Classification and Identification of Lie Algebras

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Outline

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The monograph: Classification and Identification of Lie Algebras

Title: Classification and Identification of Lie Algebras

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Why to be interested in identification and classification of Lie algebras?

Example: Consider two systems of PDEs, namely shallow water equations in the flat infinite basin

$$U_T + UU_X + VU_Y + H_X = 0, \; V_T + UV_X + VV_Y + H_Y = 0,$$
$$H_T + (UH)_X + (VH)_Y = 0 \tag{1}$$

and in the circular paraboloidal basin subjected to a Coriolis force due to the movement of the fluid inside the basin together with the Earth

$$u_t + uu_x + vu_y + (Z+h)_x = fv, \quad Z(x,y) = \frac{\omega^2 - f^2}{8}(x^2+y^2),$$
$$v_t + uv_x + vv_y + (Z+h)_y = -fu, \quad h_t + (uh)_x + (vh)_y = 0 \tag{2}$$

and compute their algebras of infinitesimal point symmetries.

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Symmetry algebras of shallow water equations, flat basin

One finds two seemingly different 9–dimensional symmetry algebras $\mathfrak{g}_A$ and $\mathfrak{g}_B$ spanned by the following vector fields, respectively

$$
\begin{gathered}
P_T = \partial_T,\ P_X = \partial_X,\ P_Y = \partial_Y,\ G_X = T\partial_X + \partial_U, \\
G_Y = T\partial_Y + \partial_V, D_1 = T\partial_T + X\partial_X + Y\partial_Y, \\
D_2 = -T\partial_T + U\partial_U + V\partial_V + 2H\partial_H, \\
L_1 = -Y\partial_X + X\partial_Y - V\partial_U + U\partial_V, \\
\Pi = T^2\partial_T + TX\partial_X + TY\partial_Y + (X - TU)\partial_U \\
+(Y - TV)\partial_V - 2TH\partial_H
\end{gathered}
\tag{3}
$$

Symmetry algebra of shallow water equations, flat basin, $\mathfrak{g}_A$

	P_T	P_X	P_Y	G_X	G_Y	D_1	D_2	L_1	Π
P_T	0	0	0	P_X	P_Y	P_T	$-P_T$	0	$D_1 - D_2$
P_X	0	0	0	0	0	P_X	0	P_Y	G_X
P_Y	0	0	0	0	0	P_Y	0	$-P_X$	G_Y
G_X	$-P_X$	0	0	0	0	0	G_X	G_Y	0
G_Y	$-P_Y$	0	0	0	0	0	G_Y	$-G_X$	0
D_1	$-P_T$	$-P_X$	$-P_Y$	0	0	0	0	0	Π
D_2	P_T	0	0	$-G_X$	$-G_Y$	0	0	0	$-\Pi$
L_1	0	$-P_Y$	P_X	$-G_Y$	G_X	0	0	0	0
Π	$-D_1 + D_2$	$-G_X$	$-G_Y$	0	0	$-\Pi$	Π	0	0

Symmetry algebra of shallow water equations, paraboloidal basin, Coriolis force

$$P_0 = \partial_t, \quad D = x\partial_x + y\partial_y + u\partial_u + v\partial_v + 2h\partial_h,$$
$$Y_1 = \cos(R_1 t)\partial_x - \sin(R_1 t)\partial_y - R_1 \sin(R_1 t)\partial_u - R_1 \cos(R_1 t)\partial_v,$$
$$Y_2 = \sin(R_1 t)\partial_x + \cos(R_1 t)\partial_y + R_1 \cos(R_1 t)\partial_u - R_1 \sin(R_1 t)\partial_v,$$
$$Y_3 = \cos(R_2 t)\partial_x + \sin(R_2 t)\partial_y - R_2 \sin(R_2 t)\partial_u + R_2 \cos(R_2 t)\partial_v,$$
$$Y_4 = \sin(R_2 t)\partial_x - \cos(R_2 t)\partial_y + R_2 \cos(R_2 t)\partial_u + R_2 \sin(R_2 t)\partial_v,$$
$$R = y\partial_x - x\partial_y + v\partial_u - u\partial_v,$$
$$K_1 = \frac{1}{2}\cos(\omega t)\left(x\partial_x + y\partial_y - u\partial_u - v\partial_v + f(y\partial_u - x\partial_v) - 2h\partial_h\right) +$$
$$+\frac{1}{2\omega}\sin(\omega t)\left(f(y\partial_x - x\partial_y + v\partial_u - u\partial_v) - \omega^2(x\partial_u + y\partial_v) + 2\partial_t\right),$$
$$K_2 = -\frac{1}{2}\sin(\omega t)\left(x\partial_x + y\partial_y - u\partial_u - v\partial_v + f(y\partial_u - x\partial_v) - 2h\partial_h\right)$$
$$+\frac{1}{2\omega}\cos(\omega t)\left(f(y\partial_x - x\partial_y + v\partial_u - u\partial_v) - \omega^2(x\partial_u + y\partial_v) + 2\partial_t\right) \quad (4)$$

Symmetry algebra of shallow water equations, paraboloidal basin, Coriolis force, $\mathfrak{g}_B$

where[1]

$$R_1 = \frac{1}{2}(\omega + f), \qquad R_2 = \frac{1}{2}(\omega - f).$$

	P_0	K_1	K_2	D	R	Y_1	Y_2	Y_3	Y_4
P_0	0	ωK_2	$-\omega K_1$	0	0	$-\frac{f+\omega}{2} Y_2$	$\frac{f+\omega}{2} Y_1$	$\frac{f-\omega}{2} Y_4$	$\frac{\omega-f}{2} Y_3$
K_1	$-\omega K_2$	0	$\frac{-1}{\omega}(P_0 + \frac{f}{2} R)$	0	0	$-\frac{1}{2} Y_3$	$\frac{1}{2} Y_4$	$-\frac{1}{2} Y_1$	$\frac{1}{2} Y_2$
K_2	ωK_1	$\frac{1}{\omega}(P_0 + \frac{f}{2} R)$	0	0	0	$\frac{1}{2} Y_4$	$\frac{1}{2} Y_3$	$\frac{1}{2} Y_2$	$\frac{1}{2} Y_1$
D	0	0	0	0	0	$-Y_1$	$-Y_2$	$-Y_3$	$-Y_4$
R	0	0	0	0	0	Y_2	$-Y_1$	$-Y_4$	Y_3
Y_1	$\frac{f+\omega}{2} Y_2$	$\frac{1}{2} Y_3$	$-\frac{1}{2} Y_4$	Y_1	$-Y_2$	0	0	0	0
Y_2	$-\frac{f+\omega}{2} Y_1$	$-\frac{1}{2} Y_4$	$-\frac{1}{2} Y_3$	Y_2	Y_1	0	0	0	0
Y_3	$\frac{\omega-f}{2} Y_4$	$\frac{1}{2} Y_1$	$-\frac{1}{2} Y_2$	Y_3	Y_4	0	0	0	0
Y_4	$\frac{f-\omega}{2} Y_3$	$-\frac{1}{2} Y_2$	$-\frac{1}{2} Y_1$	Y_4	$-Y_3$	0	0	0	0

[1] D. Levi, M.C. Nucci, C. Rogers, P. Winternitz 1989 *J. Phys. A* **22** 4743–4767

Symmetry algebra of shallow water equations, paraboloidal basin, Coriolis force, $\mathfrak{g}_B$

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$$R_1 = \frac{1}{2}(\omega + f), \qquad R_2 = \frac{1}{2}(\omega - f).$$

	P_0	K_1	K_2	D	R	Y_1	Y_2	Y_3	Y_4
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K_1	$-\omega K_2$	0	$\frac{-1}{\omega}(P_0 + \frac{f}{2} R)$	0	0	$-\frac{1}{2} Y_3$	$\frac{1}{2} Y_4$	$-\frac{1}{2} Y_1$	$\frac{1}{2} Y_2$
K_2	ωK_1	$\frac{1}{\omega}(P_0 + \frac{f}{2} R)$	0	0	0	$\frac{1}{2} Y_4$	$\frac{1}{2} Y_3$	$\frac{1}{2} Y_2$	$\frac{1}{2} Y_1$
D	0	0	0	0	0	$-Y_1$	$-Y_2$	$-Y_3$	$-Y_4$
R	0	0	0	0	0	Y_2	$-Y_1$	$-Y_4$	Y_3
Y_1	$\frac{f+\omega}{2} Y_2$	$\frac{1}{2} Y_3$	$-\frac{1}{2} Y_4$	Y_1	$-Y_2$	0	0	0	0
Y_2	$-\frac{f+\omega}{2} Y_1$	$-\frac{1}{2} Y_4$	$-\frac{1}{2} Y_3$	Y_2	Y_1	0	0	0	0
Y_3	$\frac{\omega-f}{2} Y_4$	$\frac{1}{2} Y_1$	$-\frac{1}{2} Y_2$	Y_3	Y_4	0	0	0	0
Y_4	$\frac{f-\omega}{2} Y_3$	$-\frac{1}{2} Y_2$	$-\frac{1}{2} Y_1$	Y_4	$-Y_3$	0	0	0	0

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The Lie algebra $\mathfrak{g}_B$ of (4)

The Lie algebra $\mathfrak{g}_B$ of (4) has the radical (maximal solvable ideal)

$$\mathrm{R}(\mathfrak{g}_B) = \mathrm{span}\{D, R, Y_1, Y_2, Y_3, Y_4\},$$

the nilradical (maximal nilpotent ideal)

$$\mathrm{NR}(\mathfrak{g}_B) = \mathrm{span}\{Y_1, Y_2, Y_3, Y_4\},$$

and the Levi factor (semisimple subalgebra complementing the radical)

$$\mathfrak{p} = \mathrm{span}\{P_0 + \frac{f}{2}R, K_1, K_2\}.$$

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The Lie algebra $\mathfrak{g}_B$ of (4), continued

In the adjoint representation of $\mathfrak{g}_B$ the element D acts on the nilradical $\mathrm{NR}(\mathfrak{g}_B)$ diagonally as a multiple of a unit matrix whereas R acts on it as a rotation. Both elements commute with the Levi factor.

From the indefinite signature of the Killing form of the Levi factor $\mathfrak{p}$ it follows that $\mathfrak{p}$ is isomorphic to the simple algebra $\mathfrak{sl}(2,\mathbb{R})$. The adjoint action of the Levi factor $\mathfrak{p}$ on the nilradical $\mathrm{NR}(\mathfrak{g}_B)$ corresponds to a direct sum of two 2–dimensional irreducible representations of $\mathfrak{sl}(2,\mathbb{R})$.

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The Lie algebra $\mathfrak{g}_A$ of (3)

Somewhat surprisingly, the Lie algebra $\mathfrak{g}_A$ of (3) has the same structure. When expressed in suitable bases which make the structure transparent, the two algebras $\mathfrak{g}_A$ and $\mathfrak{g}_B$ turn out to be isomorphic as Lie algebras. Namely, the Lie brackets expressed in following two bases of $\mathfrak{g}_A$ and $\mathfrak{g}_B$, respectively,

$$e_1 = P_T,\ e_2 = D_1 - D_2,\ e_3 = -\Pi,\ e_4 = -(D_1 + D_2),$$
$$e_5 = L_1,\ e_6 = P_Y,\ e_7 = P_X,\ e_8 = G_Y,\ e_9 = G_X,$$

$$\tilde{e}_1 = -\frac{1}{\omega}\left(P_0 + \frac{f}{2}R\right) + K_2,\ \tilde{e}_2 = -2K_1,$$
$$\tilde{e}_3 = \frac{1}{\omega}\left(P_0 + \frac{f}{2}R\right) + K_2,\ \tilde{e}_4 = -D,\ \tilde{e}_5 = R,\ \tilde{e}_6 = Y_1 - Y_3,$$
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The Lie algebras $\mathfrak{g}_A \simeq \mathfrak{g}_B$

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9
e_1	0	$2e_1$	$-e_2$	0	0	0	0	e_6	e_7
e_2	$-2e_1$	0	$2e_3$	0	0	$-e_6$	$-e_7$	e_8	e_9
e_3	e_2	$-2e_3$	0	0	0	e_8	e_9	0	0
e_4	0	0	0	0	0	e_6	e_7	e_8	e_9
e_5	0	0	0	0	0	e_7	$-e_6$	e_9	$-e_8$
e_6	0	e_6	$-e_8$	$-e_6$	$-e_7$	0	0	0	0
e_7	0	e_7	$-e_9$	$-e_7$	e_6	0	0	0	0
e_8	$-e_6$	$-e_8$	0	$-e_8$	$-e_9$	0	0	0	0
e_9	$-e_7$	$-e_9$	0	$-e_9$	e_8	0	0	0	0

Isomorphisms between vector field realizations

That does not by itself imply that the two sets of vector fields (3) and (4) are related to each other by a point transformation but it is a necessary condition for it and a hint that such a transformation may exist.

Indeed, using computer algebra we find a locally invertible map

$$\Phi : \mathbb{R}^6[t, x, y, u, v, h] \to \mathbb{R}^6[T, X, Y, U, V, H]$$

which transforms the algebra of vector fields (3) into (4).

Explicitly, the transformation Φ reads

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Explicitly, the transformation Φ reads

Mapping the algebra of vector fields $\mathfrak{g}_A$ to $\mathfrak{g}_B$

$$
\begin{aligned}
T &= \cot\left(\frac{\omega}{2}t\right), \qquad X = \frac{1}{2\sin\left(\frac{\omega}{2}t\right)}\left(\cos\left(\frac{f}{2}t\right)x - \sin\left(\frac{f}{2}t\right)y\right), \\
H &= C h \sin\left(\frac{\omega}{2}t\right)^2, \quad Y = -\frac{1}{2\sin\left(\frac{\omega}{2}t\right)}\left(\sin\left(\frac{f}{2}t\right)x + \cos\left(\frac{f}{2}t\right)y\right), \\
U &= \frac{1}{2\omega}\left(-2\sin\left(\tfrac{\omega}{2}t\right)\cos\left(\tfrac{f}{2}t\right)u + 2\sin\left(\tfrac{\omega}{2}t\right)\sin\left(\tfrac{f}{2}t\right)v + \right. \\
&\quad + \left(\sin\left(\tfrac{\omega}{2}t\right)\sin\left(\tfrac{f}{2}t\right)f + \cos\left(\tfrac{f}{2}t\right)\cos\left(\tfrac{\omega}{2}t\right)\omega\right)x + \\
&\quad \left. + \left(\sin\left(\tfrac{\omega}{2}t\right)\cos\left(\tfrac{f}{2}t\right)f - \sin\left(\tfrac{f}{2}t\right)\cos\left(\tfrac{\omega}{2}t\right)\omega\right)y\right), \\
V &= \frac{1}{2\omega}\left(2\sin\left(\tfrac{\omega}{2}t\right)\sin\left(\tfrac{f}{2}t\right)u + 2\sin\left(\tfrac{\omega}{2}t\right)\cos\left(\tfrac{f}{2}t\right)v + \right. \\
&\quad + \left(\sin\left(\tfrac{\omega}{2}t\right)\cos\left(\tfrac{f}{2}t\right)f - \sin\left(\tfrac{f}{2}t\right)\cos\left(\tfrac{\omega}{2}t\right)\omega\right)x - \\
&\quad \left. - \left(\sin\left(\tfrac{\omega}{2}t\right)\sin\left(\tfrac{f}{2}t\right)f + \cos\left(\tfrac{f}{2}t\right)\cos\left(\tfrac{\omega}{2}t\right)\omega\right)y\right),
\end{aligned}
\tag{5}
$$

where C is an integration constant.

Equivalence of the two shallow water equations

What is more, for a particular choice of the parameter C, namely

$$C = \frac{1}{\omega^2},$$

the two shallow water equations (1) and (2) are mapped one into the other by the change of dependent and independent variables (5). Thus, mathematically they are locally equivalent although their physical interpretation is different; any solution of one of them gives rise to a (local) solution of the other. This equivalence of equations (1) and (2) would be very difficult, if not impossible, to discover without understanding the structure of the two Lie algebras involved.

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What can be found in our book

The intention of the authors:

- The purpose of the monograph is to serve as a tool for practitioners of Lie algebra and Lie group theory, i.e., for those who apply Lie algebras and Lie groups to solve problems arising in science and engineering.
- The main motif: how to transform a randomly obtained basis of a Lie algebra into a "canonical basis" in which all basis independent features of the Lie algebra are directly visible.
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The layout of the book

The book is divided into four parts:

- General Theory
- Recognition of a Lie Algebra Given by Its Structure Constants
- Nilpotent, Solvable and Levi Decomposable Lie Algebras
- Low-Dimensional Lie Algebras

General Theory

- an introductory review of definitions and notions,
- a more detailed introduction into the computation of invariants of the coadjoint representation of a Lie algebra (a.k.a. generalized Casimir invariants), including the method of moving frames (Cartan, Fels, Olver, Boyko, Patera, Popovych).

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Recognition of a Lie Algebra Given by Its Structure Constants

- various invariant characteristics of a Lie algebra which can be helpful in its identification are reviewed. Among others dimensions of ideals in the characteristic series, the number and structure of generalized Casimir invariants, (α, β, γ)–derivations and twisted cocycles (Hrivnák, Novotný) etc.
- the algorithms for establishing decomposability (and explicit decomposition), Levi decomposition and the nilradical. These chapters are in essence a revision and correction of the paper D. Rand, P. Winternitz and H. Zassenhaus 1988 *Linear algebra and its applications* **109** 197–246, supplemented by numerous new examples.

Recognition of a Lie Algebra Given by Its Structure Constants

- various invariant characteristics of a Lie algebra which can be helpful in its identification are reviewed. Among others dimensions of ideals in the characteristic series, the number and structure of generalized Casimir invariants, (α, β, γ)–derivations and twisted cocycles (Hrivnák, Novotný) etc.
- the algorithms for establishing decomposability (and explicit decomposition), Levi decomposition and the nilradical. These chapters are in essence a revision and correction of the paper D. Rand, P. Winternitz and H. Zassenhaus 1988 *Linear algebra and its applications* **109** 197–246, supplemented by numerous new examples.

Nilpotent, Solvable and Levi Decomposable Lie Algebras

- general structure of nilpotent algebras and approaches to their classification,
- general structure of a solvable algebra, in particular in relation with its nilradical,
- classification of solvable Lie algebras with the nilradicals of the given specific form including the determination of their generalized Casimir invariants,
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Types of nilradicals considered in arbitrary finite dimensions

- Nilradicals with low degree of nilpotency (J.C. Ndogmo, J. Rubin, P. Winternitz)

  The algebras investigated in this class are the Abelian and Heisenberg algebras. These algebras possess large algebras of derivations that have well-understood properties. E.g., for an Abelian nilradical, any linear transformation is a derivation and any regular linear map is an automorphism. Consequently, the construction of solvable extensions is reduced to the consideration of Abelian subalgebras in $\mathfrak{gl}(n)$ and their equivalence. Similarly, for Heisenberg algebras $\mathfrak{h}(n)$, the task is reduced to the study of Abelian subalgebras of $\mathfrak{sp}(2n)$.

Types of nilradicals, continued

- Nilradicals of Borel subalgebras of simple Lie algebras (L. Šnobl, S. Tremblay, P. Winternitz)

  Nilpotent algebras in this class have a very particular structure given by the corresponding root diagram. Consequently, all derivations of such algebras can be found in explicit form using cohomological arguments. This was done by G.F. Leger and E.M. Luks. A prime example of a nilradical in this class is the algebra of strictly upper triangular matrices.

Types of nilradicals, continued

- Nilradicals with high degree of nilpotency

  The structure of Lie brackets of such algebras usually significantly restricts the algebra of derivations. Therefore the algebras of derivations can often be written down explicitly in arbitrary dimension and similarly for the automorphisms. Many explicit lists of solvable algebras with nilradicals in this class are known (J.M. Ancochea, R. Campoamor–Stursberg, L. Garcia Vergnolle, D. Karásek, L. Šnobl, P. Winternitz and others), we describe in detail our results concerning three such classes of nilradicals and indicate briefly the results of the others.

Levi Decomposable Algebras

- general structure of Levi decomposable algebras, the important role of the nilradical,
- convenient description of Levi decomposable algebras with nilpotent radicals, structural explanation of the low–dimensional classifications obtained previously by an explicit calculation (Turkowski),
- description of Levi decomposable algebras with nonnilpotent radicals,
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Low-Dimensional Lie Algebras

A complete list of Lie algebras up to dimension 6 (up to our best knowledge and effort :-), including the identification of the radical, nilradical, Levi factor, the dimensions of their upper, lower and derived series and the generalized Casimir invariants. Thus it contains *inter alia* the results of J. Patera, R.T. Sharp, P. Winternitz and H. Zassenhaus 1976 *J. Math. Phys.* **17** 986.

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Low-Dimensional Lie Algebras, continued

The lists are ordered in such a way as to make the identification of any given low-dimensional Lie algebra written in an arbitrary basis as simple as possible, i.e. they ordered by structural properties of the algebras (rather than by the way they were originally obtained, as was the case in the older classifications, e.g. Morozov, Mubarakzyanov, Turkowski).

The lists are significantly more refined compared to the above mentioned ones, i.e. special values of parameters implying different properties are split off as particular cases. E.g. in dimension 6 there are 242 indecomposable classes of solvable nonnilpotent Lie algebras.

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Conclusions

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Thank you for your attention