if $\mathrm{Re}\alpha_1 = \mathrm{Re}\beta_1$ then $\alpha > \beta$ if $\mathrm{Im}\alpha_1 > \mathrm{Im}\beta_1$...etc.

In particular α is a positive root $\alpha > 0$ if the first non zero number in the set (18) is positive and α is a negative root $\alpha < 0$ otherwise.

Thus, we can write the canonical decomposition in a symmetric form:

$$X = E_- + H + E_+ \tag{19}$$

where

$$E_- = \sum_{\alpha<0} X_\alpha \qquad E_+ = \sum_{\alpha>0} X_\alpha \tag{20}$$

(separate sums over negative and positive roots, as defined above).

Obviously: Positive root + Positive root = Positive root

Negative root + Negative root = Negative root

It follows, that although X_α or $X_{-\alpha}$ is in general not a subalgebra, the subspaces E_- and E_+ are subalgebras (since we have shown $[x_\alpha, x_\beta] \subset X_{\alpha+\beta}$ and e.g. $\alpha>0, \beta>0 \Rightarrow \alpha+\beta>0$)

Example: The algebra of the general linear group $GL(n,C)$. The Cartan subalgebra H can be chosen as the subalgebra of all diagonal matrices of the order n. As usual, denote e_{ik} a matrix with a 1 on the intersection of the i-th row and k-th column, and all other elements zero:

$$e_{ik} = \begin{array}{cc} & k \\ i & \begin{pmatrix} 0 & \dots & \dots & 0 \\ 0 & & & 0 \\ 0 & \dots & 1 \dots & 0 \\ 0 & & & 0 \end{pmatrix} \end{array}$$

Then $h \in H$ is

$$h = \xi_1 e_{11} + \dots + \xi_n e_{nn}$$

Further

$$[he_{ij}] = \sum_r \xi_r [e_{rr}, e_{ij}] = \sum_r \xi_r (\delta_{ri} e_{rj} - \delta_{rj} e_{ir}) =$$

$$= (\xi_i - \xi_j) e_{ij} \qquad (21)$$

Thus, all vectors e_{ij} are eigenvectors of h (the eigenspaces X_α are one-dimensional!) The roots are

$$\alpha_{ij} = \xi_i - \xi_j \qquad (22)$$

In this case E_+ consists of all upper triangular matrices, E_- of all lower ones. Indeed:

Put $i < j$

Then $\alpha_{ij} = \begin{array}{ccccccc} 1 & 2 & & i & j & & n \\ (0 & 0 & 0\ldots & 1 & \ldots -1 & \ldots 0 & 0) \end{array}$ so that it is a positive root (the first non-zero number is positive). However e_{ij}, $i<j$ is upper triangular.

Remark: G L (n,C) is not semisimple, however SL(n,C) is. Thus...we should exclude $e = e_{11}+\ldots+e_{nn}$ from the basis.

All the general features which we have so far proved can be seen in this example. However several new features appear:

1. The roots $\alpha_{ij} = \xi_i - \xi_j$ have real coordinates:

$$\alpha_{ij} = (0\ldots0,\pm1\ldots\mp1,0\ldots0) \tag{23}$$

2. All roots are different ($\alpha_{ik} = \alpha_{rs} \Longrightarrow i = r,\ k = s$); all root spaces are one-dimensional .

3. We can introduce $\omega_1 = \xi_1 - \xi_2$, $\omega_2 = \xi_2 - \xi_3 \ldots \omega_{n-1} = \xi_{n-1} - \xi_n$ as independent roots. All roots α_{ij} are combinations of these.

4. Construct a matrix out of the basis vectors:

$$\xi = \begin{pmatrix} e_{11} & e_{12} & e_{13} & \cdots & e_{1n} \\ \boxed{e_{21}} & e_{22} & e_{23} & & e_{2n} \\ & & & & e_{n-1n} \\ e_{n1} & & & \boxed{e_{n-1,n}} & e_{nn} \end{pmatrix}$$

The whole algebra ξ_+ can be obtained by taking multiple commutators of e_{ii+1} (elements in circles), similarly ξ_- by commuting $e_{k,k-1}$ (elements in squares).

We shall prove that these features are true in general, (for all semisimple algebras).

IV. The Root Spaces are One-Dimensional

Theorem 2: All operators $\hat{h}$, $h \varepsilon H$ can be diagonalized simultaneously.

Thus: there are actually no Jordan submatrices in (1).

Proof: Take $h \varepsilon H$ and define an operator $\delta(h)$

$$\delta(h) x_\alpha = \alpha x_\alpha \qquad x_\alpha \varepsilon X$$

($\delta(h)$ is the "diagonal part" of operator $\hat{h}$)

We have

$$\delta(h)\ [x_\alpha, x_\beta] = [\delta(h) x_\alpha, x_\beta] + [x_\alpha, \delta(h) x_\beta]$$

so that $\delta(h)$ is a differentiation. In a semisimple algebra every differentiation is an inner differentiation (we have not really proved this) - so that there exists an $x \varepsilon X$ such that

$$\delta(h) = \hat{x}$$

Since $\delta(h)$ must commute with H ($\hat{h} x_\alpha \subset X_\alpha$) and H is a maximal commutative subgroup, we have: $x \varepsilon H$. Take the operator $\theta = h - x$. We have

$$\theta x_\alpha = (\hat{h} - \hat{x})\ x_\alpha = 0$$

Thus all the roots $\alpha(\theta) = 0$ so that $\theta = 0$.

Finally we obtain:

$$\hat{h} = \delta(h)$$

so that $\hat{h}$ is diagonal. Q.E.D.

Let us now establish the <u>commutation relations</u>.

Take $e_\alpha \varepsilon X_\alpha$, $e_\alpha \neq 0$. Then there exists $e_{-\alpha} \varepsilon X_{-\alpha}$ such that $B(e_\alpha, e_{-\alpha}) \neq 0$. Let us normalize $e_{-\alpha}$ so that

$$B(e_{-\alpha}, e_\alpha) = 1 \tag{24}$$

Take three elements: $e_\alpha, e_{-\alpha}$ and $h_\alpha \equiv [e_\alpha, e_{-\alpha}]$; they form a subalgebra $E_\alpha \subset L$.

<u>Theorem 3:</u> The algebra E_α is isomorphic to the Lie algebra of $SL(2,C)$

Remark: We are considering $SL(2,C)$ as a three-dimensional algebra over a complex field.

<u>Proof</u>: We have

$$[e_\alpha, e_{-\alpha}] = h_\alpha \quad [h_\alpha, e_\alpha] = \alpha(h_\alpha)e_\alpha \qquad [h_\alpha, e_{-\alpha}] = -\alpha(h_\alpha)e_{-\alpha}$$

1) Let us show that $\alpha(h_\alpha) \neq 0$.

We know that at least one root $\beta(h_\alpha) \neq 0$ must exist.

Take the space enveloping all the spaces

$$\ldots\ldots X_{\beta-\alpha}, X_\beta, X_{\beta+\alpha}, \ldots, X_{\beta+k\alpha}, \ldots$$

This space is invariant under E_α. We have $\mathrm{Tr}\hat{h}_\alpha = \mathrm{Tr}[e_\alpha, e_{-\alpha}] = 0$. On the other hand we can calculate the trace of $\hat{h}_\alpha$ directly in the above space:

$$\mathrm{Tr}\hat{h}_\alpha = \sum_k n_{\beta+k\alpha} [\beta(h_\alpha) + k\,\alpha(h_\alpha)]$$

where $n_{\beta+k\alpha} = \dim X_{\beta+k\alpha}$.

Thus:
$$\alpha(h_\alpha) \sum_k n_{\beta+k\alpha}\, k = -\beta(h_\alpha) \sum_k n_{\beta+k\alpha} \tag{25}$$

R.h.s. $\neq 0 \implies \alpha(h_\alpha) \neq 0$

Introduce:

$$e_+ = \frac{e_\alpha}{\sqrt{\alpha(h_\alpha)}} \qquad e_0 = \frac{h_\alpha}{\alpha(h_\alpha)} \qquad e_- = \frac{e_{-\alpha}}{\sqrt{\alpha(h_\alpha)}}$$

(we are in an algebraically closed field, so extracting square roots is no problem).

Obviously we have

$$[e_+ e_-] = e_0 \qquad [e_0 e_+] = e_+ \qquad [e_0 e_-] = -e_- \tag{26}$$

Q.E.D.

<u>Corollary:</u> Every semisimple complex Lie algebra contains an $E^* = \{e_-, e_0, e_+\}$ subalgebra, isomorphic to SL(2,C) (i.e. to the complex extension of SU(2)).

<u>Remark:</u> Since (25) is true for arbitrary β, we have

$$\beta(h_\alpha) = r\alpha(h_\alpha) \tag{27}$$

where r is a rational number.

<u>Theorem 4:</u> All Root Spaces X_α are One-Dimensional.

<u>Proof:</u> Consider the subspace

$$\{e_{-\alpha}\} + \{h_\alpha\} + X_\alpha + X_{2\alpha} + \dots$$

invariant under the subalgebra $\hat{E}_\alpha$. Take the trace of h_α, acting in this space

$$0 = \mathrm{Tr}\hat{h}_\alpha = \alpha(h_\alpha)\{-1 + 0 + n_\alpha + 2n_{2\alpha} + 3n_{3\alpha} + \dots\}$$

$$n_\alpha \neq 0 \qquad n_{k\alpha} = 0 \quad k \geq 2, \quad \underline{\underline{n_\alpha = 1}}$$

Q.E.D.

Corollary: If α is a root, then $k\alpha$ with $k > 1$ is not a root.
Thus, the only roots proportional to α are: $-\alpha$, 0, α.

V. The System of Roots

Let us again introduce the basis $\{h_i\}$, $i = 1,\dots,r$ in the Cartan subalgebra, so that

$$h = \sum_{i=1}^{r} \xi_i h_i \tag{28}$$

The Cartan-Killing form provides us with a scalar product (in general indefinite) and we denote

$$B(xy) \equiv (x,y) \tag{29}$$

So far the number $\alpha(h)$ simply denoted a diagonal element of the matrix $\hat{h}$. We have already noticed that we can write:

$$\alpha(h) = \alpha(\xi_i h_i) = \xi_i \alpha(h_i) \equiv \alpha_i \xi_i \equiv (h,\alpha) \tag{30}$$

Thus: a vector with the components $\{\alpha_i\}$ in the Cartan algebra H corresponds to each root $a(h)$. We have a finite system of roots (vectors) $\Sigma = \{\alpha\}$. Usually the number of roots is larger than the dimension of H so that they cannot all be linearly independent (e.g. α and $-\alpha$ are both roots).

We have already established that $[h_\alpha, e_\alpha] = \alpha(h_\alpha)e_\alpha$ (remember that all matrices h_α are diagonal) and we can now write this relation as

$$[h, e_\alpha] = (h,\alpha)e_\alpha \tag{31}$$

or for each of the basis vectors

$$[h_i, e_\alpha] = \alpha(h_i)e_\alpha = \alpha_i e_\alpha \tag{32}$$

We have shown that $[X_\alpha, X_\beta] \subset X_{\alpha+\beta}$, in particular $[X_\alpha, X_{-\alpha}] \subseteq X_0 = H$. Thus, we must have

$$[e_\alpha, e_{-\alpha}] = r^i(\alpha) h_i \tag{33}$$

Let us show that with the correct normalization of $e_{-\alpha}$ we can put the $r_i(\alpha)$ of (31) equal to the α_i of (30) so that (33) can also be written as

$$[e_\alpha, e_{-\alpha}] = \alpha_i h_i \equiv \alpha \tag{34}$$

(where $\alpha \varepsilon H$ is a root (vector)). Indeed, consider the symmetric tensor

$$b_{AB} = B(e_A, e_B) = \mathrm{Tr}\ \hat{e}_A \hat{e}_B = C^S_{AR}\ C^R_{BS} \tag{35}$$

where $\{e_A\}$ is a basis of the algebra and we put

$$e_A = h_i \qquad i = A = 1, \ldots r$$

$$e_A = e_\alpha \qquad r < A \leq n$$

(to each k corresponds one A), and C^D_{AB} are the original structure constants $C^A_{ik} = 0$, $C^i_{\alpha,-\alpha} = r^i(\alpha)$, etc).

We already know that $B(e_\alpha, h_i) = 0$ and $B(e_\alpha, e_\beta) = 0$ for $\beta \neq -\alpha$. Let us normalize $e_{-\alpha}$ so that

$$b_{\alpha-\alpha} = B(e_\alpha, e_{-\alpha}) = C^B_{\alpha A}\ C^A_{-\alpha\beta} = 1$$

The tensor b_{AB} can be written as

$$
b = \begin{pmatrix}
b_{ik} & & & & \mathcal{O} \\
 & 0 & 1 & & \\
 & 1 & 0 & & \\
\mathcal{O} & & & \ddots & \\
 & & & & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}
\end{pmatrix}
$$

The algebra is semisimple, thus det $b \neq 0$ and also det $b_{ik} \neq 0$. This nonsingular symmetric matrix $b_{ik} = b_{ki}$ can be diagonalized and actually (by choosing appropriate lengths for the root vectors) brought to the form

$$b_{ik} = \delta_{ik}. \tag{36}$$

We have:

$$
\begin{aligned}
b_{ij} r^j &= C^S_{iR} C^R_{jS} C^j_{\alpha-\alpha} = -C^S_{iR}[C^R_{j-\alpha} C^j_{S\alpha} + C^R_{j\alpha} C^j_{-\alpha S}] \\
&= -C^S_{iR} C^R_{j-\alpha} C^j_{S\alpha} + C^R_{j\alpha} [C^S_{-\alpha i} C^j_{RS} + C^S_{R-\alpha} C^j_{iS}] \\
&= C^R_{j\alpha} C^j_{RS} C^S_{-\alpha i} = \delta_{S-\alpha} C^S_{-\alpha i} = C^{-\alpha}_{-\alpha i} \\
&= -C^{-\alpha}_{i-\alpha} = \alpha_i
\end{aligned}
$$

Thus, we have

$$r^j(\alpha) = \alpha_j \tag{37}$$

Let us now collect the results.

VI. The Cartan-Weyl Basis

Theorem 5: In any semisimple algebra L we can choose a basis consisting of elements of the Cartan subalgebra H and the root vectors e_α. The commutation relations can be written as:

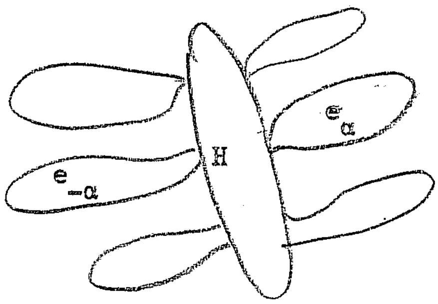
$$\begin{aligned}
[h_i, h_k] &= 0 \\
[h_i, e_\alpha] &= \alpha_i e_\alpha \\
[e_\alpha, e_{-\alpha}] &= \alpha_i h_i \\
[e_\alpha, e_\beta] &= N_{\alpha\beta} e_{\alpha+\beta}
\end{aligned} \tag{38}$$

where $N_{\alpha\beta} = 0$ if $\alpha+\beta$ is not a root.

This theorem has already been proven above.

Remarks: 1) We have imposed $(e_\alpha, e_{-\alpha}) = 1$, but the norm (e_α, e_α) is still arbitrary.

2) We have shown that any semisimple complex Lie algebra consists of individual E_α algebras which are "glued" together



(each E_α is isomorphic to the algebra of SL(2,C).

We still have to determine the structure constants $N_{\alpha\beta}$ and investigate the system of roots $\{\alpha\} = \Sigma$

Obviously we have

$$N_{\beta\alpha} = -N_{\alpha\beta} \tag{39}$$

With an appropriate normalization of e_α we can achieve

$$N_{-\alpha-\beta} = -N_{\alpha\beta} \tag{40}$$

To obtain $N_{\alpha\beta}$ explicitly is a simple, if somewhat tedious task. It is sufficient to consider the subalgebra E_α, acting on the space spanned by

$$e_{\beta+n\alpha} \quad -p \leq n \leq q \qquad p,q \geq 0 \tag{41}$$

(the finite range of n follows from the finiteness of the number of roots). Either by making use of the representation theory of E_α (we have a finite dimensional representation of E_α in space (41)) or just by making full use of the Jacobi identities and other algebraic properties of the system (38) we can show that

$$N_{\alpha\beta}{}^2 = \frac{q(1+p)}{2}(\alpha,\alpha) \tag{42}$$

Formula (42) shows that if $\alpha+\beta$ is a root $N_{\alpha\beta} \neq 0$, since then $q \neq 0$. (Since $\alpha \subset H$ (α is a vector) we always have $(\alpha\alpha) \neq 0$).

For more complete proofs and further information, see

1) N. Jacobson, Lie Algebras

2) E. B. Dynkin: Semisimple Lie Algebras. Am. Math. Soc. Transl. No. 17 (1950) [Uspehi Mat. Nauk (N.S.), 2, 59-127 (1947)].

3) E. B. Dynkin: Maximal Subgroups of the Classical Groups, Am. Math. Soc. Transl. Ser. 2, 6, 245-378 (1957) [Trudy Moskov. Mat. Obshchestva 1, 39-166 (1952)].

4) C. Fronsdal: Group Theory and Applications to Particle Physics, In: Elementary Particle Physics and Field Theory, Brandeis Summer Institute 1962; Ed. K. W. Ford; W. A. Benjamin, Inc., N.Y. (1965).

5) The book I am following most closely is D.P. Zhelobenko, Lectures on Lie Groups, Dubne (1965) (in Russian).

VII. Geometric Properties of the Root System

We have

$$\Sigma \in H$$

...a finite system of roots.

Theorem 6: The matrix of scalar products

$$M = ||(\alpha,\beta)|| \qquad \alpha,\beta \in \Sigma$$

consists of rational numbers and is positive definite. In particular $(\alpha\alpha) > 0$.

Proof:

We have

$$B(h,h) = (h,h) = \sum_\alpha \alpha^2(h) = \sum_\alpha (h\alpha)^2. \qquad (43)$$

Making use of (27) we can put

$$(h\alpha) = r_\alpha(h,h)$$

with r_α rational. Thus

$$1 = (h,h)\sum_\alpha r_\alpha^2 \qquad \text{i.e.} \quad (h,h) = \frac{1}{\sum_\alpha r_\alpha^2} \qquad (44)$$

Formula (44) shows that (h,h) is rational and positive. Using (27) again, we have

$$(\alpha,\beta) = r_\beta(\alpha\alpha)$$

so that (α,β) is a rational number. Further, for any $h,k\varepsilon H$ we have

$$(h,k) = \sum_{\alpha=\text{roots}} \alpha(h)\,\alpha(k)$$

In particular:

$$(\beta,\gamma) = (h_\beta,h_\gamma) = \sum_\alpha \alpha(h_\beta)\alpha(h_\gamma) = \sum_\alpha (\beta\alpha)(\gamma\alpha).$$

Thus

$$M = MM^T$$

where M^T means the transposed matrix and if any matrix is real then RR^T is positive definite (A matrix M is positive definite if the quadratic form $(x,My) = \sum_{i,j=1}^{n} M_{ij}x_iy_j$ is positive definite).

Q.E.D.

Remarks: 1) Relation (27), following from (25) can be further specified. Since we now know that all $n_\alpha = 1$ we have

$$\alpha(h_\alpha) \sum_{k=-p}^{q} n_{\beta+k\alpha}k = -\beta(h_\alpha) \sum_{k=-p}^{q} n_{\beta+k\alpha}$$

Summing, we have

$$\alpha(h_\alpha) \frac{(-p+q)(p+q+1)}{2} = -\beta(h_\alpha)(p+q+1)$$

Thus: $$\boxed{\frac{-2(\alpha,\beta)}{(\alpha,\alpha)} = -p+q} \tag{45}$$

where $-p$ and q ($p,q \geq 0$) are the minimal and maximal numbers k in the series of roots $\beta+k\alpha$

2) Since $M = M^T$ in view of the symmetry of the scalar product $(\alpha\beta) = (\beta\alpha)$, and since $MM^T = M$, we obtain

$$M = M^2 . \tag{46}$$

so that M is a real projection matrix.

<u>Theorem 7:</u> The complex linear envelope of Σ coincides with the entire algebra H.

<u>Proof:</u> We already know that for a semisimple algebra X we have

$$[X,X] = X$$

Hence the collection of vectors $[e_\alpha e_{-\alpha}] = \alpha_i h_i \equiv \alpha$ must span H, i.e. the set of roots $\Sigma = \{\alpha\}$ spans H. Q.E.D.

VII. <u>Simple Roots</u>

<u>Definition:</u> A <u>simple root</u> ω is a positive root that cannot be written as the sum of any other two positive roots.

Obviously any positive root can be written as a linear combination of simple roots. What is more, we have

Theorem 8: Let Π be the collection of simple roots $\{\omega\}$ (in some basis in H). Then

(i) $\alpha,\beta\in\Pi \Longrightarrow \alpha-\beta$ is not a root.

(ii) $\alpha,\beta\in\Pi$ and $\alpha \neq \beta \Rightarrow (\alpha,\beta) \leq 0$

(iii) The set Π is a basis for H. If α is a positive root then

$$\alpha = \sum_{i=1}^{r} k_i \omega_i \tag{47}$$

where k_i are non-negative integers.

Proof: (i) Let $\alpha-\beta$ be a positive root. Then $\alpha = \beta+(\alpha-\beta)$ is not a simple root.

Let $\alpha-\beta$ be a negative root. Then $\beta=(\beta-\alpha) + \alpha \Rightarrow \beta$ is not a simple root.

(ii) It follows from (i) that any "α-string" of roots obtained from β must start with β :

$$\beta,\beta+\alpha,\ldots \beta+q\alpha$$

(since $\beta-\alpha$ is not a root). Thus: $p = 0$ in (45) and we have:

$$\boxed{\frac{2(\alpha,\beta)}{(\alpha\alpha)} = -q \leq 0} \tag{48}$$

(We already know that $(\alpha\alpha) > 0$).

(iii) Let us first prove by induction that the k_i in (47) are non-negative integers. Any simple root ω_i can be written in form (47). Let β and γ be positive roots and $\beta > \gamma > 0$. Assume that (iii) is true for γ and take $\beta\notin\Pi$. Then $\beta= \beta_1+\beta_2$; $\beta_i > 0$. We have $\beta > \beta_i \Longrightarrow$ $\beta_1 = \Sigma k_{1i}\omega_i$, $\beta_2 = \Sigma k_{2i}\omega_i$ and we obtain the result:

$$\beta = \sum_{i=1}^{r} (k_{1i} + k_{2i})\omega_i$$

Now let us show that the simple roots are linearly independent.
Assume the opposite:

$$\Sigma\, \lambda^i \omega_i = 0$$

and take a scalar product with ω_j, $j=1\ldots r$. We obtain a system of linear equations for λ^i with real coefficients $(\omega_i\omega_j)$ $\Longrightarrow$ we can proceed as if the λ^i were real (since $\mathrm{Im}\lambda$ and $\mathrm{Re}\lambda$ satisfy the same equations).
Call the positive $\lambda^i \ldots a^i$, the negative ones $(-b^j)$. We obtain

$$a^i\omega_i = b^j\omega_j \qquad\qquad a^i \geq 0,\ b^j \geq 0$$

Put $h = a^i\omega_i = b^j\omega_j$. We have

$$(h,h) = a^i b^j (\omega_i\omega_j)$$

The ℓ.h.s. is $(h,h) \geq 0$, the r.h.s. is ≤ 0. Thus $h = 0$ so that $a^i = b^j = 0$.

Q.E.D.

Thus we have r linearly independent simple roots in the Cartan subalgebra H of any semisimple algebra and they form a basis of H.

Lecture 8

Example: SU(3): The algebra has 6 roots in the root system Σ:

$$\alpha_{ij} = \xi_i - \xi_j \qquad i,j = 1,2,3 \quad i \neq j$$

Positive roots: α_{ij} for $i < j$

Simple roots: $\omega_1 = \xi_1 - \xi_2 = \alpha_{12}$, $\omega_2 = \xi_2 - \xi_3 = \alpha_{23}$

$$\alpha_{13} = \xi_1 - \xi_1 = \omega_1 + \omega_2 \text{ ... positive, but not simple}$$

Remark: The root vectors e_ω generate E_+ since

$$[e_{\omega_1}, e_{\omega_2}] = N \; e_{\omega_1 + \omega_2} \qquad N \neq 0 .$$

Thus, the vectors e_ω and $e_{-\omega}$ generate the whole algebra X. In particular:

$$[e_{\omega_1}, e_{-\omega}] = \omega \subset H.$$

From here we can obtain the following assertions:

1. Given the system Π we obtain the algebra X up to an isomorphism.
2. If the system Π can be written as an orthogonal sum of two subsystems

$$\Pi = \Pi' \oplus \Pi'' \qquad (49)$$

then L decomposes into two subalgebras:

$$L = L' \oplus L''$$

3. If we multiply all elements of Π by the same real number we obtain an algebra isomorphic to X. Thus

the system Π is only defined up to dilatations.

<u>Definition:</u> Π is indecomposable if no splitting of the type (49) exists.

Assertion: The algebra L is simple iff the corresponding system of simple roots Π is indecomposable.

Since all semisimple algebras can be obtained as direct products of simple ones, it is sufficient to classify all simple Lie algebras.

IX. <u>Classification of Simple Lie Algebras</u>

<u>Definition:</u> The number of elements in Π (equal to the dimension of the Cartan subalgebra H) is called the <u>rank</u> r of the algebra L.

All we have to do in order to describe all simple Lie algebras of rank r is to consider all possible systems of simple roots in an r-dimensional linear space which cannot be decomposed into orthogonal subspaces.

We already know that simple roots satisfy the equation

$$-\frac{2(\alpha,\beta)}{(\alpha\alpha)} \equiv q_{\alpha\beta} \geq 0 \qquad (50)$$

where $q_{\alpha\beta}$ is an integer. This is a very strong restriction on possible configurations of the system Π. Indeed:

$$q_{\alpha\beta}\cdot q_{\beta\alpha} = \frac{4(\alpha\beta)^2}{(\alpha\alpha)(\beta\beta)} = 4\cos^2\theta < 4 \qquad \alpha \neq \beta \quad (51)$$

Thus:

$$4\cos^2\theta = 0,1,2 \text{ or } 3 \qquad (52)$$

Since $(\alpha,\beta) \nleq 0$ we have

$$\theta = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6} \qquad (53)$$

Let us now use notations, introduced by Dynkin. A simple root is represented by a circle and two simple roots with angle $\theta = \frac{\pi}{2}$ between them are not connected, with $\theta = \frac{2\pi}{3}, \frac{3\pi}{4}$ or $\frac{5\pi}{6}$ are connected by 1,2 or 3 lines respectively. Thus:

θ: 90° 120° 135° 150°

Performing a lot of quite tedious manipulations we find that the only geometrically possible configurations of simple roots fall into four series A_n, B_n, C_n and D_n and five individual algebras E_6, E_7, E_8, F_4 and G_2, called the "exceptional" Cartan algebras.

A_n

B_n

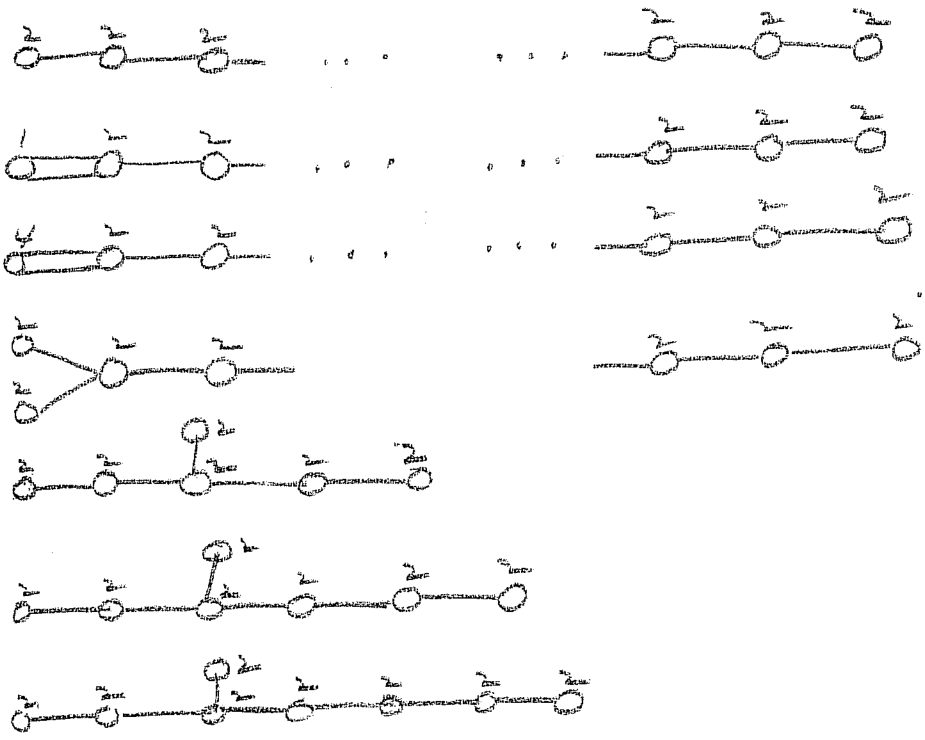
C_n

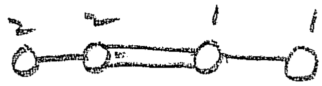
D_n

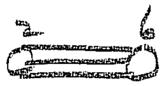
E_6

E_7

E_8



F_4 

G_8 

The numbers above the circles denote the relative squared lengths of the individual simple roots (ω,ω).

<u>Remark</u>: The following factor can be inferred from the Dynkin diagrams

1. There are certain isomorphisms between the Lie algebras of low rank

$$A_1 \sim B_1 \sim C_1$$

$$B_2 \sim C_2 \qquad (54)$$

$$A_3 \sim D_3$$

2. D_1 does not exist, D_2 is represented by two disconnected points

and is thus not simple. Indeed

$$D_2 \sim A_1 \oplus A_1 \text{ (or } B_1 \oplus B_1) \qquad (55)$$

Thus the only nonequivalent simple complex Lie algebras are

$$A_n,\ n \geq 1 \qquad B_n,\ n \geq 2 \qquad C_n,\ n \geq 3 \qquad D_n,\ n \geq 4$$

and the exceptional ones

$$E_6,\ E_7,\ E_8,\ F_4,\ G_2\ .$$

Thus we have classified all simple Lie algebras and also all semisimple ones, which are just direct sums of those listed above. We still wish to construct them explicitly. For the algebras A_n, B_n, C_n and D_n this is easy - namely it can be shown quite simply that these are the Lie algebras of the classical linear groups.

Let us summarize some of their properties.

Algebra	Lie Group	Number of real parameters	Properties of Matrices in Lie Group	Property of Matrices in Algebra
A_n	SL(n+1,c) Special linear groups	$2(n+1)^2-2$	Complex matrices of order n+1 with det G=1	Complex (n+1) (n+1) matrice with TrX=0
B_n	SO(2n+1,c) Special orthogonal groups	(2n+1)(2n)	Complex orthogonal of order (2n+1), satisfy $O^TO = OO^T = 1$ det O = 1	Antisymmetric complex matrices of order 2n+1: $X^T = -X$
C_n	Sp(2n,c) Complex symplectic group	2n(2n+1)	Complex Symplectic Matrices of order 2n	Matrices of type $\begin{pmatrix} A_1 & A_2 \\ A_3 & -A_1^T \end{pmatrix}$ A_1-complex matrices of order n A_2, A_3-symmetr
D_n	SO(2n,c)	2n(2n-1)	Same as A_n	Same as A_n

Remarks: 1. The orthogonal groups leave the symmetric (complex) form

$$z_1^2 + z_2^2 + \ldots z_{2n(+1)}^2 \text{ invariant.}$$

2. The symplectic groups leave the antisymmetric (complex) bilinear form

$$(xy) = g_{ik} x_i y_k \qquad g_{ik} = -g_{ki}$$

invariant, i.e.

$$(xy) = x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3 + \ldots \quad y_{2n-1}y_{2n} - x_{2n}y_{2n-1} \; ,$$

A symplectic matrix satisfies

$$S^T J_{2n} S = J_{2n}$$

where

$$J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

and I_n is a unit matrix of order n.

The algebras G_2, F_4 and E_6, as well as the corresponding Lie Groups have been constructed explicitly in the literature. I do not know about E_7 and E_8, but their existence has been proved. The dimensions are:

Algebra	E_6	E_7	E_8	F_4	G_2
Dimension	78	133	248	52	14

From the isomorphisms of algebras we find that the following groups are locally isomorphic:

$$SL(2,c) \sim SO(3,c) \sim Sp(2,c)$$

$$SO(5,c) \sim Sp(4,c)$$

$$SL(4,c) \sim SO(6,c)$$

Further: $SO(2,c)$ is Abelian and thus not semisimple

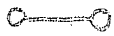
$$SO(4,c) \sim SO(3,c) \otimes SO(3,c)$$

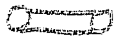
Actually: $SL(2,c)$ and $Sp(2,c)$ are even globally isomorphic

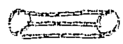
$SO(3,c) = SL(2,c)/D$ $\quad$ D = discrete 2 dimensional centre

Thus: 1) One algebra of rank 1: $\quad$ 0 $\quad$ $A_1 = B_1 = C_1$

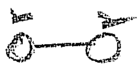
2) Three algebras of rank 2:

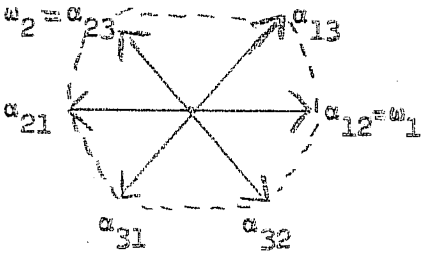
 A_2

 $B_2 = C_2$

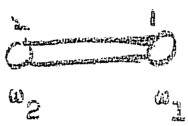
 G_2

The system Π of simple roots is represented completely by the Dynkin diagrams. Let us look at the system Σ of all roots for algebras of rank 2. Since all roots are linear combinations of the two simple ones, they can be represented by vectors in a plane (more generally in a Euclidean space of dimension r, equal to the rank of the algebra).

A_2:  : the lengths are the same, the angle between simple roots is 120°.

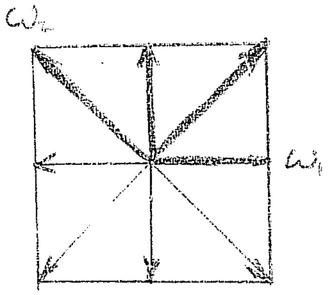


$\left.\begin{array}{l} \alpha_{12}, \alpha_{23} \ldots \text{simple} \\ \alpha_{13} \end{array}\right\}$ positive

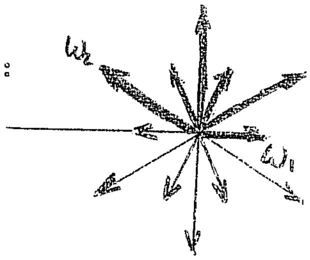
B_2: 

$\omega_2^2 = 2\omega_1^2$

Angle between simple roots = 135°



Emphasized lines correspond to positive roots.

G_2: 

$\omega_2^2 = 3\omega_1^2$

Angle between ω_1 and ω_3 is 150°

Problem: 1) Consider the algebra of SL(n,c) and find the Cartan-Weyl basis, check the "canonical" commutation rules.

2) Optionally: do the same for the other classical algebras.

Having completed the classification and description of all complex simple Lie algebras, let us look at the real simple Lie algebras.

The Real Lie Algebras

We repeat the whole procedure performed for complex algebras. Complications come from the fact that the field of real numbers is not algebraicall closed, so that the problem of finding eigenvalues and eigenfunctions of a matrix $\hat{x}$ is much more complicated (you can no longer, in general, reduce a matrix to the Jordan canonical form).

A more convenient approach is to start from complex Lie algebras, the structure of which we already know, and then to find all possible ways of splitting off real forms. This was done to a large degree by Cartan, the final and complete version is due to Gantmakher (contained in two Russian articles, to my knowledge not translated:

F. Gantmakher, Mat. Sbornik 5, 101-146 (1939) and

5, 217-249 (1939))

The results can be extracted from e.g. S. Helgason: Differential Geometry and Symmetric Spaces, A.P. New York 1962.

Let us here list the results, then discuss them.

Real Forms of the Simple Lie Algebras

Cartan Algebra	Lie Group	Number of Real Parameters	Properties of Matrices of Lie Group	Properties of Matrices in Lie Algebras	Remarks
A_n	$SU(n+1)$ Special Unitary Groups	$(n+1)^2-1$	Unitary, Unimodular $U^+U=UU^+=1$, det $U=1$	Real, antihermitean Trace $X=0$	Compact. Leaves $\lvert z_1\rvert^2+\dots+\lvert z_{n+1}\rvert^2$ invariant
	$SU(p,q)$ $pq\neq 0$ $p+q=n+1$ Special pseudo-unitary groups	$(n+1)^2-1$	$U^+I_{pq}U=I_{pq}$ det $U=1$	$\begin{pmatrix} A_1 & A_2 \\ A_2^+ & A_3 \end{pmatrix}$ A_1, A_3 anti-hermitean of order p and q. $\mathrm{Tr}A_1+\mathrm{Tr}A_3=0$ A_2 arbitrary	Leaves $\lvert z_1\rvert^2+\dots+\lvert z_p\rvert^2-\lvert z_{p+1}\rvert^2-\dots-\lvert z_{p+q}\rvert^2$ invariant
	$SL(n+1,R)$ Special Real Groups	$(n+1)^2-1$	Real, unimodular	Real, Trace $X=0$ order $(n+1)$	
	$SU^*(2k)$, $2k=n+1$ Unimodular Quaternion Group $SL(k,Q)$	$(2k)^2-1$	Matrices of $SL(2k,C)$ which commute with the transformatio $(x_1,\dots x_k,x_{k+1}\dots x_{2k})$ $(x^*_{k+1},\dots,x^*_{2k},-x^*_1,\dots x^*_k)$	$\begin{pmatrix} A_1 & A_2 \\ -A_2^* & A_1^* \end{pmatrix}$ A_1 and A_2 -complex of order k $\mathrm{Tr}A_1+\mathrm{Tr}A_1^*=0$	
B_n, D_n $B_n: k=2n+1$ $D_n: k=2n$	$R_k=SO(k)=SO(k,R)$ Special real orthogonal groups (Rotation groups)	$\frac{k(k-1)}{2}$	$O^TO=1$, O..real det $O=1$	Antisymmetric Real matrices of order k	Compact, Leaves $x_1^2+\dots+x_k^2$ invariant

Cartan Algebra	Lie Group	Number of Real Parameters	Properties of Matrices of Lie Group	Properties of Matrices in Lie Algebras	Remarks
	$SO(p,q)^{pq\neq 0}_{k=p+q}$ special real pseudo-orthogonal groups	$\frac{k(k-1)}{2}$	$O^T I_{pq} O = I_{pq}$ $\det O = 1$	$\begin{pmatrix} A_1 & A_2 \\ A_2^T & A_3 \end{pmatrix}$ A_1, A_2, A_3 - real A_1, A_3 - anti-symmetric of order p and q	Leaves $x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2$ invariant
	$SO^*(2n)$ Quaternion Orthogonal group	$\frac{2n(2n-1)}{2}$	Orthogonal, $SO^*(2k) \subset SO(2k,C)$ Leave the antihermitean form $-x_1 x_{k+1}^* + x_{k+1} x_1^* \ldots - x_k x_{2k}^* + x_{2k} x_k^*$ invariant	$\begin{pmatrix} A_1 & A_2 \\ -A_2^* & A_1^* \end{pmatrix}$ A_1, A_2 .. complex of order k A_1 .. antisymmetric A_2 .. hermitean	Leaves Two complex forms invariant
C_n	$Sp(2n) = Sp_u(2n)$ Unitary symplectic groups	$\frac{2n(2n+1)}{2}$	$Sp_u(2n) = Sp(2n,C) \cap U(2n)$ $S^+ K_{n,o} S = K_{n,o}$	$\begin{pmatrix} A_1 & A_2 \\ -A_2^* & A_1 \end{pmatrix}$ A_1, A_2 complex of order k A_1 .. antisymmetric A_2 .. symmetric	Compact
	$Sp(2n,R)$ - Real symplectic groups	$\frac{2n(2n+1)}{2}$	Real symplectic matrices or order 2n	$\begin{pmatrix} A_1 & A_2 \\ A_3 & -A_1^T \end{pmatrix}$ A_1, A_2, A_3 real order n A_2, A_3 - symmetric	
	$Sp(p,q), p+q=2n$ $pq \neq 0$ Pseudounitary sympletic groups	$\frac{2n(2n+1)}{2}$	$Sp(p,q) \subset Sp(p+q,C)$ and satisfy $S^+ K_{pq} S = K_{pq}$	$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{12}^+ & A_{22} & A_{14}^T & A_{24} \\ -A_{13}^* & A_{14}^* & A_{11}^* & -A_{12}^* \\ A_{14}^+ & -A_{24}^* & -A_{12}^T & A_{22}^* \end{pmatrix}$ continued	A_{ij} - complex matrices A_{11}, A_{13} .. order p A_{11}, A_{22} .. antihermitean A_{13}, A_{24} .. symmetric

We shall not go into the details of the structure of the exceptional real Lie algebras. Let us just note that one and only one real compact form exists for each complex simple Lie algebra, including the exceptional ones.

Above we have:

$$I_{p,q} = \begin{pmatrix} +I_p & 0 \\ 0 & -I_q \end{pmatrix}$$

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

$$K_{pq} = \begin{pmatrix} I_p & 0 & 0 & 0 \\ 0 & -I_q & 0 & 0 \\ 0 & 0 & I_p & 0 \\ 0 & 0 & 0 & -I_q \end{pmatrix}$$

where I_p is a unit matrix of order p.

Among the real simple Lie algebras of low order we again have certain isomorphisms, leading to local isomorphisms for the groups. Let us just list them:

1) $SU(2) \sim SO(3) \sim Sp_u(2)$
$SL(2,R) \sim SU(1,1) \sim SO(2,1) \sim Sp(2,R)$

2) $SO(5) \sim Sp_u(4)$

$SO(3,2) \sim Sp(2,R)$

$SO(4,1) \sim Sp(2,2)$

3) $SO(4) \sim SO(3) \otimes SO(3) \sim SU(2) \otimes SU(2)$

$SO(3,1) \sim SL(2,C)$

$SO(2,2) \sim SL(2,R) \times SL(2,R)$
$SO^*(4) \sim SU(2) \times SL(2,R)$

4) $SU(4) \sim SO(6)$

$SL(4,R) \sim SO(3,3)$

$SU^*(4) \sim SO(5,1)$

$SU(2,2) \sim SO(4,2)$ (the conformal group of space-time)

$SU(3,1) \sim SO^*(6)$

5) $SO^*(8) \sim SO(6,2)$

Lecture 9

In the previous lectures we have completed the classification of all complex semisimple Lie algebras and written down the commutation relations for these algebras in the Cartan-Weyl basis. In particular we have shown that the total information about complex semisimple Lie algebras can be expressed in terms of the properties of simple roots.

We have also considered real semisimple Lie algebras and have shown how each complex algebra splits into several different (non-isomorphic) real ones. So far we have only presented the results, let us now discuss the real Lie algebras.

Connection between Real and Complex Lie Algebras and Lie Groups

Formulation of the problem: Let L be a given semisimple complex Lie algebra. We wish to find all possible bases of L in which the structure constants are real numbers.

Obviously, if we take such a basis $\{e_i\}$ and consider its real envelope

$$\sum_{i=1}^{n} c_i e_i$$

where c_i are real numbers, we obtain a real semisimple Lie algebra.

One basis satisfying the above criterion has already been found - namely the Cartan-Weyl basis with the simple roots taken as a basis for the Cartan subalgebra H.

Indeed, we have

$$[\omega_i, e_\alpha] = (\omega_i, \alpha) e_\alpha$$

$$[e_\alpha, e_{-\alpha}] = \alpha = \Sigma n_i \omega_i$$

$$[e_\alpha, e_\beta] = N_{\alpha\beta} e_{\alpha+\beta}$$

and all the structure constants are real.

The algebra L_r

$$x = \sum_{i=1}^{r} \xi^i \omega_i + \sum_{\alpha \gtrless 0} x^\alpha e_\alpha$$

with ξ^i and x^α real is a real form of algebra L, namely in this fashion we obtain the restrictions

$$SL(n,C) \longrightarrow SL(n,R)$$

$$SO(n,C) \longrightarrow SO(n,R)$$

$$Sp(n,C) \longrightarrow Sp(n,R)$$

Thus, one solution of the above problem always exists. Another solution that is of great importance is:

The Compact Form of the Algebra L

The scalar length of a vector in the canonical basis is

$$(x,x) = f_{ik}\xi^i\xi^k + \delta_{\alpha\beta}x^\alpha x^{-\beta}$$

where $f_{ik} = (\omega_i, \omega_k)$

We already know that the metric tensor f_{ik} in the algebra H is positive definite.

Introduce

$$C_\alpha = \frac{1}{\sqrt{2}}(e_\alpha + e_{-\alpha})$$

$$S_\alpha = \frac{1}{i\sqrt{2}}(e_\alpha - e_{-\alpha}) .$$

Obvious $(c_\alpha, s_\alpha) = 0$ so that

$$(x,x) = f_{ik}\xi^i\xi^k + \sum_{\alpha>0} \{ (a^\alpha)^2 + (b^\alpha)^2 \}$$

where $\quad x = \sum_i \xi^i \omega_i + \sum_{\alpha>0} (a^\alpha c_\alpha + b^\alpha s_\alpha).$

Thus, the "metric tensor" of (x,x) in the whole algebra L is positive definite. It is easy to check that

$$\{ i\omega_k, ic_\alpha, i\, s_\alpha \}$$

form the basis of a subalgebra with real structure constants. Let us call this algebra L_u and we have

$$(x,x) < 0.$$

The group, corresponding to this algebra, will by a previous theorem, be compact.

Thus we obtain:

Theorem: Any semisimple complex group G has a compact real form.

Namely:

$$SL(n,C) \rightarrow SU(n)$$
$$SO(n,C) \rightarrow SO(n,R)$$
$$Sp(n,C) \rightarrow Sp_u(n)$$

Example: $L = SL(2,c)$

$$e_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad e_o = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad e_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$c = \sigma_1 = e_+ + e_- = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_3 = 2e_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$s = \sigma_2 = \frac{e_+ - e_-}{i} = \sigma_4\begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}$$

$x = i(\xi\sigma_3 + a\sigma_1 + b\sigma_2)$ with ξ,a,b real is an element of $L_u = SU(2)$.

Quite similarly: $SL(n,C) \longrightarrow SU(n)$

The picture we now have is the following. Given a real semisimple Lie group G_R we can find its complex extension G_C and then restrict G_C, to a real compact group U. This is called Weyl's unitary trick:

$$G_R \to G_C \to U.$$

<u>Theorem</u>: The subgroup U is determined uniquely (upto automorphisms) as the maximal compact subgroup of G_C.

Let us show how to find all real subalgebras V of L. If V is a real form of L then:

$$L = V + iV$$

i.e. $z \epsilon L \Rightarrow z = x+iy \quad x,y \epsilon V.$

Introduce the operation of "<u>conjugation</u>":

$$z \to \sigma(z) = x-iy$$

The real algebra V is invariant under conjugation.

$$\sigma(z) = z \Leftrightarrow z\epsilon V.$$

Example: Take X as the algebra of $SL(n,C)$:

1) $\sigma_1(z) = z^*$ (complex conjugation)
 $\Rightarrow V =$ algebra of $SL(n,R)$
2) $\sigma_2(z) = -z^+$ (z^+ is the hermitean conjugate of z)
 $\Rightarrow V = L_u =$ algebra of $SU(n)$.

In general the mapping $z \to \sigma(z)$ is an <u>involution</u>, i.e.:

$$\sigma(z_1+z_2) = \sigma(z_1) + \sigma(z_2)$$

$$\sigma([z_1,z_2]) = [\sigma(z_1), \sigma(z_2)]$$

$$\sigma(\lambda z) = \lambda^* \sigma(z)$$

$$\sigma^2(z) = z$$

Thus: the problem of finding all real forms of L reduces to that of finding all (in some sense) different involutions.

Construction of All Real Forms of a Complex Algebra L, Starting from the Compact one L_u.

$L_u \subset L$ is determined up to a choice of a basis in L_u (up the inner automorphisms).

<u>Theorem:</u> Let σ be an involution of L, leaving the compact form L_u invariant;

$$L_u = K + N$$

where K and N are eigenspaces of σ, corresponding to eigenvalues ± 1 (we have $\sigma^2 = 1$). Then $V = K + iN$ is a real form of L and all real forms of L can be obtained in this manner.

Thus: all we have to do is find all involutions leaving L_u invariant.

<u>Theorem:</u> Let σ be an involution conserving the form L_u. Then there exists a Cartan subalgebra $H \subset L$, invariant with respect to σ.

(We drop the proof).

<u>Example:</u> $SL(n,c) \rightarrow$ maximal compact subgroup $SU(n)$. Cartan subalgebra: H...diagonal matrices of order n. Put:

$$S = \left(\begin{array}{ccc|ccc} 1 & & & & & \\ & 1 & & & & \\ & & \ddots \; 1 & & & \\ \hline & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{array}\right)$$

where the separation into $+1$ and -1 terms is arbitrary. Then

$$\sigma(z) = -S^{-1}z^{+}S$$

is an involution preserving $SU(n)$. The invariant subspace is $v \varepsilon V$

$$S^{-1}v^{+}S = -v$$

The group corresponding to V is $SU(p,q)$.

<u>Cartan Decomposition</u>

$$L = X + iX$$

where X is a subalgebra, invariant under an involution. (If X is a subalgebra, iX in general is not one). If $X = L_u$ then we can "integrate" the Cartan decomposition to obtain

$$G = U.R$$

where U is the maximal compact subgroup and R is the supplementary subspace.

Example: $GL(n) = U(n)R(n)$ polar decomposition of matrix $g \varepsilon GL(n)$

$R(n)$ - positive definite hermitean matrices of order n.

For SL(2,C):

$$\begin{pmatrix} ab \\ cd \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha^* \end{pmatrix} \begin{pmatrix} r & s \\ s^* & r \end{pmatrix}$$

$$ad-bc = 1 \qquad |\alpha|^2 + |\beta|^2 = 1 \qquad r = \text{real}$$

Analogy: $z = e^{i\theta}|z|$ ... polar decomposition of complex numbers.

The Gauss Decomposition of a Complex Group G.

We have for a semisimple Lie algebra

$$L = E_- + H + E_+$$

For the corresponding group we have

$$G = \overline{Z_- DZ_+}$$

Here Z_-, D and Z_+ are Lie groups corresponding to the Lie algebras E_-, H and E_+. The bar means topological closure. In other words not every $g \varepsilon G$ can be written as

$$g = z_- \delta \, z_+$$

but such g do form a dense set in G.

Theorem: Every complex semisimple Lie Group allows a Gauss decomposition.

Remark: If G is a matrix group, then Z_- and Z_+ can be considered to be lower and upper triangular matrices with ones on the main diagonal. The matrices D are diagonal.

The Gauss Decomposition for a Real Lie Group G.

The algebra of G can be written as

$$L_R = K + iN$$

where K and N are eigensubspaces of the involution σ , determining L_R in the first place.

Theorem: The real semisimple Lie Group G_R with the algebra L_R can be decomposed as

$$G_R = \overline{Y_- \ F \ Y_+}$$

where F is locally isomorphic to a direct product of an abelian group D_o and a compact one U_K with the algebra K. Further $Y_{\pm}$ are subgroups of $Z_{\pm}$ in the complex group G.

The Centre of a Simple Lie Group

The centre of a simple Lie algebra is (0), thus the centre of a simple group is discrete.

Theorem: The centre of a complex simple Lie group and of a compact real simple Lie group are finite.

(No proof given).

Corollary: The universal covering groups of the groups mentioned above consist of finite numbers of individual sheets.

Indeed, if L is a simple complex Lie algebra and C(L) is the centre of its universal covering group then it can be shown that the following table summarizes the relation between L and C(L):

L	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
C(L)	Z_{n+1}	Z_4	Z_2	$Z_1 x Z_2$	Z_3	Z_2	Z_1	Z_1	Z_1

where Z_n is the discrete group of divisors of unity:

$Z_n = \{e\}$ $\lambda^n = 1$ e = unit matrix
λ = complex number

Thus: $Z_2 = (+1, -1)$

$Z_4 = (1, i, -1, -i)$

$Z_n = \{ e^{i\frac{2k\pi}{n}} \}$ $o \leq k \leq n-1$

This table completes the classification of simple complex Lie algebras, in that it indicates the number of different, locally isomorphic groups, corresponding to a given algebra.

For the compact groups it can be shown that:

A_n: SU(n+1) is simply connected, so that it is its own universal covering group.

C_n: $S_{pu}(2n)$ is simply connected (and is it's own universal covering group).

B_n and D_n: The groups SO(2n,R) and SO(2n+1,R) are not simply connected.

Let us construct the universal covering group of SO(ℓ,R):

Consider the real vector $X = (x_1, \ldots x_\ell)$

and introduce some abstract elements of an algebra:

$$\gamma_1, \ldots, \gamma_\ell$$

Put: $x = x^i \gamma_i$

and demand

$$(x^i \gamma_i)^2 = x_1^2 + x_2^2 + \ldots + x_\ell^2$$

This implies the anticommutation relations:

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu}$$

All monomials

$$\gamma_{\mu_1} \ldots \gamma_{\mu_s} \qquad s \leq \ell$$

form an algebra of dimension 2^ℓ, called a Clifford algebra K (for $\ell = 4$ the γ_μ are the Dirac γ-matrices). Consider the ℓ-dimensional subspace $X \subset K$

$$x = x^i \gamma_i \qquad x \in X$$

Introduce left and right multiplication

$$k \to k_o k \qquad k \to k k_o$$

by k_o, for which $\det k_o \neq 0$ (k_o is a $2^\ell \times 2^\ell$ matrix). Let G_K be the group of all automorphisms

$$k \to k_o \, k \, k_o^{-1} \qquad \det k_o \neq 0$$

Let $G_x \subset G_k$ be the subgroup, leaving X invariant:

$$k_o \, x \, k_o^{-1} = x'$$

and let G_x^0 be the sheet of G_x, connected to the identity. G_x leaves the length $(x_i\gamma_i)^2$ invariant and it is easy to see that G_x is locally isomorphic to $SO(\ell)$ (it has the same number of parameters, leaves the same quadratic form invariant).

Definition: $G_x = \mathrm{Spin}(\ell)$ is called the spinor group.

Theorem: The group $G_x^0 = \mathrm{spin}\ (\ell)$ is simply connected. It is the universal covering group of $SO(\ell)$ and we have

$SO(2n+1) = \mathrm{spin}(2n+1)/Z_2$ and $SO(2n) = \mathrm{Spin}(2n)/C$ where $C = Z_4$ for n = odd and $C = Z_2 \times Z_2$ for n = even.

The importance of knowing whether a group is simply connected is due to the relation between simple connectivity and representation theory. Indeed, if we have a group G that is not simply connected, we can consider its universal covering group G and the single-valued representations of G will provide multivalued representations of G (e.g. the half integer spin representations of SO(3)). A simply connected group has only single valued representations.

Remark: The properties of the real noncompact Lie groups with respect to simple connectivity are much more complicated. E.g. SU(1,1) is a covering group of O(2,1), not however a universal covering group. It can be shown that the universal covering group has infinitely many sheets and thus an infinite dimensional discrete centre. We can thus consider not only single-valued and double-valued representations of O(2,1), but "arbitrary-valued" ones.

This completes our survey of some basical properties of Lie Groups and Lie Algebras and we go over to the second part of the course, namely to a consideration of the theory of representations.

Group Representation Theory

Definition: A representation of a group G in a linear space E is a mapping

$$g \longrightarrow T_g$$

of the group G into a group of linear transformations of the space E, such that

$$T_{g_1 g_2} = T_{g_1} T_{g_2}$$
$$T_e = I$$

where I is the identity operator.

If G is a topological group then we demand that T_g depends continuously on g.

Definition: Two representations T_g and S_g are equivalent if there exists a mapping A from the space of one to the space of the other such that:

$$S_g = A\, T_g\, A^{-1}$$

Definition: A subspace $E_o \subset E$ is called invariant with respect to the representation T_g if

$$T_g\, E_o \subset E_o$$

for all $g \in G$.

An invariant subspace is nontrivial if $E_o \neq \{0\}, E_o \neq E$.

Definition: A representation T_g is called irreducible if no nontrivial invariant subspaces in E exist.

If a nontrivial invariant subspace E_o exists, then T_g is <u>reducible</u>.

A representation is called <u>completely reducible</u> if every invariant subspace E_1 has a complementary invariant subspace E_2 such that

$$E = E_1 + E_2$$

i.e. the space E can be decomposed into a direct sum of invariant subspaces.

The space of a completely reducible representation can be decomposed into irreducible components.

Lecture 10

Some Applications of Group Representation Theory

Before we continue with our exposition of some general features of group representation theory, let us just give some examples of the applications of group representations in quantum physics.

1. Degeneracy of energy levels in non-relativistic quantum mechanics.

Consider the Schrodinger equation

$$H \psi_E(x) = E\psi_E(x) \tag{1}$$

and assume that the Hamiltonian H is invariant with respect to a certain group G. This means that for every element $g \in G$ there exists an operator T_g acting in the space of wave functions ψ

$$T_g \psi_E(x) = \psi(x) \tag{2}$$

and satisfying

$$T_g^{-1} H T_g = H \tag{3}$$

Obviously, if $\psi(x)$ satisfies (1), then so does $T_g\psi(x)$. Indeed:

$$H\{T_g\psi(x)\} = T_g T_g^{-1} H T_g \psi(x) = T_g H\psi(x) = E\{T_g\psi(x)\}$$

If

$$T_g\psi(x) \neq \mathbb{C}\, \psi(x)$$

for all g, then the energy level E is degenerate and we denote the eigenfunction $\psi_{Ek}(x)$. The operators T_g form a representation of the invariance group G. If the operators T_g transform all functions $\psi_{Ek}(x)$ with E fixed amongst each other, leaving no subspace of functions ψ_{Ek} invariant, then the representation is irreducible. The representation theory of G then provides us with a lot of information, e.g. it tells us what possible degrees of degeneracy can occur (namely it must coincide with possible dimensions of representations of the group G), it provides us with a means of classifying and labeling different functions, corresponding to the same energy, etc.

If the functions $\psi_{Ek}(x)$ correspond to a reducible representation of G, then the group G does not describe the degeneracy completely and either there exists a larger invariance group $\tilde{G} \supset G$ for which the functions ψ_{Ek} do transform irreducibly, or we say that the degeneracy is accidental.

Examples: Put

$$H = -\frac{1}{2}\Delta + V(\vec{r}) \qquad (4)$$

If the potential is spherically symmetric $V = V(r)$, then we always have a "geometric" invariance group $G = SO(3)$, leading to a degeneracy with respect to the magnetic quantum number m. In two special cases we have a higher degeneracy, than the one described by $SO(3)$, namely for the Coulomb potential $V(r) = 1/r$ when the group $\tilde{G} = SO(4)$, and for the isotropic harmonic oscillator $V(r) = \alpha r^2$, when the group $\tilde{G} = SU(3)$.

2. Selection Rules for Transitions Between Energy Levels

We are interested in matrix elements of the type

$$T^{\mu\nu}_{ij} = \int \overline{\psi_{\mu i}} T \psi_{\nu j} d\tau \qquad (5)$$

where μ,ν enumerate representations of the invariance group G and i,j enumerate basis functions in each representation. If we know something about the transformation properties of the transition operator T, we can use group representation theory to investigate (5), in particular to find when do we have $T^{\mu\nu}_{ij} = 0$, etc.

3. Consequences of Relativistic Invariance (or invariance of a theory with respect to any group).

Consider the quantum mechanics of a free particle. Experimental quantities are the moduli of scalar products of wave functions

$$|(\psi_f,\psi_i)^2|$$

The requirement of special relativity is that such experimental quantities (transition probabilities, etc.) should be invariant under Lorentz transformations. Thus, a Lorentz transformation

$$x' = \Lambda x+a$$

corresponds to

$$\psi(x') = U(\Lambda,a)\psi(x)$$

and $\psi(x)$ must then transform under a representation of the inhomogeneous Lorentz group (Poincare' group).

By definition a physical system is elementary, if it transforms according to an irreducible representation of the group. Thus: a classification of irreducible representations of the Poincare' group is a classification of all possible elementary physical systems.

4. Classification of Particle States with Respect to the Representations of an Internal Symmetry Group, (e.g. SU(2), SU(3), etc.).

5. Partial Wave Analysis of Scattering Amplitudes and its Generalizations.

Consider a function $f(x)$ on some homogeneous manifold X on which a group G acts as

$$T_g f(x) = f(xg).$$

It is often useful to find components of $f(x)$ with definite transformation properties with respect to G, e.g. to expand $f(x)$ in terms of the basis functions of irreducible representations of G.

Thus: we have a lot of physical motivation for performing a careful study of group representation theory.

The Basical Problems of Group Representation Theory

For most physical applications we need a quite detailed knowledge about the Lie group of interest itself and about the group representations. Basically, we have to know the following:

A. Knowledge about the Lie Group

1. Definition of the group, its structure (semisimple, solvable, etc.), its isomorphisms, local isomorphisms, covering groups, universal covering group, it's centre, its connectivity properties, etc.

2. The Lie algebra of the group, its universal enveloping algebra, the invariants of the algebra (the Casimir or Laplace operators).

Remarks: a) The universal envelloping algebra K of a Lie algebra $L = \{e_i, \ldots e_n\}$ is obtained as the ring of all polynomials in e_i. Two polynomials are considered equal to each other if they can be obtained from one another by a finite number of commutations $[e_i, e_k] = c^j_{ik}\, e_j$. [A ring is a linear space of elements a_i, which is an abelian group with respect to addition $a_i + a_k$ and in which we have a distributive multiplication $a_i(a_k a_\ell) = (a_i a_k)a_\ell$. The ring is a field if it has a multiplicative identity and an inverse for every $a_i \neq 0$).

b) The invariants of the algebra, or the Casimir operators, are the operators of the centre of the universal envelloping algebra, i.e. polynomials in e_i, commuting with all e_i (and thus with all polynomials in e_i).

3. A complete study of the subgroup structure of the given group, namely a classification of all continuous subgroups into equivalence classes. The Lie algebras of the subgroups and their invariants, if such exist.

4. A systematic study of different possible parametrizations of the group, i.e. all possible ways of representing a general group element $g \in G$ as a product of elements of subgroups of G and eventually as a product of elements of one parameter subgroups.

5. A list of all homogeneous spaces X on which the group acts transitively. A systematic study of all (in some sense) types of coordinates in the space X, some of which, but not all are related to various chains of subgroups of G.

<u>Remark</u>: A linear space X is a homogeneous manifold with respect to the group G, if G acts transitively on X, i.e. if for any $x,y \in X$ there exists a $g \in G$ such that $y = gx$.

6. The left and right invariant measure on the group (the Haar measures) in a general form and in different forms, corresponding to each parametrization. The invariant measures on each homogeneous manifold.

<u>Remark:</u> Haar has shown, that for an arbitrary Lie group (and even for a larger class of topological groups) one can introduce invariant integration over the group, i.e. write

$$\int f(g) d\mu_L(g) = \int f(g_o g) d\mu_L(g) \qquad (6)$$

and

$$\int f(g)\, d\mu_R(g) = f(gg_o) d\mu_R(g) \qquad (7)$$

where $\mu_L(g) = \mu_L(g_o g)$ and $\mu_R(g) = \mu_R(gg_o)$ are the left and right invariant measures. The measures are determined uniquely, up to a constant factor, and for a large class of Lie groups, called "unimodular" groups, the left and right invariant measures coincide.

Similarly, on a homogeneous manifold we have a uniquely determined (up to a constant factor) invariant measure

$$\int f(x)d\mu(x) = \int f(gx)d\mu(x) \qquad (8)$$

B. Knowledge about the group representations

1. A classification and explicit construction of all unitary irreducible representations of the group, all finite dimensional representations and usually also certain classes of non-unitary infinite dimensional representations (for non-compact groups).

2. A consideration of various specific realizations of the representation spaces. A systematic approach to the problem of classifying and finding all possible different bases for the representations. This is directly related to the problem of finding all nonequivalent complete sets of commuting operators in the enveloping algebra of the Lie algebra and is in part related to the classification of all chains of subgroups of the given group.

3. An explicit construction of the different complete sets of basis functions for each representation.

4. An explicit construction of the infinitesimal operators (as differential operators) and of their matrix elements in each different basis for all representations.

5. An explicit construction of the matrix elements of the finite transformation operators in the different bases.

6. The construction of the operators realizing the transformations from one type of basis to another (the overlap functions).

7. The reduction of useful reducible representations to irreducible ones (e.g. the regular representation, the quasi-regular one). An investigation of representations that are not irreducible, but not completely reducible.

8. The reduction of the representations of the group to representations of each of its subgroups.

9. The Clebsch-Gordan series, telling us which irreducible representations of the group are contained in the direct product of two irreducible representations and with which multiplicity

10. The Clebsch-Gordan coefficients of the group, connecting the basis functions of the irreducible representations contained in the direct product of two representations, with the products of basis functions of these two representations. These coefficients should be obtained for each of the bases under consideration.

11. Formulae generalizing classical Fourier analysis to non-Abelian and non-compact groups, i.e. formulae for the expansion of functions defined on the group (or on a homogeneous space) and square-integrable with respect to the corresponding invariant measure, in terms of the matrix elements of finite transformation operators (or in terms of the basis functions) of irreducible unitary representations of the group. These expansions would be various generalizations of the Plancherel formula, they depend crucially not only on the group under consideration, but also on the chosen group representation basis.

12. Generalizations of the above expansions to wider classes of functions and thus to non-unitary representations. Analytic continuation of group representations.

This is by no means a complete list of the mathematical problems, important for physical applications.

The program, as listed above, has not really been fulfilled completely for any non-abelian group, not even for SO(3). We shall treat some of the problems in general, first for compact, then for noncompact groups. In the third part of this course we shall go over to the Poincare' group and it's subgroups and little groups and treat their representation theory in some detail.

We have already defined a representation and the concepts of equivalence, of irreducibility and of complete reducibility. Let us introduce some further concepts.

Definition:

Representation T_g and $\hat{T}_g$ are contragradient to each other, if a nondegenerate bilinear form

$$(x,\hat{x}) \qquad x\varepsilon E \quad \hat{x}\varepsilon\hat{E} \tag{9}$$

exists, which is invariant in the following sense

$$(T_g x, \hat{T}_g\hat{x}) = (x,\hat{x}) \tag{10}$$

Obviously we must have

$$\dim E = \dim\hat{E}$$

We can consider E and $\hat{E}$ as one space, we can choose a basis, in which $(x,\hat{x})$ is diagonal. Condition (10) implies

$$\hat{T}_g = (T_g{}^T)^{-1} \tag{11}$$

(superscript T means transposed).

Definition: The Representation T_g is called a tensor product of two representations

$$T_g = A_g \times B_g \tag{12}$$

if it acts in the tensor product of two spaces $E(A) \times E(B)$;

$$T_g e_i \varepsilon_\alpha = (A_g e_i) . (B_g \varepsilon_\alpha)$$

Remark: We shall be considering representations both in finite-dimensional and infinite-dimensional spaces. Strictly speaking, we should be more careful with some of the above definitions. In particular, for infinite dimensional representations (of non-compact groups) we should specify the types of spaces in which the definitions make sense, e.g. Banach spaces. (A Banach space is a complete normed space; a normed space is a linear space R, in which every element $x \varepsilon R$ has a norm $|x|$, satisfying: (i) $|x| \geq 0$, $|x| = 0 \leftrightarrow x = 0$. (ii) $|\alpha x| = |\alpha| . |x|$ for $x \varepsilon R$, α = complex number, (iii) $|x+y| \leq |x| + |y|$ for $x, y \varepsilon R$. We shall mainly be working in a Hilbert space (a Hilbert space is a special case of a Banach space, in which we have a scalar product (x,y), satisfying the usual conditions (1) $(x,x) > 0$ $(x,x) = 0 \leftrightarrow x = 0$, (2) $(y,x) = \overline{(x,y)}$ (3) $(\alpha x, y) = \alpha(x,y)$ (4) $(x_1 + x_2, y) = (x_1, y) + (x_2, y)$ and the norm $|x| = \sqrt{(x,x)}$). We shall not go into the necessary refinements here.

Tensors

Let G be a linear group (a group of matrices in an n-dimensional space $E; n < \infty$). Introduce a basis $\{e_i\}$ in E. Then the action of $g \varepsilon G$ on e_i can be considered as a transformation to a new basis:

$$e'_i = g_i^k e_k \tag{13}$$

A vector $x \varepsilon E$ can be written as

$$x = x^i e_i = x'^i e'_i$$

Thus, the trace $x^i e_i$ is invariant so that

$$x'^i = h^i_k x^k$$

where

$$h = \hat{g} = (g^T)^{-1} \tag{14}$$

We shall call transformations g <u>covariant</u> (e.g. transformations of basis vectors), transformations $\hat{g}$ <u>contravariant</u> (e.g. transformations of coordinates).

Let us introduce multiplication in E, defining:

$$x.y = x^i y^j e_i e_j \tag{15}$$

Thus we obtain the quantities $e_i e_j$, transforming as

$$e'_i e'_j = g^k_i g^\ell_j e_k e_\ell \qquad \ldots g \otimes g \tag{16}$$

whereas the coordinates transform as:

$$x'^i x'^j = h^i_k h^j_\ell x^k x^\ell \qquad \ldots \hat{g} \otimes \hat{g} \tag{17}$$

<u>Definition:</u>

We shall call any quantity t_{ij} transforming according to $g \otimes g$ a covariant tensor of rank two, a quantity t^{ij} transforming according to $\hat{g} \otimes \hat{g}$ a contravariant tensor of rank two. Similarly we introduce covariant and contravariant tensors of rank n and mixed tensors of arbitrary rank:

$$t^{i_1 \ldots i_p}_{j_1 \ldots j_q} \tag{18}$$

In this manner we obtain finite-dimensional representations of the group G.

$$g \to A_g \qquad g \to B_g \quad \text{and } g \to A_g \otimes B_g \tag{19}$$

where

$$\begin{aligned} A_g &= g \otimes g \otimes \cdots \otimes g && \text{p factors} \\ B_g &= \hat{g} \otimes \hat{g} \otimes \cdots \otimes \hat{g} && \text{q factors} \end{aligned} \tag{20}$$

<u>Remark</u>: It is sometimes convenient to consider certain <u>multilinear forms</u>, instead of tensors. Thus, we can replace a covariant tensor of rank p by a form

$$P(x,y,\ldots w) = t_{i_1 i_2 \ldots i_p} x^{i_1} y^{i_2} \ldots w^{i_p}$$

where the x^i etc transform contravariantly. The operators T_g

$$T_g P(x,y,\ldots,w) = P(g^{-1}x, g^{-1}y, \ldots g^{-1}w)$$

form a representation of G (we consider the g's as matrices the x's as vector-columns). Obviously the coefficients of

$$P' = T_g P$$

are given by the tensor

$$t_{j_1 \ldots j_p} = g^{i_1}_{j_1} g^{i_2}_{j_2} \ldots g^{i_p}_{j_p} t_{i_1 \ldots i_p}$$

so that this representation is equivalent to A_g.

Similarly, we have

$$T_g P(x,y,\ldots w) = P(xg,\ldots,wg)$$

where $x, y, \ldots$ are written as rows and transform covariantly. This representation is equivalent to B_g.

Thus: instead of tensors we can consider linear functions of vector variables $x, \ldots, w$.

Symmetries of the Tensors

Introduce an operator S the action of which is to permute some of the indices of a tensor:

$$S \; t_{i_1 \ldots i_n} = t_{j_1 \ldots j_n}$$

Since one matrix g acts on each tensor index separately a rearrangement of these matrices is irrelevant. Thus:

$$Sg = gS \qquad g \in G.$$

We can classify tensors according to their symmetry properties, which are invariant under the group G. If a tensor changes sign under the permutation of two indices it is antisymmetric with respect to these indices, if it stays invariant it is symmetric.

As we know the permutations of m elements themselves form a (discrete) group S_m - the symmetric group. For information on this group see e.g. Hammermesh.

The symmetry properties will be used to split the tensor representations into irreducible ones. We shall return to this in a future lecture. Let us now make some more remarks, necessary to prove some powerful general theorems on the representations of compact groups.

We already know that the group G itself is a <u>homogeneous manifold</u> with respect to left and right multiplication:

$$g_1, g_2 \varepsilon G \quad \text{there exist } g_o \varepsilon G \text{ and } \tilde{g}_o \varepsilon G \text{ such that}$$

$$g_1 = g_o g_2 \qquad g_1 = g_2 \tilde{g}_o$$

<u>Definition</u>: <u>Right regular representation</u> R_g:

$$R_{g_o} f(g) = f(gg_o)$$

<u>Left regular representation</u> L_g

$$L_{g_o} f(g) = f(g_o^{-1} g)$$

<u>Problem</u> Check that R_g and L_g are indeed representations of g.

Lecture 11

Reference to previous lecture:

A. O. Barut, R. Raczka: Classification of non-compact real simple Lie groups and groups containing the Lorentz group. Proc. Roy. Soc. 287A, 519-532 (1965) (Essentially they reproduce Gantmakher's articles of 1939).

We already know that a group G is a homogeneous manifold with respect to left and right multiplication. Let us put $X_o \equiv G$ and prove that:

Any homogeneous space X can be fitted in a standard manner into the space X_o, i.e. the group G is a universal homogeneous space.

a) Consider $x_o \in X$ and assume that the equation

$$x_o g = x \tag{1}$$

has only one solution $g \in G$ for every $x \in X$. Then we can simply identify X and G (we have a one-to-one correspondence $x \leftrightarrow g$, $x \in X$, $g \in G$.

b) Let $x_o \in X$, and assume that g in (1) is not unique . Consider a subgroup $H \subset G$ such that

$$x_o h = x_o \qquad h \in E \tag{2}$$

H is the stationary subgroup of G, corresponding to vector x_o. Let us establish the degree of non-uniqueness in (1). Assume

$$x = x_o g_1 = x_o g_2, \quad g_1 \neq g_2$$

Then $x_o = x_o g_2 g_1^{-1}$, i.e $g_2 g_1^{-1} \in H$.

Thus, for each $x \in X$ we can choose g_x such that

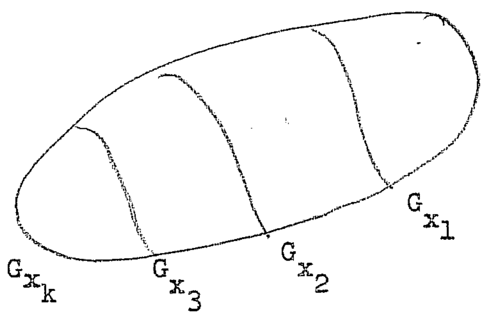
$$x = x_o g_x$$

Then any transformation

$$g = hg_x \quad h\varepsilon H$$

also transforms x_o into x. We thus split the group G into layers

$$G_x = Hg_x$$



Only the layer $G_{x_o} = H$ is a subgroup, since the others do not contain the identity e.

We can now write

$$g = hg_x \quad x\varepsilon X,\ h\varepsilon H,\ g\varepsilon G$$

and, formally

$$G = HX$$

(to determine an element in G we must determine a vector $x\varepsilon X$ and an element $h\varepsilon H$)

Instead of functions $f(x)$, $x\varepsilon X$ we can consider functions $f(g)$, $g\varepsilon G$ which do not depend on h. Consider:

$$g = hg_x$$

and perform a right translation

$$gg_o = h(g_x g_o) = hg_{xg_o}$$

(since $xg_o = x_o g_x g_o = g_{xg_o}$)

Thus, the transformation $g \longrightarrow gg_o$ induces the transformation $x \longrightarrow xg_o$. Thus, we can replace $f(x)$ by functions $f(g_o)$ constant on the classes G_x:

Thus:

$$Rg_o f(g) = f(gg_o) \text{ and } f(g) = f(hg)$$

<u>Example:</u> $G = O(3)$, X = two-dimensional sphere.

Put: $x = (x_1, x_2, x_3) \quad x^2 = 1$

North Pole: $x_o = (0,0,1)$

H = rotation about O_3: $g_3(\psi)$

In Euler angles: $g = g_3(\psi)\, g_1(\Theta)\, g_3(\phi) = h \circ g(\Theta, \phi)$

where $h = g_3(\psi)$, $g(\Theta,\phi) = g_1(\theta) g_3(\phi)$

The condition $f(g) = f(hg)$ obviously implies that $f(g)$ does not depend on ψ Thus: functions $f(g)$ in general are expanded in terms of the Wigner D-functions, however if they satisfy $f(g) = f(hg)$ then they get expanded in terms of $Y_{\ell m}(\Theta, \phi)$.

<u>Remarks on Matrix Elements of Finite Rotations</u>

Consider a group G and an irreducible finite-dimensional representation T_g. In a finite dimensional space we can always choose a basis

$$\{ e_i \} \qquad i = 1 \ldots n$$

and calculate the matrix elements of each operator T_g.

$$T_g = ||T_{ij}(g)||$$

Consider the space $L_G = \{f(g)\}$ of all continuous functions on the group. Obviously $C_T = \{T_{ij}(g)\} \subset L_G$

We have: $T_{ij}(gg_o) \varepsilon L_G$, $T_{ij}(g_og) \varepsilon L_G$

$$T_{ij}(gg_o) = T_{i\alpha}(g)T_{\alpha j}(g_o)$$

Thus: each row of T_g forms a subspace of L_G, invariant under right translations, each column a subspace, invariant under left translations.

Take: $e_j(g) = T_{1j}(g)$

We have

$$e_j(gg_o) = T_{1j}(gg_o) = T_{1\alpha}(g)\ T_{\alpha j}(g_o) = e_\alpha(g)T_{\alpha j}(g_o)$$

Thus: a) Elements of each row get transformed amongst each other only.

b) If T_g is irreducible then the $e_j(g)$ are linearly independent (or T_g would act in a space of lower dimension).

We obtain the following result;

Theorem: The right regular representation contains every finite dimensional representation of the group G with a multiplicity equal to the number of rows in $T_{ij}(g)$, i.e. equal to the dimension of the representation.

If we can also show the opposite, namely: all irreducible components of the regular representation are generated by a row $T_{ij}(g)$ in a chosen basis, then we can easily decompose the regular representation. We would simply have to consider all irreducible nonequivalent representations $E^{(\ell)}(g)$ of G with ℓ running through some set and find how the matrix elements $E^{\ell}_{ij}(g)$ are contained in the space of functions $f(g)$

The functions $E^{\ell}_{ij}(g)$ are called elementary harmonics of the group G, we shall show that they satisfy equations like

$$\Delta f(g) = \lambda f(g)$$

where Δ is a generalized Laplace operator and that functions $f(g)$ can be expanded in terms of elementary harmonics.

<u>Remark</u>: Essentially all the special functions used in mathematical physics are elementary harmonics of some group in some basis.

<u>Compact Lie Groups</u>

We have already given definitions of a compact Lie group and also topological and algebraical criteria of compactness.

For a linear Lie group (Matrix group), a simple way to find out whether the group is compact is to consider some parametrization of the group and see whether the range of these parameters is finite. A linear Lie group is compact iff this range - the "group volume" is finite.

<u>Example:</u> a) $SO(3)$: $g = g_{12}(\psi) g_{13}(\theta) g_{12}(\phi)$

The volume $\quad 0 \leq \psi < 2\pi$

$0 \leq \theta \leq \pi$

$0 \leq \phi < 2\pi$

is finite; the group is compact.

b) $SO(2,1)$: The invariant is $x_0^2 - x_1^2 - x_2^2$; the analogue of the Euler angle parametrization for $O(3)$ is: $g = g_{12}(\psi) g_{01}(\beta) g_{12}(\phi$.

The group volume is determined by the bounds:

$$0 \leq \psi < 2\pi \ , \quad 0 \leq \phi < 2\pi \qquad 0 \leq \beta < \infty$$

and is infinite: the group is noncompact.

We already know that the real orthogonal groups $O(n,R)$, the matrices of which are real and satisfy $O^T O = 1$, are compact. Similarly the groups $U(n)$ of complex unitary matrices satisfying $U^+ U = 1$ are compact.

<u>Theorem:</u> Any compact group G can be realized as a subgroup of $O(n)$ and as a subgroup of $U(m)$ if n and m are large enough. We shall not give a proof here.

<u>Problem:</u> Show that $O(n) \subset U(m)$ and $U(n) \subset U(m)$. For a given n find as small as possible m.

<u>Corollary:</u> A classification of all compact Lie groups is equivalent to a classification of all subgroups of $O(n)$.

<u>Classification of Compact Lie Groups</u>

Whenever we talk of a classification, we mean classification up to isomorphisms.

<u>Definition:</u> The <u>group</u> G is the direct product of two subgroups G_1 and G_2 if there exists a faithful representation $g \to T_g$ such that

$$T_g = \begin{pmatrix} Tg_1 & 0 \\ 0 & Tg_2 \end{pmatrix} \qquad g_1 \varepsilon G_1, \ g_2 \varepsilon G_2$$

for all $g \varepsilon G$.

Example:

$$U(n) \quad g = \begin{pmatrix} \det g & 0 \\ 0 & g_o \end{pmatrix} \quad \det g_o = 1$$

Definition: A group is indecomposable if it cannot be written as a direct product of subgroups.

All we have to do is give a list of all indecomposable groups. This we have already done. A group that is indecomposible has no invariant subgroups, i.e. it is simple. Thus, we obtain:

Theorem: Any compact connected indecomposable Lie group is locally isomorphic to one of the classical groups $SO(n,R)$, $SO(n)$, $Sp_u(n)$ or to one of the five exceptional compact Cartan groups E_6, E_7, E_8, F_4 or G_2. Thus: Any compact Lie group is a direct product of compact simple Lie groups and compact Abelian groups.

Remark: More generally a group (compact or noncompact) is a reductive group if its adjoint representation is completely reducible. Thus, every reductive group is the direct product of a semisimple Lie group and an Abelian one. There is a one-to-one correspondence between complex reductive Lie groups and compact Lie groups. Every complex reductive group has a compact real form, every compact group has a reductive complex extension.

Abelian Groups

A one-parameter compact Abelian group is always isomorphic to the group of rotation of a circle $\{e^{i\theta}\}$. The direct product of n such groups forms an n-dimensional torus. For $n = 2$:

A noncompact Abelian group e^x is isomorphic to the group of motions of a straight line. The direct product of n such groups forms an n-dimensional space R^n.

<u>Theorem</u>: Any connected Abelian Lie group G is the direct product of a torus and a Euclidean space

$$g = \begin{pmatrix} e^{i\phi_1} & & & & \\ & \ddots & & & \\ & & e^{i\phi_n} & & \\ & & & e^{x_1} & \\ & & & & \ddots \\ & & & & & e^{x_m} \end{pmatrix} \qquad \begin{matrix} 0 \leq \phi_i < 2\pi \\ -\infty < x_j < \infty \end{matrix}$$

Let us now consider some general features of the representation theory of compact groups.

<u>Schur's Lemma</u>

Let T_g and S_g be two irreducible non-equivalent finite-dimensional representations of a group G. Then:

1) If there exists a constant operator A such that

$$AT_g = S_gA$$

then $A = 0$.

2) If there exists an operator A commuting with all operators T_g:

$$AT_g = T_gA \qquad g\varepsilon G$$

then A is a multiple of the unit operator:

$$A = \lambda I.$$

<u>Proof:</u> 1. Consider the representation spaces $E(T)$ and $E(S)$ and let A provide a mapping

$$E(T) \xrightarrow{A} E(S)$$

Denote $E_o(T) \subset E(T)$ the set of all vectors mapped by A into zero. $E_o(T)$ is an invariant subspace since

$$Ax_o = 0 \Rightarrow A(T_g x_o) = S_g(Ax_o) = 0$$

The representation is irreducible $\Rightarrow E_o(T) = E(T)$ or (0). However: if $A \neq 0$ then $E_o(T) \neq E(T) \Rightarrow E_o(T) = (0)$. Now denote: $E_o(S) \subset E(S)$ the set of vectors obtained by the transformation A. $E_o(S)$ is an invariant subspace, S_g is irreducible $\Rightarrow E_o(S) = E(S)$.

Thus, the mapping A is a one-to-one mapping $\Rightarrow \dim E(T) = \dim E(S) \Rightarrow A^{-1}$ exists $\Rightarrow$

$$AT_gA^{-1} = S_g$$

This means T_g and S_g are equivalent, which is against the assumptions

$$A = 0 \qquad\qquad \text{Q.E.D.}$$

2. Put $T_g = S_g$ and consider the mapping A:

$$E \xrightarrow{A} E$$

Let λ be an eigenvalue of A and define $E_o \subset E$ as the eigensubspace such that

$$Ax_o = \lambda I x_o \qquad x_o \varepsilon E_o$$

E_o = invariant subspace since

$$A(T_g x_o) = T_g A x_o = T_g \lambda I x_o = \lambda I(T_g x_o)$$

$\Rightarrow$ $E_o = E$ so that

$$Ax = \lambda I x \qquad x \varepsilon E \qquad\qquad \text{Q.E.D.}$$

<u>Corollary 1:</u> Let T_g be a direct sum of two representations U and V:

$$T_g = \begin{pmatrix} U_g & 0 \\ 0 & V_g \end{pmatrix}$$

where U and V are irreducible and nonequivalent. Then: the only operators commuting with T_g are

$$A = \begin{pmatrix} \lambda I_1 & 0 \\ 0 & \mu I_2 \end{pmatrix}$$

where I_1 and I_2 are identity operators in the representation spaces of U_g and V_g.

The generalization to a sum of an arbitrary number of irreducible representations is obvious.

<u>Corollary 2:</u> The matrix elements of an irreducible representation

$$T_g = ||T_{ij}(g)||$$

form a system of linearly independent functions on G.

From Schur's lemma we see that either $\omega = 0$ or T_g is equivalent to $\hat{S}_g$. If $T_g \sim \hat{S}_g$ then

$$\omega = \lambda I \qquad \text{Q.E.D.}$$

<u>Remark</u>: Let T_g and S_g be contragradient: $S_g = \hat{T}_g = (T_g^T)^{-1}$, then their tensor product acts in the space of square matrices p:

$$T_g \times \hat{T}_g p \quad = T_g p \, T_g^{-1}$$

This leaves

$$\mathrm{Tr}p = p_{11} + \ldots + p_{nn} \text{ invariant.}$$

If p is the tensor product of two vectors x and y

$$p_{ik} = x_i y_k$$

then the invariant is

$$\mathrm{Tr}p = (x,y) = x_1 y_1 + \ldots x_n y_n,$$

i.e. that bilinear form which connects the two contragradient representations.

<u>Remark</u>: It is far from simple to generalize Schur's lemma from finite-dimensional spaces to infinite-dimensional ones, but it is possible and important. We shall only return to this in special cases.

Lecture 12

One of the central problems of group representation theory is a generalization of Fourier analysis, namely the expansion of continuous functions $f(x)$ where $x \in X$ and X is some vector space in terms of some system of "elementary" functions. Let us here state without proof an important theorem.

<u>The Stone-Weierstrass Theorem</u>: Let X be a compact space and let $\{F\}$ be a system of continuous functions of $x \varepsilon X$ satisfying the conditions:

1) $\{F\}$ is an algebra, closed under addition, multiplication and multiplication by a number
2) $\{F\}$ contains an identity
3) $\{F\}$ is symmetric under complex conjugation, i.e. if it contains $f(x)$ it also contains $f^*(x)$.
4) $\{F\}$ separates the points in X, i.e. if $x_1 \varepsilon X$, $x_2 \varepsilon X$, $x_1 \neq x_2$, then there exists an $f(x) \varepsilon F$ such that $f(x_1) \neq f(x_2)$.

Then the closure of $\{F\}$ coincides with all continuous functions on X, i.e any continuous function $f(x)$ can be approximated with arbitrary accuracy by functions contained in $\{F\}$.

<u>Example:</u> $X = [0,1]$, $\{F\} = \{1, x, x^2, \ldots, x^n, \ldots\}$

The application of the Stone-Weierstress theorem to group representation theory will be in the following. Consider a group G and a fixed chosen representation T_g. Consider the series

$$1, T_g, T_g \otimes T_g \ , \ T_g \otimes T_g \otimes T_g \otimes \cdots \otimes T_g, \ldots$$

The matrix elements of these representations form an algebra. If G is a matrix group, we can choose.

$$T_g = g$$

The "elementary harmonics" of a matrix group G can be constructed by considering

$$1,\ g,\ g \otimes g, \ldots, g \otimes g \otimes \ldots \otimes g, \ldots$$

and reducing out all the irreducible components. If G is compact, then any continuous function f(g) can be expanded in terms of the above system.

Global Theorem on the Representations of Compact Groups

Let G be a compact Lie group and

$$S = \{T^{(\ell)}(g)\}$$

the system of all it's irreducible representations. Let G be realized as a group of matrices. Then

1. All representations $T^{(\ell)}(g)$ are finite-dimensional and unitary (with an appropriate choice of a scalar product).
2. All representations $T^{(\ell)}(g)$ can be obtained from the tensor powers

$$1,\ g,\ g \otimes g, \ldots g \otimes g \otimes \ldots \otimes g, \ldots$$

of the group G.

3. The system of matrix elements $T_{ij}^{(\ell)}(g)$ for all possible values of ℓ, i and j form a complete orthogonal system of functions on G with respect to the scalar product

$$(f_1,f_2) = \int f_1(g)f_2(g)^* dg$$

where $dg = dg_L = dg_R$ is the invariant measure on the group.

4. For fixed ℓ the functions $T_{ij}^{(\ell)}(g)$ all have the same norm

$$||T_{ij}^{(\ell)}(g)|| = N_\ell = \frac{1}{\sqrt{d(\ell)}}$$

where $d(\ell)$ is the dimension of the representation (the order of the matrix $T^{(\ell)}$). (The norm is defined as $||f|| = \sqrt{(f,f)}$).

5. If the function $f(g)$ is square integrable, then the Fourier series

$$f(g) \sim \sum_{i,j,\ell} C_{ij}^{(\ell)} T_{ij}^{(\ell)}(g)$$

converges in norm. If $f(g)$ is smooth enough, then the Fourier series converges uniformly.

Remarks:

1) The left and right invariant measures (Haar measures) are defined by the relations

$$\int f(g_o g)d_L g = \int f(g)d_L g$$

$$\int f(gg_o)d_R g = \int f(g)d_R g$$

The group volume for a compact group is finite and we can normalize

$$\int d_L g = 1, \qquad \int d_R g = 1$$

For a compact group we have $d_L g = d_R g = dg$. In general for any Lie group $d_L g$ and $d_R g$ exist and are determined uniquely (up to a constant factor).

A group for which $d_L g = d_R g$ is called <u>unimodular</u>. All semisimple groups, all connected nilpotent Lie groups, all compact groups and many others are unimodular.

Proof that $d_L g = d_R g$ for a compact group:

We have: $d_L(gg_o)$ and $d_L(g)$ are both left invariant measures

$$d_L(gg_o) = C(g_o)d_L(g)$$

$C(g_o)$ is a Jacobian, satisfying $C(g_1g_2) = C(g_1)C(g_2)$

We have

$$\int f(gg_o)d_L(gg_o) = \int f(gg_o)C(g_o)d_L(g)$$

so that: iff $C(g_o) = 1$, then $d_L g = d_R g$.

However:

$$\int d_L(gg_o) = C(g_o) \int d_L g$$

$$1 = C(g_o).1 \qquad C(g_o) = 1$$

We have

$$d(gg_o) = d(g_o g) = d(g^{-1}) = dg$$

(We shall not prove the above assertions).

2. We say that a function $f(g)$ is differentiable if the infinitesimal operators for left and right translations

$$f(g) \to f(g_o g) , \ f(g) \to f(gg_o)$$

exist.

3. Convergence in norm means:

A sequence of elements x_n in a normed space R converges in norm to x if $|x-x_n| \to 0$ as $n \to \infty$.

In the proof of the global theorem use is made of the procedure of averaging over the group.

Let T_g be a representation of G in the linear space E. For each $x \varepsilon E$ introduce

$$x_o = \int_G T_g \, x \, dg \qquad \int_G dg = 1$$

(G must be compact!)

The vector x_o is invariant:

$$T_g x_o = \int T_{g_o} T_g \, x \, dg = \int T_{g_o g} \, x \, d(g_o g) = x_o$$

If there is no invariant vector $x \varepsilon E$ (except the vector 0), then

$$\int T_g dg = 0$$

Schur's lemma

$$\int T_g \otimes \hat{S}_g dg = 0$$

if T_g and S_g are finite-dimensional, irreducible and nonequivalent.

Proof of Global theorem:

1) Unitarity of finite dimensional representations. Let f(x,y) be a bilinear form in the space of representation T_g. Let us average it over the group:

$$f_o(x,y) = \int f(T_g x, T_g y) dg$$

The form $f_o(x,y)$ is obviously invariant

$$f_o(T_{g_o} x, T_{g_o} v) = \int f(T_g T_{g_o} \, T_g T_{g_o} v) dg =$$

$$\int f(T_{gg_o} x, T_{gg_o} y) dgg_o = f_o(x,y)$$

In a finite dimensional space we can always choose a positive definite bilinear form, e.g.

$$f(x,y) = x_1 y_1^* + \ldots + x_n y_n^*$$

(in a chosen basis). The stars mean complex conjugation. If $f(x,y)$ is positive definite, then so is $f_o(x,y)$. Thus, for a finite dimensional representation of a compact group we can always construct a positive definite invariant bilinear form:

$$(x,y) = f_o(x,y) = (T_g x, T_g y)$$

$$(x,x) \geq 0 \quad , \; (x,x) = 0 \Longleftrightarrow x = 0$$

Thus, every finite dimensional representation of a compact group has a positive definite invariant bilinear form and is thus unitary.

2) Orthogonality and normalization of matrix elements.

Consider two irreducible representations of G:

$$g \to T_g \qquad \text{and } g \to S_g.$$

Consider the matrix elements $T_g = ||T_{ij}(g)||$, $S_g = ||S_{\alpha\beta}(g)||$. Let the basis in each representation be so chosen that the matrices T_{ij} and $S_{\alpha\beta}$ are unitary. Consider the tensor product

$$\pi_g = T_g \times \hat{S}_g$$

where $\hat{S}_g = \left(S_g^T\right)^{-1} = S_g^*$ (the representation is unitary). The matrix element can be written as:

$$\langle i\alpha|\pi_g|j\beta\rangle = T_{ij}(g)S^*_{\alpha\beta}(g)$$

a) If T_g and S_g are not equivalent, then it follows from one of the corollaries of Schur's lemma, that π_g has no invariants. Thus

$$\int \pi_g dg = 0$$

i.e. in terms of matrix elements

$$\int T_{ij}(g)\, S^*_{\alpha\beta}(g) dg = 0$$

Thus: the matrix elements of two non-equivalent irreducible representations are mutually orthogonal.

b) Put $S_g = T_g$. Choose a basis matrix

$$e_{st} = \begin{array}{c c} t & \\ \begin{pmatrix} 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 \cdots & 1 & \cdots 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{pmatrix} & s \end{array}$$

and apply the averaging operation. We obtain an invariant matrix

$$||e^0_{st}||_{i\alpha} = \int \pi_g e_{st} dg||_{i\alpha}$$

By Schur's lemma we must have

$$||e^0_{st}||_{i\alpha} = ||\int \pi_g e_{st} dg||_{i\alpha} = \lambda(s,t)\delta_{i\alpha}$$

Take matrices with zero trace, e.g.:

$$e_{st} \text{ for } s \neq t \quad , \; e_{ss} - e_{tt}$$

We find:

$$\lambda(s,t) = 0 \text{ for } s \neq t, \quad \lambda_{ss} = \lambda_{tt}$$

Finally:

$$\int T_{is}(g)T^{*}_{\alpha t}(g)dg = \lambda \delta_{i\alpha}\delta_{st}$$

Thus: the matrix elements are mutually orthogonal and all have the same norm.

c) Calculate the norm.

For a unitary matrix we have $UU^{+} = 1$, i.e.

$$U_{ik}U^{+}_{k\ell} = U_{ik}U^{*}_{\ell k} = \delta_{i\ell}, \text{ in particular}$$

$$\sum_{j=1}^{d} |U_{ij}(g)|^2 = 1 \qquad \text{(no summation over i)}$$

For our matrix elements:

$$\sum_{j=1}^{d} |T_{ij}(g)|^2 = 1$$

Integrate over the group:

$$\int \sum_{j=1}^{d} |T_{ij}(g)|^2 = \lambda d = \int dg = 1$$

Thus: $\lambda = \frac{1}{d} = N^2$ where d is the dimension of the representation.

Finally, we can put

$$\int T_{is}(g)T^*_{\alpha t}(g) = \frac{1}{d}\ \delta_{i\alpha}\delta_{st}$$

3. Fourier Series on the Group G.

Let G be a compact matrix group. Consider all tensor powers of G and let us separate out from among them a system of nonequivalent irreducible representations $E^{(\ell)}(g)$, where ℓ runs through a discrete set. Introduce an orthogonal system of functions, consisting of the matrix elements

$$e^{\ell}_{ij}(g) = \frac{1}{N_\ell}\ E^{\ell}_{ij}(g)$$

Take a function f(g) and introduce a Fourier series

$$f(g) \sim \sum_{\ell,i,j} C^{\ell}_{ij} e^{\ell}_{ij}$$

where

$$C^{\ell}_{ij} = (f, e^{\ell}_{ij}).$$

Let f(g) be continuous. It follows from the Stone-Weierstress theorem that there does exist a linear combination $\theta(g)$ of $e^{\ell}_{ij}(g)$, which approximates f with arbitrary chosen accuracy:

$$\max|f(g) - \theta(g)| < \varepsilon$$

From here one can prove the "mean square convergence" of

$$\sum_{i,j,\ell} C^{(\ell)}_{ij}\ T^{(\ell)}_{ij}(g)$$

to f(g).

More precisely: An arbitrary function $f(g)$, $g \in G$ (G-compact), satisfying

$$\int |f(g)|^2 dg < \infty$$

can be expanded into a Fourier series:

$$f(g) \sim \sum_{\ell \in A} \sum_{i,j=1}^{d_\ell} C_{ij}^{\ell} E_{ij}^{\ell}(g)$$

where A is the complete set of all pair-wise nonequivalent irreducible unitary representations of G, d_ℓ is the dimension of the representation (ℓ). We have

$$C_{ij}^{(\ell)} = d_\ell \int f(g)\, E_{ij}^{\ell\,*}(g) dg$$

and the "mean square convergence" means that the Parceval identity holds:

$$\int |f(g)|^2 dg = \sum_{\ell \in A} \frac{1}{d_\ell} \sum_{i,j=1}^{d_\ell} |C_{ij}^{\ell}|^2.$$

For a proof see N. J. Vilenkin, Special Functions and the Theory of Group Representations, Chapter 1 paragraph 4.

4. Finite-dimensionality of Irreducible Representations

We shall show that every irreducible representation T_g of a compact group G is contained amongst the system of "elementary harmonics" E^ℓ, introduced above. The E^ℓ were obtained from the tensor powers of the group G and are thus finite-dimensional by construction. We shall make use of the fact that every irreducible representation of a compact group is contained in the right regular representation.

In the proof we make use of a powerful technique, having many other applications, namely that of projection operators, which project out a chosen irreducible representation from an arbitrary reducible one.

Indeed, we already know that we can expand any function $f(g)\varepsilon D(G)$, where $D(G)$ is a Hilbert space of square-integrable functions over the group:

$$f(g) = \sum_{\ell,i,j} C^{\ell}_{ij}E^{\ell}_{ij}(g); \; C^{\ell}_{ij} = d_{\ell} \int f(g)E^{\ell\,*}_{ij}(g)dg$$

Now introduce an operator

$$P^{\ell}_{ij} \equiv d_{\ell} \int_{G} dg \; E^{\ell *}_{ij}(g)R_{g}$$

where R_g is the right regular representation of G acting in D(G). When acting on $f(g)\;\varepsilon D(G)$ the operator P^{ℓ}_{ij} projects out a set of functions, transforming according to the irreducible representation T^{ℓ}_{g}. Indeed:

$$P^{\ell}_{ij}f(g) = d_{\ell} \int_{G} d\tilde{g} \; E^{\ell *}_{ij}(\tilde{g})R_{\tilde{g}}f(g) = d_{\ell} \int_{G} d\tilde{g} \; E^{\ell *}_{ij}(\tilde{g})f(g\tilde{g})$$

Put $g\tilde{g} = g'$ and use $dg' = d\tilde{g}$ to obtain

$$P^{\ell}_{ij}f(g) = d_{\ell} \int dg' E^{\ell *}_{ij}(g^{-1}g')f(g') = d_{\ell} \sum_{p} E^{\ell *}_{ip}(g^{-1}) \int dg' E^{\ell *}_{pj}(g')f(g')$$

$$= \sum_{p} E^{\ell}_{pi}(g)C^{\ell}_{pj}$$

Thus, $P^{\ell}_{ij}f(g)$ is a linear combination of the matrix elements of the irreducible representation E^{ℓ} with one fixed ℓ. Operator P^{ℓ}_{ij} is a projection operator in the following sense:

$$P^{\ell_1}_{i_1j_1}P^{\ell_2}_{i_2j_2}f(g) = P^{\ell_1}_{i,j_1}\sum_p C^{\ell_2}_{pj_2}\ E^{\ell_2}_{pi_2}(g) =$$

$$= d_{\ell_1}\sum_p C^{\ell_2}_{pj_2}\int d\tilde{g}\ E^{\ell_1 *}_{i_1j_1}(\tilde{g})E^{\ell_2}_{pi_2}(g\tilde{g}) =$$

$$= d_{\ell_1}\sum_p C^{\ell_2}_{pj_2}\sum_q E^{\ell_2}_{pq}(g)\int d\tilde{g}\ E^{\ell_1 *}_{i_1j_1}(\tilde{g})\ E^{\ell_2}_{qi_2}(\tilde{g}) =$$

$$= \delta_{\ell_1\ell_2}\delta_{j_1i_2}\sum_p C^{\ell_2}_{pj_2}E^{\ell_2}_{pi_1}(g) = \delta_{\ell_1\ell_2}\delta_{j_1i_2}\ P^{\ell_1}_{i_1j_2}$$

Now let T_g be some abstract irreducible representation of G. We know that $T_g \epsilon R_g$. This means that there exists a subspace $G_o \epsilon G$, such that $D(G_o) \subset D(G)$ is a subspace of square-integrable functions, such that T_g acts on $D(g_o)$ irreducibly. We have

$$R_g D(G_o) = D(G_o)$$

i.e. $$f(g_o)\epsilon\ D(G_o) \Longrightarrow R_g f(g_o) = f(g_o g)\epsilon\ D(G_o)$$

Now put $f(g_o)\in D(g_o)$ and act upon it with our projection operator

$$P^{\ell}_{ij}f(g_o) = d_{\ell}\int d\tilde{g}\ E^{\ell *}_{ij}(\tilde{g})f(g_o\tilde{g})$$

Since $f(g_o\tilde{g}) \subset D(G_o)$, P^{ℓ}_{ij} either project $D(G_o)$ into zero or into itself. In other words, since this space $D(G)$ is invariant and irreducible, it can only contain one of the harmonics E^{ℓ} and our chosen T_g must be equivalent to E .

To summarize: the irreducible representations of a compact group are finite-dimensional, unitary and contained in the tensor powers, generated by the matrices $g \varepsilon G$.

5. Complete reducibility

Any unitary representation of a group G and thus any finite dimensional representation of a compact group is completely reducible. This follows immediately from the lemma:

Lemma: Let T_g be a unitary representation with the scalar product (x,y) in the space X and let X_1 be an invariant subspace of X. Then the orthogonal complement X_2 of X_1 is also invariant.

Proof: X_2 is the orthogonal complement of X_1, i.e.

$$(x,y) = 0 \qquad x \varepsilon X_1,\ y \varepsilon X_2.$$

Take $x \varepsilon X_2$, $y \varepsilon X_1$

$$(T_g x,y) = (T_g^{-1} T_g x, T_g^{-1} y) = (x, T_g^{-1} y)$$

However X_1 is invariant $\Longrightarrow$ $T(g^{-1}) y \in X_1$ $\Longrightarrow$ r.h.s. is equal to zero
$\Longrightarrow$

$$(T_g x,y) = 0$$

Thus, $T_g x \in X_2$ $\Longrightarrow$ X_2 is an invariant subspace.

Q.E.D.

A succesive application of the lemma will reduce a reducible representation into irreducible components.

<u>Remark:</u> There are complications for infinite dimensional representations, even if they are unitary. In general one has to introduce the concept of a continuous direct sum, etc.

Thus, No <u>Jordan matrices</u> can occur in the representations of a compact group:

$$z(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

Lecture 13

Example

As an example, let us consider the representation theory of the group SU(2). Everything to be said here is of course well known, we just wish to demonstrate some general results and prepare the ground for generalization to arbitrary compact groups.

SU(2): Group of second order matrices satisfying:

$$U^+U = UU^+ = 1 \qquad \det U = 1 \tag{1}$$

1) The algebra of SU(2):

Any unitary matrix can be written as

$$U = e^{ith} \tag{2}$$

where hs is hermitean and traceless

$$h = h^+ \qquad \mathrm{Tr}h = 0 \tag{3}$$

We can use the Pauli matrices as a basis for the space h for SU(2):

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4}$$

The algebra of SU(2) is spanned by $a_k = -i\sigma_k$ (k=1,2,3) satisfying

$$[a_i, a_k] = \varepsilon_{ik\ell} a_\ell \tag{5}$$

($\varepsilon_{ik\ell}$ is the totally antisymmetric third order tensor satisfying $\varepsilon_{123} = 1$)

Introduce complex linear combinations of a_i, namely

$$e_+ = \frac{1}{2}(\sigma_1 + i\sigma_2) \quad e_- = \frac{1}{2}(\sigma_1 - i\sigma_2) \quad e_o = \frac{1}{2}\sigma_3 \tag{6}$$

$$[e_+, e_-] = 2e_o \quad [e_o e_+] = e_+ \quad [e_o e_-] = -e_- \tag{7}$$

The algebra E of SU(2) consists of the complex linear combinations of e_+, e_-, e_o

$$a = z_+ e_+ + z_- e_- + z_o e_o$$

with

$$z_- = -z_+^* \qquad z_o = -z_o^*$$

2) The diagonal basis element a_3 (or e_o) generates a one-parameter subgroup of SU(2) consisting of diagonal matrices

$$\gamma = \begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix} \tag{8}$$

Introduce a matrix

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{9}$$

satisfying

$$S^{-1}\gamma S = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} = \gamma^{-1} \tag{10}$$

(it leaves a diagonal matrix diagonal and reshuffles the matrix elements)

Call S a <u>Weyl element</u>.

3) Consider a finite-dimensional representation $u \to T_u$. Call A_i the operators, representing the infinitesimal operators a_i in the representation space E:

$$[A_i A_k] = \varepsilon_{ik\ell} A_\ell \tag{11}$$

Similarly we have

$$[E_+ E_-] = 2E_o \;, \quad [E_o E_+] = E_+ \;, \quad [E_o E_-] = -E_- \tag{12}$$

Let us start out by first looking for a representation of the Lie algebra and only afterwards consider the question, whether this representation can be extended to a single-valued representation of the group.

Thus: We wish to find all irreducible realizations of the algebra E by linear operators E_o, E_+ and E_-, satisfying (12).

We know that any commuting set of unitary matrices T_u can be simultaneously diagonalized, in particular the matrices T_γ, representing the subgroup of matrices γ. In other words - we can always consider the matrix E_o to be diagonal.

4. Consider an eigenvector x_λ of E_o:

$$E_o x_\lambda = \lambda x_\lambda \tag{13}$$

Lemma: If λ is an eigenvalue of E_o satisfying (13) then we also have

$$\begin{aligned} E_o x_{-\lambda} = -\lambda x_{-\lambda} \;, \quad E_o x_{\lambda+1} &= (\lambda+1)x_{\lambda+1} \\ E_o x_{\lambda-1} &= (\lambda-1)x_{\lambda-1} \end{aligned} \tag{14}$$

where $x_{\lambda+1}$ and $x_{\lambda-1}$ can, in particular, be noll vectors.

Proof: Consider the vectors

$$x_{\lambda+1} = E_+ x_\lambda \quad x_{\lambda-1} = E_- x_\lambda \quad \text{and } x_{-\lambda} = x_\lambda^s = T_s x_\lambda \tag{15}$$

where T_s represents the element s in the group, so that $T_s^{-1} T_\gamma T_s = T_\gamma^{-1}$ from which follows

$$T_s^{-1} E_o T_s = -E_o \tag{16}$$

Using (16) and the commutation relations (12), it is easy to check that the vectors (15) satisfy (14). Q.E.D.

Corollary: If T_u is an irreducible representation, then the eigenvalues of E_o are nondegenerate and can be written as a chain

$$-\ell, -\ell+1, \ldots, \ell-1, \ell$$

where ℓ is integer or half-integer.

Proof: Define $\ell = \lambda_{max}$ (the largest eigenvalue) and x_ℓ as the corresponding eigenvector. Then $\ell, \ell-1, \ell-2$ are also eigenvalues, and so is $-\ell$. (The quantity $-\ell-1$ is not an eigenvalue, since then $\ell+1$ would be one too, contradicting the assumption that ℓ is maximal). Thus $\ell-(-\ell) = 2\ell$ is integer. None of the eigenvalues can be degenerate, if T_u is irreducible, since if two functions x_1 and x_2 corresponded to one eigenvalue λ, then the representation would have two invariant subspaces, the ones generated by applying E_+ and E_- and powers thereof to x_1 and x_2. Q.E.D.

We can choose a basis in E consisting of the eigenvectors of E_o

$$E_o x_\lambda = \lambda x_\lambda \qquad \lambda = -\ell, -\ell+1, \ldots, \ell-1, \ell \tag{17}$$

Choosing an appropriate normalization of x_λ we can arrange that

$$\begin{aligned} E_+ x_\lambda &= (\ell-\lambda) x_{\lambda+1} \\ E_- x_\lambda &= (\ell+\lambda) x_{\lambda-1} \end{aligned} \tag{18}$$

(Check that the operators thus defined satisfy the correct commutation relations).

We have obtained all representations of the algebra of SU(2), each one of them corresponding to a definite highest eigenvalue ℓ of E_o (we shall also call ℓ "the highest weight"). We must now find out whether representations T_u of the group exist, which correspond to the found representations of the algebra.

6. The <u>generators</u> as <u>differential operators</u>.

Consider the space R of homogeneous polynomials $f(x_1,x_2)$ of order 2ℓ. The monomials

$$Z_\mu = x_1^{\ell-\mu} x_2^{\ell+\mu} \qquad \mu = -\ell, -\ell+1, \ldots, \ell \tag{19}$$

form a basis in R_ℓ ($\dim R_\ell = 2\ell+1$). It is easy to check that the differential operators

$$D_+ = x_2 \frac{\partial}{\partial x_1} \qquad D_- = x_1 \frac{\partial}{\partial x_2} \qquad D_o = \frac{1}{2}\left(x_2 \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_1}\right) \tag{20}$$

form a Lie algebra, isomorphic to that of SU(2).

7. <u>Realization of representations of the group</u>

Consider

$$g = \begin{pmatrix} \alpha & \beta \\ \sigma & \delta \end{pmatrix} \qquad \det g = \alpha\delta - \beta\sigma = 1 \tag{21}$$

i.e. $g \varepsilon$ SL(2,C)

The equation

$$T_g f(x_1,x_2) = f(\alpha x_1 + \gamma x_2, \beta x_1 + \delta x_2) \tag{22}$$

determines a representation of SL(2,C) in the space R_ℓ. Restricting the group SL(2,C) to SU(2), i.e. considering only matrices (21), satisfy the additional conditions

$$|\alpha|^2 + |\gamma|^2 = |\beta|^2 + |\delta|^2 = 1 \qquad \alpha\beta^* + \gamma\delta^* = 0 \tag{23}$$

we obtain a representation of SU(2), which we denote D_ℓ or $\Delta^{2\ell}$. The corresponding representation of the algebra D_+, D_- and D_o, coincides with (20). The two-dimensional representation $D_{1/2}=\Delta^{2\ell}$ is the group itself and is spanned by the monomials

$$x_1 \text{ and } x_2.$$

Representation $\Delta^{2\ell}$ can be realized using symmetric tensors of order 2ℓ. Indeed, every polynomial $f\in R_{2\ell}$ can be written as:

$$f = t^{i_1\ldots i_\ell}\, x_{i_1} x_{i_2} \ldots x_{i_\ell} \qquad (i_3 = 1,2) \tag{24}$$

where the tensor $t^{i_1\ldots i_\ell}$ is totally symmetric. Thus $\Delta^{2\ell}$ is the symmetric part of the tensor product of ℓ terms

$$g \otimes g \otimes \ldots \otimes g$$

8. <u>Normalization</u> of the basis:

The basis

$$Z_\mu = x_1^{\ell-\mu} x_2^{\ell+\mu}$$

is orthogonal, since z_{μ_1} and z_{μ_2} are both eigenvectors of E_o and thus

$$(z_{\mu_1}, z_{\mu_2}) = 0 \quad \text{for } \mu_1 \neq \mu_2$$

Consider the normalization. We can check that the vectors

$$e_\mu = \frac{z^\mu}{\sqrt{(\ell-\mu)!(\ell+\mu)!}} \tag{25}$$

satisfy

$$(e_\mu, e_\mu) = 1$$

9. Alternative realization of the representation D_ℓ. The homogeneity condition on $f(z_1,z_2)$ is

$$f(az_1,az_2) = a^{2\ell} f(z_1,z_2) \tag{26}$$

Thus:

$$f(z_1,z_2) = z_2^{2\ell}\, f(\frac{z_1}{z_2}, 1) = z_2^{2\ell}\, \phi(z) \tag{27}$$

and instead of the homogeneous functions $f(z_1,z_2)$ we can consider functions of one complex variable $\phi(z)$. We have

$$\begin{aligned} T_g f(z_1,z_2) &= f(\alpha z_1 + \gamma z_2,\ \beta z_1 + \delta z_2) \\ &= (\beta z_1 + \delta z_2)^{2\ell} f(\frac{\alpha z_1+\gamma z_2}{\beta z_1+\delta z_2}, 1) = (\beta z_1+\delta z_2)^{2\ell} \phi\left(\frac{\alpha z_1+\gamma z_2}{\beta z_1+\delta z_2}\right) \\ &= (z_2)^{2\ell} (\beta \frac{z_1}{z_2} + \delta)^{2\ell} \phi\left(\frac{\alpha \frac{z_1}{z_2} + \gamma}{\beta \frac{z_1}{z_2} + \delta}\right) \end{aligned} \tag{28}$$

Thus: the action of T_g in the space $\{\phi(z)\}$ is:

$$T_g\phi(z) = (\beta z+\delta)^{2\ell}\phi\left(\frac{\alpha z+\gamma}{\beta z+\delta}\right) \tag{29}$$

Using the basis $z_\mu = z^{\ell-\mu}$, i.e

$$1,\ z,\ z^2 \ldots\ z^{2\ell} \tag{30}$$

one can now calculate e.g. the matrix elements of T_g. We have obtained the following result:

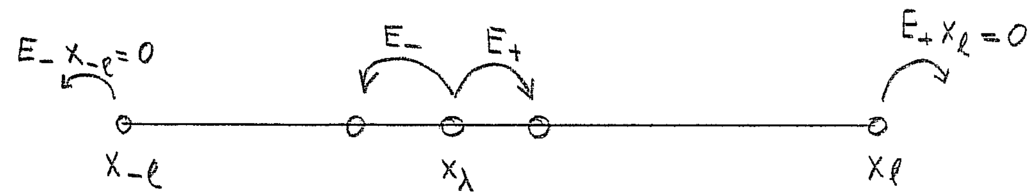
<u>Theorem:</u> Any irreducible representation of SU(2) is given by one parameter ℓ, which is integer or half-integer. The operators of the representation are given explicitly by (29). The dimension of the representation is $(2\ell+1)$ and the representation of the algebra is

$$E_+z_\mu = (\ell-\mu)z_\mu \qquad E_-z_\mu = (\ell+\mu)z_\mu \qquad E_oz_\mu = \mu z_\mu \tag{31}$$

The basis vectors z_μ, $\mu=-\ell,-\ell+1,\ldots\ell$ are the powers $z^{\ell-\mu}$, $\mu = -\ell,\ldots,+\ell$
In particular, we have:

$$E_+1 = 0 \qquad E_-2^{2\ell} = 0 \tag{32}$$

Schematically, we have



<u>Remark</u>: For every representation of the algebra (31) we have a single-valued representation (29) of the group SU(2). Had we been considering SO(3) we would have found that only representations with ℓ = integer are single valued representations of SO(3).

<u>Complex Extension of Lie Algebras and Complex Lie Groups</u>

In the SU(2) example we considered a complex extension of the Lie algebra to E_-, E_0, E_+ and we saw that as a complex algebra these operators generate SL(2,C)

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad \det g = 1$$

where $\alpha,\beta,\gamma,\delta$ are complex. SL(2,C), as we know is compact. However, if we restrict ourselves to representations depending analytically on the parameters (depending on $\alpha,\ldots,\delta$, not however explicitly on $\alpha^*,\ldots\delta^*$), then we have a simple correspondence with the representation theory of SU(2).

Definition: A <u>Complex Lie Group</u>. A Lie group satisfying: (1) The elements are parametrized by a finite number of complex parameters $t_1\ldots t_n$. 2) The group operations $g_1 g_2$ and g^{-1} are determined as complex analytic functions (of t_i, not depending on t_i^*).

In a complex Lie group any one-parameter subgroup g^λ can be continued to complex λ. It follows that the Lie algebra is a <u>complex Lie algebra</u>. Given a real Lie algebra with a basis $e_1,\ldots,e_n$

$$[e_i e_k] = C_{ik}^{\ell} e_\ell \qquad C_{ik}^{\ell} \text{ - real}$$

we can complete it by adding the independent vectors:

$$ie_1, \ldots, ie_n$$

We have already discussed complex semisimple Lie groups and know that in general any complex Lie group has several non-isomorphic real forms.

<u>Example</u>: $GL(n,C)$. In the neighborhood of the identity put $g = e^x$, $x \in X$ where X is the algebra of all complex nxn matrices:

Consider two subspaces of X:

X_1 - real nxn matrices : $x = x^*$

X_2 - antihermitean nxn matrices $x = -x^+$

Complex extension of X_1 = complex extension of X_2 = X

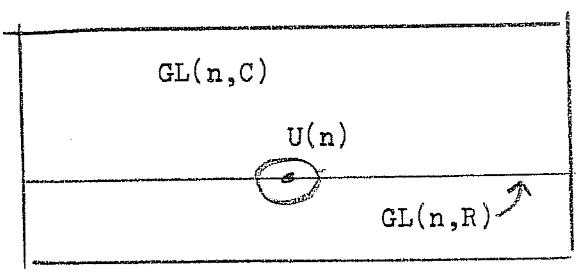
The corresponding groups are : $X_1 \ldots GL(n,R)$

$X_2 \ldots U(n)$

(We know there are other real forms).

If n is equal to one, then X corresponds to the group of all complex numbers (with respect to multiplication) $z = \rho e^{i\phi}$; $\rho \neq 0$. X_1 corresponds to the group of all real numbers $x \neq 0$ and X_2 to the group of unimodular numbers $e^{i\phi}$.

Schematically, in general



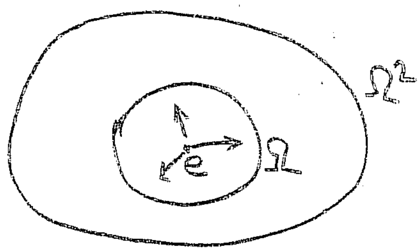
Lecture 14

Complexification of a Real Group

Let $G(R)$ be a real matrix group. In the neighborhood Ω of the identity we can use "canonical" coordinates θ_i, provided by the Lie algebra:

$$g = \exp\{\theta_1 e_1 + \ldots + \theta_n e_n\}$$

and we can now continue analytically to complex values of θ_i, i.e. to the group $G(C)$. Further, the powers Ω, Ω^2, Ω^3,... will cover the whole connected group $G(R)$ and can serve to continue the complexification out of Ω:



The group $G(C)$ does not depend on the choice of the basis e_i and is called the <u>complexification</u> of $G(R)$.

A similar extension is possible even when $G(R)$ is not a matrix group.

We shall not prove the above statements.

Symbolically:

$$G(R) = e^{X(R)} \longrightarrow G(C) = e^{X(R)+iX(R)} = e^{X(C)}$$

The Principle of Analytic Continuation.

Let $g \rightarrow T_g$ be a representation of $G(R)$ in a finite dimensional space E. In "canonical" coordinates

$$T_g = \exp.\{\theta_1 E_1 + \ldots + \theta_n E_n\}$$

where $E_1,\ldots,E_n$ are the generators of T_g, (representing $e_1,\ldots,e_n$). Replacing the real numbers θ_i by complex ones we obtain a representation of the group G(C). However, different paths for extending the neighborhood Ω to G(R) can lead to different T_g, i.e. the representation T_g of G(C) is not necessarily single valued.

We can deal with this difficulty in two ways: a) Agree to consider multivalued representations on the same footing as single-valued ones. b) Instead of using the group G(C) consider its <u>universal covering</u> group. Indeed, that group which is the universal covering group of all Lie groups with the same Lie algebra can only have single-valued representations. Thus we shall define G(C) to be the universal covering group (the complexification $G(R) \to G(C)$ has the same degree of arbitrariness as the reconstruction of a Lie group from a Lie algebra). Then any representation T_g of G(R) can be uniquely continued to a single valued representation of G(C). The finite-dimensionality of E was not essential. It is however essential to assume that T_g depends analytically on the real parameters in G(real analyticity).

<u>The Principle of Continuation:</u> Any real analytic representation of the group G(R) can be continued to a complex analytic representation of G(C). If G(C) is chosen as the universal covering group of all complex extensions of G(R), then the continuation is unique.

Symbolically:

$$G(R) \to T_g = e^{D(X(R))} \qquad G(C) \to T_g = e^{D(X(R))+i\,D(X(R))}$$

where D(X(R)) is the Lie algebra of the representation T_g. Analytic continuation preserves the irreducibility, reducibility and complete reducibility of representations.

<u>Example:</u> $G(R) = SU(2)$, $G(C) = SL(2,C)$

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad \det g = 1$$

$$\delta = \alpha^{*},\ \gamma = -\beta^{*} \implies g\in SU(2)$$

Representations of $SU(2)$

$$T_g f(z) = (\beta z+\delta)^{2\ell} f\left(\frac{\alpha z+\gamma}{\beta z+\delta}\right)$$

Dropping the conditions $\delta = \alpha^{*}$, $\gamma = -\beta^{*}$ we obtain a complex analytic representation of $SL(2,C)$ (T_g does not depend on $\alpha^{*},\dots\delta^{*}$).

<u>Definitions:</u> Let $G(C)$ be a complex Lie group with parameters $t_1,\dots,t_n$. The representation $g \to T_g$ is:

<u>Analytic</u>, if it depends analytically on t_i.

<u>Antianalytic</u>, if it depends analytically on t_i^{*}.

<u>Real</u>, if it represents $G(C)$ as a real Lie group with parameters $\mathrm{Re}\,t_1, \mathrm{Im}\,t_1 \dots \mathrm{Re}\,t_n, \mathrm{Im}\,t_n$

<u>Example 1:</u> Let $G(C)$ be a matrix group:

Representation: $g \to g$ is analytic

$g \to g^{*}$ is antianalytic

$g \to g \otimes g^{*}$ is real

<u>Example 2:</u> Let $f(z)=f(x,y)$ be analytic functions, (i.e. they satisfy the Cauchy-Rieman condition:

$$f(x,y) = u(x,y) + iv(x,y), \quad u \text{ and } v \text{ - real}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x})$$

The representation of the additive group of complex numbers

$$T_{z_0} f(z) = f(z+z_0)$$

is analytic.

Remark: the Cauchy-Riemann analyticity conditions are equivalent to the condition

$$\frac{\partial f}{\partial z^*} = 0$$

since we have
$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$
$$\frac{\partial}{\partial z^*} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Infinitesimal Operators

If $G(R)$ is a real Lie group, we can calculate the infinitesimal operators in any representation according to the formula:

$$D(x) = \frac{\partial}{\partial\lambda}\left[T_{g^\lambda}\right]_{\lambda=0}$$

where $g^\lambda = e^{\lambda x}$ is a one-parameter subgroup, generated by the vector x. We thus have a mapping $x \rightarrow D(x)$ satisfying

$$D(\alpha x + \beta y) = \alpha D(x) + \beta D(y)$$

$$[D(x), D(y)] = D([x,y])$$

i.e. $D(x)$ is a representation of the Lie algebra of $G(R)$.

Now let $G(C)$ be a complex group and $g \to T(g)$ an analytic representation. The above construction is also valid in this case and λ, α, β are now complex. If $g \to T_g$ is a real representation, then we put $\lambda = \tau + i\sigma$ and consider g^τ and $g^{i\sigma}$ separately. Thus, every vector x is represented by two differential operators

$$D(x) \text{ and } D(ix)$$

where

$$D(x) = \frac{\partial}{\partial \tau} [T_g \tau]_{\tau=0}$$

$$D(ix) = \frac{\partial}{\partial \sigma} [T_g \sigma]_{\sigma=0}$$

In general $D(x)$ and $D(ix)$ are independent. However the representation is <u>analytic</u> if

$$D(ix) = i\, D(x)$$

and <u>antianalytic</u> if

$$D(ix) = -i\, D(x)$$

We can also introduce the linear combinations

$$A(x) = \frac{1}{2} [D(x) - i\, D(ix)]$$

$$\bar{A}(x) = \frac{1}{2} [D(x) + i\, D(ix)]$$

and now we have:

1) $A(x)$ is an analytic representation

2) $\bar{A}(x)$ is an antianalytic representation

3) $[A(x), \bar{A}(x)] = 0$

4) $D(x) = A(x) + \bar{A}(x)$

Now let $G(C)$ be a complex group and $g \to T(g)$ an analytic representation. The above construction is also valid in this case and λ, α, β are now complex. If $g \to T_g$ is a real representation, then we put $\lambda = \tau + i\sigma$ and consider g^{τ} and $g^{i\sigma}$ separately. Thus, every vector x is represented by two differential operators

$$D(x) \text{ and } D(ix)$$

where

$$D(x) = \frac{\partial}{\partial\tau} [T_g\tau]_{\tau=0}$$

$$D(ix) = \frac{\partial}{\partial\sigma} [T_g\sigma]_{\sigma=0}$$

In general $D(x)$ and $D(ix)$ are independent. However the representation is <u>analytic</u> if

$$D(ix) = i\, D(x)$$

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We can also introduce the linear combinations

$$A(x) = \frac{1}{2} [D(x) - i\, D(ix)]$$

$$\bar{A}(x) = \frac{1}{2} [D(x) + i\, D(ix)]$$

and now we have:

1) $A(x)$ is an analytic representation
2) $\bar{A}(x)$ is an antianalytic representation
3) $[A(x), \bar{A}(x)] = 0$
4) $D(x) = A(x) + \bar{A}(x)$

<u>Result:</u> Every real representation of the complex algebra X(C) can be decomposed into the direct sum of two components, one analytic, the other antianalytic.

<u>Example:</u> 1) the group of <u>complex translations:</u>

$$T_{z_o} f(x,y) = f(x+x_o, y+y_o)$$

The infinitesimal operator is

$$D(z_o) = x_o \frac{\partial}{\partial x} + y_o \frac{\partial}{\partial y} = z_o \frac{\partial}{\partial z} + \bar{z}_o \frac{\partial}{\partial \bar{z}}$$

and we have $A(z) = z_o \frac{\partial}{\partial z}$, $\bar{A}(z) = \bar{z}_o \frac{\partial}{\partial \bar{z}}$

2) <u>The group SL(2,C)</u>

Let us take the basis

$$e_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \; e_o = \begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \quad e_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The operators E_-, E_o and E_+ which have already been introduced, generate analytic representations of SL(2,C). To consider real representations of SL(2,C) we must double the number of parameters, i.e. introduce

$$\bar{E}_-, \bar{E}_o, \bar{E}_+$$

Satisfying the same commutation relations and commuting with E_-,E_+ and E_o.

In terms of tensors, we consider not only the tensor series:

$$1, g, g \otimes g, \ldots \; g \otimes g \otimes \ldots \otimes g, \ldots$$

but also

$$1, \bar{g}, \bar{g} \times \bar{g}, \ldots \bar{g} \times \bar{g} \times \ldots \times \bar{g}$$

and also terms like $\qquad g \times \bar{g}.$

In relativistic quantum theory one usually introduces analytic representations of $SL(2,C)$ as "dotted tensors", the antianalytic ones as "undotted tensors".

The Principle of Unitary Reduction

Take a group G with a subgroup $H \subseteq G$. If we consider a representation $g \to T_g$ and then consider T_g for only those g, which satisfy $g = h \epsilon H$, then we say that we are reducing the representation of G to a representation of the subgroup H. The representation T_h in this reduction can be reducible, even if T_g is irreducible.

<u>Example:</u> Consider two subgroups of $SL(2,C)$: the group of triangular matrices H and the group of diagonal matrices D:

$$h \epsilon H \qquad h = \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}$$

$$d \epsilon D \qquad d = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$$

Thus, if we take an irreducible two dimensional representation of $SL(2,C)$

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad \det g \neq 0$$

and reduce it to H, the representation T_h is reducible. Similarly the representation T_d, $d \epsilon D$ is completely reducible.

Reduction Theorem: Let $g \to T_g$ be a complex analytic representation of a complex group G and let $h \to T_h$ be the reduction of T_h to a real form H of G. If T_g is irreducible or completely reducible, then so is T_h.

Proof: Let T_g be finite-dimensional i.e. T_g is a matrix

$$T_g = \begin{pmatrix} T_{11} \cdots T_{1n} \\ \cdots\cdots\cdots \\ T_{n1} \cdots T_{nn} \end{pmatrix}$$

If T_h is reducible then its matrix is

$$T_h = \left(\begin{array}{c|c} T_{11} \cdots & 0 \\ \hline T_{n1} & T_{nn} \end{array}\right)$$

However, each element of T_g is an analytic function of some parameters $t_1, \ldots, t_n$. If an analytic function is equal to zero for all real values of its arguments it is identically zero. Thus T_g would also be reducible, contrary to the assumptions.

The theorem can also be proven for infinite dimensional representations, if we give an appropriate definition of the matrix elements.

Corollary: We have a one-to-one correspondence between irreducible real analytic representations of a real group and an irreducible complex analytic representations of the complex extension of this group.

If we consider a compact Lie group G and it's complex extension, $\tilde{G}$ we obtain:

1) $\tilde{G}$ can be realized as a group of matrices

2) All <u>irreducible</u> <u>analytic</u> representations of the group $\tilde{G}$ are finite dimensional and are contained in the tensor algebra

3)All finite dimensional analytic representation of $\tilde{G}$ are completely reducible (or irreducible).

Complex Extension of the Group U(n)

We already know that if

$$G = U(n) \quad \text{then } \tilde{G} = GL(n,C)$$

and the algebra $\tilde{X}$ of GL(n,C) consists of all complex nxn matrices. Choose the usual basis of $\tilde{X}$, namely the matrices e_{pq}. They satisfy

$$e_{pq} e_{rs} = \delta_{qr} e_{ps}$$

and

$$[e_{pq}, e_{rs}] = \delta_{qr} e_{ps} - \delta_{ps} e_{rq}$$

We know that

$$\tilde{X} = E_+ + E_o + E_-$$

where

$$E_+ = \{e_{ij}, i < j\} \; ; \; E_o = \{e_{ii}\} \; ; \; E_- = \{e_{ij}, i < j\}$$

E_o is commutative and corresponds to the diagonal subgroup

$$\delta = \begin{pmatrix} \delta_1 & & & 0 \\ & \delta_2 & & \\ & & \ddots & \\ 0 & & & \delta_n \end{pmatrix}$$

It is sometimes convenient to replace the basis $\{e_{ii}\}$ in E_o by

$$h_1 = e_{11}-e_{22},\ h_2 = e_{22}-e_{33},\ \ldots\ h_{n-1} = e_{n-1,n-1}-e_{nn}$$

$$\varepsilon = e_{11}+\ldots+e_{nn}$$

Then eliminating ε from the algebra corresponds to going over to the group $SL(n,C)$ (or $SU(n)$).

Analytic Representations of GL(n,C)

Let $g \to T_g$ be an analytic representation of the complex group $GL(n,C)$. The generators form a representation of the algebra of $GL(n,C)$

Theorem: 1) The generators of the representation E_{ij} can be split into three subalgebras E_+, E_- and E_o, corresponding to e_+, e_- and e_o. They satisfy:

$$[E_{ij},E_{pq}] = \delta_{jp}E_{iq} - \delta_{qi}E_{pj}$$

2) All operators E_{ii} can be simultaneously diagonalized

3) The eigenvalues of the operators E_{ii} are all integers.

Proof: 1) The first assertion is obvious

2) Consider a commutative compact subgroup $\gamma \subset GL(n,C)$

$$\gamma = \begin{pmatrix} e^{i\phi_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\phi_n} \end{pmatrix} \qquad 0 \le \phi_i \le 2\pi$$

A set of commuting unitary operators T_γ, representing matrices γ, can always be simultaneously diagonalized. Now consider the subgroup of matrices $\delta \subset GL(n,C)$

$$\delta = \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix} \qquad \lambda_i\text{-complex}$$

The operators T_δ will be the analytic continuation of a real analytic representation of a compact group. It follows, that if the T_δ are chosen to be diagonal, then T_δ will also be diagonal.

3) Consider a common eigenvector of all operators T_δ:

$$T_\delta\, x = \alpha(\delta)x$$

We have:

$$T_{\delta'}T_{\delta''}x = \alpha(\delta)\alpha(\delta'')x$$

$$T_{\delta'\delta''}x = \alpha(\delta'\delta'')x \qquad \alpha(\delta'\delta'') = \alpha(\delta')\alpha(\delta'')$$

We also have $\alpha(e) = 1$, so that $\alpha(\delta)$ is a numeric representation (a one-dimensional representation) of the Abelian group D, also called the <u>character of</u> the Abelian group D. Such representations of an Abelian group are just ordinary exponentials.

Thus:

$$\alpha(\delta) = \exp i\{\lambda_1 c_1 + \ldots + \lambda_n c_n\}$$

where $c_1,\ldots,c_n$ are fixed numbers. Restricting ourselves to $\alpha(\gamma)$, we have

$$\alpha(\gamma) = \exp i\ \{\phi_1 c_1 + \ldots + \phi_n c_n \qquad 0 \le \phi_i < 2\pi$$

Since we demand that the representation should be single-valued we must have

$$c_1 \ldots c_n \quad \text{integer}$$

Denoting the eigenvalues of the matrix δ :

$$e^{i\lambda k} = \delta_k$$

we have

$$\alpha(\delta) = \delta_1^{C_1}\delta_2^{C_2}\cdots\delta_n^{C_n}$$

We have thus shown that in the space of the analytic representation T_g there exists a basis in which the matrices T_δ are diagonal

$$T_\delta = \begin{pmatrix} \alpha_1(\delta) & & & 0 \\ & \alpha_2(\delta) & & \\ & & \ddots & \\ 0 & & & \alpha_N(\delta) \end{pmatrix}$$

where N is the dimension of the representation T_g and each eigenvalue can be written as

$$\alpha(\delta) = \delta_1^{C_1}\cdots\delta_n^{C_n} \qquad C_i = \text{integer}$$

It follows that all eigenvalues are enumerated by a set of integers $(C_1,\ldots,C_n)$.

In order to construct the representation theory, we must

a) Find all irreducible systems of operators E_{ij}, satisfying the given commutation relations, such that all E_{ii} are diagonal and have only integer eigenvalues.

b) In each irreducible representation find the common spectrum of the operators contained in E_o, i.e. find all eigenvalues:

$$\lambda(\phi) = C_1\phi_1 + C_2\phi_2 + \ldots + C_n\phi_n$$

corresponding to

$$H(\phi) = \phi_1 E_{11} + \ldots + \phi_n E_{nn}$$

The Weyl Subgroup

The Weyl group W is a finite subgroup of GL(n,C) consisting of all possible permutations of the coordinate axes (of the basis). If we decide to keep the orientation of the axes (keep det g = 1), then, we can write a system of matrices of the type

$$S = \begin{pmatrix} 1 & & & & & \\ & 1 & & & 0 & \\ & & 1 & & & \\ & 0 & & \sigma & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \qquad \sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Their products form the Weyl group with n! parameters. The importance of these matrices is in that

$$S^{-1}\delta\, S = \delta'$$

for $\delta\varepsilon D$, i.e. they take a diagonal matrix into a diagonal matrix with permuted matrix elements.

Above we have shown that if $\alpha(\delta)$ is an eigenvalue of T_δ, corresponding to an eigenvector x, common to all matrices T_δ (for all δ), then

$$\alpha(\delta) = \exp i(\lambda_1 C_1 + \ldots + \lambda_n C_n) = \delta_1^{C_1}\delta_2^{C_2}\ldots\delta_n^{C_n}$$

Let us call the set of integers

$$C = \{C_1 \ldots C_n\}$$

a spectral point.

We now have: if $C = \{C_1 \ldots, C_n\}$ is a spectral point for a representation T_g of $GL(n,C)$, then so is

$$C^S = \{C_{i_1}, C_{i_2}, \ldots, C_{i_n}\}$$

obtained from C by a permutation S.

Indeed if

$$\alpha(\delta) = \exp i \{\lambda_1 C_1 + \ldots \lambda_n C_n\}$$

is an eigenvalue corresponding to x, then $y = T_s x$ is also an eigenvector:

$$T_\delta y = T_\delta T_s x = T_{\delta s} x = T_s T_{s^+ \delta s} x = T_s \alpha(\delta') x =$$

$$\alpha(\delta') T_s x = \alpha(\delta') y$$

where $\alpha(\delta')$ is obtained from $\alpha(\delta)$ by the permutation S. The operators T_s have inverse operators, so they preserve the dimension of the eigenspace the eigenvalues determined by C and by C^S have the same multiplicity.

Remark: The noncompact group $GL(n,C)$ only figures in an auxiliary form - we wish to obtain all representations of $SU(n)$ but we only study a very specific class of representations of $GL(n,c)$, not all of them.

Lecture 15

As a further example, consider:

The Group SU(3)

Put the generators of U(3) into a table:

	E_{11}	E_{12} E_{13}
E_{21}	E_{22}	E_{23}
E_{31} E_{32}	E_{33}	

We have:

$$E_- = \{E_{21}, E_{31}, E_{32}\} \quad ; \; [E_{32}, E_{21}] = E_{31}, \; [E_{21}, E_{31}] = [E_{31}, E_{32}] = 0$$

$$E_+ = \{E_{12}, E_{13}, E_{23}\} \qquad [E_{12}, E_{23}] = E_{13}, \; [E_{13}, E_{23}] = [E_{12}, E_{13}] = 0$$

$$E_0 = \{E_{11}, E_{22}, E_{33}\}$$

As we know E_-, E_+ and E_0 are subalgebras.

Putting

$$H = \phi_1 E_{11} + \phi_2 E_{22} + \phi_3 E_{33}$$

we have

$$[H, E_{ij}] = (\phi_i - \phi_j) E_{ij}$$

All operators H can be chosen diagonal and their eigenvalues are

$$\lambda(\phi) = c_1\phi_1 + c_2\phi_2 + c_3\phi_3$$

(For the group we have

$$\alpha(\gamma \;= \exp i\lambda(\phi)).$$

Restricting ourselves to SU(3), we have det $g = 1$, i.e. Tr$H = 0$, i.e. $\phi_1+\phi_2+\phi_3 = 0$. Thus: $\{C_1,C_2,C_3\}$ and $\{C_1+t,\ C_2+t,\ C_3+t\}$ determine the same eigenvalue $\lambda(\phi)$. In other words, the parameters C_1,C_2 and C_3 are defined up to a common additive constant.

<u>Lemma</u>: If $C = \{C_1,C_2,C_3\}$ is a spectral point, then the points

$$C + \alpha_{ij} \qquad i,j = 1,2,3$$

where $\alpha_{ij} = -\alpha_{ji}$, $\alpha_{12} = (1,-1,0)$, $\alpha_{23} = (0,1,-1)$ and $\alpha_{13} = \alpha_{12}+\alpha_{23} = (1,0,-1)$ can also be eigenvectors. If the eigenvector x_λ corresponds to the eigenvalue $\lambda(\phi)$, then the vector $x_\lambda^{ij} = E_{ij}x_\lambda$ corresponds to the eigenvalue $\lambda(\phi) + \phi_i-\phi_j$ with the spectral point $C+\alpha_{ij}$. Further, $\lambda(\phi) + \phi_i-\phi_j$ is indeed an eigenvalue, unless $E_{ij}x_\lambda = 0$.

<u>Proof</u>:

$$Hx_\lambda^{ij} = HE_{ij}x_\lambda = E_{ij}Hx_\lambda + [HE_{ij}]x_\lambda = E_{ij}\lambda x_\lambda + (\phi_i-\phi_j)E_{ij}x_\lambda = (\lambda+\phi_i-\phi_j)x_\lambda^{ij}$$

The operator E_{ij} does not necessarily have an inverse, so we can have $x_\lambda^{ij} = E_{ij}x_\lambda = 0$.

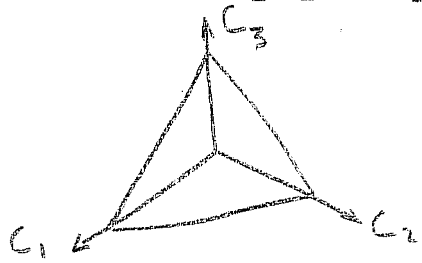
We now have two operations acting on the set of points C, namely

$$C \to C^S \quad \text{and } C \to C+\alpha_{ij}.$$

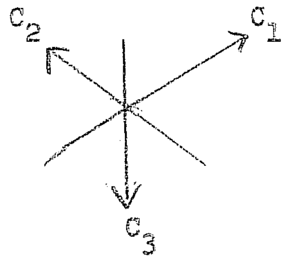
Both the Weyl transformation and the addition of α_{ij} preserve the sum of C_i, so we can normalize

$$C_1 + C_2 + C_3 = m$$

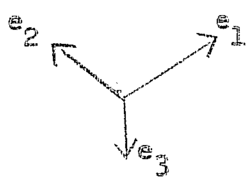
which determines a plane in the C_1, C_2 and C_3 space.



All points of our spectrum are on this plane. Project the axes onto this plane.



For U(3) the numbers C_i are integers, for SU(3), with the appropriate normalization, too. Thus the whole spectrum will be in the gridpoints of a lattice, generated by the basis vectors $\{e_1, e_2, e_3\}$:



Represent the points as:

$$e_1 = (1 \quad 0 \quad 0)$$
$$e_2 = (0 \quad 1 \quad 0)$$
$$e_3 = (0 \quad 0 \quad 1)$$

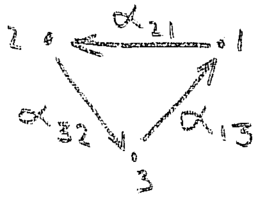
and remember that:

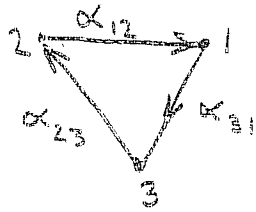
$$\alpha_{12} = (1 \quad -1 \quad 0) = -\alpha_{21}$$
$$\alpha_{23} = (0 \quad 1 \quad -1) = -\alpha_{32}$$
$$\alpha_{13} = (1 \quad 0 \quad -1) = -\alpha_{31}$$

We have: $e_1 + \alpha_{21} = e_2, \quad e_2 + \alpha_{12} = e_1$

$$e_2 + \alpha_{32} = e_3 \,, \quad e_3 + \alpha_{23} = e_2$$

$$e_3 + \alpha_{13} = e_1 \,, \quad e_1 + \alpha_{31} = e_3$$





Now consider a general point $C = (C_1, C_2, C_3)$, representing an eigenvalue. There is only a finite number of such points. Choose a set of C, namely all those for which C_1 is maximal; amongst these that, for which m_2 is maximal ($C_1+C_2+C_3$ is fixed). Call this point the <u>maximal</u> (or <u>highest</u>) weight

$$C^o = (m_1, m_2, m_3)$$

It follows symmetry (with respect to the Weyl group), that

$$m_1 \geq m_2 \geq m_3$$

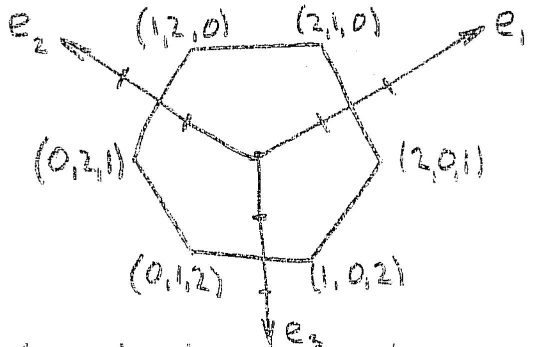
It follows from the maximality of C^o that if ξ is the corresponding eigenvector, then

$$E_{12}\xi = E_{23}\xi = E_{13}\xi = 0$$

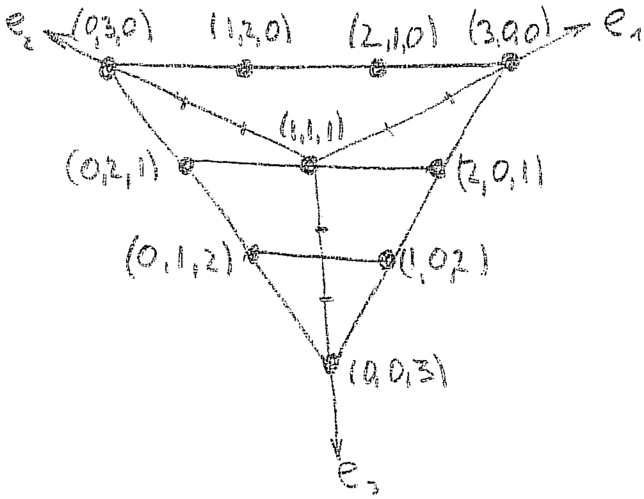
Now consider an irreducible representation. All products of the infinitesimal operators, E_{ik}, acting on ξ produce a set of vectors, the envelope of which is an invariant subspace $E_o \subseteq E$. Since the representation is irreducible, we must have $E_o = E$. Thus: all the basis vectors of the representation can be obtained by applying the lowering operators E_{21}, E_{32} and E_{31} to the vector ξ, corresponding to the highest weight. Correspondingly, all points on the spectral diagram can be obtained by successively applying α_{21}, α_{32} and α_{31} to C^o.

<u>Examples:</u>

1) $C^o = (2,1,0)$ (The octet)

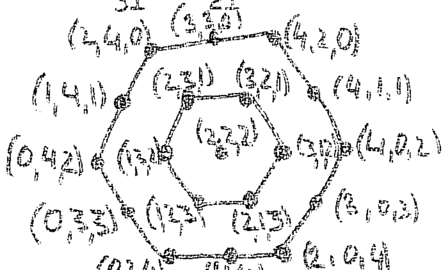


2) $C^o = (3,0,0)$ (The decuplet)



$$(3,0,0)+\alpha_{31} = (2,0,1)$$
$$(3,0,0)+2\alpha_{31} = (1,0,2)$$
$$(3,0,0)+\alpha_{21} = (2,1,0)$$
$$(3,0,0)+2\alpha_{21} = (1,2,0)$$
$$(3,0,0)+3\alpha_{21} = (0,3,0)$$
$$(3,0,0)+3\alpha_{21}+\alpha_{32} = (0,2,1)$$
$$(3,0,0)+3\alpha_{21} + 2\alpha_{32} = (0,1,2)$$
$$(3,0,0)+\alpha_{31} + \alpha_{21} = (+1,1,1)$$

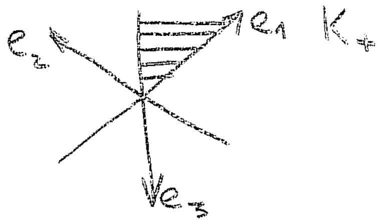
3) $C^o = (4,2,0)$ (A 27-plet)



The above examples indicate the following result:

<u>Theorem:</u> The spectral points for the irreducible representations of SU(3) lie on a hexagon in the plane $C_1 + C_2 + C_3 = m$ and on a system of hexagons, inserted into the largest one, at definite distances. Some or all of the hexagons sometimes degenerate into triangles (if one of the points lies on

one of the three axes). The vector ξ, corresponding to the highest weight, is determined uniquely (up to normalization). All the highest weights lie in the Weyl chamber K_+, where $m_1 \geq m_2 \geq m_3$.



Further: the multiplicities of the eigenvalues are constant along each hexagon (triangle) and are equal to one on the outmost hexagon. Moving inwards, the multiplicity either increases linearly, or stays constant (starting from the first hexagon, which degenerates into a triangle).

We shall return to this problem, but at present we drop the proof.

Irreducible Representations of U(n)

Let us take n general and to avoid algebraic complications, use a global approach, rather than the infinitesimal one. Thus, we shall actually construct representations, generalizing formulae like

$$T_g f(z) = (\beta z + \delta)^{2\ell} f\left(\frac{\alpha z + \gamma}{\beta z + \delta}\right)$$

for SU(2).

An important step is the parametrization of the group. We shall use a decomposition, already mentioned above.

The Gauss Decomposition

For $G = SL(2,C)$ we can write any matrix as

$$g = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \quad \det g = \delta_1 \delta_2 = 1$$

Now consider $G = GL(2,n)$ and consider three subgroups Z_-, D and Z_+, where

$$z_- = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ * & & 1 \end{pmatrix} \quad \delta = \begin{pmatrix} \delta_1 & & & 0 \\ & \delta_2 & & \\ & & \ddots & \\ 0 & & & \delta_n \end{pmatrix} \quad z_+ = \begin{pmatrix} 1 & & * \\ & 1 & \\ 0 & & \ddots & 1 \end{pmatrix}$$

($z_- \varepsilon Z_-$, $z_+ \varepsilon Z_+$ and $\delta \varepsilon D$).

We know that these matrices generate the algebras

$$E_-,\ E_o \text{ and } E_+$$

It is (supposedly) well-known that if the diagonal subdeterminants of a matrix

$$g = \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ & & \\ g_{n1} & \cdots & g_{nn} \end{pmatrix}$$

are not equal to zero, i.e. if

$$\Delta_i \neq 0 \quad i = 1,\ldots,n$$

where

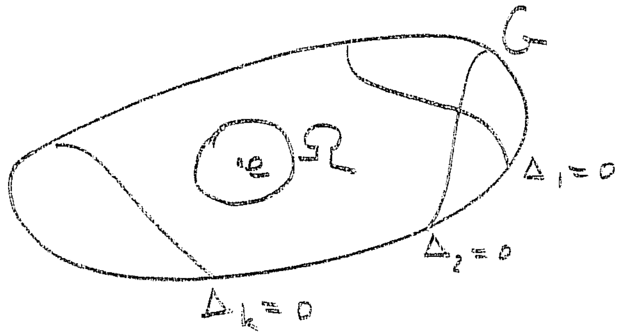
$$\Delta_1 = g_{11} \qquad \Delta_2 = \begin{vmatrix} g_{11} g_{12} \\ g_{21} g_{22} \end{vmatrix} \qquad \Delta_n = \det g$$

then g can be written as

$$g = z_- \delta z_+ \qquad z_- \varepsilon Z_-,\ z_+ \varepsilon Z_+, \quad \delta \varepsilon D$$

where z_-, δ and z_+ are determined uniquely.

We can exclude a set $\theta \varepsilon G$ from the group, for which at least one $\Delta_i = 0$. θ will not contain the identity e and there exists a neighborhood Ω of e where Ω is not contained in θ. Excluding the set θ, which is of lower dimension than G, we obtain a parametrization of the group:



Lie's Theorem

A well-known theorem, which can be checked directly is: In a finite-dimensional space over the field of complex numbers every commutative family of linear operators has at least one common eigenvector.

Sophus Lie generalized this theorem to some classes of non-commutative operators. We shall neither prove Lie's theorem, nor give it's most several formulation but only use a special case (since we are not going into the theory of solvable groups).

Thus, consider the group H of triangular matrices:

$$h = \begin{pmatrix} h_{11} & h_{12} & \cdots & h_{1n} \\ & h_{22} & \cdots & h_{2n} \\ & 0 & \ddots & \\ & & & h_{nn} \end{pmatrix} \qquad \det h \neq 0$$

Special case of Lie's theorem: In any finite-dimensional representation of the group H there exists at least one non-zero vector x, which is the eigenvector of all operators of the group H. Thus:

$$h \longrightarrow T_h$$

$$T_h x = \lambda(h) x$$

where $\lambda(h)$ is a continuous function, which itself is a one-dimensional representation of H

$$\lambda(h_1 h_2) = \lambda(h_1)\lambda(h_2)$$

$$\lambda(e) = 1$$

<u>Corollary:</u> Every finite-dimensional representation of T_h can be brought to a triangular form.

<u>Proof:</u> One common eigenvector exists $\Rightarrow$ all matrices T_h can be reduced to the form

$$T_h = \left(\begin{array}{c|cccc} \lambda(h) & * & * & \cdots & * \\ \hline 0 & & & & \\ \vdots & & & & \\ \vdots & & R_h & & \\ 0 & & & & \end{array}\right)$$

The matrices R_h again form a representation of H, so that we can again apply Lie's theorem, thus gradually reducing T_h to triangular form.

<u>Remark:</u> In the infinitesimal language this corresponds to choosing a basis in the representation space diagonalizing the operators E_0 and then enumerating the basis vectors in such a fashion that E_+ operators act in one direction, E_- in the opposite one.

It follows, that the groups Z_-, D and Z_+ "keep" their structure:

$$T_{z-} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} \quad T_\delta \quad \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix} \quad T_{z+} = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

where $\alpha_i(\delta)$ are the characters of the diagonal group D.

When considering the right regular representation

$$R_{g_o}f(g) = f(gg_o)$$

we saw that each row of the matrix

$$T_g = \|T_{ij}(g)\|$$

forms an invariant subspace with respect to P_{go} and that the representation T_g can be realized in this subspace. Put $G = z_-\delta z_+$ and find the first row of $T_g = T_{z-}T_\delta T_{z+}$.

We obtain the first row as:

$$\alpha(\delta),\ \alpha(\delta)T_{12}(z),\ldots,\alpha(\delta)T_{1N}(z)$$

where

$$\alpha(\delta) \equiv \alpha_1(\delta) \qquad z \equiv z_+$$

Thus: $T_{11}(g)$ depends on $\delta\epsilon D$ only, $T_{1k}(g)$, $k = 2,\ldots,N$ depend on δ in the same simple manner and do not depend on z_- at all.

<u>Note</u>: The Lie algebra of the group Z_+ is E_+, consisting of matrices

$$x = \begin{pmatrix} 0 & * & * \\ & 0 & * \\ 0 & & \ddots \ 0 \end{pmatrix}$$

with the basis e_{ij} $\quad i < j$.

We shall combine the infinitesimal and the global method.

The maximal eigenvector

For SU(2) we had a highest weight ℓ and to each ℓ corresponded one definite eigenvector x_ℓ. Call it the maximal eigenvector.

More generally: A maximal eigenvector of a representation T_g of G is a non-zero solution x of the set of equations

$$E_{ij}x = 0 \qquad i < j$$

for all infinitesimal operators $E_{ij} \in E_+$. In terms of the group this corresponds to the equation

$$T_z x = x$$

for all $z \in Z_+$. Thus, a maximal eigenvector is an invariant of the subgroup Z_+.

Existence of a maximal eigenvector: Put $H = DZ_+$. It follows from Lie's theorem that there exists a common eigenvector $x \neq 0$ of T_h, for which

$$E_{ij}x = \lambda_{ij}x \qquad i \leq j$$

However

$$E_{ij} = [E_{ii}, E_{ij}] \qquad i < j$$

so that $\lambda_{ij} = 0$ for $i < j$. Thus, x is maximal

$$e_{ij}x = 0 \qquad i < j.$$

and is also an eigenvector:

$$\boxed{E_{ii}x = m_i x} \qquad m_i \equiv \lambda_{ii}$$

Using canonical parameters in the group, i.e.

$$\delta_k = e^{it_k} \quad , \quad t_1 \dots t_n \text{ complex}$$

we have

$$T_\delta x = \alpha(\delta)x$$

where

$$\alpha(\delta) = e^{im_1t_1+\dots+im_nt_n} = \delta_1^{m_1}\delta_2^{m_2}\dots\delta_n^{m_n}$$

It follows from the analyticity of $\alpha(\delta)$, i.e. from the single-valuedness on the torus $\delta_k = e^{it_k}$ $0 \le t_k < 2\pi$ that $m_1,\dots,m_n$ are integers.

Consider the SU(2) subalgebra

$$E_+ = E_{12},\ E_0 = E_{11}-E_{22},\ E_- = E_{21}$$

We have $E_+x = 0$, so that x is the maximal eigenvector of this subalgebra too. We have

$$E_0x = (m_1-m_2)x$$

We know that the highest weight for SU(2) is non-negative. Thus we obtain $m_1 \ge m_2$ and generally

$$m_1 \ge m_2 \ge \dots \ge m_n$$

Put: $\alpha = (m_1,m_2,\dots m_n)$ and call it the <u>highest weight</u> of representation T_g. Weyl calls the set $(m_1\dots m_n)$ the <u>signature</u>.

We have proved the following:

<u>Theorem</u>: For every analytic representation of the group G there exist an invariant of the subgroup Z_+:

$$T_z x = x$$

called the maximal eigenvector, satisfying

$$T_\delta(x) = \alpha(\delta)x.$$

The eigenvalue is an exponential.

$$\alpha(\delta) = \delta_1^{m_1} \dots \delta_n^{m_n}$$

and the numbers $m_1, \dots, m_n$ are integers satisfying:

$$m_1 \geq m_2 \geq \dots \geq m_n.$$

Similarly we can define a minimal eigenvector of T_g as an invariant with respect to the subgroup Z_-.

Uniqueness of the maximal eigenvector. Let us show that if T_g is irreducible then the maximal eigenvector x is unique (up to normalization).

Let x be a maximal eigenvector in the space E of T_g and let ε be a minimal eigenvector in the space $\hat{E}$ of the dual representation. (Remember that T_g is contragredient to $\hat{T}_g$ if there exists a nondegenerate bilinear invariant form

$$(x,\hat{x}) = (T_g x, \hat{T}_g \hat{x}) \qquad x \in E, \ \hat{x} \in \hat{E}.$$

We have $\quad \hat{T}_g = (T_g^T)^{-1}.)$

These vectors satisfy

$$(\varepsilon, x) \neq 0$$

since otherwise we would have

$$(\varepsilon, T_g x) = (\varepsilon, T_{z-} T_\delta T_{z+} x) = (\hat{T}_{z-}^{-1} \varepsilon, T_\delta T_{z+} x) = (\varepsilon, T_\delta x) = \alpha(\delta)(\varepsilon, x) = 0$$

Irreducibility of T_g implies that the vectors $T_g x$ generate the whole space E. Thus $(\varepsilon, E) = 0$ which contradicts the assumption that $(x,\hat{x})$ is nondegenerate.

Let us assume there are two highest weights in E, x_1 and x_2. Normalize so that

$$(\varepsilon, x_1) = 1 \qquad (\varepsilon, x_2) = 1$$

Since x_1 and x_2 are invariants of Z_+, so is $x_1 - x_2$. However

$$(\varepsilon, x_1 - x_2) = 0$$

$\Longrightarrow \quad x_1 - x_2 = 0$

We obtain the theorem:

Theorem: If T_g is an irreducible representation, then the maximal vector x is determined uniquely up to normalization. Thus, the highest weight is also determined uniquely.

Corollary: If two different representations T_g and S_g of G have different highest weights, then they are not equivalent.

Lecture 16

Let us now consider realizations of T_g in various spaces.

Realization on the Group G

Consider the right regular representation.

$$R_g f(g) = f(gg_o) \qquad (1)$$

For the matrix elements of any representation we have:

$$T_{ik}(gg_o) = T_{i\alpha}(g)T_{\alpha k}(g_o) \qquad (2)$$

Put

$$e_i = T_{1i}(g) \qquad (3)$$

(elements of the first row are chosen as a basis). Then:

$$T_{go}e_i = T_{1i}(gg_o) = \sum_\alpha T_{1\alpha}(g)T_{\alpha i}(g_o) = \sum_\alpha e_\alpha T_{\alpha i}(g_o) \qquad (4)$$

Let us construct the basis vectors e_i explicitly. Take a representation T_g and denote it's maximal eigenvector e_1. Let ε be the minimal eigenvector of the contragradient representation T_g. Normalize e_1 (for fixed ε) so that

$$(\varepsilon e_1) = 1 \qquad (5)$$

The equation $(\varepsilon,x) = 0$ determines an N-1 dimensional hyperplane in the space E. Choose the other basis vectors in this hyperplane, so that

$$(\varepsilon,e_k) = 0 \quad \text{for } 2 \le k \le N \qquad (6)$$

Now apply this bilinear form to eq. (4) for i=1

$$(\varepsilon, T_g e_1) = T_{11}(g) = e_1(g) \qquad (7)$$

In terms of the Gauss parameters: $g = z_- \delta z_+$

$$e_1(g) = (\varepsilon, T_{z-} T_\delta T_{z+} e_1) = (\hat{T}_{z-}^{-1}\varepsilon, T_\delta e_1) = (\varepsilon, T_\delta e_1) = e_1(\delta) \qquad (8)$$

We also have $T_\delta e_1 = \alpha(\delta) e_1$, so that

$$\boxed{e_1(\delta) = \alpha(\delta)} \qquad (9)$$

where

$$\alpha(\delta) = \delta_1^{m_1} \ldots \delta_n^{m_n} \qquad (10)$$

is the highest weight.

Let us express the parameters of δ in terms of those of g. Consider again the diagonal subdeterminants of g:

$$|g| = \begin{vmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ g_{n1} & & \cdots & g_{nn} \end{vmatrix}$$

One can use the expression $g = z_- \delta z_+$ to check that these subdeterminants are:

$$\Delta_1(g) = \delta_1 \qquad \Delta_2(g) = \delta_1\delta_2 \qquad \Delta_n(g) = \delta_1\delta_2\ldots\delta_n \qquad (11)$$

Putting

$$\Delta_0 = 1$$

We have

$$\delta_k = \frac{\Delta_k(g)}{\Delta_{k-1}(g)} \tag{12}$$

We can now write the basis vector e_1 as:

$$e_1(g) = g_{11}^{\;m_1-m_2} \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix}^{m_2-m_3} \ldots (\det g)^{m_n} \tag{13}$$

where the exponents

$$p_1 = m_1-m_2,\ p_2 = m_2-m_3 \qquad p_{n-1} = m_{n-1}-m_n,$$

satisfy

$$\begin{array}{l} p_k > 0 \qquad 1 \leq k \leq n-1 \\ p_n = m_n \ \ldots \text{ arbitrary} \end{array}$$

If T_g is irreducible, then all other basis vectors can be obtained by applying right translations to e_1:

$$R_{go}\, e_1(g) = e_1(gg_o)$$

We obtain the theorem:

<u>Theorem:</u> All irreducible analytic representations T_g of the group $G = GL(n,C)$ are determined by a set of n numbers (the signature) $\alpha = (m_1,\ldots,m_n)$

$$m_1 \geq m_2 \geq m_3 \geq \ldots \geq m_n$$

where m_i are integers. The representation T_g^α can be realized in a space E_α, spanned by the function

$$e_1(g) = g_{11}^{m_1-m_2} \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix}^{m_2-m_3} \ldots (\det g)^{m_n}$$

and all functions $e_1(gg_o)$ obtained by right translations. The signature α determines the representation uniquely (up to equivalence). The action of the operators T_g in the space E_α is given as

$$T_{go}f(g) = f(gg_o)$$

and $e_1(g)$ is the maximal eigenvector of the representation (invariant under Z_+).

Remark: The restriction to SL(n,C) corresponds to putting det g = 1. We obtain all analytic irreducible representations of G. For SL(n,C) the signature is no longer unique and has to be normalized, e.g. by putting $m_n = 0$ or $m_1+m_2+\ldots+m_n = 0$. The space E_α for SL(n,C) consists of polynomials on the group. (For GL(n,C) not necessarily, since we can have $m_n < 0$.)

Realization on the Maximal Compact Subgroup U

If we are interested only in the representations of the maximal compact subgroup U = U(n), then we can make use of the principle of unitary restriction to obtain the following.

Theorem: All irreducible representations of U = U(n) are determined (up to equivalence) by a set of integers

$$m_1 \geq m_2 \geq \ldots \geq m_n$$

and can be realized in a linear space of functions f(u), spanned by the function:

$$e_1(u) = u_{11}^{m_1-m_2} \begin{vmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{vmatrix}^{m_2-m_3} \ldots (\det u)^{m_n} \qquad (14)$$

and all functions $e_1(uu_o)$ obtained by right translations. All irreducible representations of SU(n) are obtained here by setting det u = 1. The operators of the representation act as

$$R_{u_o} f(u) = f(uu_o).$$

<u>Remark:</u> If we construct the space E_α as above by applying right translations to $e_1(u)$ and then use also left translations on the group U, then the space E_α gets extended to M_α spanned by all matrix elements $T_{ij}(u)$. The representation acting in this space is reducible.

<u>Realization on the Group Z</u>

Above the signature α did not enter explicitly into the formula for the action of T_g, but only into the definition of the representation space E_α. Let us construct a different realization in which the operators depend explicitly on α. Since the vector ε is defined as the minimal eigenvector of the contragradient representation the basis vectors

$$e_i(g) = (\varepsilon, T_g e_i)$$

do not depend on z_- in the Gauss decomposition. Thus we have $e_i(g) = T_{1i}(g) = T_{1i}(z_-\delta z_+) = T_{1\alpha}(z_-)T_{\alpha\beta}(\delta)T_{\beta i}(z_+) = \delta_{1\alpha}\delta_{\alpha\beta}T_{11}(\delta)T_{\beta i}(z_+)$ $= \alpha(\delta)T_{1i}(z_+)$. Finally

$$e_i(g) = \alpha(\delta)e_i(z)$$

where we put $z_+ \equiv z$. The same must then be true for all elements of E_α, so that

$$f(g) = \alpha(\delta)f(z) \qquad g = z_-\delta z_+ \tag{15}$$

Thus, to each function $f(g)$ on the group G there corresponds a function $f(z)$ on the subgroup $Z = (Z_+)$ namely the restriction of $f(g)$ from G to Z. As can be seen from (15), the correspondence between $f(z)$ and $f(g)$ is one-to-one.

Denote R the obtained space of functions $f(z)$ and consider the action of T_g in this space. We have $Tg_o f(g) = f(gg_o)$. Restricting $f(g)$ to $f(z)$, we have:

$$Tg_o f(z) = f(zg_o) \tag{16}$$

Put

$$zg_o = \tilde{z}_- \tilde{\delta} \tilde{z}, \tag{17}$$

then

$$Tg_o f(z) = \alpha(\tilde{\delta}) f(\tilde{z})$$

To find the multiplier $\alpha(\tilde{\delta}) = \alpha(z, g_o)$ we notice that $\tilde{\delta}$ is the diagonal part in the decomposition of zg_o. Using (9) and (13) we have:

$$\alpha(z, g_o) = (zg)_{11}^{m_1 - m_2} \begin{vmatrix} (zg)_{11} & (zg)_{12} \\ (zg)_{21} & (zg)_{22} \end{vmatrix}^{m_2 - m_3} \ldots (\det g)^{m_n} \tag{18}$$

(we have $\det zg = \det g$).

We have obtained:

<u>Theorem:</u> The irreducible representation with signature $\alpha = (m_1, \ldots m_n)$ of G can be realized in the space of polynomials of the matrix

$$z = \begin{pmatrix} 1 & z_{12} z_{13} \ldots z_{1n} \\ & 1 \quad z_{23} \ldots z_{2n} \\ 0 & \ddots \\ & \quad 1 \end{pmatrix} \qquad z \in Z \tag{19}$$

The operators of $g \to T_g$ are given explicitly as

$$T_g f(z) = \alpha(z,g) f(z_g) \tag{20}$$

where the multiplier $\alpha(z,g)$ is

$$\alpha(z,g) = \Delta_1^{p_1}(zg)\,\Delta_2^{p_2}(zg)\ldots\Delta_n^{p_n}(zg) \tag{21}$$

$p_i = m_i - m_{i+1}$, $i = 1,\ldots n$, $(m_{n+1} = 0)$. Further z_g is the right triangular matrix in the Gauss decomposition for zg. The maximal eigenvector is

$$e_i(z) = 1$$

and the space R_α is spanned by the functions

$$f_g(z) = T_g . 1 = \alpha(z,g)$$

for all $g \in G$.

<u>Remark 1</u>: Formula (20) simplifies in special cases:

a) $g = z_o \in Z$

Obviously $\alpha(z,z_o) = 1$, so that

$$T_{z_o} f(z) = f(zz_o) \tag{22}$$

b) $g = \delta$

$$T_\delta f(z) = f(z\delta) = f[\delta(\delta^{-1} z\delta)]$$

We have $\delta^{-} z\delta \in Z$, hence

$$T_\delta f(z) = \alpha(\delta) f(\delta^{-1} z\delta) \tag{23}$$

Remark 2: The realization in the space $f(z)$ is convenient in that it involves a minimal number of variables. For n=2 this realization reduces to

$$T_g f(z) = (\beta z+\delta)^p f\left(\frac{\alpha z+\gamma}{\beta z+\delta} \right) .$$

Explicit Expression for the Gauss Parameters

Put

$$g = \xi \delta z \qquad \xi \varepsilon Z_-, \delta \varepsilon D, \; z \varepsilon Z_+ \tag{24}$$

Call

$$g^{i_1 \ldots i_p}_{j_1 \ldots j_p}$$

the subdeterminant of $|g|$ obtained from the rows $i_1, \ldots, i_p$ and columns $j_1, \ldots, j_p$. Making use of standard multiplication rules for determinants, we find

$$g^{12\ldots p-1p}_{12\ldots p-1q} = \Delta_p \cdot z^{12\ldots p-1p}_{12\ldots p-1q} = \Delta_p z_{pq} \tag{25}$$

where Δ_p is a diagonal subdeterminant of $|\delta|$. For p=q we have:

$$g^{12\ldots p-1p}_{12\ldots p-1p} = \Delta_p \tag{26}$$

(in agreement with (11)).

For $p > q$ we get 0=0. For $p \leq q$ we have

$$z_{pq} = \frac{\Delta_{pq}}{\Delta_p} \qquad p \leq q \tag{27}$$

where $\Delta_{pq} = g^{12\ldots p-1p}_{12\ldots p-1q}$.

Similarly:

$$\xi_{pq} = \frac{\Delta'_{pq}}{\Delta_p} \qquad p \geq q \tag{28}$$

where $\Delta'_{pq} = g^{12\ldots q-1p}_{12\ldots q-1q}$

We can now put:

$$f(z) = f(z_{\mu\nu}) \quad \mu, \nu = 1,2,\ldots n, \quad \mu < \nu \tag{29}$$

and the transformation T_g of (20), representing $g \epsilon G$, can be written as:

$$T_g f(z_{\mu\nu}) = \Delta_1^{p_1} \Delta_2^{p_2} \ldots \Delta_n^{p_n} f(\frac{\Delta_{\mu\nu}}{\Delta_\mu}) \tag{30}$$

where the subdeterminants $\Delta_{\mu\nu}$ and Δ_μ are calculated for the matrix zg.

Remark 1: The transformation $\tilde{z}_{\mu\nu} = \frac{\Delta_{\mu\nu}(zg)}{\Delta_\mu(zg)}$ is a generalization of the transformation

$$\tilde{z} = \frac{\alpha z + \gamma}{\beta z + \delta}$$

for n = 2.

Remark 2: For GL(2,C) (or U(2))we have also constructed a basis in the representation space, namely the monomials

$$1, z, z^2, \ldots z^p \qquad p = 2\ell$$

In the general case we could also construct a complete set of linearly independent basis vectors for each signature α, but we shall not go into that here.

Lecture 17

Fundamental Representations

Let us consider a certain set of signatures, namely:

$$\begin{aligned}
\Delta_1 &= (1 \;\; 0 \;\; 0 \ldots 0) \\
\Delta_2 &= (1 \;\; 1 \;\; 0 \ldots 0) \\
&\ldots\ldots\ldots \\
\Delta_n &= (1 \;\; 1 \;\; 1 \ldots 1)
\end{aligned} \tag{1}$$

<u>Theorem</u>: Let E be an n-dimensional space in which $G = GL(n,C)$ acts as a group of linear transformations. Consider all completely antisymmetric tensors

$$p^{i_1 i_2 \ldots i_k} \qquad\qquad k \leq n$$

over E. They form an irreducible set and transform according to the irreducible representation of G with signature Δ_k.

<u>Proof:</u> Take a set of numbers

$$x = (x_1 \ldots x_n)$$

transforming covariantly under G. In terms of matrices we have

$$(x_1', \ldots, x_n') = (x_1, \ldots, x_n) \begin{pmatrix} g_{11} \ldots g_{1n} \\ g_{n1} \ldots g_{nn} \end{pmatrix}$$

The determinants

$$e_{i_1 i_2 \ldots i_k} = \begin{vmatrix} x_{i_1} \ldots x_{i_k} \\ y_{i_1} \ldots y_{i_k} \\ \\ w_{i_1} \ldots w_{i_k} \end{vmatrix} \tag{2}$$

formed out of k such "rows" of numbers provide a basis in the space of antisymmetric tensors. Take

$$\delta = \begin{pmatrix} \delta_1 & & 0 \\ & \delta_2 & \\ 0 & & \ddots \ \delta_n \end{pmatrix} \tag{3}$$

and act upon each row in (2). We obtain:

$$T_\delta e_{i_1 i_2 \ldots i_k} = \mu e_{i_1 i_2 \ldots i_k} \tag{4}$$

i.e. $e_{i_1 \ldots i_k}$ is an eigenvector of T_δ, corresponding to the eigenvalue

$$\mu = \delta_{i_1} \delta_{i_2} \ldots \delta_{i_k}$$

The antisymmetry implies that each set $i_1, \ldots, i_k$ appears only once so that all eigenvalues μ are different. Thus, one of the determinants (2) must be the maximal eigenvector. The only one of them invariant with respect to $Z = Z_+$ is $e_{12 \ldots k}$, with the weight

$$\delta_1 \delta_2 \ldots \delta_k = \Delta_k .$$

Hence, since there is only one maximal eigenvector, the antisymmetric tensors transform irreducibly according to the representatio Δ_k. Q.E.D.

Young Patterns

We write the signature as

$$\alpha = (m_1, \ldots m_n) \qquad m_1 \geq m_2 \geq \ldots \geq m_n$$

and denote the corresponding representation D_α (or just α). Sometimes it is convenient to introduce a different set of integers:

$$p_i = m_i - m_{i+1} \quad i = 1,\ldots,n, \qquad m_{n+1} = 0 \tag{5}$$

and write

$$[p_1,\ldots,p_n] \qquad p_i \geq 0$$

(in square brackets). Now p_i are arbitrary nonnegative integers. In these parameters the signatures of the fundamental representations will be

$$\begin{aligned} \Delta_1 &= [1,0,\ldots 0] \\ \Delta_2 &= [0,1,\ldots 0] \\ &\text{-------------} \\ \Delta_n &= [0,0,\ldots 1] \end{aligned} \tag{6}$$

If the signature of a representation is

$$(m_1,\ldots,m_n) = [p_1,\ldots,p_n]$$

then the highest weight is

$$\alpha(\delta) = \delta_1^{m_1} \cdots \delta_n^{m_n} = \Delta_1^{p_1} \Delta_2^{p_2} \cdots \Delta_n^{p_n}$$

where δ_i are the diagonal elements of δ and Δ_i are the corresponding subdeterminants. We know that in the realization of representations T_g on the group G, the maximal eigenvector is

$$e_1(g) = \Delta_1^{p_1}(g)\ \Delta_2^{p_2}(g)\ \ldots\ \Delta_n^{p_n}(g) \tag{7}$$

(see previous lecture).

Let us introduce the Young patterns. For each representation Δ_k we draw a pattern:

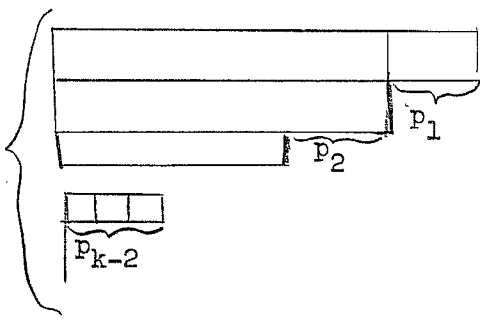
$\Delta_k \sim$

consisting of k boxes. The pattern indicates that Δ_k corresponds to a tensor $e_{i_1 \ldots i_k}$ with k indices, that it is completely antisymmetric and can be obtained as the "antisymmetric product" of k vectors according to (2).

Let us generalize to arbitrary signatures:

$[p_1, \ldots, p_n] \sim$

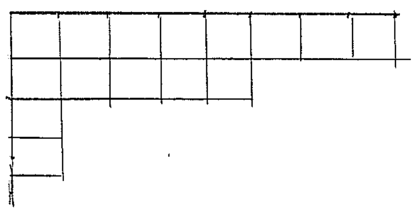
n



We have n rows (the bottom ones can be empty), getting shorter (or staying equal) as we go down. The numbers $(m_1 \ldots m_n)$ are the lengths of the rows, the numbers $[p_1, \ldots, p_n]$, the differences between successive lengths.

Example:

$(8,5,1,1,0) = [3,4,0,1,0]$



The Young patterns will be used to study symmetry and antisymmetry properties of irreducible representations realized with the help of tensors.

Tensors and Young Tableaux

We already know that all irreducible representations of any compact group can be realized in the class of tensors.

For $U(n)$ we shall consider the contravariant tensor;

$$t = t^{i_1 i_2 \ldots i_m} \tag{8}$$

of rank m, and decompose the corresponding representations of G into irreducible ones:

$$T_g = \begin{pmatrix} \boxed{\alpha_1} & & 0 \\ & \boxed{\alpha_2} & \\ 0 & & \boxed{\alpha_k} \end{pmatrix}$$

Obviously an operator U_s, permuting the superscripts of t, will commute with any operator T_g (since T_g acts independently on each superscript). It follows from Schur's lemma, that equations like

$$U_s t = \lambda(s)t.$$

can be used to obtain irreducible subspaces, (or subspaces that are multiples of an irreducible one). We shall show that we can get all irreducible representations in this fashion. Actually, we shall show that all operators P_α, projecting out irreducible representations with a given signature $\alpha = (m_1, \ldots, m_n)$ can be written as

$$P_\alpha = \sum_s c(s) U_s \tag{9}$$

where the c(s) are numbers. We have

$$t_\alpha = P_\alpha t$$

where t_α transforms according to the irreducible representation α and

$$\sum_\alpha P_\alpha = I ,$$

where I is the identity operator.

We could proceed by investigating the structure of the symmetric group π_m, but shall first consider a more direct method.

The Method of Z-invariants

Let $g \to T_g$ be a representation of G in a finite-dimensional space E. We know that T_g is completely reducible (unless it is irreducible), so that the space E can be decomposed into a direct sum of invariant subspaces

$$E = E_1 + \ldots + E_k$$

In each space E_i we have one highest weight $\alpha_i = \alpha_i(\delta)$ and corresponding maximal eigenvector ω_i.

Thus, we can decompose T_g into irreducible components by the following procedure:

1) Find all invariants of the group Z in the space E, i.e. all vectors that could be maximal eigenvectors.

2) Amongst these invariants find all eigenvectors ω_i:

$$T_\delta \omega_i = \alpha_i \omega_i$$

and enumerate all highest weights $\alpha_i(\delta)$.

Once we find ω_i we can immediately obtain the whole space E_i by applying elements of Z^- to ω_i.

Criterion: A finite dimensional representation of G is irreducible if and only if it contains one and only one non zero invariant of the subgroup Z.

Thus: We shall consider contravariant tensors, transforming according to the representation

$$T_g = \hat{g} \otimes \hat{g} \otimes \dots \otimes \hat{g} \tag{10}$$

Instead of $t^{i_1 i_2 \dots i_m}$ we can use the multilinear form

$$\Phi(x,y,\dots,w) = t^{i_1 \dots i_m} x_{i_1} y_{i_2} \dots w_{i_m} \tag{11}$$

where $x,y,\dots w$ are covariant vectors of dimension n

$$x = (x_1,\dots,x_n)$$

$$\text{------------}$$

$$w = (w_1,\dots,w_n)$$

We can write these vectors as rows, their transformation is right multiplication by g. Thus:

$$T_g \Phi(x,y,\dots w) = \Phi(xg,yg,\dots wg) \tag{12}$$

Denote the space of all multilinear forms of order m (all tensors $t^{i_1 i_2 \dots i_m}$) Φ_m.

Our aim is to decompose T_g acting in Φ_m into irreducible components.

The System of Basis Invariants

Consider $z \in Z$ acting on x:

$$xz = (x_1, x_2, \ldots x_n) \begin{pmatrix} 1 & z_{12} & z_{13} & \cdots & z_{1n} \\ 0 & 1 & z_{23} & \cdots & z_{2n} \\ 0 & 0 & 1 & & z_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & 1 & z_{n-1n} \\ & & & & 1 \end{pmatrix} = (x_1, x_1 z_{12} + x_2, \; x_1 z_{13} + x_2 z_{23} + x_3, \ldots)$$

The only Z-invariant in the space Φ_1 is

$$\omega_1(x) = x_1$$

More generally, the subdeterminants

$$\omega_1 = x_1$$

$$\omega_2 = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \tag{13}$$

$$\omega_n = \begin{vmatrix} x_1 & \cdots & x_n \\ & & \\ w_1 & \cdots & w_n \end{vmatrix}$$

are Z-invariants. We already know that the determinants

$$\begin{vmatrix} x_{i_1} & \cdots & x_{i_k} \\ & & \\ w_{i_1} & \cdots & w_{i_k} \end{vmatrix}$$

form a basis in the spaces of antisymmetric tensors of order k and that the Z-invariants ω_k are the maximal eigenvectors for the corresponding representations. The highest weight Δ_k corresponding to ω_k is

$$\Delta_k = \delta_1\delta_2\ldots\delta_k \qquad k = 1,\ldots,n$$

No other totally antisymmetric tensors can be constructed, out of n-dimensional vectors.

However, more general symmetries can be considered. Thus, any vector of the type

$$\omega = \omega_1^{p_1}\omega_2^{p_2}\ldots\omega_n^{p_n} \tag{14}$$

is a Z-invariant. Note that ω is a multilinear form of order

$$m = p_1 + 2p_2 + 3p_3+\ldots+np_n \tag{15}$$

Applying T_δ we find

$$T_\delta\omega = \alpha\omega$$

with

$$\alpha = \Delta_1^{p_1}\Delta_2^{p_2}\ldots\Delta_n^{p_n} = \tag{16}$$

$$= \delta_1^{m_1}\delta_2^{m_2}\ldots\delta_n^{m_n}$$

where

$$p_i = m_i - m_{i+1} \quad i = 1,\ldots,n \qquad m_{n+1} = 0 \tag{17}$$

(This is of course just an alternative way of viewing the formulas, obtained previously).

The rank of the tensors under consideration is obviously

$$m = m_1+m_2+\ldots+m_n$$

where m_i are the entries in the signature

$$(m_1,m_2,\ldots,m_n)$$

The representation with signature α will in general figure several times in a decomposition, since we can choose different orderings of the vectors $x,y,\ldots$ when writing the system of determinants ω_i. This, however is the only arbitrariness in the problem. Indeed, we have

Lemma on Z-invariants: Any multilinear form, invariant with respect to the group Z can be written as a linear combination of the monomials

$$\omega = \omega_1^{p_1}\ldots\ \omega_n^{p_n}$$

with all possible orderings of the arguments in the determinants $\omega_1,\ldots,\omega_n$ and with all possible exponents p_i, satisfying $p_1+2p_2+..np_n = m$, where m is the rank of the form.

Proof: A (classical) proof of this (classical) theorem is given by Weyl in his (classical) book:

H. Weyl: The Classical Groups

(Princeton University Press, Princeton, 1946)

From the lemma one can readily obtain the fundamental theorem:

<u>Theorem:</u> The space Φ_m of all tensors of rank n can be split into the direct sum of irreducible subspaces

$$E_\sigma(m_1,\ldots,m_n)$$

in which the maximal eigenvectors are the monomials

$$\omega_\sigma = \omega_1^{m_1-m_2}\omega_2^{m_2-m_3}\ldots\omega_n^{m_n}$$

The index σ runs through all possible substitutions of the vector arguments $x,y,\ldots w$ into the determinants $\omega_1,\ldots,\omega_n$. The multiplicity with which the representation $(m_1,\ldots,m_n)$ occurs in the space Φ_m is equal to the number of linearly independent monomials

$$\omega_{\sigma_1},\omega_{\sigma_2},\ldots,\omega_{\sigma_k}$$

with fixed exponents $m_1,\ldots,m_n$. Those and only those signatures appear, for which

$$m_1+m_2+\ldots+m_n = m \qquad m_i \geq 0.$$

Actually, the complete information on the reducibility of tensors is contained in this theorem (we do not give the proof). However, we shall still look at the symmetries of the tensor representations.

Let us return to the <u>Young Tableaux</u>.

Consider a table of boxes, corresponding to a given signature

$$\alpha = (m_1,\ldots,m_n) \qquad m_1 \geq m_2 \geq m_3\ldots\geq m_n \geq 0.$$

with the properties:

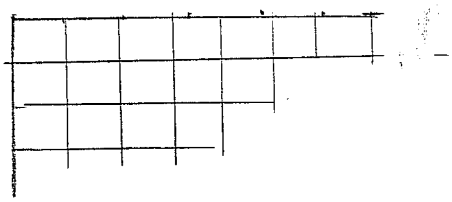
1) The lengths of the rows decrease or stay constant in the downward direction.

2) The total number of boxes m is equal to the rank of the tensor under consideration.

3) For the group $U(n)$ the number of rows is less or equal to n.

<u>Example:</u>

$\alpha = (7,5,4,0) = [2,1,4,0]$



(If n is a fixed known number, then we do not have to draw the empty rows, otherwise they are essential).

Besides the round and square brackets symbolisms, we use the symbolism of totally antisymmetric tensors (of fundamental representations). Thus:

$$\alpha = (m_1,\dots,m_n) = [p_1,\dots,p_n] = \Delta_1^{p_1}\Delta_2^{p_2}\dots\Delta_n^{p_n} = \delta_1^{m_1}\delta_2^{m_2}\dots\delta_n^{m_n} \tag{18}$$

Consider the basis invariants $\omega_1\dots\omega_n$ of (13) and write them in box form:

$$\omega_p = \begin{vmatrix} x_1 x_2 & x_p \\ y_1 y_2 & y_p \\ \multicolumn{2}{c}{\dots\dots} \\ v_1 v_2 & v_p \end{vmatrix} \equiv \begin{bmatrix} x \\ y \\ \vdots \\ v \end{bmatrix} \equiv \begin{array}{|c|} \hline x \\ \hline y \\ \hline \vdots \\ \hline v \\ \hline \end{array} \tag{19}$$

An arbitrary ordering of the arguments $x,y,\dots$ in the determinants ω_i in the product

$$\omega = \omega_1^{m_1-m_2} \dots \omega_n^{m_n}$$

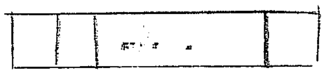
can be represented on the Young pattern, e.g.

$$\omega = \omega_1\omega_2\omega_3 = x_1 \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} \begin{vmatrix} s_1 s_2 s_3 \\ t_1 t_2 t_3 \\ u_1 u_2 u_3 \end{vmatrix} =$$

s	y	x
t	z	
u		

Let us consider some examples.

Example 1.

One row only:  (m boxes).

The corresponding maximal eigenvector is

$$\omega = [x][y]\ldots[w] = \omega_1^m$$

The maximal eigenvector is totally symmetric under all permutations. Thus, the representation

$$\alpha = \Delta_1^m$$

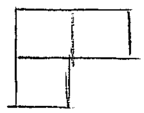
is realized in the class of totally symmetric tensors of order m. The only totally symmetric Z-invariant is ω_1^m. It follows that in the decomposition of a general tensor the symmetric tensor representation Δ_1^m occurs once and only once.

<u>Example 2:</u>

One column only  (m boxes)

The representations $\Delta_m = \begin{bmatrix} x \\ y \\ \vdots \\ w \end{bmatrix}$, as we know, correspond to totally anti-symmetric tensors. The representation Δ_n can also occur only once in the decomposition.

<u>Example 3:</u> For m = 3 we can consider e.g.

$\alpha =$ 

corresponding to 3! = 6 maximal eigenvectors $\omega_1 \circ \omega_2$:

$$\omega = \begin{array}{|c|c|}\hline y & x \\ \hline z & \\ \cline{1-1}\end{array} \qquad \omega' = \begin{array}{|c|c|}\hline x & z \\ \hline y & \\ \cline{1-1}\end{array} \qquad \omega'' = \begin{array}{|c|c|}\hline z & y \\ \hline x & \\ \cline{1-1}\end{array}$$

$$\tilde{\omega} = \begin{array}{|c|c|}\hline z & x \\ \hline y & \\ \cline{1-1}\end{array} \qquad \tilde{\omega}' = \begin{array}{|c|c|}\hline y & z \\ \hline x & \\ \cline{1-1}\end{array} \qquad \tilde{\omega}'' = \begin{array}{|c|c|}\hline x & y \\ \hline z & \\ \cline{1-1}\end{array}$$

However a permutation of arguments within a column just changes the sign of the tensor, so we may consider only the upper row. Further, the corresponding vectors are not independent. Indeed:

$$0 = \begin{vmatrix} x_1 & x_1 & x_2 \\ y_1 & y_1 & y_2 \\ z_1 & z_1 & z_2 \end{vmatrix} = x_1 \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} - y_1 \begin{vmatrix} x_1 & x_2 \\ z_1 & z_2 \end{vmatrix} + z_1 \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$$

Thus:

$$\omega - \omega'' + \omega' = 0$$

Finally two of the maximal eigenvectors $\omega = \omega_1 \circ \omega_2$ are linearly independent so that the representation

$$\alpha = \Delta_1 \Delta_2$$

figures twice in the decomposition of T^{ijk}.

<u>Remark:</u> The rank of the tensor m figured crucially in all examples, the order of the group n (we are considering U(n) did not figure at all. We shall show that this is true in general.

Lecture 18

Remarks on the Symmetric Group

We shall use the symmetric group S_m (the group of permutations of n elements). Denote an element of S_m by the symbol:

$$s = \begin{pmatrix} i_1 & i_2 \dots i_m \\ s(i_1) s(i_2) \dots s(i_m) \end{pmatrix} \qquad (1)$$

where s(i) is a function of an integer argument i and runs through all values r = 1,2,...m in some order. The multiplication law is:

$$\begin{pmatrix} i_1 \dots i_m \\ k_1 \dots k_m \end{pmatrix} \begin{pmatrix} k_1 \dots k_m \\ s_1 \dots s_m \end{pmatrix} = \begin{pmatrix} i_1 \dots i_m \\ s_1 \dots s_m \end{pmatrix} \qquad (2)$$

The number of elements in S_m is (m!).

We can now consider a representation of the group S_m in the space of multilinear forms Φ_m.

$$\tilde{\Phi}(u_1 \dots u_m) = S\, \Phi(u_1, u_2 \dots u_m) = \Phi(u_{s_1}, u_{s_2}, \dots, u_{s_m}) \qquad (3)$$

The coefficient tensor of $\tilde{\Phi}$ is obviously:

$$\tilde{t}^{\nu_1 \nu_2 \dots \nu_m} = t^{\nu_{s_1} \nu_{s_2} \dots \nu_{s_m}} \qquad (4)$$

(latin indices run through 1,...,m; greek ones through 1...n)

<u>Example</u>:

$$s = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \qquad \tilde{t}^{\nu_1 \nu_2 \nu_3 \nu_4} = t^{\nu_4 \nu_3 \nu_2 \nu_1}$$

We already know that

$$[T_g, S] = 0 \quad \text{for all } g\varepsilon G \tag{5}$$

It can be shown that <u>all</u> linear operators A, commuting with all operators T_g, can be written as

$$A = \sum_s a(s)\, S \tag{6}$$

Young Symmetrizers

Consider a Young pattern and introduce a standard numbering of the boxes:

$$\alpha = \begin{array}{|c|c|c|c|c|c|c|} \hline 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ \hline 12 & 11 & 10 & 9 & 8 & \multicolumn{2}{c}{} \\ \cline{1-5} 15 & 14 & 13 & \multicolumn{4}{c}{} \\ \cline{1-3} 18 & 17 & 16 & \multicolumn{4}{c}{} \\ \cline{1-3} \end{array} \qquad (\,m = 18)$$

Put one vector $x, y, \ldots w$ in each box.

Denote: p ... a permutation, acting horizontally, i.e. interchanging two objects (vectors) in one row.

q ... a permutation, acting vertically, i.e. interchanging vectors in one column.

The operator

$$Y = \sum (\pm\, q) p \tag{7}$$

is called a Young symmetrizer. The sum is taken over all possible p and q. The sign of a term is "+" if q is an even permutation, "-" of q is odd.

We can write:

$$Y = \hat{Q}\hat{P} \tag{8}$$

where

$$\hat{P} = \sum p$$
$$\hat{Q} = \sum(\pm\, q) \tag{9}$$

Here $\hat{P}$ is a horizontal "Symmetrizer", $\hat{Q}$ a vertical "Antisymmetrizer".

It can be shown, that if we normalize Y properly, putting

$$d = \frac{1}{\mu} Y \tag{10}$$

then

$$d^2 = d \tag{11}$$

i.e. d is a projection operator.

(See M. Hammermesh, Group Theory, for proofs and further details).

<u>Remark</u>: In general $\hat{P}$ and $\hat{Q}$ do not commute.

If we perform a permutation s in the space Φ_m, then any Young symmetrizer Y acting in this space, gets transformed into

$$Y_s = sYs^{-1}$$

We can construct a central symmetrizer, averaged over the symmetric group:

$$\varepsilon = \frac{1}{\mu^2} \sum_s sYs^{-1} \tag{12}$$

The normalization is such that

$$\varepsilon^2 = \varepsilon. \tag{13}$$

A central symmetrizer, being an averaged quantity, commutes with any permutation S_o. In particular all central symmetrizers, constructed using different Young patterns, commute amongst each other:

$$\varepsilon_1\varepsilon_2 = \varepsilon_2\varepsilon_1 \tag{14}$$

Notation: We shall write

$$Y_\alpha,\ d_\alpha \text{ and } \varepsilon_\alpha$$

to stress that the symmetrizers depend on the Young diagram α (on the representation which we are considering).

Theorem: The tensors

$$\tau_\alpha = \varepsilon_\alpha t \tag{15}$$

form a maximal subspace in Φ_m in which the representation is a multiple of an irreducible representation with signature

$$\alpha = (m_1, m_2, \ldots, m_n).$$

An arbitrary tensor can be written as a sum of projections:

$$t = \sum \tau_\alpha \tag{16}$$

where the sum is over all α satisfying

$$m_1 + m_2 + \ldots + m_n = m \qquad m_i \geq 0$$

The tensor

$$t_\alpha = T_\alpha t \tag{17}$$

transforms according to an irreducible representation with signature α and equivalent symmetrizers $sY\,s^{-1}$ project out equivalent irreducible spaces. The normalization μ_α, figuring in the central symmetrizer

$$\varepsilon_\alpha = \frac{1}{\mu_\alpha^2} \sum_s sY_\alpha s^{-1} \tag{18}$$

is

$$\mu_\alpha = \frac{m!}{k_\alpha} \tag{19}$$

where $k_\alpha = k(m_1, \ldots m_n)$ is the multiplicity of the signature α in space Φ_m.

Remark: We obviously have

$$Yp = Y \qquad qY = \pm\, Y$$

(the sign depends on the parity of the permutation q).

Example: Consider a third rank tensor t^{ijk} and act upon it with the various symmetrizers. We know that the possible independent Young patterns are:

$$\alpha_1 \sim \begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline\end{array} \qquad \alpha_2 \sim \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array} \qquad \alpha_3 \sim \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad \alpha_4 \sim \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

We have

$$t_\alpha^{ijk} = d_\alpha t^{ijk} = \frac{1}{\mu_\alpha} Y_\alpha t^{ijk}$$

Thus:

$$t_{\alpha_1}^{ijk} = \frac{1}{6}\{t^{ijk} + t^{kij} + t^{jki} + t^{ikj} + t^{jik} + t^{kji}\}$$

$$t_{\alpha_2}^{ijk} = \frac{1}{6}\{t^{ijk} + t^{kij} + t^{jki} - t^{ikj} - t^{jik} - t^{kji}\}$$

$$t_{\alpha_3}^{ijk} = \frac{1}{3}[1 - (23)][1 + (12)]t^{ijk} =$$

$$\frac{1}{3}\{t^{ijk} + t^{jik} - t^{ikj} - t^{kij}\}$$

$$t_{\alpha_4}^{ijk} = \frac{1}{3}[1 - (12)][1 + (23)]t^{ijk} =$$

$$\frac{1}{3}\{t^{ijk} + t^{ikj} - t^{jik} - t^{jki}\}$$

and

$$\sum_\alpha d_\alpha t^{ijk} = \sum_\alpha t_\alpha^{ijk} = t^{ijk}$$

<u>Remark:</u> When we speak of a certain symmetry of a tensor t_α we have in mind that t_α satisfies

$$Yt_\alpha = t_\alpha$$

This does imply that t_α is antisymmetric with respect to columns

$$qt_\alpha = \pm t_\alpha$$

but does not in general imply that it is symmetric with respect to rows:

$$pt_\alpha \neq t_\alpha .$$

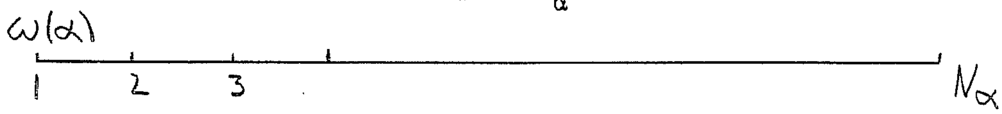
(This can be seen in the above example.)

We shall not prove the above fundamental theorem. The idea of the proof is to show that the space E_α of tensors $t_\alpha = Y_\alpha t$ is invariant and contains only one maximal eigenvector $\omega(\alpha) = Y\gamma$, where γ is a certain vector in Φ_m. Further one can show that the subspace $\tau_\alpha = \varepsilon_\alpha t$ contains all maximal eigenvectors with signature α and that $t = \sum_\alpha \tau_\alpha$.

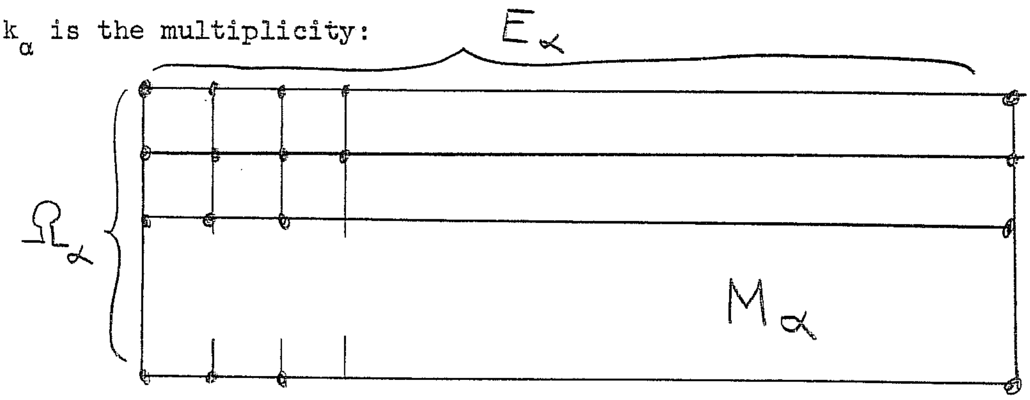
We still have to settle the question of degeneracy, i.e. how many times does each signature α occur in the given tensor $t^{i_1 \ldots i_m}$.

The picture we have is the following:

Let us represent the irreducible space E_α by a line:



The extreme left point corresponds to the maximal eigenvector $\omega_\alpha(\alpha)$, the other points are reached by applying the lowering operators. All representations corresponding to the same α form a maximal subspace $M(\alpha)$ and we can represent it by a rectangular diagram with $k = k(\alpha)$ rows, while k_α is the multiplicity:



The group G acts horizontally, the group of permutations S vertically. It can be shown that Ω_α - the space of all tensors, obtained from the maximal eigenvector by all permutations, is an irreducible space with respect to S. We can denote the representations of G by the symbol $\alpha(G)$, those of S by $\alpha(S)$ and we have a "reciprocity" relation:

Multiplicity of $\alpha(G)$ = dimension of $\alpha(S)$

Multiplicity of $\alpha(S)$ = dimension of $\alpha(G)$

Without proof we give a recursion formula for the multiplicity

$$k(m_1,m_2,\dots m_n) = \sum_{i=1}^{n} k(m_1,m_2,\dots m_i-1,\dots m_n) \qquad (20)$$

Thus: multiplicity in Φ_m in terms of multiplicity in Φ_{m-1}. Only those $k(m_1\dots m_n)$ figure, which are admissable, i.e. the m's must not increase to the right and the lengths of all Young columns are less or equal to than n.

Examples:

$m = 1 \qquad k(1) = 1 \qquad (1) = [1] = \Delta_1 \sim$ 

$m = 2 \qquad k(1,1) = k(1,0) = 1 \qquad \Delta_1 \times \Delta_1 = \Delta_1^2 + \Delta_2$

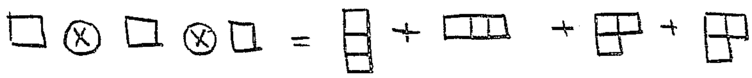
$\qquad\qquad k(2,0) = k(1,0) = 1 \qquad$ or 

$m = 3 \qquad k(1,1,1) = k(110) = k(1,0,0) = 1$

$\qquad\qquad k(3,0,0) = k(2,0,0) = k(1,0,0) = 1$

$\qquad\qquad k(2,1,0) = k(2,0,0) + k(1,1,0) = k(1,0,0) + k(1,0,0) = 2$

$$\Delta_1 \times \Delta_1 \times \Delta_1 = \Delta_3 + \Delta_1^3 + 2\Delta_1\Delta_2$$

or 

$m = 4 \qquad k(1111) = k(1110) = k(1100) = k(1000) = 1$

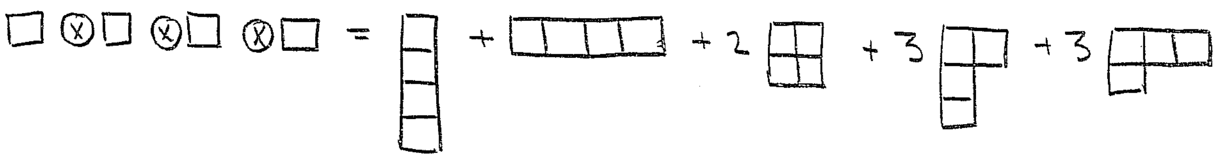
$\qquad\qquad k(4000) = k(3000) = k(2000) = k(1000) = 1$

$\qquad\qquad k(2200) = k(2100) = k(2000) + k(1100) = k(1000) + k(1000) = 2$

$\qquad\qquad k(2110) = k(2100) + k(1110) = k(2000) + k(1100) + k(1100) =$

$\qquad\qquad\qquad = k(1000) + k(1000) + k(1000) = 3$

$\qquad\qquad k(3100) = k(3000) + k(2100) = k(2000) + k(2000) + k(1100) = 3$

Thus: 

(Remark: For SU(3) the signature (1,1,1,1) is excluded, since the column has 4 boxes and 4 > 3).

This completes the construction of irreducible representations of the group G (GL(n,C) or U(n)) using three different methods - realizations on the group G, on the subgroup Z and on the class of tensors.

A problem which we have so far solved only for n=2 is that of finding a basis for each irreducible representation α. We are looking for a natural basis, in which all operators representing the subalgebra E_o and the subgroup D are diagonal

$$\delta \to T_\delta = \begin{pmatrix} \alpha_1(\delta) & & 0 \\ & \alpha_2(\delta) & \\ 0 & & \alpha_N(\delta) \end{pmatrix} \qquad (21)$$

For SU(2) all the $\alpha_i(\delta)$ where simple eigenvalues, so that there was no multiplicity problem.

Let us now consider the question of a basis and this multiplicity problem for U(n).

The Algebra of Z-multipliers.

Return to the realization of an irreducible representation in the class of functions $f(z)$, $z \in Z = Z_+$. We have

$$T_g f(z_{\mu\nu}) = \alpha(z,g) f((zg)_{\mu\nu}) \qquad (22)$$

where

$$z = \begin{pmatrix} 1 & z_{12} & z_{13} & \cdots & z_{1n} \\ & 1 & z_{23} & & z_{2n} \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix} \qquad (23)$$

$$\alpha(z,g) = (zg)_{11}^{m_1-m_2} \begin{vmatrix} (zg)_{11} & (zg)_{12} \\ (zg)_{21} & (zg)_{22} \end{vmatrix}^{m_2-m_3} \ldots (\det g)^{m_n} =$$

$$= \Delta_1^{p_1}(zg)\, \Delta_2^{p_2}(zg) \ \ldots \ \Delta_n^{p_n}(zg) \ , \qquad (24)$$

$$(z_g)_{\mu\nu} = \frac{\Delta_{\mu\nu}(zg)}{\Delta_\mu(zg)} \qquad (25)$$

and

$$\Delta_{\mu\nu} = g_{12\ldots\mu-1\nu}^{12\ldots\mu-1\nu}$$

is the indicated subdeterminant of $||g||$.

The space R_α of functions $f(z)$, in which the irreducible representation acts can be characterized by a system of "indicators" $I_1,\ldots,I_{n-1}$, where

$$I_1 = \sum_{k=1}^{n} z_{2k} \frac{\partial}{\partial z_{1k}} \equiv z_{2k} \frac{\partial}{\partial z_{1k}} \qquad (26)$$

$$I_2 = z_{3k} \frac{\partial}{\partial z_{2k}} \qquad (27)$$

$$I_{n-1} = z_{nk} \frac{\partial}{\partial z_{n-1k}} \qquad (28)$$

It is easy to check that

$$I_1\Delta_1(zg) \neq 0 \quad I_1\Delta_k(zg) = 0 \quad k = 2,\ldots,n \quad (29)$$

(The operator I_1 acting on Δ_2,Δ_3 etc. replaces the first row by the second one, so that the determinants have two identical rows and are thus equal to zero). Since Δ_1 depends linearly on z_{1k} we have

$$I_1^2\Delta_1 = 0 \qquad (30)$$

In general

$$I_k\Delta_1 = \dots I_k\Delta_{k-1} = I_k\Delta_{k+1} = \dots = I_k\Delta_n = 0 \qquad (31)$$

$$I_k\Delta_k \neq 0 \qquad I_k^2\Delta_k = 0$$

We already know that the space R_α is spanned by the functions

$$\alpha(z,g) = \Delta_1^{p_1}\Delta_2^{p_2}\dots\Delta_n^{p_n}$$

It follows that if $f(z)\in R_\alpha$ then

$$I_\alpha^{p_{\alpha+1}} f(z) = 0 \qquad \alpha = 1,2,\dots n-1 \qquad (32)$$

This is called the <u>indicator system</u> and we have n-1 equations for $\frac{n(n-1)}{2}$ variables $\Rightarrow$ infinitely many solutions.

<u>Theorem</u>: The space R_α of the irreducible representation of G coincides with the class of solutions of the indicator system.

We drop the proof.

<u>Remark:</u> Actually, we have

$$I_\alpha = E_{\alpha\;\alpha+1}$$

where $E_{\alpha\;\alpha+1}$ is the operator representing the generators

$$e_{\alpha\;\alpha+1} = \begin{array}{c c} & \begin{array}{ccc} & \alpha+1 & \end{array} \\ \alpha & \begin{pmatrix} & \vdots & \\ \cdots & 1 & \cdots \\ & \vdots & \end{pmatrix} \end{array}$$

in Z.

In the realization on the group we also have

$$I_\alpha^{p_\alpha+1} f(g) = 0$$

and we must add the condition for f(g) not to depend on z_-:

$$E_{ik} f(g) = 0 \qquad i > k$$

and the condition

$$E_{ii} f(g) = m_i f(g)$$

(i.e. the homogeneity condition $f(\delta g) = \alpha(\delta) f(g)$)

Thus we have a space R_α of functions f(z), being solutions of the indicator systems. We wish to find a basis in R_α, consisting of weight vectors

$$T_\delta f = \gamma(\delta) f \tag{33}$$

and to enumerate all the weights $\alpha(\delta)$. We know how T_g acts for any g. In particular for $g = \delta$ we have

$$T_\delta f(z) = \alpha(\delta) f(\delta^{-1} z \delta) \tag{34}$$

where

$$\alpha(\delta) = \delta_1^{m_1} \dots \delta_n^{m_n} \text{ is the highest weight.}$$

Let us fix the maximal eigenvector e_1, corresponding to the highest weight $\alpha(\delta)$, putting

$$e_1 = 1 \tag{35}$$

and obtain the other eigenvectors.

<u>Definition</u>: The polynomial $\theta(z)$ is a weight multiplier on Z if it is an eigenvector with respect to the transformation $z \to \delta^{-1}z\delta$ i.e.

$$\theta(\delta^{-1}z\delta) = \mu(\delta)\theta(z). \tag{36}$$

If $\theta(z)$ is a solution of the indicator system, then $\theta(z)\varepsilon R_\alpha$ and we have

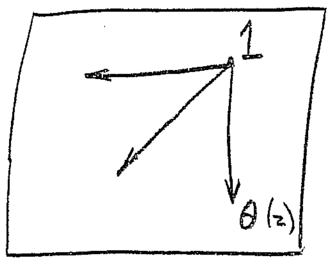
$$T_\delta\theta(z) = \alpha(\delta)\theta(\delta^{-1}z\delta) = \alpha(\delta)\mu(\delta)\theta(z) \tag{37}$$

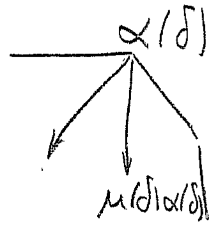
i.e. $\theta(z)$ is an eigenvector corresponding to the eigenweight

$$\gamma(\delta) = \mu(\delta)\alpha(\delta). \tag{38}$$

Similarly, it follows from (34) that any weight vector in R_α is a Z-multiplier.

<u>Thus</u>: We have vectors $\theta(z)\varepsilon R_\alpha$ acting as multipliers on the group and multipliers $\mu(\delta)$, acting in the weight space:





Remember that the highest weight is

$$\alpha(\delta) = \delta_1^{m_1}\ldots\delta_n^{m_n} \qquad m_1 \geq m_2 \geq \ldots m_n \ \ldots \text{ integers}$$

It can be shown that

$$\mu(\delta) = \delta_1^{k_1}\delta_2^{k_2}\ldots\delta_n^{k_n}$$

so that the action of $\mu(\delta)$ on $\alpha(\delta)$ corresponds to the addition:

$$m_i \to m_i + k_i$$

The multipliers $\theta(z)$ and $\mu(\delta)$ can be used to construct a basis in R_α and to obtain all weights and their multiplicities.

Let us again consider the special case of SU(3).

Spectrum and Basis for the Group SU(3)

In this case we have

$$z = \begin{pmatrix} 1 & z_{12} & z_{13} \\ 0 & 1 & z_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

The elements of z are weight multipliers in R_α.

Indeed

$$\delta^{-1} z \delta = \begin{pmatrix} 1 & \delta_1^{-1} z_{12}\delta_2 & \delta_1^{-1} z_{13}\delta_3 \\ 0 & 1 & \delta_2^{-1} z_{23}\delta_3 \\ 0 & 0 & 1 \end{pmatrix}$$

so that z_{ik} is a weight multiplier corresponding to a multiplier

$$\mu_{ik} = \delta_i^{-1}\delta_k$$

i.e. $\mu_{12} = \delta_1^{-1}\delta_2$, $\mu_{23} = \delta_2^{-1}\delta_3$ $\mu_{13} = \delta_1^{-1}\delta_3$

and

$$\mu_{13} = \mu_{12}\,\mu_{23}$$

Consider a representation $\alpha = (m_1, m_2, m_3)$, say $\alpha = (7,3,0)$ and the space $R_\alpha = R_{(m_1 m_2 m_3)}$. In particular:

$$R_{(7,3,0)}$$

The signature is

$$\alpha = (7,3,0) = [4,3,0] = \Delta_1^4\Delta_2^3$$

Let us find the corresponding multipliers:

The vector Δ_1 corresponds to two multipliers:

$$\Delta_1 = \delta_1 \leftrightarrow \{z_{12}, z_{13}\}$$

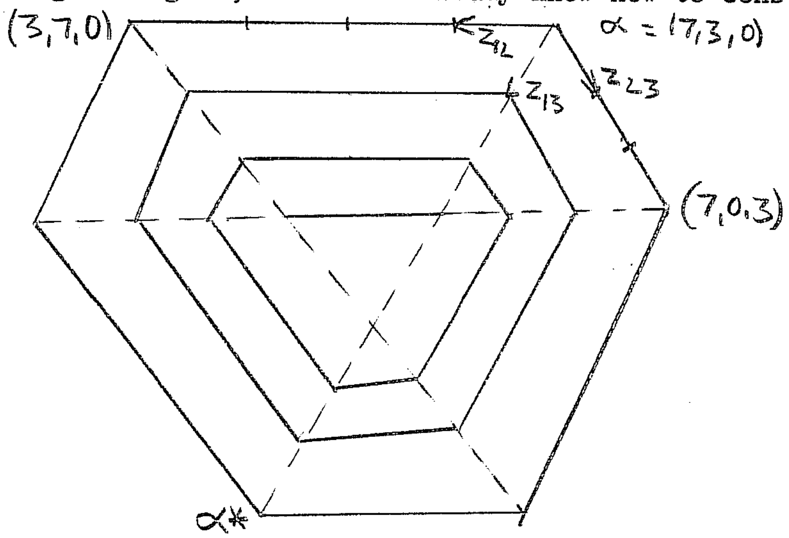
(first row in z)

The bivector Δ_2 corresponds to two other multipliers

$$\Delta_1 = \delta_1\delta_2 \leftrightarrow \{ z_{23}, \hat{z}_{13} = \begin{pmatrix} z_{12} & z_{13} \\ 1 & z_{23} \end{pmatrix} \}$$

Let us show how the multipliers act.

The weight diagram, which we already know how to construct is:



Let us first discuss just this example:

1) Consider z_{12}: $\mu_{12} = \delta_1^{-1}\delta_2$ $k_1 = -1, k_2 = 1, k_3 = 0$

Thus we can let

$$1, z_{12}, z_{12}^2, z_{12}^3 \text{ and } z_{12}^4$$

act on α (because Δ_1^4 figures in α). Higher powers are forbidden.

We get

$$(m_1m_2m_3) \longrightarrow (m_1,m_2,m_3) + n(k_1,k_2,k_3)$$

i.e. $(7,3,0) \rightarrow (6,4,0) \rightarrow (5,5,0) \rightarrow (4,6,0) \rightarrow (3,7,0)$

Thus, we reach five points along the horizontal. Each has multiplicity one

2) Consider z_{23}: $\mu_{23} = \delta_2^{-1}\delta_3$ $k_1 = 0,\ k_2 = -1\ \ k_3 = 1$

We have:

$$1,\ z_{23},\ z_{23}^2,\ z_{23}^3$$

(No higher powers, since Δ_2 figures in the power 3: Δ_2^3).

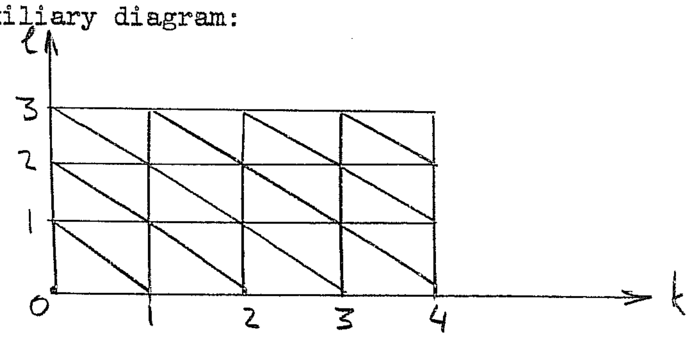
$$(7,3,0) \rightarrow (7,2,1) \rightarrow (7,1,2) \rightarrow (7,0,3)$$

We have four points with multiplicity one.

3) Along the line $\alpha \rightarrow \alpha^*$ we use

$$z_{13}^k\ \hat{z}_{13}^\ell \qquad 0 \le k \le 4$$
$$0 \le \ell \le 3$$

Draw an auxiliary diagram:



Points (k,ℓ) on a line $k+\ell$ = const. correspond to the same eigenvalue $\Rightarrow$

The multiplicities are:

$k + \ell$:	multiplicity	$k + \ell$:	multiplicity
0	1	4	4
1	2	5	3
2	3	6	2
3	4	7	1

From here we can easily show, that the multiplicities are constant along each hexagon.

Lecture 19

Now let us return to general representations of SU(3) and consider the spectrum and the basis in R_α.

We are considering the special group

$$\det g = 1$$

so we can normalize the signature $(m_1 m_2 m_3)$. Indeed let us put $m_3 = 0$ and write:

$$\alpha = [p,q] = \Delta^p \hat{\Delta}^q$$

Here $\Delta = \Delta_1$ represents the vector representation, $\tilde{\Delta} = \Delta_2$ the bivector, which in this particular case of SU(3) happens to be dual to Δ.
(In terms of Young patterns we have:

$\Delta \sim \square$ $\qquad\qquad$ $\tilde{\Delta} \sim \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$.)

<u>Theorem:</u> The basis in R with signature $\alpha = [p,q]$ consists of three series of monomials:

$$\begin{array}{llll} s_{12} & : & e_{abc} = z_{13}^a \, z_{12}^b \, \hat{z}_{13}^c & 0 \le a + b \le p,\ 0 \le c \le q,\ b \ne 0 \\ s_0 & : & e_{k\ell} = z_{13}^k \, \hat{z}_{13}^\ell & 0 \le k \le p,\ 0 \le \ell \le q \\ s_{23} & : & \hat{e}_{abc} = z_{13}^a \, z_{23}^b \, \hat{z}_{13}^c & 0 \le a \le p,\ 0 \le b + c \le q,\ b \ne 0 \end{array}$$

The series S_0 contains all multipliers acting along the line $\alpha \to \alpha^*$ (from the highest to the lowest weight). The multipliers in the series generate the multiplicities

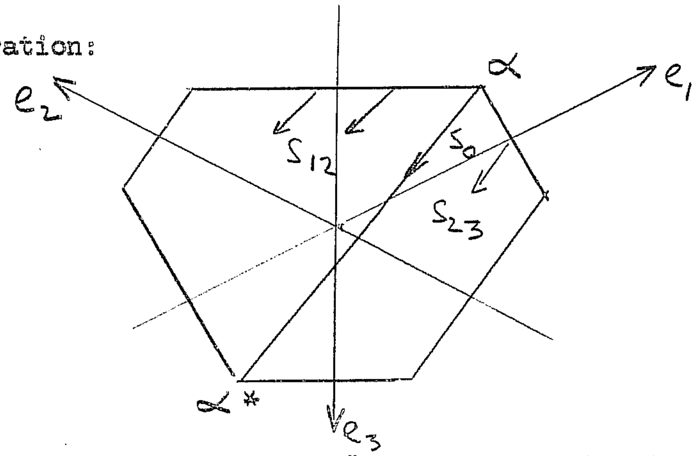
$$1,\ 2,\ \ldots\ k_0-1,\ \underbrace{k_0,\ k_0 \ldots, k_0}_{|p-q|\ +\ 1\ \text{entries}},\ k_0-1 \ldots 2, 1$$

where

$$k_0 = \min(p,q) + 1.$$

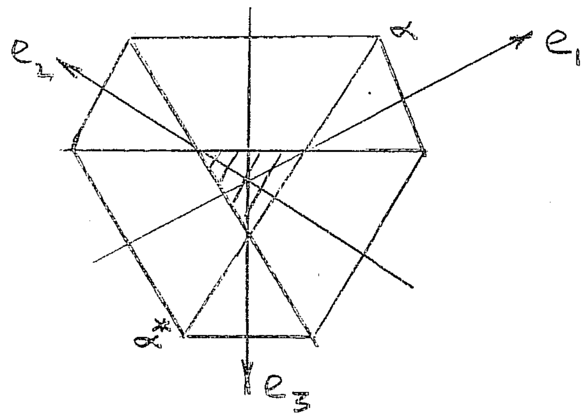
Proof: Dropped for lack of time.

Graphic illustration:



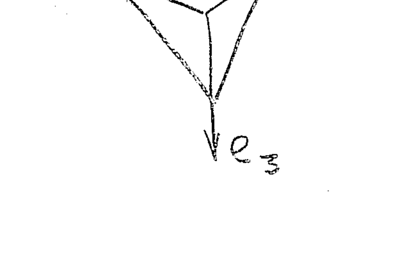
The series S_o acts along the line $\alpha\alpha^*$, S_{12} above this line, S_{23} below it.

Corollary: To construct the weight diagrams it is sufficient to find all the vertices of the largest hexagon (which can of course degenerate into a triangle). Connect the vertices as shown below

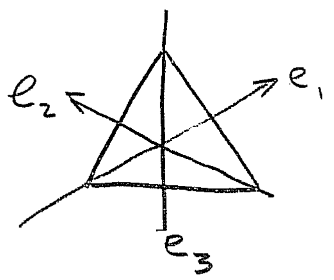


The multiplicities of eigenvalues on the outmost figure is 1, it increases by steps of one as we move parallel to the line $\alpha\alpha^*$ till we hit the triangle, where the multiplicity reaches its maximum value $k_o = \min(p,q) + 1$. Beyond the triangle the multiplicity again decreases to 1 by steps of one

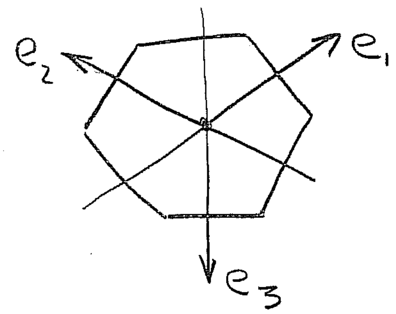
Examples:



vector (= quark)



bivector (= antiquark)



octet

Separation of multiple points in the spectrum

The weight vectors corresponding to degenerate (multiple) weights in the spectrum obviously cannot be characterized by the weight they correspond to and must be enumerated differently.

The weights themselves were obtained by considering the representation

$$g \to T_g$$

and reducing it to the subgroup of diagonal matrices δ:

$$\delta = \begin{pmatrix} \delta_1 & & 0 \\ & \delta_2 & \\ 0 & & \delta_3 \end{pmatrix} \to T_\delta$$

We can perform such a reduction by steps, i.e. first reduce the group G to a subgroup G_o, where

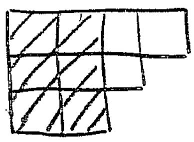
$$g_o = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{a} & 0 \\ & & 0 \\ \hline 0 & 0 & \delta_3 \end{array}\right) \qquad (\det a)\delta_3 = 1$$

for all $g_o \in G_o$. We can then restrict ourselves to the case when <u>a</u> is diagonal. This makes it possible to use the highest weights for the representations of G_o to label eigenvectors.

Let us first consider an auxiliary problem, namely the <u>reduction of U(3) to U(2)</u>.

We drop the condition det g = 1 and the normalization of the signature (m_1, m_2, m_3).

<u>Remark:</u> In the language of Young patterns the restriction $U(3) \rightarrow SU(3)$ corresponds to deleting all columns of length 3 (n for SU(n)):



(each such column corresponds to a factor Δ_n (Δ_3 for SU(3)) in the highest weight

$$\alpha = \Delta_1^{p_1} \Delta_2^{p_2} \ldots \Delta_n^{p_n} ;$$

however Δ_n = det g = 1).

Take G = U(3), $G_o \sim U(2)$, i.e.

$$g_o \in G_o \rightarrow \quad g_o = \begin{pmatrix} a & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix} \qquad a \in U(2)$$

<u>Problem:</u>

Consider a representation of G given by the signature $\alpha = (m_1, m_2, m_3)$. We wish to restrict G to G_o and find all signatures of the representations of G_o, realized in the space R_α. Thus, we are looking for a decomposition

$$\alpha\Big|_{G_o} = \beta_o + \beta_1 \ldots + \beta_s$$

where β_i are the highest weights (the signatures) of irreducible representations of G_o.

<u>Solution</u>: We must find the z-invariants of the subgroup Z_o:

$$z_o = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad z_o \varepsilon Z_o$$

In R_α we have

$$T_{z_o} f(z) = f(zz_o).$$

Thus, the maximal eigenvectors must satisfy

$$\omega(zz_o) = \omega(z) \qquad z_o \varepsilon Z_o.$$

An arbitrary $z\varepsilon Z$ can be written as

$$z = \begin{pmatrix} 1 & z_{12} & z_{13} \\ 0 & 1 & z_{23} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_1 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \zeta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & z_{13} \\ 0 & 1 & z_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & z_{12} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus $\omega(zz_o) = \omega(z)$ if $\omega(z)$ does not depend on ζ but only on t_1 and t_2.

$$\omega = \omega(t_1, t_2)$$

The maximal eigenvector for G was $\omega_o = 1$. This will certainly be a maximal eigenvector for a subgroup, so

$$\underline{\omega_o = 1}$$

is one solution of our problem. The other solutions can be obtained by applying certain Z-multipliers

$$\omega_i = \omega_i(t_1, t_2) \text{ to } \omega_o.$$

The Z-multipliers $t_1 = z_{13}$ and $t_2 = z_{23}$ correspond to the multiplier

$$\mu_1 = \delta_1^{-1}\delta_3 \text{ and } \mu_2 = \delta_2^{-1}\delta_3$$

in the weight space. For the subgroup U(2) we must put $\delta_3 = 1$. Thus the application of μ_1 and μ_2 corresponds to decreasing the number m_1 and m_2 in the signature by steps of one.

We obtain the theorem:

If we reduce the representation $\alpha = (m_1, m_2, m_3)$ of U(3) to U(2), then we always have the signature

$$\beta_o = (m_1, m_2)$$

for U(2) and further all signatures obtained as

$$= \mu_1^{k_1}\mu_2^{k_2}\beta_0 = (m_1-k_1,\ m_2-k_2)$$

Putting $\alpha = (m_1, m_2, m_3) = [p, q, r] = [m_1-m_2,\ m_2-m_3,\ m_3]$ we have

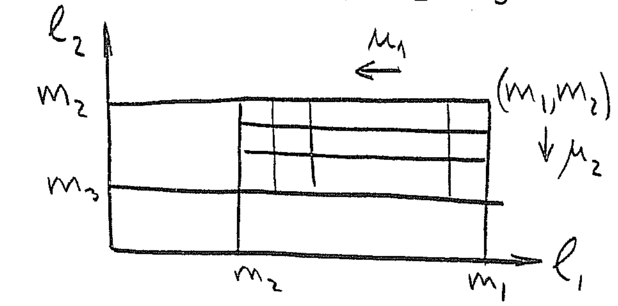
$$0 \le k_1 \le p \qquad 0 \le k_2 \le q$$

The spectrum of the subgroup U(2) in a representation of U(3) is simple (there is no multiplicity problem).

In other words:

All signatures (ℓ_1,ℓ_2) of U(2) will figure once and only once, for which

ℓ_2 $$m_1 \geq \ell_1 \geq m_2 \geq \ell_2 \geq m_3$$



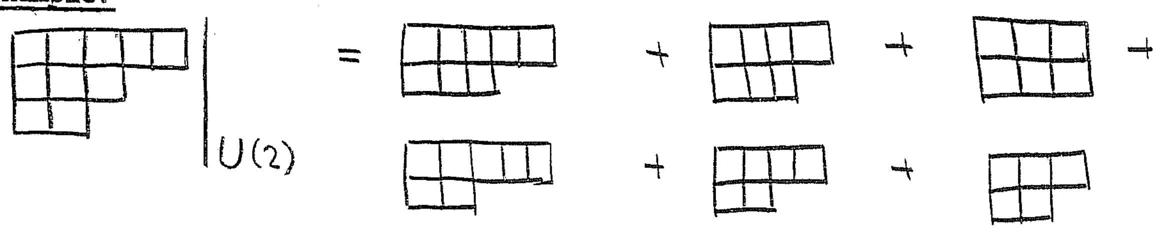
In terms of Young patterns:

We obtain the reduction $U(3) \supset U(2)$ for a given representation of U(3), by:

a) Eliminating the third row

b) Taking the obtained diagram and gradually decreasing the length of each row till its length coincides with the length of the following row in the U(3) diagram.

Example:



$$(5,3,2)\Big|_{U(2)} = (5,3) + (4,3) + (3,3) + (5,2) + (4,2) + (3,2)$$

Thus a basis for representation of U(3) (or SU(3) can be characterized by a pattern

$$\gamma = \begin{pmatrix} m_1 & & m_2 & & m_3 \\ & \ell_1 & & \ell_2 & \\ & & s_1 & & \end{pmatrix} \qquad \begin{matrix} m \geq \ell_1 \geq m_2 \geq \ell_2 \geq m_3 \\ \ell_1 \geq s_1 \geq \ell_2 \end{matrix}$$

(a Gelfand-Tseitlin pattern).

Here m_1, m_2 and m_3 are fixed and determine the representation (or its highest weight), whereas ℓ_1, ℓ_2 and s_1 run through all possible values and characterize the highest weights of the representations of the subgroups in the chain of groups

$$U(3) \supset U(2) \supset U(1)$$

The generalization to U(n) is now straightforward.

The Gelfand-Tseitlin Basis for U(n)

Reduction $U(n) \supset U(n-1)$.

Choose the basis in such a way that the relevant U(n-1) subgroup is realized by the U(n) matrices

$$g = \left(\begin{array}{c|c} a & \begin{matrix} 0 \\ 0 \\ \vdots \end{matrix} \\ \hline 0\,0 \cdots & 1 \end{array}\right) \qquad a \in U(n-1)$$

Consider an irreducible representation of $U(n): \alpha = (m_1, \ldots, m_n)$. We can again show that in the reduction of α to U(n-1) we have

1) The signature

$$\beta_o = (m_1 \ldots m_{n-1})$$

is always contained in $\alpha\big|_{U(n-1)}$.

2) The other signatures can be obtained by applying the z multipliers in the last column of

$$z = \left(\begin{array}{c|c} & \begin{matrix} z_{1n} \\ z_{n-1n} \end{matrix} \\ \hline 0\ 0 \ldots 0 & 1 \end{array}\right)$$

and z_{kn} corresponds to the weight $\delta_k^{-1}\delta_n = \delta_k^{-1}$, so that the reduction $\alpha\big|_{U(n-1)}$ contains each signature

$$\beta = (m_1 - s_1, \ldots m_n - s_n) \qquad s_i \geq 0$$

once and only once (subject to the condition that β is indeed a signature, i.e. satisfies the usual ordering conditions).

Theorem: The restriction $(m_1 m_2 \ldots m_n)\big|_{U(n-1)}$ contains all signatures $(\ell_1, \ell_2 \ldots \ell_{n-1})$ for $U(n-1)$ once and only once, for which:

$$m_1 \geq \ell_1 \geq m_2 \geq \ldots\ldots \geq \ell_{n-1} \geq m_n.$$

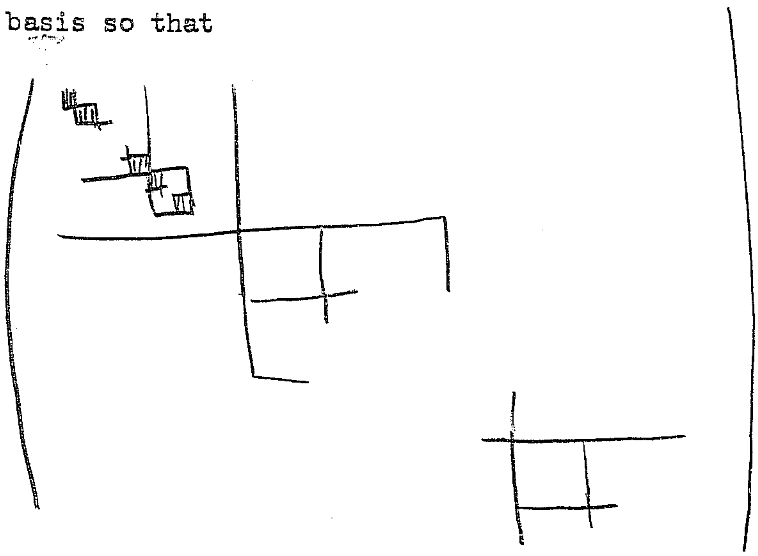
The Gelfand-Tseitlin Basis:

Consider the reduction

$$U(n) \supset U(n-1) \supset \ldots\ldots \supset U(1)$$

and classify a basis so that

$T_g =$



The largest boxes correspond to irreducible representations of U(n-1), the smaller ones to U(n-2), etc, down to U(1).

Each basis vector corresponds to a pattern

$$\gamma = \begin{pmatrix} m_1 \ldots \ldots m_{n-1} m_n \\ \ell_1 \ldots \ell_{n-2}\ \ell_{n-1} \\ s_1 \ldots s_{n-2} \\ \\ u_1 u_2 \\ v_1 \end{pmatrix}$$

with

$$m_1 \geq \ell_1 \geq m_2 \geq \ell_2 \ldots\ldots \ell_{n-1} \geq m_n$$

$$\ell_1 \geq s_1 \geq \ell_2 \qquad\qquad \geq s_{n-2} \geq \ell_{n-1}$$

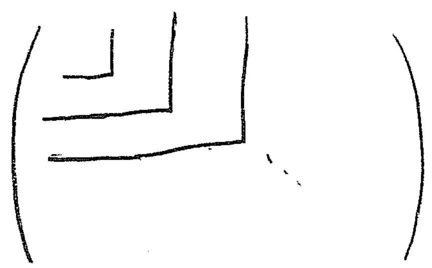
$$u_1 \geq v_1 \geq u_2$$

Again $m_1, \ldots, m_n$ are fixed, all other signatures (of the subgroups) run through all admissable values.

Thus: a vector e_γ is a basis vector if it is contained in the representation of $U(n-1)$ with signature $(\ell_1 \ldots \ell_{n-1})$, in that of $U(n-2)$ with signature $(s_1 \ldots s_{n-2})$, etc.

Remarks: The chain of subgroups $U(n) \supset U(n-1) \supset \quad \supset U(1)$ is called an enumerating cone. Its choice is not unique. The cone corresponding to the chain

g =



leads to the Gelfand-Tseitlin basis. The eigenvector e_γ corresponds to the weight (the eigenvalue):

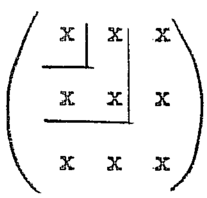
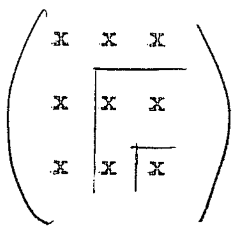
$$\gamma(\delta) = \delta_1^{v_1}\delta_2^{(u_1+u_2)-v_1}\cdots\delta_n^{(m_1+m_2+\dots+m_n)-(\ell_1+\ell_2\dots+\ell_{n-1})}$$

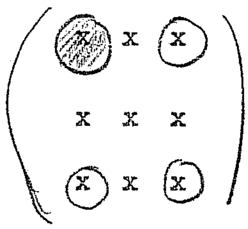
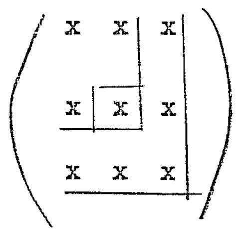
The number of parameters determining γ, if we exclude the signature $(m_1\dots m_n)$ is

$$1 + 2+\dots+n-1 = \frac{n(n-1)}{2}.$$

equal to the number of parameters in the group Z. The basis vectors e_γ can be given explicitly, but we shall not do that.

2) We could have chosen different cones, e.g. by fitting the subgroups into the group differently. For U(3) the possibilities are


and also

etc.

Mathematically all these schemes are equivalent, but their physical interpretations can be quite different. Thus in the eightfold way the first three diagrams correspond to the splitting out of isotopic spin (I-spin), the next two to so-called U and V spin.

It is an interesting mathematical problem (with a lot of physical content) to construct the operators connecting different bases with other and this can be done quite simply in terms of z-multipliers.

3) The Gelfand-Tseitlin basis is mathematically very simple and pretty, unfortunately it is not always that basis which is of physical interest. Thus, for SU(3) in particle physics it is directly relevant, since the SU(2) in the chain corresponds say to isospin and the U(1) specifically to the third projection. However, already in SU(6) we are interested in the subgroup:

$$SU(6) \supset SU(3) \times SU(2)$$

rather than in the Gelfand-Tseitlin chain, containing SU(5), etc.

For SU(3) applications in low energy nuclear physics (the shell model), or in a quantum mechanical treatment of the harmonic oscillator, we are interested in an O(3) subgroup corresponding to angular momentum and this O(3) has nothing to do with the SU(2) of isospin.

4) Similar Gelfand-Tseitlin bases exist for the other classical groups - the orthogonal and sympletic ones.

5) For non compact groups there is a much greater variety of possible subgroup chains. E.G. for U(2,1) we can consider two obviously non-equivalent chains $U(2,1) \supset U(2) \supset U(1)$ and $U(2,1) \supset U(1,1) \supset U(1)$, as well as others.

<u>Characters of Irreducible Representations</u>

Let us, without elaborating, introduce a concept useful in many applications, namely the character of an irreducible representation (see Weyl's book).

$$X(\delta) = \mathrm{Tr}T_\delta$$

Thus: the character of T_g is the sum of all weights of T_g, each entering as many times as is its multiplicity.

Weyl derived two different expressions for $X(\delta)$:

(1)

$$X(\delta) = \frac{D(\ell_1,\ell_2,\ldots,\ell_n)}{D(n-1,n-2,\ldots 0)}$$

where:

$$\begin{aligned}
\ell_1 &= m_1 + (n-1) \\
\ell_2 &= m_2 + (n-2) \\
\\
\ell_{n-1} &= m_{n-1} + 1 \\
\ell_n &= m_n
\end{aligned}$$

and

$$D(\ell_1,\ell_2,\ldots\ell_n) = \begin{vmatrix} \delta_1^{\ell_1} \delta_1^{\ell_2} & \ldots \delta_1^{\ell_n} \\ \delta_2^{\ell_1} \delta_2^{\ell_2} & \ldots \delta_2^{\ell_n} \\ \\ \delta_n^{\ell_1} \ldots\ldots & \delta_n^{\ell_n} \end{vmatrix}$$

(2)

$$X(\delta) = \begin{array}{l} \sigma_{m_1}, \sigma_{m_1+1}, \sigma_{m_1+n-1} \\ \\ \sigma_{m_n-(n-1)} \quad \sigma_{m_n} \end{array}$$

$$\sigma_m(\delta) = \sum_{s_1+\ldots+s_n=m} \delta_1^{s_1} \ldots \delta_n^{s_n}$$

Dimensions of Irreducible Representations of SU(n)

Using Weyls formulas we can derive an expression for the dimension of the representation

$$\alpha = (m_1, \ldots, m_n)$$

namely

$$\dim \alpha = \frac{\prod_{i<j} (\ell_i - \ell_j)}{\prod_{i<j} (\ell_i^o - \ell_j^o)}$$

where $\quad \ell_j = m_j + (n-j) \qquad j = 1 \ldots n$

$$\ell_j^o = n-j$$

The simplest way of understanding this formula is in terms of Young diagrams. Indeed, the method is the following: draw the Young pattern corresponding to α and eliminate all columns of length n (for SU(n)). Write the numbers $\ell_i^o = n-1, n-2 \ldots, 2, 1, 0$ next to the pattern. Then $\ell_i = (\ell_i^o + \text{length of row})$. Write dim α as a fraction with $\ell_1 . \ell_2 \ldots \ell_n$ and all differences $\ell_i - \ell_j$, $i < j$, in the numerator and $\ell_1^o \ldots \ell_n^o$ and all differences $\ell_i^o - \ell_j^o$, $i < j$, in the denominator.

Examples:

A. SU(3) $\qquad \ell_1^o = 2, \ \ell_2^o = 1, \ \ell_3^o = 0$

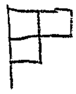
1) 2
1
0 

$$\ell_1 = 2+1 = 3$$

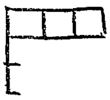
$$\ell_2 = 1+0 = 1$$

$$\ell_3 = 0+0 = 0$$

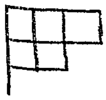
$$\dim(1,0,0) = \frac{3.1 \quad 3-1)}{2.1\ (2-1)} = 3$$

2) 2 1 0 

$$\dim(2,1,0) = \frac{4.2\,.\,(4-2)}{2.1.(2-1)} = 8$$

3) 2 1 0 

$$\dim(3,0,0) = \frac{5.1\,(5-1)}{2.1} = 10$$

4) 2 1 0 

$$\dim(3,2,0) = \frac{5.3}{2.1}\,.2 = 15$$

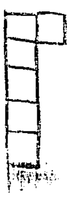
B. <u>SU(6)</u>

1) 5 4 3 2 1 0 

$$\dim(1,0,0,0,0,0) = \frac{6.4.3.2.1}{5.4.3.2.1}\;\frac{2.3.4.5.}{1.2.3.4.}\;\frac{1.2.3.1.2.1}{1.2.3.1.2.1} = 6$$

2) 5 4 3 2 1 0 

$$\dim(2,1,0,0,0,0) = \frac{7.5.3.2.1}{5.4.3.2.1}\;\frac{2.4.5.6}{1.2.3.4}\;\frac{2.3.4}{1.2.3.}\;\frac{1.2}{1.2}\;\frac{1}{1} = 70$$

3) 5 4 3 2 1 0 

$$\dim(2,1,1,1,1,0) = \frac{7.5.4.3.2}{5.4.3.2.1}\;\frac{2.3.4.5}{1.2.3.4}\;\frac{1.2.3}{1.2.3}\;\frac{1.2}{1.2}\;\frac{1}{1} = 35$$

<u>Remark</u>: The reduction of a group to subgroups, as we see, plays a fundamental role in group representation theory. In physics this reduction corresponds to the breaking of a symmetry. Thus: We have e.g. an SU(3) symmetry, then we introduce medium strong interactions reducing the SU(3) symmetry to an SU(2) one.

Casimir Operators

Above we have used the highest weights of a group and of its subgroups to separate multiple points in a spectrum (to remove a degeneracy). In physics this is usually done by introducing a complete set of commuting operators (a complete set of observables).

The corresponding problem in representation theory is: given a representation $g \to T_g$ find all functions of the operators T_g (or of their generators), which commute with all operators T_g.

These operators are called Casimir operators, or Laplace operators, or Beltrami operators.

Definition: Enveloping Algebra.

Let M be a set of matrices, not necessarily an algebra. Add to M all products of elements $m_i \varepsilon M$ and all linear combinations of products. We obtain an algebra U(M), called the enveloping algebra.

Example: The Pauli matrices σ_1, σ_2 and σ_3 do not form an associative algebra (with respect to multiplication), since $\sigma_1{}^2 = e$. Add $\lambda e \Rightarrow$ we get the enveloping algebra.

The Universal Envelloping Algebra of a Lie Algebra

Let X be an abstract Lie algebra(over the field of complex or real numbers). We know the commutators of all elements $x,y \varepsilon X$:

$$[x,y]$$

We can define a new operation - multiplication: xy- so that

$$[x,y] = xy - yx$$

It is sufficient to do this for the basis vectors:

$$e_1, e_2, \dots e_n$$

and write all products

$$e_{i_1} e_{i_2} \cdots e_{i_3}$$

remembering that

$$[e_i, e_j] = e_i e_j - e_j e_i = C^k_{ij} e_k$$

We obtain an infinite-dimensional algebra A(X), called the <u>universal enveloping algebra</u> of X. A general element of A(X) can be written as

$$t^{(s)} = t^{i_1 i_2 \cdots i_s} \, e_{i_1} e_{i_2} \cdots e_{i_s}$$

The tensor $t^{i_1 \cdots i_s}$ can be taken to be symmetric (a tensor which is antisymmetric with respect to a pair of indices can be reduced to a lower order tensor in view of the commutation relations for e_i).
If we have a representation of the Lie algebra X

$$e_i \longrightarrow E_i$$

we can continue it to a representation of the envelloping algebra A(X):

$$e_{i_1} e_{i_2} \cdots e_{i_s} \longrightarrow E_{i_1} E_{i_2} \cdots E_{i_s}$$

The obtained representation is itself a finite-dimensional envelloping algebra for the given representation of the Lie algebra X (since each E_i is a finite-dimensional matrix) (at least for finite dimensional representations of X).

The Casimir operators:

Consider a commutative subalgebra $C(X)$ of $A(X)$

$$C(X) \subset A(X)$$

such that the elements

$$C_i \varepsilon C(X)$$

commute with all elements of $A(X)$. We call $C(X)$ the centre of the enveloping algebra $A(X)$ of the Lie algebra X.

Consider a representation of the algebra:

$$e_i \rightarrow E_i$$

An element $c_i \varepsilon C(X)$ gets represented by an operator K_c

$$c \rightarrow K_c$$

If T_g is irreducible, then it follows from Schur's lemma that K_c is a multiple of the identity operator:

$$K_c = \lambda(c)I.$$

The operators K_c are called the Casimir operators for the representation T_g.

Lecture 20

Construction of the Centre of an Enveloping Algebra

Put

$$C = c^{i_1 \ldots i_s} e_{i_1} \ldots e_{i_s}$$

and let

$$[C, e_i] = 0 \qquad i = 1, 2, \ldots n$$

This is a system of equations for the tensor $c^{i_1 \ldots i_s}$. However, let us use a more global approach, using the adjoint representation of the algebra and the group. The representation

$$e_i \rightarrow E_i \qquad E_i e_j = [e_i, e_j]$$

is the adjoint representation of X. Exponentiating, we get the adjoint representation of the group:

$$g \rightarrow \rho(g)$$

acting in the algebra X:

$$\rho(g)\; e_j = \rho_j^s(g) e_s.$$

The monomials

$$e_{i_1} \ldots e_{i_s}$$

transform according to the tensor product

$$\rho(g) \otimes \rho(g) \otimes \ldots \otimes \rho(g) \qquad \text{(s factors)}$$

since it is easy to check that:

$$E_i(e_{i_1}\dots e_{i_s}) = [e_i, e_{i_1}\dots e_{i_s}]$$

We obtain the rule:

In order to find all elements of the centre of the universal enveloping algebra of the Lie algebra X, it is sufficient to find all symmetric tensors over X, invariant under the adjoint representations of the group $\rho(g)$.

Let us simplify further. Consider a row-vector

$$u = (u_1, u_2, \dots u_n)$$

transforming covariantly.

We have

$$E_i e_j = [e_i, e_j] = c_{ij}^k e_k$$

Thus:

$$E_i u_j = c_{ij}^k u_k$$

Take a symmetric tensor $C^{i_1\dots i_s}$ and construct the polynomial:

$$\phi(u) = C^{i_1\dots i_s} u_{i_1}\dots u_{i_s}$$

The invariance of $\phi(u)$ corresponds to

$$\phi(E_i u) = 0$$

or in terms of the adjoint representation of the group

$$\phi\{\rho(g)u\} = \phi(u)$$

We obtain a <u>simpler rule</u>:

In order to find all elements of the centre of $A(X)$ it is sufficient to find all solutions of the equation:

$$\phi(\rho u) = \phi(u)$$

where $\phi(u)$ is a homogeneous polynomial of a covariant vector $u = (u_1, u_2, \ldots u_n)$ and $\rho = \rho(g)$ is the adjoint representation of G, acting on X. Once we have $\phi(u) = \phi(u_1 \ldots u_n)$, then

$$\xi = \phi(e_1, e_2, \ldots e_n)$$

is an element of C (remember that $c^{i_1 \ldots i_s}$ is a symmetric tensor).

<u>Example:</u> Consider $SO(3)$:

$$\begin{array}{lll} [a_1 a_1] = 0 & [a_2 a_1] = -a_3 & [a_3 a_1] = a_2 \\ [a_1 a_2] = +a_3 & [a_2 a_2] = 0 & [a_3 a_2] = -a_1 \\ [a_1 a_3] = -a_2 & [a_2 a_3] = a_1 & [a_3 a_3] = 0 \end{array}$$

We replace these commutation relations by the transformations of a row vector

$$(\, u_1 \; u_2 \; u_3)$$

Thus:

$$A_1(\, u_1 u_2 u_3) = (0 \;\; u_3 \; - u_2)$$

$$A_2(u_1 u_2 u_3) = (-u_3 \;\; 0 \;\; u_1)$$

$$A_3(u_1 u_2 u_3) = (u_2 \; -u_1 \;\; 0)$$

Thus: The action of A_i on $(u_1 u_2 u_3)$ corresponds to infinitesimal rotations about the axes Ou_1, Ou_2 and Ou_3, respectively $\Longrightarrow$ the adjoint representation coincides with rotations in the $(u_1 u_2 u_3)$ space. Now consider the equation

$$\phi(\rho u) = \phi(u)$$

We know that the only independent quantity that is invariant under rotations in a three dimensional space, namely

$$u^2 = u_1^2 + u_2^2 + u_3^2$$

Thus we have

$$\phi(u) = F(u^2)$$

where F is an arbitrary function.

It follows that the centre of the enveloping algebra for O(3) consists of all operators

$$F(\Delta) \ , \qquad \Delta = a_1^2 + a_2^2 + a_3^2$$

Thus, O(3) has only one independent Casimir operator

$$\Delta = a_1^2 + a_2^2 + a_3^2$$

Adjoint Representation for a Matrix Group

For a matrix group, we have

$$\rho(g)\, x = g\, x\, g^{-1}$$

Indeed, for the algebra we have

$$(1 + t\hat{a})x = e^{ta}\, x\, e^{-ta} = (1 + ta)\, x\, (1-ta) = x + t[ax]$$

so that

$$\hat{a}x = [a,x]$$

Remark: $\dim_{\rho}(g) = \dim G =$ finite. Do not confuse with the regular representation which is infinite dimensional

Consider the group $G = GL(n,C)$ (or $G = GL(n,R)$). It can be easily shown that all polynomials satisfying

$$p(x) = p(x_{ij}) = p(g\, x\, g^{-1}) \qquad g \in G$$

can be expanded in terms of powers of the traces

$$p_1(x) = \mathrm{Tr}x, \ldots, p_n(x) = \mathrm{Tr}x^n$$

(possibly also including terms like $\mathrm{Tr}x^n(\mathrm{Tr}x^m)$)*.

Casimir operators for U(n):

Using the commutation relations for e_{ik} we can check that the operators

$$C_1 = e_{ii} \qquad C_2 = e_{ij}e_{ji} \qquad C_n = e_{i_1 i_2} e_{i_2 i_3} \cdots e_{i_n i_1}$$

form a complete basis of the centre C.

From here we can directly obtain the theorem: All Casimir operators for the group U(n) in any representation can be obtained as functions of the n Casimir operators K_s:

$$K_1 = E_{ii}$$

$$K_2 = E_{i_1 i_2} E_{i_2 i_1}$$

$$K_n = E_{i_1 i_2} E_{i_2 i_3} \dots E_{i_n i_1}$$

Remark: For SU(n) we must exclude K_1.

Eigenvalues of the Casimir operators

Consider an irreducible representation $\alpha = (m_1, \dots m_n)$ of U(n). From Schur's lemma we have

$$K_i = k_i I$$

for all Casimir operators. We are interested in the relation between k_i and m_i (the eigenvalues of the Casimir operators and the highest weight).

Consider the maximal eigenvector ω. By definition (of ω) we have

$$E_+ \omega = 0$$

(E_+ is the subalgebra spanned by E_{ij}, $i < j$).

We also know that

$$E_{ii}\omega = m_i \omega \qquad i = 1,2,\dots n$$

For the linear Casimir operator $K_1 = E_{11} + E_{22} + \dots + E_{nn}$ we have

$$k_1 = m_1 + m_2 + \dots + m_n.$$

The quadratic Casimir operator is

$$K_2 = \sum_{i>1} E_{ij}E_{ji} + \sum_i E^2_{ii} + \sum_{i<j} E_{ij}E_{ji}$$

The first term annihilates ω. Commute the entries in the last term:

$$E_{ij}E_{ji} = E_{ji}E_{ij} + E_{ii} - E_{jj} \qquad i < j$$

Thus:

$$k_2 = (m_1^2 + m_2^2 + \ldots + m_n^2) + \sum_{i<j} (m_i - m_j)$$

In general

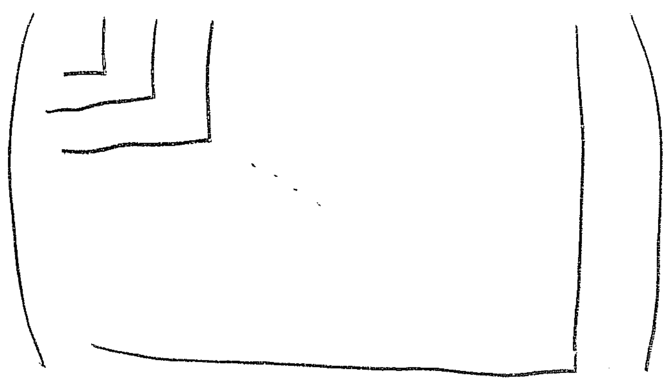
$$k_s = (m_1^s + m_2^s + \ldots + m_n^s) + X(m)$$

where X(m) is a polynomial of lower order than s. It is not difficult to construct X(m) explicitly, but let us leave it at that.

Complete Set of commuting operators

A complete set of commuting operators can be constructed in the following manner. Take the Gelfand-Tseitlin "numbering cone", i.e. the set of subgroups:

$U(n) \supset U(n-1) \supset \ldots \supset U(1)$:



A complete set of commuting operators, which removes all degeneracies, consists of the Casimir operators of the group U(n) and those of all subgroups

$$U(n-k)\ ,\quad k = 1,\dots,n-1$$

Notation:

$U(n)$: $K_{1n}, K_{2n} \dots K_{nn}$

$U(n-1)$: $K_{1n-1} \dots\dots K_{n-1,n-1}$

$U(1)$: K_{11}

The diagonal subgroup T is automatically included as

$$K_{1n-1} = E_{11}+\dots+E_{n-1,n-1}$$

$$K_{1n-2} = E_{11}+\dots+E_{n-2,n-2}$$

$$K_{11} = E_{11}$$

Let us hereby finish our exposition of the representation theory of compact groups (and of the analytic representations of non-compact ones). There are of course many important questions, which we have not even mentioned, like the reduction of direct products of representations (Clebsch-Gordan series and coefficients), the transformation matrices (the Wigner D-functions for U(n)) and many others.

We have actually already considered representations of non-compact groups, however only analytic representations. A consideration of more general finite dimensional representations containing analytic and antianalytic parts would be very similar. Indeed all above considerations can be directly generalized to arbitrary "real" representations.

Instead of one signature α we shall have two

$$\alpha = [p_1, p_2, \ldots p_r]$$

$$\beta = [q_1, q_2 \ldots q_r]$$

and in the space of functions over the subgroup Z we have

$$T_g f(z) = \alpha(z,g)\beta(z,g)f(zg)$$

where $\alpha(z,g)$ and $\beta(z,g)$ themselves form an analytic and an anti-analytic one-dimensional representation of the subgroup Z. Let us now consider unitary representations of noncompact groups.

Unitary Representations of Non-Compact Groups

In this chapter we concentrate only on one series of noncompact groups, namely SL(n,C) and wish to consider, in some sense, all representations. The methods are a straightforward generalization of the method of highest weights used for analytic representations. They can be and indeed have been directly applied to all semisimple groups. We shall also devote one lecture to more general Lie groups.

References:

1) I.M. Gelfand, M. A. Najmark: Unitary Representations of Classical Groups (In Russian or German)

2) M. A. Najmark, Unitary Representation of Noncompact Groups. (Lectures at Summer School on High Energy Physics and the Theory of Elementary Particles, Ed. V.P. Shelest, Kiev, 1967) (In Russian).

1. Definitions:

A representation of a Lie group G is a mapping $g \rightarrow T_g$ of a group element onto a group of linear operators acting in a linear space E. The mapping satisfies

1) $T_e = I$

2) $T_{g_1} \cdot T_{g_2} = T_{g_1 g_2}$

3) T_g depends continuously on g

The representation is finite dimensional or infinitely dimensional depending on whether E is finite or infinitely dimensional.

A representation is irreducible if there are no closed invariant subspaces in E, except for {0} and E, (to make this meaningful we must have some topology on E for infinite dimensional spaces).

A representation is unitary if there exists a positive definite scalar product (x,y) for $x,y \in E$, invariant under all transformations of the group:

$$(x,y) = (T_{g_x}, T_{g_y}).$$

2. Induced Representations

The method we are using is directly related to Mackey's theory of induced representations, which we shall come back to.

In this particular case (of complex semisimple groups) what is going on is the following:

We have a group G and a subgroup K. Consider a representation $k \rightarrow V_k$ of the subgroup K in space L. Consider a set F of vector functions f(g) defined over the group G and with values in L, satisfying

1. F is a linear space with respect to the addition of the functions f(g) and multiplication by a number.

2. $f(kg) = V_k f(g)$ for $k \in K$, $g \in G$ (some sort of homogeneity or covariance condition).

3. F is invariant under right translations, i.e. if $f(g)\varepsilon F$ then $f(gg_o)\varepsilon F$ for all $g, g_o \varepsilon G$.

Define an operator $T g_o$ acting in F as

$$T_{g_o} f(g) = f(gg_o) \qquad g_o \in G \tag{1}$$

The mapping $g \to T_g$ is a representation of G since

$$T_{g_1} T_{g_2} f(g) = T_{g_1} f(gg_2) = T_{g_1} \phi(g) = \phi(gg_1) = f(gg_1 g_2)$$

$(\phi(g) = f(gg_2)\varepsilon F$ by above condition 3).

The representation $g \twoheadrightarrow T_g$ of the group G, given by (1) is called a representation of G induced by the representation $k \twoheadrightarrow V_k$ of the subgroup $K \subset G$.

Notation: $\quad T_g^V$

3. <u>Application to the Group SL(n,C)</u>

Take G to be $SL(n,C)$ and $K \subset G$ as the group of upper triangular matrices:

$$k = \begin{pmatrix} k_{11} & k_{12} \cdots k_{1n} \\ & k_{22} k_{23} \cdots k_{2n} \\ 0 & \ddots \\ & & k_{nn} \end{pmatrix}$$

Let us take the representation V_k of K to be one-dimensional, given by a complex valued function $\alpha(k)$, defined on K, such that

$$\alpha(e) = 1 \tag{2}$$

$$\alpha(k_1 k_2) = \alpha(k_1)\alpha(k_2) \tag{3}$$

The condition $f(kg) = V_k f(g)$ now is

$$f(kg) = \alpha(k)f(g) \tag{4}$$

In this case F consists of complex valued functions $f(g)$ (since L is one-dimensional), satisfying

1) F is a linear space
2) $f(kg) = \alpha(k)f(g) \qquad k\varepsilon K,\ g\varepsilon G$
3) If $f(g)\varepsilon F$ then $f(gg_o)\varepsilon F \quad g,\ g_o\varepsilon G$.

The representation is given by the formula:

$$T_{g_o} f(g) = f(gg_o) \tag{5}$$

It is easy to show that $\alpha(k)$ can only depend on the diagonal elements k_{ii} (the condition det $k = 1$ can be used to eliminate one of the k_{ii}). Actually (2) and (3) can be used to show that

$$\alpha(k) = |k_{22}|^{m_2+\sigma_2} k_{22}^{-m_2} \ \ldots\ldots\ |k_{nn}|^{m_2+\sigma_n} k_{nn}^{-m_n} \tag{6}$$

where m_k are integers, σ_k-complex numbers.

This construction actually leads us, as we shall see, to, irreducible representations of SL(n,C).

4. <u>The triangular decomposition</u>

We shall use a slight modification of the Gauss decomposition which we have been using, namely we shall write

$$g = kz \qquad k\varepsilon K,\ z\varepsilon Z \tag{7}$$

where Z is the group of nilpotent lower triangular matrices:

$$Z = \begin{pmatrix} 1 & & & 0 \\ z_{21} & 1 & & \\ z_{31} & z_{32} & 1 & \\ \cdots & & & \\ z_{n1} & z_{n2} & z_{n3} & 1 \end{pmatrix}$$

Similarly as previously denote

$$\begin{pmatrix} p_1 p_2 \ldots p_m \\ q_1 q_2 \ldots q_m \end{pmatrix} \qquad \begin{matrix} p_1 < p_2 \ldots < p_m \\ q_1 < q_2 \ldots < q_m \end{matrix}$$

a subdeterminant of g, constructed out of the elements g_{ik} on the intersections of rows $p_1 \ldots p_n$ and columns $q_1 \ldots q_n$. Then:

$$k_{pq} = \frac{\begin{pmatrix} p\,q+1 \ldots n \\ q\,q+1 \ldots n \end{pmatrix}}{\begin{pmatrix} q+1 & q+2 \ldots n \\ q+1 & q+2 \ldots n \end{pmatrix}} \qquad p \leq q \tag{8}$$

$$z_{pq} = \frac{\begin{pmatrix} p & p+1 \ldots n \\ q & p+1 \ldots n \end{pmatrix}}{\begin{pmatrix} p & p+1 \ldots n \\ p & p+1 \ldots n \end{pmatrix}} \tag{9}$$

These formulae make sense if the denominators are non-zero. Thus we can represent "almost" all g in form (7), namely we have to exclude a manifold of lower dimension.

For $f(g) \varepsilon F$, $g = kz$, we have (in view of (4)):

$$f(g) = f(kz) = \alpha(k) f(z) \tag{10}$$

similarly as for the analytic representations considered previously.

Since $\alpha(k)$ is fixed, (10) allows us to replace $f(g)\varepsilon F$ by $f(z)\varepsilon F$, defined over the subgroup Z.

In this realization we put

$$T_{g_o} f(z) = \phi(z) \tag{11}$$

$$g = kz \qquad gg_o = k''z_{g_o} \qquad k,k''\varepsilon K, \; z\,z_{g_o} \varepsilon Z \tag{12}$$

Then we have

$$T_{g_o} f(g) = f(gg_o) = f(k''z_g) = \alpha(k'')f(z_g)$$

$$T_{g_o} f(g) = T_{g_o} \alpha(k)f(z) = \alpha(k)T_{g_o} f(z) = \alpha(k)\phi(z)$$

Thus:

$$T_{g_o} f(z) = \phi(z) = \alpha(k)^{-1}\alpha(k'')f(z_{g_o}) = \alpha(k^{-1}k'')f(z_g) \tag{13}$$

However: $\quad kzg_o = k''z_{g_o}$ so that

$$zg_o = k'k''z_{g_o} = k'z_{g_o}$$

Finally:

$$\boxed{T_g f(z) = \alpha(k)f(z_g)} \tag{14}$$

with

$$\boxed{zg = kz_g}$$

Example: SL(2,C):

$$zg = kz_g$$

$$\begin{pmatrix} 1 & 0 \\ z_{21} & 1 \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} k_{11} & k_{12} \\ 0 & k_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ (z_g)_{21} & 1 \end{pmatrix}$$

$$g_{11} = k_{11} + k_{12}(z_g)_{21} \qquad z_{21}g_{11} + g_{21} = (z_g)_{21}$$

$$g_{12} = k_{12} \qquad z_{21}g_{12} + g_{22} = k_{22}$$

The matrices Z and g are given, we solve for k and z_g:

$$k_{12} = g_{12} \qquad k_{11} = \frac{1}{g_{12}z_{21}+g_{22}}$$

$$k_{22} = g_{12}z_{21} + g_{22} \qquad (z_g)_{21} = \frac{g_{11}z_{21}+g_{21}}{g_{12}z_{21}+g_{22}}$$

Thus, the action of T_g on f(z) in (14) is the following:

1) The argument z undergoes a generalized fractional-linear transformation

2) The function $f(z_g)$ is multiplied by the multiplier $\alpha(k) = \alpha(z,g)$ (k depends on the transformation g, the point z and on the representation we are considering).

Lecture 21

5. The Invariant Scalar Product. The Principal Nondegenerate Series

Let us restrict outselves to unitary representations of SL(n,C). We must then choose such a space of functions F, that we can introduce an invariant scalar product. Let us first introduce an invariant measure on Z.

Put

$$z_{pq} = x_{pq} + iy_{pq}\ ,\ p > q\ ,\ x_{pq},\ y_{pq}\ \text{real}$$

It can be shown that the left (and right) invariant measure is

$$d_\mu(z) = \prod_{\substack{p,q=1 \\ p>q}} dx_{pq}\, dy_{pq} \tag{15}$$

Let us now consider the space of functions f(z) satisfying

$$\int |f(z)|^2 d\mu(z) < \infty$$

(Integration from $-\infty$ to ∞ with respect to all x_{pq} and y_{pq}). This space is a Hilbert space, denoted $L^2(Z)$ with the scalar product

$$(f_1,f_2) = \int f_1(z)\overline{f_2(z)}\,d\mu(z) \tag{16}$$

$$f_1,f_2 \in L^2(Z)$$

Consider the representation:

$$\begin{gathered} T_g f(z) = (k)f(z_g) \\ zg = kz_g \\ \alpha(k) = |k_{22}|^{m_2+\sigma_2} k_{22}^{-m_2} \ldots |k_{nn}|^{m_n+\sigma_n} k_{nn}^{-m_n} \end{gathered} \tag{17}$$

where $m_2 \ldots m_n$ are integers, $\sigma_2, \ldots \sigma_n$ are complex numbers. To find out which representations are unitary with respect to the scalar product (16) we must put

$$(f_1,f_2) = (T_g f_1, T_g f_2)$$

This condition poses a restriction on σ_k, namely:

$$\sigma_k = i\rho_k - 2(k-1) \qquad k = 2,3,\ldots,n \quad (18)$$

We shall prove this below for SL(2,C) only. The result is:

The <u>principal nondegenerate series of unitary irreducible representations</u> of SL(n,C) is determined by <u>two sets of numbers:</u> the integers $m_2,\ldots,m_n$ and the real numbers $\rho_2,\ldots,\rho_n$. The representations are then given by formulae (17) and (18).

We have not yet proved irreducibility. The way to do that is to use Schur's lemma and show that any bounded linear operator in $L^2(z)$, commuting with all operators T_g is a multiple of the identity.

<u>Example:</u>

SL(2,C) (locally isomorphic to the homogeneous Lorentz group O(3,1)).

Take: $z \in Z \qquad z = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \qquad z = x + iy$

$$f(z) = f(x,y)$$

$L^2(z)$ is the space of functions:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(z)|^2 dxdy < \infty$$

Put

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad \alpha\delta - \beta\gamma = 1$$

Representations of the principal series:

$$T_g f(z) = |\beta z+\delta|^{m+\sigma} (\beta z+\delta)^{-m} f\left(\frac{\alpha z+\gamma}{\beta z+\delta}\right) \tag{19}$$

(since $k_{22} = \beta z+\delta$, $z_g = \frac{\alpha z+\gamma}{\beta z+\delta}$) .

Here m is integer, $\sigma = -2+i\rho$, ρ = real. Unitarity: $dz = dxdy$

$$\int T_g f_1 \cdot \overline{T_g f_2}\, dz = \int |\beta z+\delta|^{m+\sigma} (\beta z+\delta)^{-m} f_1\left(\frac{\alpha z+\gamma}{\beta z+\delta}\right)$$

$$\times |\beta z+\delta|^{+m+\sigma^*} \overline{(\beta z+\delta)}^{-m} \overline{f_2\left(\frac{\alpha z+\gamma}{\beta z+\delta}\right)} dz =$$

$$\int |\beta z+\delta|^{\sigma+\sigma^*} f_1\left(\frac{\alpha z+\gamma}{\beta z+\delta}\right) \overline{f_2\left(\frac{\alpha z+\gamma}{\beta z+\delta}\right)} dz$$

Put $\frac{\alpha z+\gamma}{\beta z+\delta} = u$. We can check that

$$dz = \left|\frac{dz}{du}\right|^2 du$$

(See Appendix in M. A. Najmark: Linear Representations of the Lorentz Group).

We have: $\frac{dz}{du} = (\beta z+\delta)^2$ so that

$$\int T_g f_1(z)\, \overline{T_g f_2(z)} dz = \int |\beta z+\delta|^{\sigma+\sigma^*+4} f_1(u) \overline{f_2(u)} du$$

Thus: the representation is unitary, if $\sigma+\sigma^* + 4 = 0 \Rightarrow \sigma = -2+i\rho$.

Remark: 1) For SL(n,C) the terms $-2(k-1)$ in (18) are necessary precisely in order to cancel the Jacobian of the transformation $z \to z_g$.

2) Formula (19) also gives the finite dimensional representations of SL(2,C).

$$T_g p(z) = (\beta z+\delta)^M \, \overline{(\beta z+\delta)}^N \, p\left(\frac{\alpha z+\gamma}{\beta z+\delta}\right) \quad M,N \ldots \text{non-negative integers} \tag{20}$$

if we put

$$-\frac{m}{2} + \frac{i\rho}{2} - 1 = M$$

$$\frac{m}{2} + \frac{i\rho}{2} - 1 = N$$

i.e.

$$i\rho = M + N + 2$$

$$m = -M+N$$

In particular the analytic representations correspond to N=0, i.e. $i\rho = M + 2$, $m = -M$.

6. Realization of the Principal Nondegenerate Series On the Unitary Subgroup

Instead of using functions defined over the group G or the nilpotent subgroup Z, we can consider functions over the maximal compact subgroup U, in our case SU(n).

Put

$$\Gamma = SU(n) \cap K \tag{21}$$

Obviously: $\gamma \varepsilon \Gamma$ iff

$$\gamma = \begin{pmatrix} e^{i\phi_1} & & & 0 \\ & e^{i\phi_2} & & \\ & & \ddots & \\ 0 & & & e^{i\phi_n} \end{pmatrix} \tag{22}$$

and det $\gamma = 1$

$$\phi_1 + \phi_2 + \dots + \phi_n = 0 \tag{23}$$

<u>Lemma:</u> Every matrix $g \varepsilon SL(n,C)$ can be written as

$$g = ku \quad k \varepsilon K \quad u \varepsilon SU(n) \tag{24}$$

If we also have $g = k'u'$, then $k' = k\gamma$, $u' = \gamma^{-1}u$, $\gamma \varepsilon \Gamma$.

<u>Proof</u>: Take $k^{-1}g$: the last row in g is multiplied by k^{-1}_{nn}, the last but one row in g is replaced by a linear combination of itself and the last row, etc. This is the same procedure as the orthogonalization of a set of vectors and we can use it to orthonormalize the rows of $k^{-1}g$. Hence $k^{-1}g = u$ can be taken to be unitary, so that $g = ku$. Further: $g = ku = k'u' \implies k^{-1}k' = uu'^{-1} = r$. Obviously $r \varepsilon K$, $r \varepsilon SU(n) \implies r = \gamma$. Thus

$$k' = k\gamma \qquad u' = \gamma^{-1}u$$ Q.E.D.

On the group G we have $f(kg) = \alpha(k)f(g)$, so that

$$f(g) = f(ku) = \alpha(k'')f(u)$$

similarly as we had

$$f(g) = f(k'z) = \alpha(k')f(z)$$

Thus:

$$f(z) = \alpha^{-1}(k')\alpha(k'')f(u) = \alpha(k'^{-1}k'')f(u) = \alpha(k)f(u) \tag{25}$$

Also $\quad f(\gamma u) = \alpha(\gamma)f(u) \qquad \gamma\varepsilon\Gamma \qquad (26)$

In view of (17), we have

$$\alpha(\gamma) = e^{-i(m_2\phi_2+\ldots+m_n\phi_n)}$$

since $k_{\ell\ell} = \gamma_{\ell\ell} = e^{i\phi_\ell} \qquad \ell = 2,3,\ldots,n.$

Explicitly calculating Jacobians, we can check that

$$\int|f(z)|^2d\mu(z) = \int|f(\omega)|^2d\mu(u)$$

where $d\mu(u)$ is the invariant measure on the Group U.

It follows that (25) gives an isometric mapping of $L^2(z)$ onto $L^2_\alpha(u)$: the space of all functions on U, satisfying (26) and

$$\int|f(u)|^2d\mu(u) < \infty .$$

Representations of the principal nondegenerate series are given by the formula

$$T_gf(u) = \alpha(k)f(u_g) \tag{27}$$

where k and u_g are given by the formula

$$ug = ku_g$$

Proof:

$$T_{g_o} f(g) = T_{g_o} \alpha(k')f(u) = \alpha(k')T_{g_o} f(u) \qquad g = k'u$$

$$T_{g_o} f(g) = f(gg_o) = f(k''u_{g_o}) = \alpha(k'')f(u_{g_o}) \qquad gg_o = k''u_{go}$$

Thus: $T_{g_o} f(u) = \alpha(k'^{-1}k'')f(u_{g_o})$

Finally: $T_g f(u) = \alpha(k)f(u_g)$

$$ug = ku_g$$

Q.E.D.

The non-uniqueness in $g = k u$ is irrelevant, since if $g = k'u' = k u$, then

$$\alpha(k')f(u'_g) = \alpha(k\gamma)f(\gamma^{-1}u_g) = \alpha(k)\alpha(\gamma)\alpha(\gamma^{-1})f(u_g) = \alpha(k)f(u_g)$$

The realization of representations of G in the space $L^2(u)$ is convenient for solving the problem of the reduction of the representation $g \to T_g$ to the subgroup $U = SU(n)$. The representation $T_g|u$ will definitely be reducible since all irreducible representations of $SU(n)$ are finite-dimensional. Let us find the irreducible representations of $SU(n)$, contained in the reduction $T_g|u$.

We have:

$$T_{u_o} f(u) = \alpha(k)f(u_{u_o})$$

$$uu_o = ku_{u_o}$$

Put: $k = e$, $u_{u_o} = uu_o$, then

$$T_{u_o} f(u) = f(uu_o).$$

Consider an irreducible representation of U:

$u \to c^{\nu}(u)$ $\qquad \nu$ = set of numbers, e.g. the signature of a representation.

Choose such a basis, that $c^{\nu}(\gamma)$ is diagonal:

$$C^{\nu}_{jk}(\gamma) = \delta_{jk}\omega_j(\gamma)$$

The corresponding basis is called canonical and $\omega_j(\gamma)$ are the weights of the representation. Since $C^{\nu}(u)$ are the operators of a representation, we have

$$C^{\nu}_{j\ell}(uu_o) = \sum_s C^{\nu}_{js}(u)C^{\nu}_{s\ell}(u_o),$$

in particular

$$C^{\nu}_{j\ell}(\gamma u) = \omega_j(\gamma)C^{\nu}_{j\ell}(u)$$

$$C^{\nu}_{j\ell}(u\gamma) = C^{\nu}_{j\ell}(u)\omega_\ell(\gamma)$$

For SU(n) we already have expansion formulae, following from the Stone-Weierstress theorem, namely the Peter-Weyl theorem: Any function $f(u)\varepsilon L^2_\alpha(u)$ can be expanded in terms of the matrix elements of the irreducible representations of SU(n):

$$f(u) = \sum_{\nu j\ell} b^{\nu}_{j\ell}C^{\nu}_{j\ell}(u) \tag{28}$$

We have

$$f(\gamma u) = \alpha(\gamma)f(u) = \alpha(\gamma)\sum_{\nu j\ell} b^{\nu}_{j\ell}C^{\nu}_{j\ell}(u)$$

and $$f(\gamma u) = \sum_{\nu j\ell} b^{\nu}_{j\ell}C^{\nu}_{j\ell}(\gamma u) = \sum_{\nu j\ell} b^{\nu}_{j\ell}\omega_j(\gamma)C^{\nu}_{j\ell}(u)$$

so that

$$[\alpha(\gamma) - \omega_j(\gamma)]b^{\nu}_{j\ell} = 0 \tag{29}$$

i.e. $$b^{\nu}_{j\ell} \neq 0 \quad \text{only if } \alpha(\gamma) = \omega_j(\gamma).$$

The functions

$$C^{\nu}_{j1}(u),\ C^{\nu}_{j2}(u), \dots\ C^{\nu}_{jm_\nu}(u)$$

form a canonical basis for an irreducible representation $u \to C^{\nu}(u)$.

Let $q_1 \dots q_{p_\nu}$ be those numbers, amongst $1, \dots m_\nu$, for which

$$\omega_j\ (\gamma) = \alpha(\gamma)$$

[$\alpha(\gamma)$, as opposed to $\alpha(k)$ depends only on the integers $(m_2, \dots, m_n)$ in the "signature" of $SL(n,C)$].

We have

$$T_\gamma C_{j\ell}(u) = C_{j\ell}(u\gamma) = C_{j\ell}(u)\omega_\ell(\gamma)$$

$$T_\gamma C_{j\ell}(\gamma u) = \alpha(\gamma) C_{j\ell}(u)$$

It follows that there are p_ν-functions

$$C^{\nu}_{jq_1}(u),\ C^{\nu}_{jq_2}(u) \dots C^{\nu}_{jq_{p_\nu}}(u)$$

in the space of the representation $u \to C^{\nu}(u)$ with weight $\alpha(\gamma)$.

Thus, p_ν is the multiplicity of the weight $\alpha(\gamma)$ in $u \to C^{\nu}(u)$ (note that $\alpha(\gamma)$ is just a weight, not necessarily a highest or lowest one).

We obtain:

<u>Theorem</u>: The reduction $u \to T_u$ of the irreducible unitary representation $g \to T_g$ of the principal nondegenerate series for the group SL(n,C to SU(n) is given by the formula

$$T_g f(u) = \alpha(u) f(u_g)$$

and contains an irreducible representation $u \to C^{\nu}(u)$ as many times as is the multiplicity of the weight $\alpha(\gamma)$ in this representation.

Example: $SL(2,C) \supset SU(2)$.

We have:

$$\gamma = \begin{pmatrix} e^{i\phi_1} & 0 \\ 0 & e^{i\phi_2} \end{pmatrix} \qquad T_\gamma f(z) = e^{-i\phi_2 m} f(e^{i(\phi_1-\phi_2)} z)$$

Take a representation $T_\ell(u)$ of $SU(2)$, with the highest weight ℓ. This is contained in the representation (m,ρ) of $SL(2,C)$, if m is a weight in $T_\ell(u)$, i.e. if

$$m\varepsilon(-2\ell,-2\ell+1,\ldots,2\ell-1,2\ell) \quad (\ell = \text{integer or half-integer})$$

in other words if $|m| \leq 2\ell$. (For $SU(2)$ we have

$$T_u f(z) = (\beta z+\delta)^{2\ell} f\left(\frac{\alpha z+\gamma}{\beta z+\delta}\right)\Big) .$$

Since for $SU(2)$ the multiplicity of any weight, in particular the weight $-m$ (i.e. $e^{-i\phi m}$) is $p_\nu = 1$, each representation of $SU(2)$ with $\ell \geq \frac{|m|}{2}$ (and with ℓ integer or half-integer simultaneously with $\frac{|m|}{2}$) is contained in $T_g|u$.

7. Principal Degenerate Series

Let us consider further representations of $SL(n,C)$. Take n and split it into positive integers:

$$n = n_1 + n_2 + \ldots + n_r$$

$$2 \leq r < n \qquad n_i > 0, \ i = 1,\ldots,r.$$

Put

$$g = \begin{pmatrix} g_{11}\cdots g_{1r} \\ \cdots\cdots \\ g_{r1}\cdots g_{rr} \end{pmatrix}$$

Here g_{pq} are matrices with n_p rows and n_q columns, so chosen that $\det g = 1$.

Consider the subgroups of matrices

$$k = \begin{pmatrix} k_{11} & k_{12}\cdots k_{1r} \\ 0 & k_{22}\cdots k_{2r} \\ \cdots\cdots & \cdots\cdots \\ 0 & 0 \quad k_{rr} \end{pmatrix} \qquad Z = \begin{pmatrix} I_{n_1} & & & 0 \\ Z_{21} & I_{n_2} & & \\ Z_{31} & Z_{32} & I_{n_3} & \\ & \cdots\cdots & \cdots\cdots & \\ Z_{r1} & Z_{r2} & & I_{n_r} \end{pmatrix}$$

where k_{pq} and z_{pq} are the same type of matrices as g_{pq} and I_{n_k} is a unit matrix of order n_k.

<u>Lemma:</u> Almost any matrix g can be written as

$$g = kz \tag{30}$$

where $k\varepsilon K_{n_1\ldots n_r}$, $z\varepsilon Z_{n_1\ldots n_r}$

We shall not give a proof, nor even write the matrix elements of k and z in terms of g explicitly, but refer to the original articles. The formulae are very similar to those in the nondegenerate case.

Representations of the principal degenerate series are constructed like the non-degenerate ones.

Take a one-dimensional representation of $K_{n_1 n_2 \ldots n_r}$

$$k \to \alpha(k)$$

and a set F of functions f(g), satisfying

(i) F is a linear space

(ii) $f(kg) = \alpha(k)f(g) \quad k \varepsilon K_{n_1 \ldots n_r}$

(iii) $f(g) \varepsilon F \Rightarrow f(gg_o) \varepsilon F$ for g and $g_o \varepsilon G$

(f(g) is a mapping of the group manifold g onto the space of complex numbers).

A representation of G in the space F is given by

$$T_{g_o} f(g) = f(gg_o) \qquad (31)$$

Since we have:

$$f(g) = f(kz) = \alpha(k)f(z) \qquad k \varepsilon K_{n_1 \ldots n_r} \quad z \varepsilon Z_{n_1 \ldots n_r}$$

we can consider functions on the subgroup $Z_{n_1 \ldots n_r}$ only.

Then

$$T_g f(z) = \alpha(k) f(z_g) \qquad (32)$$

where

$$zg = kz_g \qquad k \varepsilon K_{n_1 \ldots n_r}; \; z_g \varepsilon Z_{n_1 \ldots n_r}$$

and $\alpha(k)$ depends only on the determinants of the matrices k_{pp}.

Unitary Representations of the principal degenerate series are constructed in the Hilbert space $L^2(Z_{n_1\ldots n_r})$ of integrable functions over $Z_{n_1\ldots n_r}$, satisfying

$$\int |f(z)|^2 d\mu(z) < \infty$$

where $\mu(z)$ is the invariant measure on $Z_{n_1\ldots n_r}$

i.e.

$$d\mu(z) = \prod_{\substack{p,q=1 \\ p > q}}^{n} dx_{pq} dy_{pq} \qquad , \qquad z_{pq} = z_{pq} + iy_{pq}$$

and only these z_{pq} figure which are matrix elements of the matrices $z_{ij} \varepsilon Z_{n_1\ldots n_t}$.

The scalar product is

$$(f_1, f_2) = \int f_1(z)\overline{f_2(z)}d\mu(z) \tag{33}$$

Putting: $\Lambda_j = \text{Det } k_{jj}$ we have

$$\alpha(k) = |\Lambda_2|^{m_2+i\rho_2 - (n_1+n_2)} \Lambda_2^{-m_2} |\Lambda_2|^{m_3+i\rho_3-(n_1+2n_2+n_3)} \Lambda_3^{-m_3} \ldots \tag{34}$$
$$|\Lambda_r|^{m_r+i\rho_r-(n_1+2n_2+2n_3+n_3+\ldots 2r_{r-1}+n_r)} \Lambda_r^{-m_r}$$

and it can be checked that this choice of the exponents ensures unitarity (invariance of (33)). Here

$$\rho_2, \ldots, \rho_r \qquad \text{are real}$$

$$m_2, \ldots, m_r \qquad \text{are integer}$$

and $n_1, n_2, \ldots, n_r$ are positive integers satisfying

$$n = n_1 + \ldots + n_r.$$

Thus: we have as many degenerate representations as we have "splittings" of n.

Examples:

n = 2: (SL(2,C)) : No splitting $\Rightarrow$ no degenerate series (in the above sense)

n = 3: (SL(3,C) : $\left.\begin{array}{l} 3 = 2+1 \\ 3 = 1+2 \end{array}\right\}$ two degenerate series (equivalent)

It can be shown that all representations of the principal degenerate series are irreducible.

Example: Take SL(n,C) and put

$$n = n_1+n_2 \qquad n_1 = n-1,\ n_2 = 1$$

This is called the maximal degenerate representation:

We have $Z_{n_1 \ldots n_r} = Z_{n-1,1}$

$$z = \left(\begin{array}{cccc|c} 1 & 0 \ldots\ldots & 0 & & 0 \\ 0 & 1 \ldots & 0 & & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 1 & & 0 \\ \hline z_1 & z_2 \ldots & & z_{n-1} & 1 \end{array}\right)$$

$f(z) = f(z_1 \ldots z_{n-1})$

$$T_g f(\ldots z_p \ldots) = \left|\sum_{i=1}^{n-1} g_{jn} z_j + g_{nn}\right|^{m+i\rho-n} \left(\sum_{j=1}^{n-1} g_{jn} z_j + g_{nn}\right)^{-m} \times$$

$$f\left(\ldots \frac{\sum\limits_{j=1} g_{jp} z_j + g_{np}}{\sum\limits_{j=1}^{n-1} g_{jn} z_j + g_{nn}} \ldots\right)$$

Remark: Thus, the degenerate representations are given by smaller number of numbers m_i, ρ_i then the nondegenerate ones. In the language of Casimir operators this would mean that only some of the Casimir operators are independent, The eigenvalues of the rest are equal to zero or functions of the non-zero ones.

8. Realization of the Principal Degenerate Series on the Unitary Subgroup

Introduce the subgroup

$$\Gamma_{n_1 \ldots n_r} = SU(n) \cap K_{n_1 \ldots n_r}$$

of matrices

$$\gamma = \begin{pmatrix} u_{n_1} & & & 0 \\ & u_{n_2} & & \\ & & \ddots & \\ 0 & & & u_{n_r} \end{pmatrix} \qquad \det \gamma = 1 \tag{35}$$

where U_{n_p} is a unitary matrix of order p. We have

$$g = ku \tag{36}$$

$$k \varepsilon K_{n_1 \ldots n_r}, \quad u \varepsilon SU(n)$$

If $g = ku = k_1 u_1$, then

$$k_1 = k\gamma^{-1}, u_1 = \gamma u \qquad \gamma \varepsilon \Gamma_{n_1 \ldots n_r}$$

Further: proceed as for nondegenerate representations, replacing Γ by $\Gamma_{n_1 \ldots n_r}$

Lecture 22

9. Supplementary Nondegenerate Series of Irreducible Unitary Representations.

The group SL(n,C) has further irreducible unitary representations - the supplementary series. Obviously - they must be constructed in different Hilbert spaces.

Example: Consider first SL(2,C) and a space L of functions f(z) falling of at infinity in such a fashion that the integral

$$\int A(z_1,z_2)f_1(z_1)\overline{f_2(z_2)}dz_1dz_2$$

converges absolutely for all $f_1,f_2 \in L$ and for $A(z_1,z_2)$ to be specified. Here

$$dz_1 = dz_1dy_1;\ dz_2 = dx_2dy_2 \quad z_1 = x_1+iy_1 \,.\quad z_2 = x_2+iy_2$$

The scalar product is introduced as

$$(f_1,f_2) = \int A(z_1,z_2)f_1(z_1)\overline{f_2(z_2)}dz_1dz_2 \qquad (37)$$

In general we can write

$$T_g f(z) = |\beta z+\delta|^{m+i\rho-2}(\beta z+\gamma)^{-m}f\left(\frac{\alpha z+\gamma}{\beta z+\delta}\right)$$

(with ρ-complex)

and $A(z_1,z_2)$,ρ and m must be chosen that (37) is invariant under T_g. It can be shown* that an invariant scalar produce is obtained if

*M. A. Najmark, Linear Representations of the Lorentz Group, Pergamon Press, N.Y. 1969.

$$m = 0 \qquad \rho = i\sigma \qquad (\sigma = \text{real}),\ A(z_1, z_2) = |z_1 - z_2|^{\sigma - 2} \tag{38}$$

For the representation to be unitary the invariant scalar product must be positive definite. This will be so if:

$$m = 0 \qquad \rho = i\sigma \qquad 0 < \sigma < 2 \tag{39}$$

It can be shown that the representations of the supplementary series are irreducible. The space L of functions $f(z)$ can be constructed explicitly.

Return to the general case of $SL(n,C)$. Introduce a set of matrices:

$$\overset{\circ}{z} = \left(\begin{array}{c|cccc} \underbrace{\left.\begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix}\right\}}_{n-2\tau} {\scriptstyle n-2\tau} & & & & \\ \hline & \begin{matrix} 1 & 0 \\ z_1 & 1 \end{matrix} & & & \\ & & \begin{matrix} 1 & 0 \\ z_2 & 1 \end{matrix} & & \\ & & & \ddots & \\ & & & & \begin{matrix} 1 & 0 \\ z_\tau & 1 \end{matrix} \end{array} \right) \tag{40}$$

Representations of the supplementary series:

$$T_g f(z) = \alpha(k) f(z_g) \tag{41}$$

$$\alpha(k) = |k_{22}|^{m_2 + \tilde{\sigma}_2} k_{22}^{-m_2} \cdots |k_{nn}|^{m_n + \tilde{\sigma}_n} k_{nn}^{-m_n}$$

$\tilde{\sigma}_k \ldots$ complex .

Write the invariant scalar product as

$$(f_1,f_2) = \int A(z,\mathring{z}z)f_1(z)\overline{f_2(\mathring{z}z)}d\mu(z)d\mu(\mathring{z}) \tag{42}$$

where $d\mu(\mathring{z}) = \prod_{p=1}^{\tau} dx_p dy_p \qquad z_p = z_p + iy_p$

The kernel $A(z,\mathring{z}z)$ must be determined.

The functions $f(z)$ lie in a linear space L, satisfying

a) for $f_1,f_2 \in L$ (42) converges absolutely.

b) L is invariant under T_g of (41).

It can be shown that the invariance of the scalar product has quite definite implications for the n_k and σ_k of (41) and for $A(z,\mathring{z}z)$.

The results can be stated as follows:

<u>Theorem</u>: The unitary representations of the supplementary nondegenerate series can be constructed in a Hilbert space L of functions $f(z)$ with a scalar product

$$(f_1,f_2) = \int A(\mathring{z})f_1(z)\overline{f_2(\mathring{z}z)}d\mathring{z}dz$$

where

$$A(\dot{z}) = \prod_{j=1}^{\tau} |\dot{z}_j|^{2(\sigma_j''-1)} \qquad 0 < \sigma_j'' < 1 \qquad j = 1,\ldots,\tau$$

The operators of the representation are

$$T_g f(z) = \alpha(k) f(z_g) \qquad zg = kz_g$$

$$\alpha(k) = \beta^{-1/2} \prod_{p=2}^{n-2\tau} |k_{pp}|^{m_p + i\rho_p} k_{pp}^{-m_p} \prod_{q=1}^{\tau} |\lambda_q|^{m_q' + i\sigma_q' + \sigma_q''} \lambda^{-m_q}$$

$$\times |\mu_q|^{m_q' + i\sigma_q' - \sigma_q''} \mu_q^{-m_q'}$$

where $\beta(k) = |k_{22}|^4 \, |k_{33}|^8 \dots |k_{nn}|^{4n-4}$

and

$$k = \left(\begin{array}{cccc|cc|cc} k_{11} & \dots & & & \dots & & & k_{1n} \\ 0 & k_{22} & & & & & & k_{2n} \\ \dots & \dots & \dots & & & & & \vdots \\ 0 & \dots & 0 & k_{n-2\tau} & & & & \\ \hline & & & & \lambda_1 & x & & \\ & & & & 0 & \mu_1 & & \\ \hline & & & & & & \lambda_\tau & x \\ & & & & & & 0 & \mu_\tau \end{array} \right)$$

Thus, a representation of the supplementary nondegenerate series is given by $n-\tau+1$ integers

$$\tau \left(0 \le \tau \le \frac{n}{2}\right),\ m_1 \dots m_{n-2\tau}, m_1' \dots m_\tau' \qquad \text{integers}$$

and n real numbers

$$\rho_1 \dots \rho_{n-2\tau},\ \sigma_1' \dots \sigma_\tau',\ \sigma_1'', \dots \sigma_\tau'' \qquad \text{real}$$

with $\qquad 0 < \sigma_p'' < 1 \qquad p = 1, \dots, \tau$

10. Supplementary Degenerate Series of Representations

These representations are given by an integer $\tau > 0$ and a partition

$$n = n_1 + n_2 + \ldots + n_r$$

where the last 2τ numbers are $n_p = 1$ for $p = r-2\tau+1, \ldots r$. The representations are realized in the space

$$f(z) \quad z \in Z_{n_1 \ldots n_r} .$$

We shall not go into this here.

11. Equivalence of representations

It can be shown that two representations, belonging to different series are never equivalent. Representations of the same series are equivalent if the sets of pairs:

$$(m_1 \rho_1) \ \ldots . \ (m_n \rho_n)$$
$$(m_1' \rho_1') \ \ldots . \ (m_1' \rho_n')$$

can be obtained from each other by a permutation of pairs and this permutation does not take us out of the given series.

12. Representations of GL(n,C)

Same as those for SL(n,C), except that the "signatures" $(m_1, \ldots m_n)$ and $(\rho_1 \ldots \rho_n)$ should not be normalized.

13. Representations of $U(p,q)$

These have also been studied by Gelfand and Graev and also by others, using similar methods as for $SL(n,C)$. The main difference is that $U(p,q)$ groups have further series of representations, namely discrete representations.

Reference: M. I. Graev: Am. Math. Soc. Transl. 66, 1 (1968).

General Theory of Representations

I. Solvable Groups

Definition: (a) The group T is solvable if the adjoint representation can be brought to triagonal form:

$$g \rightarrow \mathrm{adj}g = e^{\hat{x}} = \begin{pmatrix} \rho_{11}(t), & 0 & 0 & \cdots & 0 \\ \rho_{21}(t), & \rho_{22}(t), & 0 & \cdots & 0 \\ & & \ddots & & \\ \rho_{n1}(t) & \cdots & \cdots & \cdots & \rho_{nn}(t) \end{pmatrix}$$

(The adjoint representation of the Lie algebra is

$$x \rightarrow \mathrm{adj}x = \hat{x} \quad \text{where } \hat{x}y = [x,y])$$

(b) The group T is solvable if the set of derived groups

$$T, T', T'', \ldots, T^{(n)}$$

terminates with

$$T^{(n)} = \{e\} \qquad \text{for some finite } n \geq 0.$$

Here T' consists of all commutators

$$k = g^{-1}h^{-1}gh \qquad g,h\in T$$

and their products. T'' is similarly constructed from T', etc..

It follows from Lie's theorem that any representation of a solvable group can be brought to a triagonal form.

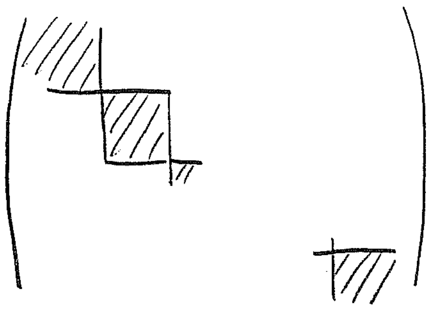
II. The Levy-Maltsev Theorem

Consider an arbitrary Lie group G and its adjoint representation

$$g \to \rho(g).$$

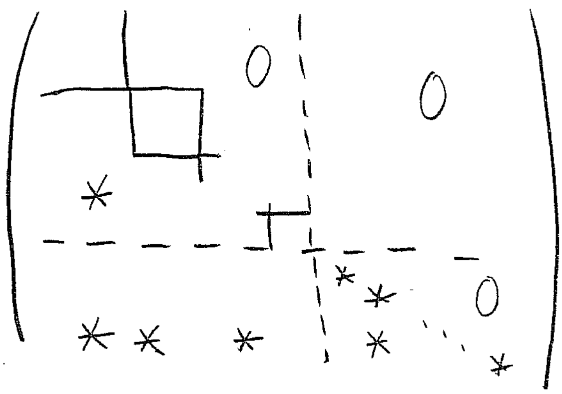
G is ~~r~~eductive, if $\rho(g)$ is completely reducible.

$\rho(g) =$



If G is not reductive, then the matrices $\rho(g)$ can be brought to the form

$\rho(g) =$



$$= \begin{pmatrix} r(g) & 0 \\ * & t(g) \end{pmatrix}$$

The space X where $\rho(g)$ acts can thus be split into the direct sum of two subspaces

$$X = A + B$$

where B is an invariant subspace

$$\rho(g)\, B \subset B$$

<u>The Levy-Maltsev Theorem</u>: Any Lie algebra X can be represented as a sum of two subalgebras

$$X = A + B \qquad (1)$$

where A is semisimple and B is solvable. B is the maximal solvable ideal in X and we have

$$[A,B] \subset B \qquad (2)$$

<u>Corollary:</u> Any connected Lie group G is locally isomorphic to the semidirect product

$$G = R\cdot T \qquad (3)$$

of a semisimple connected group R and a solvable group T. T is a radical, i.e. a maximal solvable invariant subgroup, so that

$$RTR^{-1} \subset T \qquad (4)$$

<u>Example</u>: The group of transformations of a Euclidean space E_n:

$$G = R\cdot T$$

where R is the group of rotations about a fixed point, namely the origin, and T is the group of translations:

$$g = rt = t'r$$
$$rtr^{-1} = t'$$

III. Representation Theory of General Lie Groups

We shall use the semidirect product decomposition of the Lie group G

$$G = R\cdot T \tag{5}$$

and apply the theory of induced representations.

Mackey's theory of induced representations, more general than it's predecessor due to Gelfand and Najmark, tells us how to get all irreducible unitary representations of G, given those of T and R.

We shall only consider a simple case, namely when T, in general solvable, is actually Abelian. Let us restrict ourselves to unitary representations, constructed in some Hilbert space H.

Let us go through several steps.

1) Consider the subgroup T and use it to induce representations. Being Abelian, T has only one-dimensional irreducible representations:

$$t\varepsilon T \qquad t \to U_t \qquad U_t^*\xi = \alpha(t)\xi \qquad \xi\varepsilon L \tag{6}$$

Here L is some space, in general larger than the Hilbert space H and ξ is a generalized eigenvector of the generators of T. In this case, we even know that:

$$\alpha(t) = e^{i(\lambda_1 t_1 + \dots \lambda_n t_n)} = e^{i\lambda t} \tag{7}$$

where $\lambda = (\lambda_1, \dots, \lambda_n)$ labels the representations of T and also labels the vectors $\xi = \xi(\lambda)$.

We are interested in induced representations, so now we must construct a set F of functions $f(g)$, $g\varepsilon G$, with values in L: $f(g)\varepsilon L$.

As we know, F must be a linear space, be invariant under right transformations gg_0 and must satisfy a homogenity condition

$$f(t.g) = \alpha(t)f(g) \tag{8}$$

Let us inspect this condition.

We can write an element g of G = R.T as

$$g = (r,t) \tag{9}$$

with

$$g^{-1} = (r^{-1}, -r^{-1}t) \text{ and } e = (1,0) \tag{10}$$

The condition $rtr^{-1} = t'$ corresponds to the multiplication law

$$(r_1,t_1)\ (r_2,t_2) = (r_1r_2,\ t_1+r_1t_2) \tag{11}$$

In this notation an element of T is (1,t), an element of R is (r,0). We have

$$(r,t) = (1,t)(r,0) \tag{12}$$

The homogenity condition (8) is

$$f(t'g) = f((1,t').(r,t)) = \alpha(t')f(r.t) \tag{13}$$

Using (12) and (13), we find:

$$f(g) = f((r,t)) = f((1,t)(r,0)) = \alpha(t)f((r,0))$$

i.e.:

$$f(g) = e^{i\lambda t}f(r) \tag{14}$$

According to the general method sketched previously, we now consider representations of G:

$$g = (r,t) \rightarrow U(r,t) \text{ where}$$

$$U(r_o,t_o)f((r,t)) = f((r,t)(r_o,t_o)) = f((rr_o,t + rt_o)) = e^{i\lambda(t+rt_o)}f(rr_o)$$

On the other hand

$$U(r_o,t_o)f(r,t) = U(r_o,t_o)e^{i\lambda t}f(r) = e^{i\lambda t}U(r_o,t_o)f(r)$$

Finally we obtain

$$U(r_o,t_o)f(r) = e^{i(\lambda_1 rt_o)}f(rr_o) = e^{i[\lambda_1(rt_o)_1+\ldots+\lambda_n(rt_o)_n]}f(rr_o) \tag{15}$$

In particular

$$U(1,t_o)f(r) = e^{i(\lambda_1 t_o)}f(r)$$
$$U(r_o,o)f(r) = f(rr_o) \tag{16}$$

so that the functions $f(r)$ should be labelled by the index λ.

To proceed further it is convenient to make use of the concept of a "little group" and to consider functions over a different manifold than the group R, on which the group R also acts transitively.

We already know that if we have a Lie group R acting transitively on a space Λ, then Λ can be "inserted" into R. Indeed, consider a standard point λ_o and a general point λ.

$$\lambda_o \varepsilon \Lambda \quad , \quad \lambda \varepsilon \Lambda$$

We then always have at least one $r \epsilon R$, such that

$$\lambda = \lambda_o r_\lambda \ . \tag{17}$$

If there is only one such r_λ, then Λ and R can be identified. If there is more than one r_λ, then we put

$$\lambda = \lambda_o r_1 = \lambda_o r_2$$

Then

$$\lambda_o = \lambda_o r_2 r_1^{-1} \tag{18}$$

so that

$$r_2 r_1^{-1} = h \varepsilon H \tag{19}$$

where H is defined to be the little group of λ_o, i.e. that subgroup of R, which leaves a chosen vector λ_o invariant.

For each vector λ let us choose one representative element $r_\lambda \varepsilon G$, satisfying (17). Every element of R, taking λ_o into λ, can be written as hr_λ. We thus obtain a family of left cosets

$$R_\lambda = Hr_\lambda \tag{20}$$

An arbitrary element of the group R can be written as

$$r = hr_\lambda$$

where r_λ determines a coset and h an element of the coset. Symbolically we write

$$R = H\Lambda \qquad \text{and } \Lambda = R/H \tag{21}$$

Thus, the homogeneous manifolds Λ appear as factor spaces of the group R with respect to a subgroup H, leaving a certain vector $\lambda_{\circ}$ invariant. A function over the group can be written as $f(r) = f(hr_{\lambda})$. If it satisfies an invariance condition, like $f(hr) = f(2r)$ or more generally $f(hr) = V_h f(r)$ where V_h is a linear operator, giving a representation $h \to V_h$ and transforming different functions $f(r) = f(hr_{\lambda})$, corresponding to the same λ, amongst each other then we can establish a connection between functions on the group R and functions on the space λ. Thus

$$f(r) = f(hr_{\lambda}) = V_h f(r_{\lambda}) = V_h f_i(\lambda) \tag{22}$$

where the subscript i indicates that we have in general many functions $f(\lambda)$, corresponding to one λ.

2) Let us now construct the homogeneous spaces Λ (homogeneous with respect to the semisimple subgroup R of G), in a manner close to that originally applied by Wigner for the Poincare' group.

Return to the representations of the group T of (6). We have a vector $\xi(\lambda)\varepsilon L$, we know how it transforms under U_t, representing T. Let us see how U_r, representing R acts on $\xi(\lambda)$.

Put

$$r \to U_r \qquad U_r \xi(\lambda) = \eta(\lambda) \tag{23}$$

and see how U_t acts on $\eta(\lambda)$.

$$U_t \eta(\lambda) = U_t U_r \xi(\lambda) = U_r U_{t'} \xi(\lambda) = U_r e^{i\lambda t'} \xi(\lambda) = e^{i\lambda t'} \eta(\lambda) \tag{24}$$

with

$$r^{-1} t r = t' \tag{25}$$

We can formally put

$$\lambda t' = \lambda r^{-1} t r = r \lambda r^{-1} t = \lambda' t$$

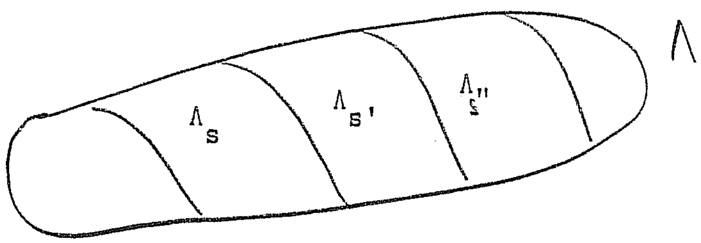
i.e. replace the transformation of the coordinates t_i by a transformation of the exponents $\lambda_1 \ldots \lambda_n$

$$\lambda' = r \lambda r^{-1} \tag{26}$$

Remark: If G is a group of matrices then (25) and (26) can be understood literally, if we suitably arrange the t_i into a matrix t (we know that the t_i correspond to one parameter subgroups of T) and the exponents λ_i into a matrix $\tilde{\lambda}$ so that $t_i \lambda_i = T_r \tilde{t} \tilde{\lambda}$. Then $\lambda t' = Tr \tilde{\lambda} \tilde{t}' = Tr \tilde{\lambda} \tilde{r}^{-1} \tilde{t} \tilde{r} = Tr \tilde{r} \tilde{\lambda} \tilde{r}^{-1} \tilde{t} = Tr \tilde{\lambda}' \tilde{t} = t_i \lambda_i'$.

Thus: We have a space Λ of points $\lambda_1 = (\lambda_1 \ldots \lambda_n)$ and the group R realizes transformations in Λ.

If the group R acts transitively in Λ, i.e. Λ is a homogeneous manifold, then we can proceed. If not, then either Λ can be decomposed into transitive subspaces Λ_s, i.e. into individual "layers", as in the figure:



or Λ cannot be thus decomposed. This last case is called the "ergodic case", is the most difficult one and we shall not go into it at all.

Thus, let us decompose Λ into subsets Λ_s on which R does act transitively. It is then clear (at least intuitively) that the space H of the representation can be decomposed into a direct (continuous) sum of irreducible subspaces

$$H = \int \oplus H_s d_s$$

Each of the subspaces H_s is spanned by the vectors $\xi(\lambda_s)$ with λ_s in Λ_s.

Each of the transitive subsets Λ_s is called an <u>orbit</u> and its structure, as a manifold, depends on the group R.

From now on we consider each orbit separately, dropping the subscript s. The vector $\xi(\lambda)$ is a vector function on the orbit Λ. Since the eigenvalues are, in general degenerate, we write a further subscripts

$$\xi = \xi_i(\lambda)$$

where i labels all eigenvectors of the generators of T_1 corresponding to one λ.

<u>Example</u>: The Proper Ortochronous Poincare' Group:

T is the group of translations, R the homogeneous Lorentz group. The set λ is the set of momenta $\lambda = (p_o, p)$, the subscript i will be a spin projection and the orbits are (for each fixed value of m^2):

1) The upper and lower sheets separately of the hyperboloid

$$p^2 = m^2 > 0 \qquad \text{a) } p_o > 0$$
$$\text{b) } p_o < 0$$

2) The one sheeted hyperboloid

$$p^2 = m^2 < 0$$

3) The upper and lower halfs of the light cone

$$p^2 = m^2 = 0 \qquad \text{a) } p_o > 0$$
$$\text{b) } p_o < 0$$

4) The vertex of the cone:

$$p^2 = m^2 = 0 \qquad p_\mu = 0 \quad \mu = 0,1,2,3$$

3. We already know how the operator U_t and U_r act on $\xi(\lambda)$. Indeed:

$$U_t\xi(\lambda) = e^{i\lambda t}\xi(\lambda)$$
$$U_r\xi(\lambda) = A(r,\lambda)\xi(\lambda_r) \qquad (28)$$

where

$$\lambda_r = r\lambda r^{-1}$$

and $A(r,\lambda)$ is an operator, acting on the subscripts i only. Now we can make use of the homogeneity of Λ to "insert" it into the group R, as discussed above. Indeed, write $r = hr_\lambda$ as in (20), where $h\varepsilon H$ and H is the little group of λ. We introduce the functions $\xi(r)$, putting

$$\xi(hr) = \xi(r)$$

and thus

$$\xi(r) = \xi(hr_\lambda) = \xi(r_\lambda) = \xi(\lambda)$$

We now have

$$U_r\xi(r_\lambda) = A(r,\lambda)\xi(r_\lambda r)$$

Putting

$$r_\lambda r = hr_\mu \qquad (29)$$

we have

$$U_r\xi(r_\lambda) = A(r,\lambda)\xi(r_\mu) \qquad (30)$$

Remark: The procedure of inducing is thus applied twice, First, the inducing group is the abelian group T, secondly the inducing group is the little group H.

4. We still have to specify what are the operators $A(r,\lambda)$ acting on the subscript i, when

$$\xi(\lambda) = \{\xi_i(\lambda)\} \qquad (31)$$

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Obviously we have:

$$U_{r_1 r_2}\ \xi(r_\lambda) = A(r_1 r_2,\lambda)\xi(r_\lambda r_1 r_2)$$

$$U_{r_1 r_2}\ \xi(r_\lambda) = A(r_1,\lambda)A(r_2,\lambda r_1)\xi(r_\lambda r_1 r_2)$$

so that

$$A(r_1 r_2,\lambda) = A(r_1,\lambda)A(r_2,\lambda r_1) \tag{32}$$

Let us again fix a definite "reference" vector $\lambda_o \varepsilon \Lambda$ and denote R_o its little group

$$\lambda_o r_o = \lambda_o \qquad \text{for all } r_o \varepsilon R_o \tag{33}$$

Consider r_1 and $r_2 \varepsilon\ R_o$. Then

$$A(r_1 r_2,\lambda) = A(r_1,\lambda)A(r_2,\lambda) \tag{34}$$

We also have

$$A(\ell,\lambda) = 1$$

Thus: The operators

$$U(r_o) = A(r_o,\lambda_o) \tag{35}$$

form a representation of the little group R_o.

However, we still have to relate $A(r,\lambda)$ for arbitrary r and λ to $U(r_o)$

Let λ_o and λ be given, then again choose (and fix) r_λ such that

$$\lambda = \lambda_o r_\lambda$$

(transitivity of Λ implies that at least one such r exists). We have

$$A(r_\lambda r,\lambda_o) = A(r_\lambda,\lambda_o)A(r,\lambda_o r_\lambda) = A(r_\lambda,\lambda_o)A(r,\lambda)$$

Put

$$B(r) = A(r,\lambda_o). \tag{36}$$

Then:

$$A(r,\lambda) = B^{-1}(r_\lambda)B(r_\lambda r) \tag{37}$$

We can always write (see (29))

$$r_\lambda r = r_o r_\mu \qquad \mu\varepsilon\Lambda \tag{38}$$

Thus:

$$B(r_\lambda r) = A(r_o r_\mu,\lambda_o) = A(r_o,\lambda_o)A(r_\mu,\lambda_o) = U(r_o)B(r_\mu) \tag{39}$$

so that (37) becomes

$$A(r,\lambda) = B^{-1}(r_\lambda)U(r_o)B(r_\mu) \tag{40}$$

and we finally obtain:

$$U_r\xi(\lambda) = b^{-1}(\lambda)U(r_o)b(\mu)\xi(\mu)$$

where we have put

$$b(\lambda) = B(r_\lambda)$$

We can further simplify by introducing a new basis

$$e(\lambda) = b(\lambda)\xi(\lambda)$$

and replacing U_r by operators V_r of an equivalent representation

$$V_r = b(\lambda)U_r b^{-1}(\lambda)$$

We then have:

$$V_t e(\lambda) = e^{i\lambda t}\, e(\lambda)$$

$$V_r e(\lambda) = U(r_o) e(\mu) \qquad\qquad r_\lambda r = r_o r_\mu$$

We have shown the following:

Theorem: All irreducible unitary representations of the group $G = R \circ T$ can be realized by means of the formulae

$$V_t e(\lambda) = e^{i(\lambda,t)} e(\lambda)$$

$$V_r e(\lambda) = U(r_o) e(\mu)$$

where $e(\lambda)$ are vector-valued functions on the homogeneous manifold Λ and $U(r_o)$ are representations of the stationary subgroup R_o, defined for some point $\lambda_o \varepsilon \Lambda$. For given r and λ we can determine $r_o \varepsilon R_o$ and $\mu \varepsilon \Lambda$ from the relations

$$r_\lambda r = r_o r_\mu$$

(We have $\lambda = \lambda_o r_\lambda$, $r = r_o r_\lambda$; λ_o is a chosen fixed point).

Remark: A representation of G is thus specified by:

a) Characterizing the "orbit" Λ , i.e. the homogeneous space over which we construct representations.

b) Specifying the representation of the stationary subgroup R_o leaving a chosen vector $\lambda_o \varepsilon \Lambda$, characterizing the orbit, invariant.

References:

1) G. W. Mackey: The Theory of Group Representations. (Lectures at University of Chicago, Summer 1955).

2) W. H. Klink, Lectures in Theoret. Physics XID (Boulder) Ed. K. T. Mahanthappa, W.E. Brittin, Gordon and Breach, N.Y. 1969.

3) R. Herman: Lie Groups for Physicists, Benjamin, N.Y. 1966.

We now proceed to the final part of this series of lectures, namely the representation theory of the Poincare' Group. We shall first talk about the group as such, then about its representations.

The Poincare' Group

I. Definition

Consider a four-dimensional real vector space

$$x = (x_o, x_1, x_2, x_3) \varepsilon X \tag{1}$$

with an indefinite scalar product

$$(x,y) = x_o y_o - x_1 y_1 - x_2 y_2 - x_3 y_3 = g^{\mu\nu} x_\mu y_\nu \tag{2}$$

$$g^{oo} = 1 \qquad g^{ii} = -1 \qquad i = 1,2,3 \quad \text{(no summation)},$$

$$g^{\mu\nu} = 0 \qquad \mu \neq \nu$$

The transformation

$$x'_\mu = \Lambda_\mu^\nu x_\nu + a_\mu \tag{3}$$

is called a Lorentz Transformation if it leaves the interval τ^2 between two points x and y invariant:

$$\tau^2 = g^{\mu\nu}(x_\mu - y_\mu)(x_\nu - y_\nu) = \text{invariant} \tag{4}$$

It is easy to check that the condition (4) is satisfied if and only if Λ_μ^ν is a O(3,1) matrix, i.e. if

$$\Lambda^T g \Lambda = g \tag{5}$$

A Lorentz transformation is homogeneous if $a_\mu = 0, \mu = 0,1,2,3$, inhomogeneous otherwise.

It is easy to check that the Lorentz transformations form a group, called the Inhomogeneous Lorentz Group, or the Poincare' Group. Denote an element of the group

$$g = (\Lambda, a) \tag{6}$$

Multiplication is

$$(\Lambda,a)(\Lambda',a') = (\Lambda\Lambda', a+\Lambda a') \tag{7}$$

the identity element is

$$e = (1,0) \tag{8}$$

and the inverse is

$$g^{-1} = (\Lambda^{-1}, -\Lambda^{-1}a) \tag{9}$$

Thus, the Poincare' group is an example of a more general structure - a semidirect product of a semisimple group (the homogeneous Lorentz group), with an Abelian group (the group of translations in the vector space X).

A convenient way of writing Lorentz transformations as a matrix group is to introduce five-dimensional matrices

$$L = \begin{pmatrix} \Lambda & a \\ 0 & 1 \end{pmatrix} \tag{10}$$

where Λ is an O(3,1) matrix and a is a four-vector, written as a column. The matrices L act on five-dimensional vectors (columns), written as

$$\begin{pmatrix} X \\ 1 \end{pmatrix}$$

The Poincare' group contains the homogeneous Lorentz group Λ, the translations a and the discrete elements:

$$I_s = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \qquad \text{Space reflections} \qquad (11)$$

$$I_t = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} \qquad \text{Time reversal} \qquad (12)$$

and

$$I_{st} = I_s \circ I_t \qquad \text{Space time inversion.}$$

Taking the determinent of the left and right hand of (5) we find

$$\det \Lambda = \pm 1 \qquad (13)$$

taking the oo-component, we find

$$\Lambda^o_o \geq 1 \quad \text{or} \quad \Lambda^o_o \leq -1 \qquad (14)$$

Thus, we can split the homogeneous Lorentz group (and also the Poincare' group), into four connected components:

$L^{\uparrow}_{+}$: $\det \Lambda = +1$, sgn $\Lambda^o_o = +1$. This component contains the identity of the group, is itself a subgroup and is called the proper ortochronous Lorentz group

$L^{\uparrow}_{-}$: $\det \Lambda = -1$, sgn $\Lambda^o_o = 1$ Contains I_s.

$L^{\downarrow}_{+}$: $\det \Lambda = +1$, sgn $\Lambda^o_o = -1$ Contains I_{st}

$L^{\downarrow}_{-}$: $\det \Lambda = -1$, sgn $\Lambda^o_o = -1$ Contains I_t.

We already know that the proper ortochronous Lorentz group is the group SO(3,1), i.e. a simple group which is one of the real noncompact forms corresponding to the Cartan algebra B_2. We know that it is locally isomorphic to SL(2,C) which is simply connected and is thus the universal covering group of the Lorentz group. We also know that the complex extensions of SO(3,1), namely SO(4,C) is only semisimple, not simple.

II. Algebra of the Poincare' Group and its Invariants

Let us restrict ourselves to the proper ortochronous Poincare' group (i.e. exclude the discrete operators I_s and I_t) and consider infinitesimal transformations of the type

$$x'_\mu = \Lambda_\mu^\nu x_\nu + a_\mu$$

Let us write an infinitesimal Lorentz transformation as

$$x'_\mu = (\delta_\mu^\nu + \varepsilon_\mu^\nu)x_\nu + \alpha_\mu \tag{15}$$

Here ε_μ^ν and α_μ are first order infinitesimals. From the invariance of the quadratic form τ^2 we readily obtain

$$\begin{aligned} &\varepsilon_\nu^\mu = 0 \quad \text{for } \mu = \nu \qquad \mu,\nu = 0,1,2,3 \\ &\varepsilon_o^i = \varepsilon_i^o \qquad \varepsilon_k^i = -\varepsilon_i^k \qquad i,k = 1,2,3 \end{aligned} \tag{16}$$

Together with the α_μ we thus have 10 parameters. Writing a general element of the Poincare' group in some representation as

$$U(\Lambda,a) = \exp[i\ \alpha_\mu P_\mu - i\frac{1}{2}\varepsilon_{\mu\nu} J_{\mu\nu}] \tag{17}$$

where P_μ and $J_{\mu\nu}$ are the generators, and are hermitian operators in any unitary representation of the group, we can directly from the multiplication law (7) obtain the familiar commutation relations:

$$[P_\mu, P_\nu] = 0$$

$$[M_{\mu\nu}, P_\lambda] = i(g_{\nu\mu}P_\mu - g_{\mu\lambda}P_\nu) \tag{18}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(g_{\nu\rho}M_{\mu\sigma} + g_{\mu\sigma}M_{\nu\rho} - g_{\mu\rho}M_{\nu\sigma} - g_{\nu\sigma}M_{\mu\rho})$$

We shall also use a three dimensional notation, putting:

$$\vec{M} = (M_{23},M_{31},M_{12}) \qquad \vec{N} = (M_{01},M_{02},M_{03})$$

$$P = (P_0,\vec{P})$$

The physical interpretation of these operators is that:

M_{ik} $i,k = 1,2,3$ are generators of space rotations

M_{ok} $k = 1,2,3$ are generators of pure Lorentz transformations

P_i $i = 1,2,3$ are generators of space translations

P_o is the generator of a time translation.

Note, that under finite Lorentz transformations the generators P_μ and $M_{\mu\nu}$ transform as a vector and as a tensor, respectively

$$U^{-1}(\Lambda,0)P_\mu U(\Lambda,0) = \Lambda_\mu^\nu P_\nu$$
$$U^{-1}(\Lambda,0)M_{\mu\nu}U(\Lambda,0) = \Lambda_\mu^\rho \Lambda_\nu^\sigma M_{\rho\sigma} \tag{19}$$

We know the role that Schur's lemma plays in representation theory and thus we are very interested in the Casimir operators of the Poincare' group, i.e. in all operators from the envelloping algebra of the Lie algebra, that commute with all generators.

It is a simple matter to check that the operators

$$P^2 = g^{\mu\nu}P_\mu P_\nu = P_o^2 - \vec{P}^2 \tag{20}$$

and

$$W^2 = g_{\mu\nu} W^\mu W^\nu \tag{21}$$

where

$$W^\mu = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma} \tag{22}$$

($\varepsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor with $\varepsilon^{0123} = 1$) are always invariants of the group (are Casimir operators). It is somewhat less obvious that in general these are the only invariants. We shall show below that in many specific cases there are however additional invariants.

An important feature of the Pauli-Lyubanski vector W is that it is invariant under translations

$$[P_\mu, W_\nu] = 0 \tag{23}$$

and also that

$$g^{\mu\nu} P_\mu W_\nu = 0 \tag{24}$$

i.e. only three components are actually independent.

III. The Poincare' Group in Physics

The outstanding role of the Poincare' group becomes clear as soon as we demand that the physical theory we are considering is compatible with the special theory of relativity. In such a theory all frames of reference that can be obtained from a given frame by a Lorentz transformation are equivalent for the description of a physical system, i.e. the same observation made on a system in two such frames must give the same result.

In classical physics this relativity principle is satisfied by demanding that all equations of motion should have the same form in equivalent frames of reference.

In quantum physics the situation is much more complicated, since strictly speaking no consistent quantum theory compatible with special relativity exists. However, the relativity principle requires that experiments conducted in different systems lead to the observation of the same probability for a given result. This leads directly to restrictions imposed upon wave functions.

Indeed let us consider the quantum mechanics of a free particle. The requirements of relativity can be imposed on the equations of motion, i.e. one can write equations invariant under Lorentz transformations like, say, the Dirac equation. On the other hand the correct transformation properties can be imposed directly upon the wave functions.

Experimentally measurable quantities, like transition probabilities, in a quantum theory are given by the moduli of scalar products of wave functions $|(\psi_f, \psi_i)^2|$. The requirement of special relativity is now that the values of such quantities should be invariant under Lorentz transformations.

Thus, if the coordinates are subjected to a Lorentz transformation the wave function is transformed according to

$$U(\Lambda,a)\ \psi(X) = \psi(X') \tag{25}$$

(this is a definition of the operator $U(\Lambda,a)$), satisfying

$$|(U(\Lambda,a)\psi_f,\ U(\Lambda,a)\psi_i)^2| = |(\psi_f,\psi_i)^2| \tag{26}$$

It can be proved that if the transformations $X \to X'$ form a continuous group, connected to the identity operator, represented by a unit operator, then condition (26) is equivalent to

$$(U(\Lambda,a)\psi_f,\ U(\Lambda,a)\psi_i) = (\psi_f,\ \psi_i) \tag{27}$$

The condition that a set of transformations leading stepwise from one frame of reference to another one should be equivalent to a direct transformation between the two frames leads to the condition

$$U(\Lambda,a)\ U(\Lambda',\ a') = e^{i\phi(\Lambda,a,\Lambda',a')}U(\Lambda\Lambda',a+\Lambda a') \tag{28}$$

where $\phi(\Lambda,a,\Lambda',a')$ is a real phase. Specifically for the Poincare' group it can be proved that the phase-factor in (28) can be replaced by +1 or -1 so that

$$U(\Lambda,a)\ U(\Lambda'a') = \pm\ U(\Lambda\Lambda',\ a+\Lambda a') \tag{29}$$

(See Wigner's or Bargmann's articles, or the review by T. D. Newton)

It follows, that the wave-functions admissible in a relativistic theory transform under single-valued or double-valued unitary representations of the Poincare' group. If ϕ(in (28) is an arbitrary real phase then we are considering unitary ray representations or projective representations. For the Poincare' group this is not necessary.

We shall accept as a definition that we call a physical system elementary, if it is described by a wave function, transforming according to an irreducible representation of the Poincare' group. The classification of irreducible representations is thus a very important and physically meaningful task, corresponding to a classification of all possible elementary physical systems.

To summarize, in elementary particle physics we consider a free (non-interacting) particle to be an elementary relativistic quantum mechanical system described by a wave functions transforming under a unitary irreducible representation of the Poincare' group.

It should be stressed that the significance of the representations of the Poincare' group in particle physics is by no means limited to the classification of free particle states. Indeed this representation theory forms a basis for the relativistic kinematics of reactions amongst particles, when we have to consider many-particle states, transforming according to reducible representations.

IV. Classes of Irreducible Unitary Representations of the Poincare' Group

We have already mentioned that the invariant operators P^2 and W^2, commute with all the generators of the Poincare' group. It follows from Schur's lemma, which is applicable in this case, that the necessary and sufficient condition for a unitary representation to be irreducible is, that any operator commuting with all the generators must be the multiple of the unit operator. Thus all functions belonging to the Hilbert space of one irreducible representation must be eigenfunctions of the operators P^2 and W^2, corresponding to one and the same eigenvalue.

The problem of classifying all irreducible representations of the group thus reduces to finding the eigenvalue spectra of the complete set of invariants of the group.

Let us introduce a notation for the eigenvalues of the two invariants of the group, namely

$$P^2 = P_\mu P^\mu = m^2 \tag{29}$$

$$\begin{aligned} W^2 = W_\mu W^\mu &= -m^2\, s(s+1) \quad \text{for } m^2 \neq 0 \\ &= -\rho^2 \qquad \text{for } m^2 = 0 \end{aligned} \tag{30}$$

The irreducible representations of the Poincare' group differ principally from one another depending on whether the vector P_μ is timelike ($m^2 > 0$), spacelike ($m^2 < 0$), lightlike ($m^2 = 0$, but not all components of P_μ are equal to zero) or a null-vector ($m^2 = 0$, $P_\mu = 0$ $\mu = 0,1,2,3$). Thus, we shall distinguish and discuss below four classes of irreducible unitary representations of the Poincare' group, which we shall call.

a) Timelike representations

b) Spacelike representations

c) Lightlike representations

d) Null representations

The additional invariants which appear in the individual classes of representations, will be discussed together with the other properties of these presentations in the following paragraphs.

V. Physical Meaning of the Operators and a Classification of the States of an Elementary Relativistic Quantum System.

According to the above definitions an elementary relativistic quantum system is described by a wave function, transforming according to an irreducible unitary representation of the Poincare' group. Such awave function will clearly be the eigenfunction of the operators P^2 and W^2, corresponding to definite values of m and s (or ρ), as well as the additional invariant operators. We shall identify the invariant m with the mass of the system (e.g. an elementary particle) and s with its spin. Thus the values of the invariants of the Poincare' group specify the type of particle we are considering.

The infinitesimal operators of the group can now be identified with the quantum mechanical operators of linear momentum (P_i, $i = 1,2,3$), energy (P_o), angular momentum (M_{ik}, $i,k = 1,2,3$), centre of inertia ($g_\mu = M_{\mu\nu}P^\nu$).

We are interested not only in the "type" of particle under consideration, but also in its "state", e.g. in its momentum and in the orientation of its spin. To do this we associate a particle in a specific state not only with a certain irreducible unitary representation of the Poincare' group, but with a basis function of such a representation. This leads us to the problem of constructing and classifying all possible bases of irreducible unitary representations of the Poincare' group.

A convenient and physically meaningful way of constructing a basis of a representation is to consider the algebra of the group generators and its envelloping algebra (i.e. all powers of the generators). Using these operators we construct a complete set of commuting operators (commuting with each other, but not with all generators of the group). The complete set of common eigenfunctions of these operators, corresponding to a definite set of values of the group invariant, can then serve as the basis of a representation.

The choice of the complete set of commuting operators is, of course, not unique, and there are many physically non-equivalent possibilities. A classification of the different possible complete sets of commuting operators has not been provided for the Poincare' group.

The basis most commonly used in particle physics consists of the common eigenfunctions of the linear momenta P and of one of the spin projections, say W_3 (naturally, the basis functions, like all other functions, in the space carrying the representation, are eigenfunctions of the invariants P^2 and W^2). Thus e.g. for $m^2 > 0$ this basis, which we shall call canonical, satisfies the equations

$$P_\mu \psi_{ms\xi,p\lambda} = p_\mu \psi_{ms\xi,p\lambda}$$

$$W_\kappa \psi_{ms\xi,p\lambda} = C_\kappa \lambda \psi_{ms\xi,p\lambda}$$

$$P^2 \psi_{ms\xi,p\lambda} = m^2 \psi_{ms\xi,p\lambda} \qquad (31)$$

$$W^2 \psi_{ms\xi,p\lambda} = -m^2 s(s+1) \psi_{ms\xi,p\lambda}$$

Here ξ is a parameter, indicating a possible degeneracy which is lifted by considering the additional invariants of the Poincare' group, when they exist. The subscript κ tells us which component or combination of components of the spin W_κ we are diagonalizing. The coefficient C_κ will be specified.

Clearly, when using the canonical basis, we are treating a subgroup of the Poincare' group preferentially, namely the translations generated by P_μ and a one-dimensional rotation generated by W_κ. This basis corresponds to the reduction of the Poincare' group to the subgroup $T_4 \times O(2)$.

$$P_+^\uparrow \supset T_4 \times O(2) \qquad (32)$$

A different basis, which is also of great use in physics, corresponds to the reduction of the Poincare' group to the (homogeneous) Lorentz group and one of its subgroups, e.g.:

$$P_+^\uparrow \supset L_+^\uparrow \supset O(3) \supset O(2) \qquad (33)$$

(at least for representations of the group algebra).

The complete set of commuting operators consists of the Casimir operators of each group in the chain of subgroups, i.e. the invariants of the homogeneous Lorentz group, the square of the three dimensional angular momentum and one of its projections.

The corresponding "angular momentum basis functions" satisfy the equations (for $m^2 > 0$):

$$F\ \phi_{ms\xi,\mu\lambda MM_3} = \frac{1}{2}\left(\ 1 + \lambda^2 - \nu^2\right)\ \phi_{ms\xi,\nu\lambda MM_3}$$

$$G\ \phi_{ms\xi,\nu\lambda MM_3} = \nu\lambda\phi_{ms\xi,\nu\lambda MM_3}$$

$$M^2\phi_{ms\xi,\nu\lambda MM_3} = M(M+1)\ \phi_{ms\xi,\nu\lambda MM_3} \tag{34}$$

$$M_{12}\phi_{ms\xi,\nu\lambda MM_3} = M_3\phi_{ms\xi,\nu\lambda MM_3}$$

$$P^2\phi_{ms\xi,\nu\lambda MM_3} = m^2\phi_{ms\xi,\nu\lambda MM_3}$$

$$W^2\phi_{ms\xi,\nu\lambda MM_3} = m^2 s(s+1)\phi_{ms\xi,\nu\lambda MM_3}$$

Here we have

$$\begin{aligned} F &= +\frac{1}{2}M_{\mu\nu}M^{\mu\nu} = (\vec{M}^2 - \vec{N}^2) \\ G &= \frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}\ M_{\mu\nu}M_{\rho\sigma} = \vec{M}\vec{N} \end{aligned} \tag{35}$$

Lecture 24

Irreducible Unitary Representations of the Poincare' Group

Let us proceed with a systematic exposition of the representation theory of the Poincare' group. The treatment follows the original work of Wigner, which has been reviewed and extended in numerous articles and books.

References:

1. E. P. Wigner. On Unitary Representations of the Inhomogeneous Lorentz Group. Ann. of Math. 40, 149 (1939).
2. Yu. M. Shirokov. A Group Theoretical Consideration of the Basis of Relativistic Quantum Mechanics I-V.
   Soviet Physics JETP 6, 669, 919, 929 (1958)
   7, 493 (1958)
   9, 620 (1959).
3. H. Joos: Fortchr.d. Physik 10, 65 (1962).
4. T. D. Newton: The Inhomogeneous Lorentz Group in the Theory of Groups in Classical and Quantum Physics. Vol. 1, Ed. T. Kaham. Oliver and Boyd, Edinburgh, 1965.

The method is identical with the one discussed in a previous lecture for an arbitrary Lie group with an Abelian invariant subgroup, namely the method of induced representations.

1. Reduction of the Representations of the Poincare' group to Representations of Little Groups.

Let us first start by using the Abelian subgroup of translations, to induce representations. Thus, let us choose a basis for the representations, consisting of eigenvectors of the translations. Since this does not in general specify the eigenvectors completely, we shall also demand that they are eigenvectors of a further operator W_κ (a spin projection). Thus we have

$$P^2\psi_{ms\xi,p\lambda} = m^2\psi_{ms\xi,p\lambda}$$

$$\begin{aligned} W^2\psi_{ms\xi,p\lambda} &= -m^2 s(s+1)\psi_{ms\xi,p\lambda} \qquad & m^2 \neq 0 \\ &= -\rho^2\psi_{ms\xi,p\lambda} & m^2 = 0 \end{aligned} \tag{1}$$

$$P_\mu\psi_{ms\xi,p\lambda} = p_\mu\psi_{ms\xi,p\lambda}$$

$$W\ \psi_{ms\xi,p\lambda} = C\ \ \lambda\psi_{ms\xi,p\lambda}$$

The number m and s (mass and spin) characterize the representations, sometimes an additional invariant exists, so we may have an additional label ξ. The meaning of ρ^2 for $m^2=0$ will be specified below. The state vectors (basis functions) are also labeled by a four-vector, the momentum $p = \{p_\mu\}$ and a spin porjection λ, also to be specified. The meaning of the subscript and proportionality coefficient C_κ is discussed below.

In agreement with the general theory we can immediately write the operators representing translations in the representation $(\Lambda,a) \to U(\Lambda,a)$

$$U(1,a)\psi_{ms\xi,p\lambda} = e^{ip_\mu a_\mu}\psi_{ms\xi,p\lambda} \tag{2}$$

We know that $U(\Lambda,0)\psi_{ms\xi,p\lambda}$ will again be an eigenvector of P_μ, corresponding to a different eigenvalue of the momentum p_μ. Thus

$$U(\Lambda,0)\psi_{ms\xi,p\lambda} = \sum_{\lambda'} Q_{\lambda'\lambda}(\Lambda,p)\psi_{ms\xi,\Lambda p\lambda'} \tag{3}$$

Here $Q_{\lambda'\lambda}(\Lambda,p)$ are matrix elements of an operator acting on the "degeneracy labels" only, i.e. on the spin projection. As in the general theory, we shall relate them to the representations of certain little groups.

The operators $Q(\Lambda,p)$, depending on the Lorentz transformation Λ and on the value p of the momentum, at which they are applied, can be evaluated conveniently, using Wigner "boosts".

Thus, let us choose a "reference vector" p_R which we fix. We must now split the manifold $\{p\}$ into layers on which the Lorentz group acts transitively, i.e. into individual "orbits". Consider a proper orthchronous Lorentz transformation Λ and put

$$p_\nu = \Lambda^\mu_\nu (p_R)_\mu$$

By definition the operator Λ preserves the "length"

$$p^2 = p_R^2 = m^2$$

so each value of m^2 will correspond to an orbit.

Further, if $m^2 > 0$ or $m^2 = 0$ but $p_\mu \neq 0$ for all μ, then the sign of p_o is an invariant as well. There are no further invariants and we obtain the orbits

Timelike:	1a) $p^2 = m^2 > 0$	$p_o \geq m$	$m > 0$
	1b) $p^2 = m^2 > 0$	$p_o \leq -m$	
Spacelike:	2) $p^2 = m^2 < 0$		
Lightlike:	3) $p^2 = 0$	$p_\mu \neq 0$ for at least one $\mu = 0,1,2,3$	
Null Vector	4) $p^2 = 0$	$p_\mu = 0$, for all μ.	

For each orbit we must choose a different reference vector P_R and there will be a big difference, depending on whether P_R is timelike, spacelike, lightlike or nullvector.

Consider a chosen orbit and fix P_R. Any p in the same orbit can be obtained from p_μ by a Lorentz transformation. Choose a specific fixed Lorentz transformation L(p) (a Wigner "boost"), putting:

$$p_\nu = L(p)^\mu_\nu \, (p_R)_\mu \tag{4}$$

and choose L(p) such that

$$\psi_{ms\xi,p\lambda} = U(L(p),0)\psi_{ms\xi,p_R\lambda} \tag{5}$$

Such an operator L(p) is called a "rotationless boost": it takes p_R into p and U(L(p),0) leaves the spin labels unchanged (i.e. it corresponds to the identity element in the representation of the relevant little group).

Now consider the little group of the reference vector:

$$R^\nu_\mu (p_R)_\nu = (p_R)_\mu$$

For such a specific Lorentz transformation we have:

$$U(R,0)\psi_{ms\xi,p_R\lambda} = \sum_\lambda Q_{\lambda'\lambda}(R,p_R) \; {}_{ms\xi,p_R\lambda'} \tag{6}$$

Thus the operators $Q(R,p_R)$ realize a representation of the little group R and $Q_{\lambda'\lambda}(R,p_R)$ are simply the corresponding matrix elements.

Now let us reduce a general $U(\Lambda,0)$ to rotationless boosts and little group transformations.

Put

$$\Lambda = L(\Lambda p)R(\Lambda,p)L^{-1}(p) \tag{7}$$

where L is a rotationless boost and this is a definition of $R(\Lambda,p)$.

We have

$$R(\Lambda,p)p_R = L^{-1}(\Lambda p)\Lambda L(p)p_R = L^{-1}(\Lambda p).\Lambda p = p_R \tag{8}$$

Thus $R(\Lambda,p)$ is an element of the little group R, leaving p_R invariant. Let us denote the matrix elements of the little group transformation $R(\Lambda,p)$ by the symbol

$$D_{\lambda'\lambda}(R(\Lambda,p))$$

A general transformation of the homogeneous Lorentz group is now represented by the operator $U(\Lambda,0)$, acting on the basis functions as

$$U(\Lambda,0)\psi_{ms\xi,p\lambda} = U(L(\Lambda p),0)U(R(\Lambda,p),0)\ U(L^{-1}(p),0)\psi_{ms\xi,p\lambda} =$$

$$= U(L(\Lambda p),0)\ \sum_{\lambda'} D_{\lambda'\lambda}(R(\Lambda p)).\psi_{ms\xi,p_R\lambda'} \qquad (9)$$

and finally

$$U(\Lambda,0)\psi_{ms\xi,p\lambda} = \sum_{\lambda'} D_{\lambda'\lambda}(R(\Lambda,p)\psi_{ms\xi,\Lambda p\lambda'} \qquad (10)$$

An arbitrary element of the Poincare' group can be written as

$$U(\Lambda,a) = U(1,a)\ U(\Lambda,0) \qquad (11)$$

so that (10) and (11) completely specify the action of the operator representing the transformation (Λ,a) in the considered representation.

The results of this paragraph lead us to a theorem, which we shall only state, referring for the proof e.g. to the review article by T. D. Newton.

Theorem: A unitary irreducible representation of the Poincare' group is completely specified by giving:

a) A real number m^2, corresponding to the mass of the elementary physical system, together with the reference vector p_R satisfying $p_R^2 = m^2$.

b) A unitary irreducible representation of the little group leaving p_R invariant.

2. Realizations of the Individual Classes of Irreducible Unitary Representations.

We shall now discuss the individual classes of representations introduced previously. To do this we must in each case specify the reference vector p_R, the corresponding little group and the representations of this little group. We must also specify the boost $L(p)$ and the additional invariants ξ.

1. Time-like Representations

The reference vector p_R for timelike representations satisfies $p_R^2 = m^2 > 0$ and can be chosen as

$$p_R = (\pm m, 0, 0, 0) \tag{12}$$

where $m = +\sqrt{p_R^2} > 0$. The sign of the energy component $(p_R)_0 = \pm m$ is an invariant of the group for timelike representations (we are not considering discrete operations, like time reversal, here), so that this class of representations contains two subclasses, corresponding to $\varepsilon = \mathrm{sgn} p_o = \pm 1$.

The little group of vector (12) is clearly the three-dimensional rotation group O(3). Indeed, to understand the physical meaning of this O(3) group consider the components of the spin operator W, when acting in the subspace $p = p_R$, i.e. for particles in their rest frame:

$$W = -p_o(0, M_{23}, M_{31}, M_{12}) \tag{13}$$

Thus, for particles at rest the space components of the relativistic spin operator W are simply the generators of space rotations, satisfying

$$[M_{23}, M_{31}] = i\, M_{12},\ [M_{31}, M_{12}] = i\, M_{23},\ [M_{12}, M_{23}] = i\, M_{31} \qquad (14)$$

The irreducible unitary representations of O(3) are, of course, well known, being characterized by the eigenvalues

$$W^2 = -p^2\ ({M_{23}}^2 + {M_{31}}^2) = -m^2\ s(s+1) \qquad (15)$$

with

$$s = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots. \qquad (16)$$

(the representations corresponding to s integer are one-valued, to s half-integer two-valued). These representations are finite-dimensional, since O(3) is a compact group, the dimensionality being 2s+1.

Thus, the basis functions of timelike representations $\psi_{ms\varepsilon,p\lambda}$ are characterized by a real non-zero mass m, a finite integer or half-integer spin s, a positive or negative sign ε of the energy p_o and further by the four-momentum p and spin projection λ. These representations, corresponding to a real elementary particle state, have extensive applications in physics.

To specify the basis functions completely, it is sufficient to give the values of these functions in the reference frame (for $p^2 > 0$ this is the rest frame) and then to specify the boost operators.

Indeed, let us for brevity of writing drop the invariant quantum numbers $ms\varepsilon$ in the wave functions and use the physical "ket" notation, putting

$$\psi_{ms\varepsilon,p\lambda} = |p\lambda> \qquad (17)$$

In the reference frame let us choose $|p\lambda>$ such that

$$M_{12}|p_R\lambda> = \lambda|p_R\lambda> \tag{18}$$

and naturally also

$$P_\mu\ |p_R\lambda> = (p_R)_\mu|p_R\lambda> \tag{19}$$

$$\begin{aligned} P^2\ |p_R\lambda> &= m^2\ |p_R\lambda> \\ W^2\ |p_R\lambda> &= -m^2 s(s+1)\ |p_R\lambda> \end{aligned} \tag{20}$$

The wave function (basis function) in an arbitrary frame can be expressed as

$$|p\lambda> = U(L(p))|p_R\lambda> \tag{21}$$

with

$$L(p)^\mu_\nu\ (p_R)_\mu = p_\nu \tag{22}$$

When p and p_R (two vectors of the same orbit, in this case both timelike) are given, (22) does not specify L(p) completely but only up to a little group transformation (in this case an O(3) rotation).

We shall, give the expressions for three different boosts here, which correspond to different parametrizations of the vector p (and respectively to the reductions of the group O(3,1) to O(3), O(2,1) and to the Euclidean group in two dimensions E_2).

Indeed the components of a timelike vector $p(p^2 = m^2)$ can be written as

$$p = \pm \sqrt{p^2}\,(\mathrm{ch}a, \mathrm{sh}a \sin\theta \cos\psi, \mathrm{sh}a \sin\theta \sin\psi, \mathrm{sh}a \cos\theta) \tag{23}$$

$$p = \pm \sqrt{p^2}\,(\mathrm{ch}\alpha\, \mathrm{ch}\beta, \mathrm{ch}\alpha\, \mathrm{sh}\beta \cos\psi, \mathrm{ch}a\, \mathrm{sh}\beta \sin\psi, \mathrm{sh}\alpha) \tag{24}$$

$$p = \pm \sqrt{p^2}\,(\mathrm{ch}\gamma + \tfrac{1}{2} r^2 e^{-\gamma}, r e^{-\gamma} \cos\psi, r e^{-\gamma} \sin\psi, \mathrm{sh}\gamma + \tfrac{1}{2} r^2 e^{-\gamma}) \tag{25}$$

With p_R given by (12) it is easy to check that the corresponding boosts can be given as

$$L^+(p) = \begin{pmatrix} \mathrm{ch}a, & 0, & 0, & -\mathrm{sh}a \\ \mathrm{sh}a \sin\theta \cos\psi, & -\cos\theta\cos^2\psi - \sin^2\psi, & (1-\cos\theta)\sin\psi \cos\psi, & -\mathrm{ch}a \sin\theta \cos\psi \\ \mathrm{sh}a \sin\theta \sin\psi, & (1-\cos\theta)\sin\psi \cos\psi, & -\cos\theta \sin^2\psi - \cos^2\psi, & -\mathrm{ch}a \sin\theta \sin\psi \\ \mathrm{sh}a \cos\theta, & \sin\theta \cos\psi, & \sin\theta \sin\psi, & -\mathrm{ch}a \cos\theta \end{pmatrix} \tag{26}$$

$$L^-(p) = \begin{pmatrix} \mathrm{ch}\alpha\, \mathrm{ch}\beta, & -\mathrm{sh}\beta \cos\psi, & -\mathrm{sh}\beta \sin\psi, & -\mathrm{sh}\alpha\, \mathrm{ch}\beta \\ \mathrm{ch}\alpha\, \mathrm{sh}\beta \cos\psi, & -\mathrm{ch}\beta \cos^2\psi - \sin^2\psi, & (1-\mathrm{ch}\beta)\sin\psi \cos\psi, & -\mathrm{sh}\alpha\, \mathrm{sh}\beta \cos\psi \\ \mathrm{ch}\alpha\, \mathrm{sh}\beta \sin\psi, & (1-\mathrm{ch}\beta)\sin\psi \cos\psi, & -\mathrm{ch}\beta \sin^2\psi - \cos^2\psi, & -\mathrm{sh}\alpha\, \mathrm{sh}\beta \sin\psi \\ \mathrm{sh}\alpha & 0 & 0 & -\mathrm{ch}\alpha \end{pmatrix} \tag{27}$$

$$L^0(p) = \begin{pmatrix} \mathrm{ch}\gamma + \frac{1}{2} r^2 e^{-\gamma}, & -r \cos\psi, & -r \sin\psi, & -\mathrm{sh}\gamma + \frac{1}{2} r^2 e^{-\gamma} \\ r\, e^{-\gamma} \cos\psi, & -1, & 0, & r e^{-\gamma} \cos\psi \\ r\, e^{-\gamma} \sin\psi, & 0, & -1, & r\, e^{-\gamma} \sin\psi \\ \mathrm{sh}\gamma + \frac{1}{2} r^2 e^{-\gamma}, & -r \cos\psi, & -r \sin\psi, & -\mathrm{ch}\gamma + \frac{1}{2} r^2 e^{-\gamma} \end{pmatrix} \tag{28}$$

We shall call a, θ, ψ spherical coordinates on the hyperboloid $p^2 = m^2$, α, β, γ hyperbolic coordinates and γ, r, ψ horospheric coordinates. Their ranges are

$$\begin{array}{lll} 0 \leq \alpha < \infty & 0 \leq \theta < \pi & 0 \leq \psi < 2\pi \\ -\infty < \alpha < \infty & 0 \leq \beta < \infty & 0 \leq \psi < 2\pi \\ -\infty < \gamma < \infty & 0 \leq r < \infty & 0 \leq \psi < 2\pi \end{array} \qquad (29)$$

As shown by Boyce, Delbourgo, Salam and Strathdee the unitary operators acting on the basis functions can be written as

$$\begin{aligned} U(L_p^+) &= e^{-i\psi M_{12}} \, e^{-i\theta M_{31}} \, e^{i\psi M_{12}} \, e^{-iaM_{03}} \\ U(L_p^-) &= e^{-i\psi M_{12}} \, e^{-i\beta M_{01}} \, e^{i\psi M_{12}} \, e^{-i\alpha M_{03}} \qquad (30) \\ U(L_p^0) &= e^{-i\psi M_{12}} \, e^{ir\pi_1} \, e^{i\psi M_{12}} \, e^{-i\gamma M_{03}} \end{aligned}$$

where $\pi_1 = -M_{01} + M_{31}$.

The boosts were so chosen that the basis functions in an arbitrary system (obtained by applying the boost operators to the basis functions in the reference system)

$$\begin{aligned} |p\lambda >^+ &= U(L^+(P)) \, |p_R\lambda> \\ |p\lambda >^- &= U(L^-(p)) \, |p_R\lambda> \qquad (31) \\ |p\lambda >^0 &= U(L^0(p)) \, |p_R\lambda> \end{aligned}$$

satisfy

$$W_0 \, |p\lambda >^+ = \lambda\varepsilon \sqrt{p_1^2 + p_2^2 + p_3^2} \, |p\lambda >^+ \qquad (32)$$

$$W_3 \, |p\lambda >^- = \lambda\varepsilon \sqrt{p_0^2 - p_1^2 - p_2^2} \, |p\lambda >^+ \qquad (33)$$

$$(W_o - W_3)\,|p\lambda>^o = \mp(p_o - p_3)|\; p\lambda>^o \qquad (34)$$

Thus, e.g. the functions

$$\psi_{ms\varepsilon,p\lambda} = |p\lambda>^{\mp} = U(L^{\mp}(p))\;|p_R\lambda> \qquad (35)$$

are eigenfunctions of the operators

$$P^2,\; W^2,\; P_\mu \text{ and } W_3 \qquad (36)$$

The basis functions (35) transform under transormations of the Poincare' group according to (2), (10) and (11) where $D_{\lambda'\lambda}$ $(R(\Lambda,p))$ are matrix elements of the Wigner rotation functions. The action of the infinitesimal operators $M_{\mu\nu}$ can easily be calculated.

2. Spacelike Representations

For spacelike representations the reference vector satisfies $p_R^2 = m^2 < 0$ and we choose it in the form

$$p_R = (0,0,0,\sqrt{-p_R^2}\,) \qquad (37)$$

The root is taken to be positive and its sign has no invariant meaning. Spacelike representations do not have any additional invariant.

The little group of vector (37) can directly be seen to be O(2,1), i.e. the three-dimensional homogeneous Lorentz group acting in the (p_0,p_1,p_2) - space. Indeed, the spin operator W, when acting in the subspace $p = p_R$, reduces to

$$W = -\sqrt{-p_R^2}\,(M_{12}, M_{02}, -M_{01}, 0) \qquad (38)$$

Thus, for particles in the reference frame (37) the surviving components of the spin operator W satisfy the algebra:

$$[M_{12}, M_{20}] = iM_{01} \quad [M_{20}, M_{01}] = -iM_{12} \quad [M_{01}, M_{12}] = iM_{20}$$

The Casimir operator of this O(2,1) group can be written as

$$W^2 = -m^2(M_{12}^2 - M_{20}^2 - M_{01}^2) = -m^2\ s(s+1) \qquad (39)$$

Since O(2,1) is a non-compact group, all its irreducible unitary representations, except the trivial one, are infinite dimensional.

The irreducible representations of O(2,1) can be characterized by two numbers

$$X = (s, \varepsilon) \qquad (40)$$

where s is an arbitrary complex number and $\varepsilon = 0$ or $\frac{1}{2}$. The representation theory of this group must be treated separately; here we shall just give a classification of the unitary representations:

(a) Unitary representations of the first principal series

$$X = (-\frac{1}{2} + iq,\ 0),\ -\infty < q < \infty \qquad q - \text{real}$$

Here

$$\lambda = 0, \pm 1, \pm 2, \ldots$$

(b) Unitary representations of the second principal series

$$X = (-\frac{1}{2} + iq,\ \frac{1}{2}) \quad -\infty < q < \infty \qquad q - \text{real}$$

$$\lambda = \pm \frac{1}{2}, \pm \frac{3}{2}, \ldots$$

(c) Unitary representations of the supplementary series

$$X = (s,0) \quad -1 < s < 0 \qquad s \text{ - real}$$
$$\lambda = 0, \pm 1, \pm 2 \ldots.$$

(d) Unitary representations of the discrete series

$$X = (s,\varepsilon) \quad s - \varepsilon = \text{negative integer}$$

Depending on the sign of λ we have two types

$$D^{s+} : \lambda = -s, -s + 1, -s + 2, \ldots$$
$$D^{s-} : \lambda = s, s - 1, s - 2, \ldots$$

(e) The trivial representation

$$X = (0,0)$$

The basis functions of space-like representations are thus characterized by an imaginary mass $m^2 < 0$ and by a generally complex value of the spin s. The spin projection λ takes either an infinite or semi-infinite number of values. These representations do not have direct applications in the quantum mechanics of relativistic free particles (unless we consider "tachyons" moving at superlight velocities). However, they play a very important role in relativistic partial wave analysis.

Again we can construct basis functions making explicit use of the Wigner boost operators. Indeed we can again construct the wave function $|p_R\lambda >$ in the reference frame (which is now, however, not the rest frame of a particle). The basis function in an arbitrary frame is obtained by applying the boost operator as in (21) and (22) again using the L_p^+, L_p^- and L_p^0 boosts of (26) - (28) and (30).

The components of a spacelike vector p are then expressed as

$$p = \sqrt{-p^2}\,(\mathrm{sh}a,\ \mathrm{ch}a\ \sin\theta\ \cos\psi,\ \mathrm{ch}a\ \sin\theta\ \sin\psi,\ \mathrm{ch}a\ \cos\theta)$$

$$p = \sqrt{-p^2}\,(\mathrm{sh}\alpha\ \mathrm{ch}\beta,\ \mathrm{sh}\alpha\ \mathrm{sh}\beta\ \cos\psi,\ \mathrm{sh}\alpha\ \mathrm{sh}\beta\ \sin\psi,\ \mathrm{ch}\alpha\ ) \qquad (41)$$

$$p = \sqrt{-p^2}\,(\mathrm{sh}\gamma - \frac{r^2}{2}\,e^{-\gamma},\ -re^{-\gamma}\cos\psi,\ -re^{-\gamma}\sin\psi,\ \mathrm{ch}\gamma - \frac{r^2}{2}\,e^{-\gamma})$$

with

$$\begin{array}{lll} -\infty < a < \infty & 0 \leq \theta < \pi & 0 \leq \psi < 2\pi \\ -\infty < \alpha < \infty & 0 \leq \beta < \infty & 0 \leq \psi < 2\pi \\ -\infty < \gamma < \infty & 0 \leq r < \infty & 0 \leq \psi < 2\pi \end{array} \qquad (42)$$

The basis function

$$\psi_{ms,p\lambda} = |p\lambda\ >^- = U(L^-(p))\ |p_R\lambda\ >$$

is again an eigenfunction of P^2, W^2, P and W_3 and satisfies

$$W_3\ |p\lambda >^- = -\lambda\sqrt{p_o^2 - p_1^2 - p_2^2}\ \ |\ p\lambda >^-$$

These functions transform according to the given general formulas where the functions $D_{\lambda'\lambda}$ $(R(\Lambda,p))$ are now the finite transformation matrix elements of $O(2,1)$.

3. Lightlike representations

We can choose the reference vector for lightlike representations as

$$P_R = (w, 0, 0, +w) \qquad w \neq o. \tag{43}$$

Here w has no invariant meaning, but its sign has, so that we obtain two different subclasses of representations for $w > o$ or $w < o$.

The little group of vector P_R in (43) is isomorphic to the group E_2, i.e., the group of motions of a Euclidean plane. When acting on the subspace determined by P_R, the spin vector W reduces to

$$W = -w\ (+M_{12}, M_{23} + M_{o2}, M_{31} - M_{o1}, M_{12}) \tag{44}$$

Putting

$$\Pi_1 = -M_{o1} + M_{31} \quad \Pi_2 = -M_{o2} - M_{23} \tag{45}$$

We obtain the algebra of E_2:

$$[M_{12}, \Pi_1] = i\Pi_2, \ [\Pi_2, M_{12}] = i\Pi_1, \ [\Pi_1, \Pi_2] = 0.$$

Obviously, M_{12} generates rotations and Π_1, Π_2 are isomorphic to generators of translations.

The Casimir operator of this algebra is:

$$W^2 = w^2\ (-\Pi_1^2 - \Pi_2^2) = -\rho^2 \tag{46}$$

Here ρ^2 is clearly an invariant of the Poincare group, whereas $\frac{\rho^2}{w^2}$ only of the group E_2.

The unitary irreducible representations of E_2 are of two types.

(a) Principal series:

ρ real, $0 < \rho < \infty$

$\lambda = 0, \pm 1, \pm 2, \ldots$

or $\lambda = \pm \frac{1}{2}, \pm \frac{3}{2}, \ldots$

(b) Discrete series

$\rho = 0$

Representations (b) are not "faithful" since $\Pi_1 = \Pi_2 = 0$ in them and they are all one-dimensional, since M_{12}, as the only surviving operator, becomes the Casimir operator. Thus

$$M_{12}\ \Psi_{oo\varepsilon' p_R \lambda} \qquad \lambda\ \Psi_{oo\varepsilon' p_R \lambda} \tag{47}$$

where $\lambda = 0, \pm \frac{1}{2}, \pm 1, \ldots$ labels the one-dimensional representations, (remember that we are not considering discrete operations like space inversion).

It follows from (44) that for these lightlike representations with $\rho = o$ we have

$$W_\mu = \pm\lambda \quad P_\mu \tag{48}$$

and λ is an additional invariant of the Poincaré group (when both the mass p^2 and spin W^2 are equal to zero). These are the representations corresponding to physical particles of mass zero and (48) shows that for such particles the spin vector must be parallel or antiparallel to the linear momentum vector (so that the special theory of relativity implies that e.g. the neutrino must have a definite helicity). Note, that for a massive particle the statement, that the spin is parallel to the momentum is not Lorentz invariant and can only hold in a definite coordinate system.

The basis functions for light-like representations can again be obtained using the same boosts as for time-like and space-like representations. It is however, convenient to apply them to the vector

$$P_R' = (w, o, o, -w) \tag{49}$$

obtaining

$$\begin{aligned} P &= we^{-a}\ (1, \sin\theta \cos\Psi, \sin\theta \sin\Psi, \cos\theta) \\ &= we^{-\alpha}\ (\mathrm{ch}\beta, \mathrm{sh}\beta \cos\Psi, \mathrm{sh}\beta \sin\Psi, -1) \\ &= we^{-\gamma}\ (1 + r^2, 2r \cos\Psi, 2r \sin\Psi, -1 + r^2) \end{aligned} \tag{50}$$

The range of the parameters is

$$-\infty < a < \infty \qquad 0 \leq \theta \leq \pi \qquad 0 \leq \Psi < 2\pi$$

$$-\infty < \alpha < \infty \qquad 0 \leq \beta < \infty \qquad 0 \leq \Psi < 2\pi$$

$$-\infty < \gamma < \infty \qquad 0 \leq r < \infty \qquad 0 \leq \Psi < 2\pi$$

The basis functions in an arbitrary frame can now be written as

$$|p\lambda>^{\kappa} = U(L(p)^{\kappa})\ |p_R'\lambda> = U(L(p)^{\kappa})\ e^{-i\pi M_{31}}|p_R\lambda> \qquad (51)$$

with $\kappa = +, -, o$ and satisfy

$$\begin{aligned} W_o\ |p\lambda>^{+} &= \lambda p_o\ |p\lambda>^{+} \\ W_3\ |p\lambda>^{-} &= \lambda p_3\ |p\lambda>^{-} \\ (W_o-W_3)\ |p\lambda>^{o} &= \lambda(p_o-p_3)\ |p\lambda>^{o} \end{aligned} \qquad (52)$$

The obtained basis functions again transform according to the same general formulas but the coefficients $D_{\lambda',\lambda}(R(\Lambda,p))$ are this time the matrix elements of finite transformation operators of the group E_2.

4. Null-vector Representations

The null-vector representations are exceptional in the sense that all components of the momentum p are equal to zero in any reference frame

$$p = (0, 0, 0, 0)$$

so that we do not need a standard reference vector.

All elements of the translation subgroup of the Poincaré group are represented by the unit operator in null-vector representations. The representations of the Poincaré group in this case coincide with the representations of the homogeneous Lorentz group.

The representation theory of the homogeneous Lorentz group has been treated in great detail by Gelfand and Najmark in a number of papers, summarized in the books: M. A. Najmark "Linear Representations of the Lorentz Group" and I. M. Gelfand, R. A. Minlos, Z. Ya. Shapiro "Representations of the Rotation and Lorentz Groups and Their Applications". This group has two invariant operators F and G, given previously as

$$F = \frac{1}{2} M_{\mu\nu} M^{\mu\nu} = \vec{M}^2 - \vec{N}^2$$

$$G = \frac{1}{4} \varepsilon^{\mu\nu\rho\sigma} M_{\mu\nu} M_{\rho\sigma} = \vec{M}\vec{N} \tag{53}$$

There are many possible choices of basis functions for representations of the Lorentz group. We shall return to this problem below, but let us only note that the "canonical" basis used by Gelfand and Najmark corresponds to the reduction of the Lorentz group to its rotation subgroup

$$O(3,1) \supset O(3) \supset O(2) \tag{54}$$

The basis functions are the eigenfunctions of a complete set of operators, chosen as the Casimir operators of all the groups, figuring in the chain (54). Thus we have

$$F\, \Psi_{oo,\nu\lambda j j_3} = \frac{1}{2} (-\nu^2+\lambda^2+ 1)\, \Psi_{oo,\nu\lambda j j_3}$$

$$G\, \Psi_{oo,\nu\lambda j j_3} = \nu\lambda\, \Psi_{oo,\nu\lambda j j_3} \tag{55}$$

$$M^2 \Psi_{oo,\nu\lambda j j_3} = j\, (j+1) \Psi_{oo,\nu\lambda j j_3}$$

$$M_{12} \Psi_{oo,\nu\lambda j j_3} = j_3 \Psi_{oo,\nu\lambda j j_3}$$

There exist two series of unitary irreducible representations, namely:

a) Principal series (56)

$- \infty < \mathrm{Re}\, \lambda < \infty$ $\qquad$ $\mathrm{Im}\, \lambda = 0$

$\nu = 0, \frac{1}{2}, 1, \ldots$

The O(3) quantum numbers take the values

$j = \nu, \quad \nu + 1, \nu + 2, \ldots$

$j_3 = -j, -j+1, \ldots, +j$

b) Supplementary series

$\text{Re}\ \lambda = 0 \qquad\qquad 0 < \text{Im}\ \lambda < 1$

$\nu = 0$

The O(3) quantum numbers take the values

$j = 0, 1, 2, \ldots$

$j_3 = -j, -j + 1, \ldots j$

The null vector representations of the Poincaré group have no application to the classification of free particles, but they have very interesting applications in scattering theory.

Applications of the Representation Theory of Poincaré Group.

1. Classification of Elementary Particles.

The first and most obvious application of the representation theory of the Poincaré group is the one which we have already discussed. Namely, since elementary physical systems, by definition transform according to irreducible unitary representations of the Poincaré group (or rather its covering group, so as to include half-integer values of spins and spin projections), we have obtained a classification of possible types of elementary particles (at least of free particle states).

We notice immediately that only a few of the possible types of representations seem to correspond to particles realized in nature. Indeed, we have:

a) $p^2 = m^2 > 0,\ \varepsilon = 1 \qquad s = 0, 1, 2 \ldots$

or $s = 1/2, 3/2, \ldots$

These representations correspond to the usual elementary particles with positive real mass. Only those with very low values of spin seem to be realized as stable elementary particles. Quite possibly the non-existence of higher spin particles has something to do with the difficulties in constructing a theory of such particles including any type of interaction.

b) $p^2 = m^2 > o \quad \varepsilon = -1 \qquad s = 0, 1, 2, \ldots$

or $\qquad s = 1/2, 3/2, \ldots$

These representations correspond to the standard antiparticles of the above particles (positrons, antiprotons, etc.)

c) $p^2 = o$, $p_\mu \neq 0$ for all μ, $\varepsilon = \pm 1$.

In this case only the exceptional, discrete, non-faithful representations, for which

$$\rho = o, \quad \lambda = 0, \pm 1, \pm 2, \ldots \quad \text{or} \quad \lambda = \pm 1/2, \pm 3/2$$

correspond to elementary particles of zero rest mass. Again only the lowest spins are realized in nature.

The continuous series of representations with

$$o < \rho < \infty$$

does not correspond, as far as we know, to any particles in nature.

d) $p^2 < o$.

These space-like representations, as far as we know do not correspond to any particles observed in nature. If they do, then these particles with "imaginary rest mass", must be tachyons, which figure in many theoretical considerations. Usually they are assumed to have spin zero, corresponding to the trivial representation of the O(2,1) little group, rather than continuous spin, corresponding to, say the principal series.

e) $p^2 = o$, $p_\mu = o$.

It is rather difficult to imagine having elementary particles transforming under null-vector representations.

We shall not pursue the very important question of the representations of the extended group, including reflections of all kinds and possible internal symmetries. We would thus obtain a further classification of particles, answering questions like: When can particles be equal to their antiparticles, what are the possible behaviours under time reversal, parity, etc.

2. Relativistic Kinematics.

It should be stressed that all representations of the Poincaré group are important in physics, not only those corresponding to real elementary particles. One of the reasons for this is that when considering any sort of reaction among elementary particles

$$1 + 2 + \ldots n \to (n+1) + (n+2) + \ldots +m$$

we are interested in multiparticle, or at least two-particle states.

Relativistic kinematics is basically the problem of taking the direct product of several irreducible representations (a reducible multiparticle state) and reducing out its irreducible components. This is of course the classical Clebsch-Gordon problem for the Poincaré group.

It turns out that if we consider the direct product of two physical representations, e.g. one representation with $m^2>o$, $\varepsilon=1$, the other with $m^2>o$, $\varepsilon = -1$, then the decomposition in the irreducible representations (the Clebsch-Gordon series) can in general, contain every type of representation of the Poincaré group--time-like, space-like, light-like and null-vector, in particular those with continuous spin.

The simplest reaction of interest is two-body scattering

$$1 + 2 \to 3 + 4.$$

We can write the scattering amplitudes for such a process as the matrix elements of the scattering matrix

$$\langle p_3 s_3 \lambda_3, p_4 s_4 \lambda_4 \, |S| \, p_1 s_1 \lambda_1, p_2 s_2 \lambda_2 \rangle \tag{1}$$

where relativistic invariance implies that S is a scalar operator

$$U(\Lambda,a)\, S\, U^{-1}(\Lambda,a) = S \tag{2}$$

We cannot go into any details here, since we have not really developed the necessary mathematical tools (e.g. the Clebsch-Gordan coefficients of the Poincaré group).

Let it suffice to say that if we consider scattering in the centre-of-mass frame of reference we express the initial and final two-particle states in terms of irreducible ones as

$$|p_1 s_1 \lambda_1,\ p_2 s_2 \lambda_2 > \sum_{s \geq |\lambda_1 - \lambda_2|} (2s+1) |s_1 s_2 \lambda_1 \lambda_2,\ p_R s\ \lambda_1 - \lambda_2 >$$

$$<p_3 s_3 \lambda_3,\ p_3 s_4 \lambda_4| = \sum_{s' \geq |\lambda_3 - \lambda_4|} (2s'+1) \sum_{\lambda' = -s'}^{s'} d^{s'}_{\lambda_3 - \lambda_4, \lambda'}(\theta) < s_3 s_4 \lambda_3 \lambda_4, p_R s' \lambda'| \tag{3}$$

where

$$P_R = (\sqrt{s},\ o,\ o,\ o) = p_1 + p_2 = p_3 + p_4\ ;\ s = (p_1 + p_2)^2 \tag{4}$$

$d^j_{\mu\nu}(\theta)$ is a Wigner rotation function (figuring as a Clebsch-Gordon coefficient in this case) and the states on the right-hand side of the above formulas are two-particle irreducible states.

We can now calculate the matrix element of the S-matrix between the considered states. Making use of Schur's lemma, which tells us that the S-matrix, being a scalar operator, must be diagonal with respect to the indices, referring to irreducible representations, we find

$$<p_3 s_3 \lambda_3, p_4 s_4 \lambda_4 | S | p_1 s_1 \lambda_1, p_2 s_2 \lambda_2> =$$

$$\sum_{j = \max(|\lambda_1 - \lambda_2|, |\lambda_3 - \lambda_4|)} (2j+1) <s_3 s_4 \lambda_3 \lambda_4 | S_j(s) | s_1 s_2 \lambda_1 \lambda_2> \ d^{j*}_{\lambda_3 - \lambda_4, \lambda_1 - \lambda_2}(\theta)\ \delta(p_1 + p_2 - p_3 - p_4) \tag{5}$$

In the above formula it is easy to recognize the Jacob and Wick expansion of helicity amplitudes, which here was derived in a purely group theoretical manner. The d-functions of the rotation group O(3) figure here because of our choice of the c.m.s. i.e. because of a frame-of-reference, in which a time-like vector $p_1 + p_2$, having O(3) as a little group, was fixed. The resulting formula is of course a direct generalization of the usual formulas of partial wave analysis. Indeed, for particles with spin zero (5) would reduce to

$$<p_3 p_4 | S | p_1 p_2> = f\ (s,t) = \sum_{j=o}^{\infty} (2j+1)\ a_j(s)\ P_j(\cos\theta)\ \delta(p_1 + p_2 - p_3 - p_4) \tag{6}$$

We could just as well have performed a different reduction of the S-matrix. Indeed, an operator $\tilde{S}$ can be introduced, satisfying

$$<-p_2 s_2 -\lambda_2, p_4 s_4 \lambda_4 \ |S| \ p_1 s_1 \lambda_1, -p_3 s_3 -\lambda_3> = (-1)^{s_2+s_3-\lambda_2-\lambda_3} <p_3 s_3 \lambda_3 p_4 s_4 \lambda_4 |S| p_1 s_1 \lambda_1, p_2 s_2 \lambda_2 > \quad (7$$

If we now consider the scattering in an appropriate frame of reference, obtained by standardizing the momentum transfer p_1-p_3, and reduce the two-particle states $|p_1 s_1 \lambda_1, -p_3 s_3 -\lambda_3>$ in a similar fashion as in (3), we obtain different results, depending on the charact r of p_1-p_3. Thus, if p_1-p_3 is space-like as it usually is, we obtain an expansion in terms of the D-functions of O(2,1), if p_1-p_3 is light-like the group generating the expansions will be E_2, etc. A specially interesting case is when $p_1-p_3 =(1, o, o, o)$, (elastic forward scattering) and the relevant little group is O(3,1).

The space-like **case** $(p_1-p_3)^2<0$ is of particular importance and in that case for spinless particles, we obtain

$$f(s,t) = -\frac{1}{2i} \int_{-1/2 - i\infty}^{-1/2 + i\infty} \frac{2j+1}{\sin\pi j} \ a(j,t) \ P_j(\cosh\beta)$$

one of the fundamental formulas of Regge theory.

For further information on relativistic partial **wave** analysis and general expansions of scattering amplitudes we refer to the literature.