Structure of Levi extensions of nilpotent and solvable Lie algebras

Libor Šnobl

Department of Physics
Faculty of Nuclear Sciences and Physical Engineering
Czech Technical University in Prague

The 31st Winter School GEOMETRY AND PHYSICS, Srní, January 2011

J. Phys. A: Math. Theor. **43** (2010) 505202 (17pp)
doi:10.1088/1751-8113/43/50/505202

Outline

Outline

Outline

Outline

Review of basic notions

A Lie algebra $\mathfrak{g}$ is:

- $\mathfrak{g}$ is decomposable if nonvanishing ideals $\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_k$ exist such that

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \ldots \oplus \mathfrak{g}_k, \tag{1}$$

- semisimple $\Leftrightarrow$ no nonvanishing Abelian ideal exists,
- simple $\Leftrightarrow$ no nontrivial ideal exists.
  Any semisimple algebra $\mathfrak{p}$ is a direct sum of simple algebras

Any representation ρ of a semisimple algebra $\mathfrak{p}$ on a vector space V is fully reducible, i.e. if W is an invariant subspace

$$\rho(\mathfrak{p})W \subseteq W$$

then its invariant complement $\widetilde{W}$ of W in V exists such that

$$\rho(\mathfrak{p})\widetilde{W} \subseteq \widetilde{W}, \qquad V = W \dotplus \widetilde{W}. \tag{2}$$

Review of basic notions

A Lie algebra $\mathfrak{g}$ is:

- $\mathfrak{g}$ is decomposable if nonvanishing ideals $\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_k$ exist such that

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \ldots \oplus \mathfrak{g}_k, \tag{1}$$

- semisimple $\Leftrightarrow$ no nonvanishing Abelian ideal exists,
- simple $\Leftrightarrow$ no nontrivial ideal exists.
  Any semisimple algebra $\mathfrak{p}$ is a direct sum of simple algebras

Any representation ρ of a semisimple algebra $\mathfrak{p}$ on a vector space V is fully reducible, i.e. if W is an invariant subspace

$$\rho(\mathfrak{p})W \subseteq W$$

then its invariant complement $\widetilde{W}$ of W in V exists such that

$$\rho(\mathfrak{p})\widetilde{W} \subseteq \widetilde{W}, \qquad V = W \dot{+} \widetilde{W}. \tag{2}$$

Review of basic notions

A Lie algebra $\mathfrak{g}$ is:

- $\mathfrak{g}$ is decomposable if nonvanishing ideals $\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_k$ exist such that

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \ldots \oplus \mathfrak{g}_k, \tag{1}$$

- semisimple $\Leftrightarrow$ no nonvanishing Abelian ideal exists,
- simple $\Leftrightarrow$ no nontrivial ideal exists.

  Any semisimple algebra $\mathfrak{p}$ is a direct sum of simple algebras

Any representation ρ of a semisimple algebra $\mathfrak{p}$ on a vector space V is fully reducible, i.e. if W is an invariant subspace

$$\rho(\mathfrak{p})W \subseteq W$$

then its invariant complement $\widetilde{W}$ of W in V exists such that

$$\rho(\mathfrak{p})\widetilde{W} \subseteq \widetilde{W}, \qquad V = W \dotplus \widetilde{W}. \tag{2}$$

Review of basic notions

A Lie algebra $\mathfrak{g}$ is:

- $\mathfrak{g}$ is decomposable if nonvanishing ideals $\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_k$ exist such that

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \ldots \oplus \mathfrak{g}_k, \tag{1}$$

- semisimple $\Leftrightarrow$ no nonvanishing Abelian ideal exists,
- simple $\Leftrightarrow$ no nontrivial ideal exists.
  Any semisimple algebra $\mathfrak{p}$ is a direct sum of simple algebras

Any representation ρ of a semisimple algebra $\mathfrak{p}$ on a vector space V is fully reducible, i.e. if W is an invariant subspace

$$\rho(\mathfrak{p})W \subseteq W$$

then its invariant complement $\widetilde{W}$ of W in V exists such that

$$\rho(\mathfrak{p})\widetilde{W} \subseteq \widetilde{W}, \qquad V = W \dot{+} \widetilde{W}. \tag{2}$$

Review of basic notions

A Lie algebra $\mathfrak{g}$ is:

- $\mathfrak{g}$ is decomposable if nonvanishing ideals $\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_k$ exist such that

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \ldots \oplus \mathfrak{g}_k, \tag{1}$$

- semisimple $\Leftrightarrow$ no nonvanishing Abelian ideal exists,
- simple $\Leftrightarrow$ no nontrivial ideal exists.
  Any semisimple algebra $\mathfrak{p}$ is a direct sum of simple algebras

Any representation ρ of a semisimple algebra $\mathfrak{p}$ on a vector space V is fully reducible, i.e. if W is an invariant subspace

$$\rho(\mathfrak{p})W \subseteq W$$

then its invariant complement $\widetilde{W}$ of W in V exists such that

$$\rho(\mathfrak{p})\widetilde{W} \subseteq \widetilde{W}, \qquad V = W \dot{+} \widetilde{W}. \tag{2}$$

Review of basic notions

A Lie algebra $\mathfrak{g}$ is:

- $\mathfrak{g}$ is decomposable if nonvanishing ideals $\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_k$ exist such that

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \ldots \oplus \mathfrak{g}_k, \tag{1}$$

- semisimple $\Leftrightarrow$ no nonvanishing Abelian ideal exists,
- simple $\Leftrightarrow$ no nontrivial ideal exists.
  Any semisimple algebra $\mathfrak{p}$ is a direct sum of simple algebras

Any representation ρ of a semisimple algebra $\mathfrak{p}$ on a vector space V is fully reducible, i.e. if W is an invariant subspace

$$\rho(\mathfrak{p})W \subseteq W$$

then its invariant complement $\widetilde{W}$ of W in V exists such that

$$\rho(\mathfrak{p})\widetilde{W} \subseteq \widetilde{W}, \qquad V = W \dot{+} \widetilde{W}. \tag{2}$$

Solvable and nilpotent algebras

Three series of ideals – characteristic series of $\mathfrak{g}$:

- derived series $\mathfrak{g} = \mathfrak{g}^{(0)} \supseteq \ldots \supseteq \mathfrak{g}^{(k)} \supseteq \ldots$ defined

$$\mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}], \ \mathfrak{g}^{(0)} = \mathfrak{g}.$$

  If $\exists k \in \mathbb{N}$ such that $\mathfrak{g}^{(k)} = 0$, then $\mathfrak{g}$ is solvable.

- lower central series $\mathfrak{g} = \mathfrak{g}^1 \supseteq \ldots \supseteq \mathfrak{g}^k \supseteq \ldots$ defined

$$\mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}], \ \mathfrak{g}^1 = \mathfrak{g}.$$

  If $\exists k \in \mathbb{N}$ such that $\mathfrak{g}^k = 0$, then $\mathfrak{g}$ nilpotent. The largest value of K s.t. $\mathfrak{g}^K \neq 0$ is the degree of nilpotency.

- upper central series $\mathfrak{z}_1 \subseteq \ldots \subseteq \mathfrak{z}_k \subseteq \ldots \subseteq \mathfrak{g}$ where $\mathfrak{z}_1$ is the center of $\mathfrak{g}$, $\mathfrak{z}_1 = C(\mathfrak{g}) = \{x \in \mathfrak{g} | [x, y] = 0, \ \forall y \in \mathfrak{g}\}$ and $\mathfrak{z}_k$ are the higher centers defined recursively through

$$\mathfrak{z}_{k+1}/\mathfrak{z}_k = C(\mathfrak{g}/\mathfrak{z}_k).$$

Solvable and nilpotent algebras

Three series of ideals – characteristic series of $\mathfrak{g}$:

- derived series $\mathfrak{g} = \mathfrak{g}^{(0)} \supseteq \ldots \supseteq \mathfrak{g}^{(k)} \supseteq \ldots$ defined

$$\mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}], \;\; \mathfrak{g}^{(0)} = \mathfrak{g}.$$

  If $\exists k \in \mathbb{N}$ such that $\mathfrak{g}^{(k)} = 0$, then $\mathfrak{g}$ is solvable.

- lower central series $\mathfrak{g} = \mathfrak{g}^1 \supseteq \ldots \supseteq \mathfrak{g}^k \supseteq \ldots$ defined

$$\mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}], \;\; \mathfrak{g}^1 = \mathfrak{g}.$$

  If $\exists k \in \mathbb{N}$ such that $\mathfrak{g}^k = 0$, then $\mathfrak{g}$ nilpotent. The largest value of K s.t. $\mathfrak{g}^K \neq 0$ is the degree of nilpotency.

- upper central series $\mathfrak{z}_1 \subseteq \ldots \subseteq \mathfrak{z}_k \subseteq \ldots \subseteq \mathfrak{g}$ where $\mathfrak{z}_1$ is the center of $\mathfrak{g}$, $\mathfrak{z}_1 = C(\mathfrak{g}) = \{x \in \mathfrak{g} | [x, y] = 0, \; \forall y \in \mathfrak{g}\}$ and $\mathfrak{z}_k$ are the higher centers defined recursively through

$$\mathfrak{z}_{k+1}/\mathfrak{z}_k = C(\mathfrak{g}/\mathfrak{z}_k).$$

Solvable and nilpotent algebras

Three series of ideals – characteristic series of $\mathfrak{g}$:

- derived series $\mathfrak{g} = \mathfrak{g}^{(0)} \supseteq \ldots \supseteq \mathfrak{g}^{(k)} \supseteq \ldots$ defined

  $$\mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}], \; \mathfrak{g}^{(0)} = \mathfrak{g}.$$

  If $\exists k \in \mathbb{N}$ such that $\mathfrak{g}^{(k)} = 0$, then $\mathfrak{g}$ is solvable.
- lower central series $\mathfrak{g} = \mathfrak{g}^1 \supseteq \ldots \supseteq \mathfrak{g}^k \supseteq \ldots$ defined

  $$\mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}], \; \mathfrak{g}^1 = \mathfrak{g}.$$

  If $\exists k \in \mathbb{N}$ such that $\mathfrak{g}^k = 0$, then $\mathfrak{g}$ nilpotent. The largest value of K s.t. $\mathfrak{g}^K \neq 0$ is the degree of nilpotency.
- upper central series $\mathfrak{z}_1 \subseteq \ldots \subseteq \mathfrak{z}_k \subseteq \ldots \subseteq \mathfrak{g}$ where $\mathfrak{z}_1$ is the center of $\mathfrak{g}$, $\mathfrak{z}_1 = C(\mathfrak{g}) = \{x \in \mathfrak{g} | [x, y] = 0, \; \forall y \in \mathfrak{g}\}$ and $\mathfrak{z}_k$ are the higher centers defined recursively through

  $$\mathfrak{z}_{k+1}/\mathfrak{z}_k = C(\mathfrak{g}/\mathfrak{z}_k).$$

Solvable and nilpotent algebras

Three series of ideals – characteristic series of $\mathfrak{g}$:

- derived series $\mathfrak{g} = \mathfrak{g}^{(0)} \supseteq \ldots \supseteq \mathfrak{g}^{(k)} \supseteq \ldots$ defined

$$\mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}], \;\; \mathfrak{g}^{(0)} = \mathfrak{g}.$$

  If $\exists k \in \mathbb{N}$ such that $\mathfrak{g}^{(k)} = 0$, then $\mathfrak{g}$ is solvable.

- lower central series $\mathfrak{g} = \mathfrak{g}^1 \supseteq \ldots \supseteq \mathfrak{g}^k \supseteq \ldots$ defined

$$\mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}], \;\; \mathfrak{g}^1 = \mathfrak{g}.$$

  If $\exists k \in \mathbb{N}$ such that $\mathfrak{g}^k = 0$, then $\mathfrak{g}$ nilpotent. The largest value of K s.t. $\mathfrak{g}^K \neq 0$ is the degree of nilpotency.

- upper central series $\mathfrak{z}_1 \subseteq \ldots \subseteq \mathfrak{z}_k \subseteq \ldots \subseteq \mathfrak{g}$ where $\mathfrak{z}_1$ is the center of $\mathfrak{g}$, $\mathfrak{z}_1 = C(\mathfrak{g}) = \{x \in \mathfrak{g} | [x, y] = 0, \;\forall y \in \mathfrak{g}\}$ and $\mathfrak{z}_k$ are the higher centers defined recursively through

$$\mathfrak{z}_{k+1}/\mathfrak{z}_k = C(\mathfrak{g}/\mathfrak{z}_k).$$

A linear map $\psi : \mathfrak{g} \to \mathfrak{g}$ is

- derivation $\Leftrightarrow \quad \psi([x, y]) = [\psi(x), y] + [x, \psi(y)]$,
- inner derivation $\Leftrightarrow \exists\, z \in \mathfrak{g}$: $\psi = \mathrm{ad}_z$, i.e. $\psi(x) = [z, x]$,
- automorphism $\Leftrightarrow \quad \psi([x, y]) = [\psi(x), \psi(y)]$.

Any derivation which is not inner is outer.

The centralizer $\mathfrak{g}_{\mathfrak{h}}$ of a given subalgebra $\mathfrak{h} \subset \mathfrak{g}$ in $\mathfrak{g}$ is the set of all elements in $\mathfrak{g}$ commuting with all elements in $\mathfrak{h}$, i.e.

$$\mathfrak{g}_{\mathfrak{h}} = \{x \in \mathfrak{g} | [x, y] = 0, \ \forall y \in \mathfrak{h}\}. \tag{3}$$

Ideals in the characteristic series as well as their centralizers (and their various intersections) are invariant with respect to any derivation and automorphism, i.e. they belong among characteristic ideals.

A linear map $\psi : \mathfrak{g} \to \mathfrak{g}$ is

- derivation $\Leftrightarrow \quad \psi([x,y]) = [\psi(x), y] + [x, \psi(y)]$,
- inner derivation $\Leftrightarrow \exists\, z \in \mathfrak{g}$: $\psi = \mathrm{ad}_z$, i.e. $\psi(x) = [z, x]$,
- automorphism $\Leftrightarrow \quad \psi([x,y]) = [\psi(x), \psi(y)]$.

Any derivation which is not inner is outer.

The centralizer $\mathfrak{g}_{\mathfrak{h}}$ of a given subalgebra $\mathfrak{h} \subset \mathfrak{g}$ in $\mathfrak{g}$ is the set of all elements in $\mathfrak{g}$ commuting with all elements in $\mathfrak{h}$, i.e.

$$\mathfrak{g}_{\mathfrak{h}} = \{x \in \mathfrak{g} | [x,y] = 0,\ \forall y \in \mathfrak{h}\}. \quad (3)$$

Ideals in the characteristic series as well as their centralizers (and their various intersections) are invariant with respect to any derivation and automorphism, i.e. they belong among characteristic ideals.

A linear map $\psi : \mathfrak{g} \to \mathfrak{g}$ is

- derivation $\Leftrightarrow \quad \psi([x, y]) = [\psi(x), y] + [x, \psi(y)]$,
- inner derivation $\Leftrightarrow \exists\, z \in \mathfrak{g}$: $\psi = \mathrm{ad}_z, \text{ i.e. } \psi(x) = [z, x]$,
- automorphism $\Leftrightarrow \quad \psi([x, y]) = [\psi(x), \psi(y)]$.

Any derivation which is not inner is outer.

The centralizer $\mathfrak{g}_{\mathfrak{h}}$ of a given subalgebra $\mathfrak{h} \subset \mathfrak{g}$ in $\mathfrak{g}$ is the set of all elements in $\mathfrak{g}$ commuting with all elements in $\mathfrak{h}$, i.e.

$$\mathfrak{g}_{\mathfrak{h}} = \{x \in \mathfrak{g} | [x, y] = 0, \ \forall y \in \mathfrak{h}\}. \tag{3}$$

Ideals in the characteristic series as well as their centralizers (and their various intersections) are invariant with respect to any derivation and automorphism, i.e. they belong among characteristic ideals.

A linear map $\psi : \mathfrak{g} \to \mathfrak{g}$ is

- derivation $\Leftrightarrow \quad \psi([x, y]) = [\psi(x), y] + [x, \psi(y)]$,
- inner derivation $\Leftrightarrow \exists\, z \in \mathfrak{g}$: $\psi = \mathrm{ad}_z$, i.e. $\psi(x) = [z, x]$,
- automorphism $\Leftrightarrow \quad \psi([x, y]) = [\psi(x), \psi(y)]$.

Any derivation which is not inner is outer.

The centralizer $\mathfrak{g}_{\mathfrak{h}}$ of a given subalgebra $\mathfrak{h} \subset \mathfrak{g}$ in $\mathfrak{g}$ is the set of all elements in $\mathfrak{g}$ commuting with all elements in $\mathfrak{h}$, i.e.

$$\mathfrak{g}_{\mathfrak{h}} = \{x \in \mathfrak{g} | [x, y] = 0, \ \forall y \in \mathfrak{h}\}. \tag{3}$$

Ideals in the characteristic series as well as their centralizers (and their various intersections) are invariant with respect to any derivation and automorphism, i.e. they belong among characteristic ideals.

A linear map $\psi : \mathfrak{g} \to \mathfrak{g}$ is

- derivation $\Leftrightarrow$ $\psi([x,y]) = [\psi(x),y] + [x,\psi(y)]$,
- inner derivation $\Leftrightarrow \exists\, z \in \mathfrak{g}$: $\psi = \mathrm{ad}_z$, i.e. $\psi(x) = [z,x]$,
- automorphism $\Leftrightarrow$ $\psi([x,y]) = [\psi(x),\psi(y)]$.

Any derivation which is not inner is outer.

The centralizer $\mathfrak{g}_{\mathfrak{h}}$ of a given subalgebra $\mathfrak{h} \subset \mathfrak{g}$ in $\mathfrak{g}$ is the set of all elements in $\mathfrak{g}$ commuting with all elements in $\mathfrak{h}$, i.e.

$$\mathfrak{g}_{\mathfrak{h}} = \{x \in \mathfrak{g} | [x,y] = 0,\ \forall y \in \mathfrak{h}\}. \tag{3}$$

Ideals in the characteristic series as well as their centralizers (and their various intersections) are invariant with respect to any derivation and automorphism, i.e. they belong among characteristic ideals.

A linear map $\psi : \mathfrak{g} \to \mathfrak{g}$ is

- derivation $\Leftrightarrow \quad \psi([x, y]) = [\psi(x), y] + [x, \psi(y)]$,
- inner derivation $\Leftrightarrow \exists\, z \in \mathfrak{g}$: $\psi = \mathrm{ad}_z$, i.e. $\psi(x) = [z, x]$,
- automorphism $\Leftrightarrow \quad \psi([x, y]) = [\psi(x), \psi(y)]$.

Any derivation which is not inner is outer.

The centralizer $\mathfrak{g}_{\mathfrak{h}}$ of a given subalgebra $\mathfrak{h} \subset \mathfrak{g}$ in $\mathfrak{g}$ is the set of all elements in $\mathfrak{g}$ commuting with all elements in $\mathfrak{h}$, i.e.

$$\mathfrak{g}_{\mathfrak{h}} = \{x \in \mathfrak{g} | [x, y] = 0,\ \forall y \in \mathfrak{h}\}. \tag{3}$$

Ideals in the characteristic series as well as their centralizers (and their various intersections) are invariant with respect to any derivation and automorphism, i.e. they belong among characteristic ideals.

Any Lie algebra $\mathfrak{g}$ has a uniquely defined

- its maximal solvable ideal – radical $\mathfrak{r}$,
- its maximal nilpotent ideal – nilradical $\mathfrak{n}$.

Levi decomposition

Let $\mathfrak{g}$ denote an (indecomposable) Lie algebra. A fundamental theorem due to E. E. Levi[1] tells us that any Lie algebra can be represented as the semidirect sum

$$\mathfrak{g} = \mathfrak{p} \niplus \mathfrak{r}, \ [\mathfrak{p}, \mathfrak{p}] = \mathfrak{p}, \ [\mathfrak{r}, \mathfrak{r}] \subsetneq \mathfrak{r}, [\mathfrak{p}, \mathfrak{r}] \subset \mathfrak{r}, \tag{4}$$

where $\mathfrak{p}$ is a semisimple subalgebra, called Levi factor, and $\mathfrak{r}$ is the radical of $\mathfrak{g}$.

The Levi factor is unique up to isomorphism of $\mathfrak{g}$.

When $[\mathfrak{p}, \mathfrak{r}] = \mathfrak{r}$ the algebra $\mathfrak{g}$ is perfect, i.e. $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

[1] Levi E E 1905 Sulla struttura dei gruppi finiti e continui, *Atti Accad. Sci. Torino* **40** 551–65

Levi decomposition

Let $\mathfrak{g}$ denote an (indecomposable) Lie algebra. A fundamental theorem due to E. E. Levi[1] tells us that any Lie algebra can be represented as the semidirect sum

$$\mathfrak{g} = \mathfrak{p} \ni \mathfrak{r}, \ [\mathfrak{p}, \mathfrak{p}] = \mathfrak{p}, \ [\mathfrak{r}, \mathfrak{r}] \subsetneq \mathfrak{r}, [\mathfrak{p}, \mathfrak{r}] \subset \mathfrak{r}, \tag{4}$$

where $\mathfrak{p}$ is a semisimple subalgebra, called Levi factor, and $\mathfrak{r}$ is the radical of $\mathfrak{g}$.

The Levi factor is unique up to isomorphism of $\mathfrak{g}$.

When $[\mathfrak{p}, \mathfrak{r}] = \mathfrak{r}$ the algebra $\mathfrak{g}$ is perfect, i.e. $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

[1] Levi E E 1905 Sulla struttura dei gruppi finiti e continui, *Atti Accad. Sci. Torino* **40** 551–65

Levi decomposable algebras in applications

Levi decomposable with nontrivial Levi decomposition, in particular perfect ones, occur in many physical applications.

- The Poincaré algebra itself is of that kind, with Abelian radical generated by translations and Lorentz algebra forming the semisimple part $\mathfrak{so}(3,1)$.
- Levi decomposable algebras often occur as symmetry algebras of physically interesting PDEs.
- Such algebras also occur in the construction of higher dimensional cosmological models as the algebras of Killing vectors. In fact, that was the original motivation for P. Turkowski who did a lot of initial research in the structure of Levi decomposable algebras.

Levi decomposable algebras in applications

Levi decomposable with nontrivial Levi decomposition, in particular perfect ones, occur in many physical applications.

- The Poincaré algebra itself is of that kind, with Abelian radical generated by translations and Lorentz algebra forming the semisimple part $\mathfrak{so}(3,1)$.
- Levi decomposable algebras often occur as symmetry algebras of physically interesting PDEs.
- Such algebras also occur in the construction of higher dimensional cosmological models as the algebras of Killing vectors. In fact, that was the original motivation for P. Turkowski who did a lot of initial research in the structure of Levi decomposable algebras.

Levi decomposable algebras in applications

Levi decomposable with nontrivial Levi decomposition, in particular perfect ones, occur in many physical applications.

- The Poincaré algebra itself is of that kind, with Abelian radical generated by translations and Lorentz algebra forming the semisimple part $\mathfrak{so}(3,1)$.
- Levi decomposable algebras often occur as symmetry algebras of physically interesting PDEs.
- Such algebras also occur in the construction of higher dimensional cosmological models as the algebras of Killing vectors. In fact, that was the original motivation for P. Turkowski who did a lot of initial research in the structure of Levi decomposable algebras.

Levi decomposable algebras in applications

Levi decomposable with nontrivial Levi decomposition, in particular perfect ones, occur in many physical applications.

- The Poincaré algebra itself is of that kind, with Abelian radical generated by translations and Lorentz algebra forming the semisimple part $\mathfrak{so}(3,1)$.
- Levi decomposable algebras often occur as symmetry algebras of physically interesting PDEs.
- Such algebras also occur in the construction of higher dimensional cosmological models as the algebras of Killing vectors. In fact, that was the original motivation for P. Turkowski who did a lot of initial research in the structure of Levi decomposable algebras.

Levi decomposable algebras as symmetries of PDEs

Example: Symmetries of the heat equation

$$u_t - u_{xx} = 0.$$

The (point) symmetries, i.e. transformations of solutions into solutions arising as transformations of graphs of $f : (x, t) \to u$, are generated by the vector fields

$$4xt\partial_x + 4t^2\partial_t - (2t + x^2)u\partial_u, 2x\partial_x + 4t\partial_t + u\partial_u, \partial_t,$$

$$-2t\partial_x + xu\partial_u, u\partial_u, \partial_x.$$

The first three form the simple algebra $\mathfrak{sl}(2)$, the rest forms the radical. A simple investigation of its structure shows that it is isomorphic to the Heisenberg algebra in 1 dimension.

Levi decomposable algebras as symmetries of PDEs

Example: Symmetries of the heat equation

$$u_t - u_{xx} = 0.$$

The (point) symmetries, i.e. transformations of solutions into solutions arising as transformations of graphs of $f : (x, t) \longrightarrow u$, are generated by the vector fields

$$4xt\partial_x + 4t^2\partial_t - (2t + x^2)u\partial_u, 2x\partial_x + 4t\partial_t + u\partial_u, \partial_t,$$

$$-2t\partial_x + xu\partial_u, u\partial_u, \partial_x.$$

The first three form the simple algebra $\mathfrak{sl}(2)$, the rest forms the radical. A simple investigation of its structure shows that it is isomorphic to the Heisenberg algebra in 1 dimension.

Levi decomposable algebras as symmetries of PDEs

Example: Symmetries of the heat equation

$$u_t - u_{xx} = 0.$$

The (point) symmetries, i.e. transformations of solutions into solutions arising as transformations of graphs of $f : (x, t) \to u$, are generated by the vector fields

$$4xt\partial_x + 4t^2\partial_t - (2t + x^2)u\partial_u, 2x\partial_x + 4t\partial_t + u\partial_u, \partial_t,$$

$$-2t\partial_x + xu\partial_u, u\partial_u, \partial_x.$$

The first three form the simple algebra $\mathfrak{sl}(2)$, the rest forms the radical. A simple investigation of its structure shows that it is isomorphic to the Heisenberg algebra in 1 dimension.

Notice that we have discarded an infinite dimensional Abelian subalgebra of point symmetries which originate from the linearity of the heat equation. Its structure is described by commuting vector fields

$$F(x,t)\partial_u$$

where $F(x,t)$ is an arbitrary solution of $F_t - F_{xx} = 0$.

Similarly, the Burgers equation $u_t + uu_x + u_{xx} = 0$ has the symmetry algebra $\mathfrak{sl}(2) \ni \mathfrak{a}_2$.

Notice that we have discarded an infinite dimensional Abelian subalgebra of point symmetries which originate from the linearity of the heat equation. Its structure is described by commuting vector fields

$$F(x, t)\partial_u$$

where $F(x, t)$ is an arbitrary solution of $F_t - F_{xx} = 0$.

Similarly, the Burgers equation $u_t + uu_x + u_{xx} = 0$ has the symmetry algebra $\mathfrak{sl}(2) \niplus \mathfrak{a}_2$.

Turkowski's classifications of Levi decomposable algebras

We notice that by virtue of Jacobi identities $\mathfrak{r}$ is a representation space for $\mathfrak{p}$ and that $\mathfrak{p}$ is isomorphic to some semisimple subalgebra of the algebra of all derivations of $\mathfrak{r}$.

The Levi decomposable algebras up to dimension 8 were classified in Turkowski P 1988 Low–dimensional real Lie algebras, *J. Math. Phys.* **29** 2139-44 and in dimension 9 in Turkowski P 1992 Structure of real Lie algebras *Linear Alg. Appl.* **171** 197-212.

Turkowski's classifications of Levi decomposable algebras

We notice that by virtue of Jacobi identities $\mathfrak{r}$ is a representation space for $\mathfrak{p}$ and that $\mathfrak{p}$ is isomorphic to some semisimple subalgebra of the algebra of all derivations of $\mathfrak{r}$.

The Levi decomposable algebras up to dimension 8 were classified in Turkowski P 1988 Low–dimensional real Lie algebras, *J. Math. Phys.* **29** 2139-44 and in dimension 9 in Turkowski P 1992 Structure of real Lie algebras *Linear Alg. Appl.* **171** 197-212.

Method of Turkowski

The approach used by Turkowski was to consider a given semisimple algebra $\mathfrak{p}$ and all its possible representations ρ on a vector space V of chosen dimension. For each ρ, all solvable algebras $\mathfrak{r}$ compatible with the representation ρ were found by an explicit evaluation of Jacobi identities with unknown structure constants ${c_{ij}}^k$ of the radical $\mathfrak{r}$, and classified into equivalence classes.

Turkowski stated and used also some general properties of Levi decomposable algebras. These properties are a direct consequence of the complete reducibility of representations of semisimple Lie algebras.

Method of Turkowski

The approach used by Turkowski was to consider a given semisimple algebra $\mathfrak{p}$ and all its possible representations ρ on a vector space V of chosen dimension. For each ρ, all solvable algebras $\mathfrak{r}$ compatible with the representation ρ were found by an explicit evaluation of Jacobi identities with unknown structure constants ${c_{ij}}^k$ of the radical $\mathfrak{r}$, and classified into equivalence classes.

Turkowski stated and used also some general properties of Levi decomposable algebras. These properties are a direct consequence of the complete reducibility of representations of semisimple Lie algebras.

Turkowski's general theorems

Namely,

- if the representation $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{r}}$ of $\mathfrak{p}$ is irreducible then $\mathfrak{r}$ is Abelian.
- If $\mathfrak{r}$ is solvable non–nilpotent, then there exists a complement $\mathfrak{q}$ of $\mathfrak{n}$ in $\mathfrak{r}$, i.e.

$$\mathfrak{r} = \mathfrak{n} \dot{+} \mathfrak{q}$$

  such that $\mathrm{ad}(p)|_{\mathfrak{q}} = 0$ for all $p \in \mathfrak{p}$, i.e. $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{r}}$ must necessarily contain a copy of the trivial representation.

  In view of this property, it is of interest to study and classify Levi decomposable algebras with nilpotent radicals first.

Turkowski's general theorems

Namely,

- if the representation $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{r}}$ of $\mathfrak{p}$ is irreducible then $\mathfrak{r}$ is Abelian.
- If $\mathfrak{r}$ is solvable non–nilpotent, then there exists a complement $\mathfrak{q}$ of $\mathfrak{n}$ in $\mathfrak{r}$, i.e.

$$\mathfrak{r} = \mathfrak{n} \dotplus \mathfrak{q}$$

such that $\mathrm{ad}(p)|_{\mathfrak{q}} = 0$ for all $p \in \mathfrak{p}$, i.e. $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{r}}$ must necessarily contain a copy of the trivial representation.

In view of this property, it is of interest to study and classify Levi decomposable algebras with nilpotent radicals first.

Turkowski's general theorems

Namely,

- if the representation $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{r}}$ of $\mathfrak{p}$ is irreducible then $\mathfrak{r}$ is Abelian.
- If $\mathfrak{r}$ is solvable non–nilpotent, then there exists a complement $\mathfrak{q}$ of $\mathfrak{n}$ in $\mathfrak{r}$, i.e.

$$\mathfrak{r} = \mathfrak{n} \dot{+} \mathfrak{q}$$

such that $\mathrm{ad}(p)|_{\mathfrak{q}} = 0$ for all $p \in \mathfrak{p}$, i.e. $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{r}}$ must necessarily contain a copy of the trivial representation.

In view of this property, it is of interest to study and classify Levi decomposable algebras with nilpotent radicals first.

Turkowski's general theorems II

- The set of all elements belonging to the trivial representation

$$\{x \in \mathfrak{n} | \mathrm{ad}(p)x = 0, \forall p \in \mathfrak{p}\}$$

is a subalgebra of $\mathfrak{n}$.

Our main goal

By inspection, one immediately realizes that only few out of 5– and 6–dimensional nilpotent algebras (see Patera J, Sharp R T, Winternitz P and Zassenhaus H 1976 Invariants of real low dimension Lie algebras, *J. Math. Phys.* **17** 986–94) appear as nilradicals of Levi decomposable algebras in Turkowski's tables.

We would like to understand this feature of Levi decomposable algebras and try to find some criteria allowing us to easily predict whether a given nilpotent algebra may have a Levi extension. Turkowski's classification will be used as a useful test case for our ideas.

Our main goal

By inspection, one immediately realizes that only few out of 5– and 6–dimensional nilpotent algebras (see Patera J, Sharp R T, Winternitz P and Zassenhaus H 1976 Invariants of real low dimension Lie algebras, *J. Math. Phys.* **17** 986–94) appear as nilradicals of Levi decomposable algebras in Turkowski's tables.

We would like to understand this feature of Levi decomposable algebras and try to find some criteria allowing us to easily predict whether a given nilpotent algebra may have a Levi extension. Turkowski's classification will be used as a useful test case for our ideas.

Our main goal

By inspection, one immediately realizes that only few out of 5– and 6–dimensional nilpotent algebras (see Patera J, Sharp R T, Winternitz P and Zassenhaus H 1976 Invariants of real low dimension Lie algebras, *J. Math. Phys.* **17** 986–94) appear as nilradicals of Levi decomposable algebras in Turkowski's tables.

We would like to understand this feature of Levi decomposable algebras and try to find some criteria allowing us to easily predict whether a given nilpotent algebra may have a Levi extension. Turkowski's classification will be used as a useful test case for our ideas.

Simple no–go criterion

Because all ideals in the characteristic series and their centralizers are invariant with respect to any derivation, in particular with respect to $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{r}}$, we can use Lie's theorem to easily deduce the following proposition.

Theorem

If a complete flag

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq \ldots \subsetneq V_n = \mathfrak{n}$$

of codimension 1 subspaces can be built out of ideals in the characteristic series and their centralizers, then no Levi decomposable algebra

$$\mathfrak{g} = \mathfrak{p} \ni \mathfrak{n}$$

such that $[\mathfrak{p}, \mathfrak{n}] \neq 0$ *exists.*

Simple no–go criterion

Because all ideals in the characteristic series and their centralizers are invariant with respect to any derivation, in particular with respect to $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{r}}$, we can use Lie's theorem to easily deduce the following proposition.

Theorem

If a complete flag

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq \ldots \subsetneq V_n = \mathfrak{n}$$

of codimension 1 subspaces can be built out of ideals in the characteristic series and their centralizers, then no Levi decomposable algebra

$$\mathfrak{g} = \mathfrak{p} \ni \mathfrak{n}$$

such that $[\mathfrak{p}, \mathfrak{n}] \neq 0$ *exists.*

Using this criterion one can immediately establish, without further considerations of the structure of the representations of $\mathfrak{p}$, that out of low dimensional nilpotent algebras (dimension at most 5), the following can never appear as a nilradical of a Levi decomposable algebra.

$\dim \mathfrak{n} = 4$:

$A_{4,1}$: $[e_2, e_4] = e_1$, $[e_3, e_4] = e_2$; the characteristic flag is

$$0 \subset \mathfrak{n}^3 \subset \mathfrak{n}^2 \subset \mathfrak{cent}(\mathfrak{n}^2) \subset \mathfrak{n}.$$

Using this criterion one can immediately establish, without further considerations of the structure of the representations of $\mathfrak{p}$, that out of low dimensional nilpotent algebras (dimension at most 5), the following can never appear as a nilradical of a Levi decomposable algebra.

$\dim \mathfrak{n} = 4$:

$A_{4,1}$: $[e_2, e_4] = e_1$, $[e_3, e_4] = e_2$; the characteristic flag is

$$0 \subset \mathfrak{n}^3 \subset \mathfrak{n}^2 \subset \mathfrak{cent}(\mathfrak{n}^2) \subset \mathfrak{n}.$$

$\dim \mathfrak{n} = 5$:

$A_{5,2}$: $[e_2, e_5] = e_1$, $[e_3, e_5] = e_2$, $[e_4, e_5] = e_3$;
the characteristic flag is

$$0 \subset \mathfrak{n}^4 \subset \mathfrak{n}^3 \subset \mathfrak{n}^2 \subset \mathfrak{cent}(\mathfrak{n}^3) \subset \mathfrak{n}.$$

$A_{5,5}$: $[e_3, e_4] = e_1$, $[e_2, e_5] = e_1$, $[e_3, e_5] = e_2$;
the characteristic flag is

$$0 \subset \mathfrak{n}^3 \subset \mathfrak{n}^2 \subset \mathfrak{z}_2 \subset \mathfrak{cent}(\mathfrak{n}^2) \subset \mathfrak{n}.$$

$A_{5,6}$: $[e_2, e_5] = e_1$, $[e_3, e_4] = e_1$, $[e_3, e_5] = e_2$, $[e_4, e_5] = e_3$; with the same flag as $A_{5,2}$.

$\dim \mathfrak{n} = 5$:

$A_{5,2}$: $[e_2, e_5] = e_1$, $[e_3, e_5] = e_2$, $[e_4, e_5] = e_3$;
the characteristic flag is

$$0 \subset \mathfrak{n}^4 \subset \mathfrak{n}^3 \subset \mathfrak{n}^2 \subset \mathfrak{cent}(\mathfrak{n}^3) \subset \mathfrak{n}.$$

$A_{5,5}$: $[e_3, e_4] = e_1$, $[e_2, e_5] = e_1$, $[e_3, e_5] = e_2$;
the characteristic flag is

$$0 \subset \mathfrak{n}^3 \subset \mathfrak{n}^2 \subset \mathfrak{z}_2 \subset \mathfrak{cent}(\mathfrak{n}^2) \subset \mathfrak{n}.$$

$A_{5,6}$: $[e_2, e_5] = e_1$, $[e_3, e_4] = e_1$, $[e_3, e_5] = e_2$,
$[e_4, e_5] = e_3$; with the same flag as $A_{5,2}$.

$\dim \mathfrak{n} = 5$:

$A_{5,2}$: $[e_2, e_5] = e_1$, $[e_3, e_5] = e_2$, $[e_4, e_5] = e_3$;
the characteristic flag is

$$0 \subset \mathfrak{n}^4 \subset \mathfrak{n}^3 \subset \mathfrak{n}^2 \subset \mathfrak{cent}(\mathfrak{n}^3) \subset \mathfrak{n}.$$

$A_{5,5}$: $[e_3, e_4] = e_1$, $[e_2, e_5] = e_1$, $[e_3, e_5] = e_2$;
the characteristic flag is

$$0 \subset \mathfrak{n}^3 \subset \mathfrak{n}^2 \subset \mathfrak{z}_2 \subset \mathfrak{cent}(\mathfrak{n}^2) \subset \mathfrak{n}.$$

$A_{5,6}$: $[e_2, e_5] = e_1$, $[e_3, e_4] = e_1$, $[e_3, e_5] = e_2$,
$[e_4, e_5] = e_3$; with the same flag as $A_{5,2}$.

We find it rather intriguing that all other indecomposable nilpotent algebras $A_{3,1}$, $A_{5,1}, A_{5,3}, A_{5,4}$ of dimension $2 \leq n \leq 5$ do show up as nilradicals in Turkowski's list of Levi decomposable algebras of dimension ≤ 8, i.e. they do admit Levi extension(s).

In the case of six–dimensional nilpotent algebras, the same argument allows to immediately exclude from the list of Levi extendable nilradicals the algebras

$$A_{6,1}, A_{6,2}, A_{6,6}, A_{6,7}, A_{6,11}, A_{6,16}, A_{6,17}, A_{6,19}, A_{6,20}, A_{6,21}, A_{6,22}.$$

In this case, however, not all of the remaining algebras allow a Levi extension. According to Turkowski, only four algebras $A_{6,3}, A_{6,4}, A_{6,5}, A_{6,12}$ out of 22 indecomposable 6–dimensional nilpotent algebras allow a Levi extension. The structural reasons for that will be given below.

We find it rather intriguing that all other indecomposable nilpotent algebras $A_{3,1}$, $A_{5,1}$, $A_{5,3}$, $A_{5,4}$ of dimension $2 \leq n \leq 5$ do show up as nilradicals in Turkowski's list of Levi decomposable algebras of dimension ≤ 8, i.e. they do admit Levi extension(s).

In the case of six–dimensional nilpotent algebras, the same argument allows to immediately exclude from the list of Levi extendable nilradicals the algebras

$$A_{6,1}, A_{6,2}, A_{6,6}, A_{6,7}, A_{6,11}, A_{6,16}, A_{6,17}, A_{6,19}, A_{6,20}, A_{6,21}, A_{6,22}.$$

In this case, however, not all of the remaining algebras allow a Levi extension. According to Turkowski, only four algebras $A_{6,3}$, $A_{6,4}$, $A_{6,5}$, $A_{6,12}$ out of 22 indecomposable 6–dimensional nilpotent algebras allow a Levi extension. The structural reasons for that will be given below.

We find it rather intriguing that all other indecomposable nilpotent algebras $A_{3,1}$, $A_{5,1}$, $A_{5,3}$, $A_{5,4}$ of dimension $2 \leq n \leq 5$ do show up as nilradicals in Turkowski's list of Levi decomposable algebras of dimension ≤ 8, i.e. they do admit Levi extension(s).

In the case of six–dimensional nilpotent algebras, the same argument allows to immediately exclude from the list of Levi extendable nilradicals the algebras

$$A_{6,1}, A_{6,2}, A_{6,6}, A_{6,7}, A_{6,11}, A_{6,16}, A_{6,17}, A_{6,19}, A_{6,20}, A_{6,21}, A_{6,22}.$$

In this case, however, not all of the remaining algebras allow a Levi extension. According to Turkowski, only four algebras $A_{6,3}$, $A_{6,4}$, $A_{6,5}$, $A_{6,12}$ out of 22 indecomposable 6–dimensional nilpotent algebras allow a Levi extension. The structural reasons for that will be given below.

Decomposition of $\mathfrak{n}$ into $\mathrm{ad}(\mathfrak{p})$–invariant subspaces

The complete reducibility of representations of semisimple Lie algebras allows us to deduce the existence of complementary $\mathrm{ad}(\mathfrak{p})$–invariant subspaces $\tilde{\mathfrak{m}}_j$ of $\mathfrak{n}^{j+1}$ in $\mathfrak{n}^j$

$$\mathfrak{n}^j = \tilde{\mathfrak{m}}_j \dotplus \mathfrak{n}^{j+1}, \; \mathrm{ad}(\mathfrak{p})\tilde{\mathfrak{m}}_j \subset \tilde{\mathfrak{m}}_j, \; j = 1, \ldots, K. \tag{5}$$

We can refine this statement in the following way. Let us take

$$\mathfrak{m}_1 = \tilde{\mathfrak{m}}_1.$$

Now the commutator of two $\mathrm{ad}(\mathfrak{p})$–invariant subspaces is again an $\mathrm{ad}(\mathfrak{p})$–invariant subspace. In particular, $[\mathfrak{m}_1, \mathfrak{m}_1]$ is an $\mathrm{ad}(\mathfrak{p})$–invariant subspace of $\mathfrak{n}^2$ and we have[2]

$$\mathfrak{n}^2 = [\mathfrak{m}_1, \mathfrak{m}_1] + \mathfrak{n}^3.$$

[2] in general not a direct sum

Decomposition of $\mathfrak{n}$ into $\mathrm{ad}(\mathfrak{p})$–invariant subspaces

The complete reducibility of representations of semisimple Lie algebras allows us to deduce the existence of complementary $\mathrm{ad}(\mathfrak{p})$–invariant subspaces $\tilde{\mathfrak{m}}_j$ of $\mathfrak{n}^{j+1}$ in $\mathfrak{n}^j$

$$\mathfrak{n}^j = \tilde{\mathfrak{m}}_j \dot{+} \mathfrak{n}^{j+1}, \ \mathrm{ad}(\mathfrak{p})\tilde{\mathfrak{m}}_j \subset \tilde{\mathfrak{m}}_j, \ j = 1, \dots, K. \qquad (5)$$

We can refine this statement in the following way. Let us take

$$\mathfrak{m}_1 = \tilde{\mathfrak{m}}_1.$$

Now the commutator of two $\mathrm{ad}(\mathfrak{p})$–invariant subspaces is again an $\mathrm{ad}(\mathfrak{p})$–invariant subspace. In particular, $[\mathfrak{m}_1, \mathfrak{m}_1]$ is an $\mathrm{ad}(\mathfrak{p})$–invariant subspace of $\mathfrak{n}^2$ and we have[2]

$$\mathfrak{n}^2 = [\mathfrak{m}_1, \mathfrak{m}_1] + \mathfrak{n}^3.$$

[2] in general not a direct sum

Decomposition of $\mathfrak{n}$ into $\mathrm{ad}(\mathfrak{p})$–invariant subspaces

The complete reducibility of representations of semisimple Lie algebras allows us to deduce the existence of complementary $\mathrm{ad}(\mathfrak{p})$–invariant subspaces $\tilde{\mathfrak{m}}_j$ of $\mathfrak{n}^{j+1}$ in $\mathfrak{n}^j$

$$\mathfrak{n}^j = \tilde{\mathfrak{m}}_j \dot{+} \mathfrak{n}^{j+1}, \ \mathrm{ad}(\mathfrak{p})\tilde{\mathfrak{m}}_j \subset \tilde{\mathfrak{m}}_j, \ j = 1, \ldots, K. \qquad (5)$$

We can refine this statement in the following way. Let us take

$$\mathfrak{m}_1 = \tilde{\mathfrak{m}}_1.$$

Now the commutator of two $\mathrm{ad}(\mathfrak{p})$–invariant subspaces is again an $\mathrm{ad}(\mathfrak{p})$–invariant subspace. In particular, $[\mathfrak{m}_1, \mathfrak{m}_1]$ is an $\mathrm{ad}(\mathfrak{p})$–invariant subspace of $\mathfrak{n}^2$ and we have[2]

$$\mathfrak{n}^2 = [\mathfrak{m}_1, \mathfrak{m}_1] + \mathfrak{n}^3.$$

[2] in general not a direct sum

Since both $[\mathfrak{m}_1, \mathfrak{m}_1]$ and $\mathfrak{n}^3$ are $\mathrm{ad}(\mathfrak{p})$–invariant, so is their intersection $[\mathfrak{m}_1, \mathfrak{m}_1] \cap \mathfrak{n}^3$. By the complete reducibility of $\mathrm{ad}(\mathfrak{p})$, there is an $\mathrm{ad}(\mathfrak{p})$–invariant complement of $[\mathfrak{m}_1, \mathfrak{m}_1] \cap \mathfrak{n}^3$ in $[\mathfrak{m}_1, \mathfrak{m}_1]$ which we denote $\mathfrak{m}_2$. Altogether, we have

$$\mathfrak{n}^2 = \mathfrak{m}_2 \dot{+} \mathfrak{n}^3, \ \mathfrak{m}_2 \subset [\mathfrak{m}_1, \mathfrak{m}_1], \ \mathrm{ad}(\mathfrak{p})\mathfrak{m}_2 \subset \mathfrak{m}_2.$$

Continuing in the same way, we can construct a sequence of subspaces $\mathfrak{m}_j$ such that

$$\mathfrak{n} = \mathfrak{m}_K \dot{+} \mathfrak{m}_{K-1} \dot{+} \ldots \dot{+} \mathfrak{m}_1 \tag{6}$$

where

$$\mathfrak{n}^j = \mathfrak{m}_j \dot{+} \mathfrak{n}^{j+1}, \ \mathfrak{m}_j \subset [\mathfrak{m}_{j-1}, \mathfrak{m}_1], \ \mathrm{ad}(\mathfrak{p})\mathfrak{m}_j \subset \mathfrak{m}_j. \tag{7}$$

Since both $[\mathfrak{m}_1, \mathfrak{m}_1]$ and $\mathfrak{n}^3$ are $\mathrm{ad}(\mathfrak{p})$–invariant, so is their intersection $[\mathfrak{m}_1, \mathfrak{m}_1] \cap \mathfrak{n}^3$. By the complete reducibility of $\mathrm{ad}(\mathfrak{p})$, there is an $\mathrm{ad}(\mathfrak{p})$–invariant complement of $[\mathfrak{m}_1, \mathfrak{m}_1] \cap \mathfrak{n}^3$ in $[\mathfrak{m}_1, \mathfrak{m}_1]$ which we denote $\mathfrak{m}_2$. Altogether, we have

$$\mathfrak{n}^2 = \mathfrak{m}_2 \dotplus \mathfrak{n}^3, \; \mathfrak{m}_2 \subset [\mathfrak{m}_1, \mathfrak{m}_1], \; \mathrm{ad}(\mathfrak{p})\mathfrak{m}_2 \subset \mathfrak{m}_2.$$

Continuing in the same way, we can construct a sequence of subspaces $\mathfrak{m}_j$ such that

$$\mathfrak{n} = \mathfrak{m}_K \dotplus \mathfrak{m}_{K-1} \dotplus \ldots \dotplus \mathfrak{m}_1 \tag{6}$$

where

$$\mathfrak{n}^j = \mathfrak{m}_j \dotplus \mathfrak{n}^{j+1}, \; \mathfrak{m}_j \subset [\mathfrak{m}_{j-1}, \mathfrak{m}_1], \; \mathrm{ad}(\mathfrak{p})\mathfrak{m}_j \subset \mathfrak{m}_j. \tag{7}$$

Since both $[\mathfrak{m}_1, \mathfrak{m}_1]$ and $\mathfrak{n}^3$ are $\mathrm{ad}(\mathfrak{p})$–invariant, so is their intersection $[\mathfrak{m}_1, \mathfrak{m}_1] \cap \mathfrak{n}^3$. By the complete reducibility of $\mathrm{ad}(\mathfrak{p})$, there is an $\mathrm{ad}(\mathfrak{p})$–invariant complement of $[\mathfrak{m}_1, \mathfrak{m}_1] \cap \mathfrak{n}^3$ in $[\mathfrak{m}_1, \mathfrak{m}_1]$ which we denote $\mathfrak{m}_2$. Altogether, we have

$$\mathfrak{n}^2 = \mathfrak{m}_2 \dot{+} \mathfrak{n}^3, \ \mathfrak{m}_2 \subset [\mathfrak{m}_1, \mathfrak{m}_1], \ \mathrm{ad}(\mathfrak{p})\mathfrak{m}_2 \subset \mathfrak{m}_2.$$

Continuing in the same way, we can construct a sequence of subspaces $\mathfrak{m}_j$ such that

$$\mathfrak{n} = \mathfrak{m}_K \dot{+} \mathfrak{m}_{K-1} \dot{+} \ldots \dot{+} \mathfrak{m}_1 \quad (6)$$

where

$$\mathfrak{n}^j = \mathfrak{m}_j \dot{+} \mathfrak{n}^{j+1}, \ \mathfrak{m}_j \subset [\mathfrak{m}_{j-1}, \mathfrak{m}_1], \ \mathrm{ad}(\mathfrak{p})\mathfrak{m}_j \subset \mathfrak{m}_j. \quad (7)$$

Since both $[\mathfrak{m}_1, \mathfrak{m}_1]$ and $\mathfrak{n}^3$ are $\mathrm{ad}(\mathfrak{p})$–invariant, so is their intersection $[\mathfrak{m}_1, \mathfrak{m}_1] \cap \mathfrak{n}^3$. By the complete reducibility of $\mathrm{ad}(\mathfrak{p})$, there is an $\mathrm{ad}(\mathfrak{p})$–invariant complement of $[\mathfrak{m}_1, \mathfrak{m}_1] \cap \mathfrak{n}^3$ in $[\mathfrak{m}_1, \mathfrak{m}_1]$ which we denote $\mathfrak{m}_2$. Altogether, we have

$$\mathfrak{n}^2 = \mathfrak{m}_2 \dot{+} \mathfrak{n}^3, \ \mathfrak{m}_2 \subset [\mathfrak{m}_1, \mathfrak{m}_1], \ \mathrm{ad}(\mathfrak{p})\mathfrak{m}_2 \subset \mathfrak{m}_2.$$

Continuing in the same way, we can construct a sequence of subspaces $\mathfrak{m}_j$ such that

$$\mathfrak{n} = \mathfrak{m}_K \dot{+} \mathfrak{m}_{K-1} \dot{+} \ldots \dot{+} \mathfrak{m}_1 \tag{6}$$

where

$$\mathfrak{n}^j = \mathfrak{m}_j \dot{+} \mathfrak{n}^{j+1}, \ \mathfrak{m}_j \subset [\mathfrak{m}_{j-1}, \mathfrak{m}_1], \ \mathrm{ad}(\mathfrak{p})\mathfrak{m}_j \subset \mathfrak{m}_j. \tag{7}$$

In any basis of the nilradical $\mathfrak{n}$ which respects the decomposition (6) the matrices of $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{n}}$ have block diagonal form. If any of the blocks is 1–dimensional then it necessarily corresponds to the trivial representation $\rho(p) = 0, \forall p \in \mathfrak{p}$.

The $\mathfrak{m}_j\mathfrak{m}_j$–submatrices of the representation $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{n}}, j > 1$, i.e. the matrices of $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_j}$, are fully determined by $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$ through commutators.

For the same reason, the kernel of $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$ is also the kernel of $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{n}}$. Therefore, the representation $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{n}}$ of the Levi factor $\mathfrak{p}$ is faithful if and only if $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$ is faithful.

In any basis of the nilradical $\mathfrak{n}$ which respects the decomposition (6) the matrices of $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{n}}$ have block diagonal form. If any of the blocks is 1–dimensional then it necessarily corresponds to the trivial representation $\rho(p) = 0, \forall p \in \mathfrak{p}$.

The $\mathfrak{m}_j\mathfrak{m}_j$–submatrices of the representation $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{n}}$, $j > 1$, i.e. the matrices of $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_j}$, are fully determined by $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$ through commutators.

For the same reason, the kernel of $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$ is also the kernel of $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{n}}$. Therefore, the representation $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{n}}$ of the Levi factor $\mathfrak{p}$ is faithful if and only if $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$ is faithful.

In any basis of the nilradical $\mathfrak{n}$ which respects the decomposition (6) the matrices of $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{n}}$ have block diagonal form. If any of the blocks is 1–dimensional then it necessarily corresponds to the trivial representation $\rho(p) = 0, \forall p \in \mathfrak{p}$.

The $\mathfrak{m}_j\mathfrak{m}_j$–submatrices of the representation $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{n}}$, $j > 1$, i.e. the matrices of $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_j}$, are fully determined by $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$ through commutators.

For the same reason, the kernel of $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$ is also the kernel of $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{n}}$. Therefore, the representation $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{n}}$ of the Levi factor $\mathfrak{p}$ is faithful if and only if $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$ is faithful.

The representations on $\mathfrak{m}_1, \mathfrak{m}_j$ and $\mathfrak{m}_{j+1}$ are related in the following way. Because the commutators of $e_a \in \mathfrak{m}_j$, $e_b \in \mathfrak{m}_1$ transform under action of any block–diagonal derivation D by

$$D[e_a, e_b] = \sum_{c=n+1-\sum_{i=1}^{j} m_i}^{n-\sum_{i=1}^{j-1} m_i} D^c{}_a [e_c, e_b] + \sum_{d=n+1-m_1}^{n} D^d{}_b [e_a, e_d] \tag{8}$$

where $D^c{}_a$ are components of the matrix of $D|_{\mathfrak{m}_j} : \mathfrak{m}_j \to \mathfrak{m}_j$ and $D^d{}_b$ are components of $D|_{\mathfrak{m}_1} : \mathfrak{m}_1 \to \mathfrak{m}_1$, the commutator subspace $[\mathfrak{m}_j, \mathfrak{m}_1]$ transforms under a certain subset of irreducible factors in the tensor representation $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_j} \otimes \mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$ and the same is true also for $\mathfrak{m}_{j+1} \subset [\mathfrak{m}_j, \mathfrak{m}_1]$.

The representations on $\mathfrak{m}_1, \mathfrak{m}_j$ and $\mathfrak{m}_{j+1}$ are related in the following way. Because the commutators of $e_a \in \mathfrak{m}_j$, $e_b \in \mathfrak{m}_1$ transform under action of any block–diagonal derivation D by

$$D[e_a, e_b] = \sum_{c=n+1-\sum_{i=1}^{j} m_i}^{n-\sum_{i=1}^{j-1} m_i} D^c{}_a [e_c, e_b] + \sum_{d=n+1-m_1}^{n} D^d{}_b [e_a, e_d] \tag{8}$$

where $D^c{}_a$ are components of the matrix of $D|_{\mathfrak{m}_j} : \mathfrak{m}_j \to \mathfrak{m}_j$ and $D^d{}_b$ are components of $D|_{\mathfrak{m}_1} : \mathfrak{m}_1 \to \mathfrak{m}_1$, the commutator subspace $[\mathfrak{m}_j, \mathfrak{m}_1]$ transforms under a certain subset of irreducible factors in the tensor representation $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_j} \otimes \mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$ and the same is true also for $\mathfrak{m}_{j+1} \subset [\mathfrak{m}_j, \mathfrak{m}_1]$.

Main result

Theorem

Let $\mathfrak{g}$ be an indecomposable Lie algebra with a nilpotent radical $\mathfrak{n}$ and a nontrivial Levi decomposition

$$\mathfrak{g} = \mathfrak{p} \niplus \mathfrak{n}.$$

There exists a decomposition of $\mathfrak{n}$ into a direct sum of $\mathrm{ad}(\mathfrak{p})$–invariant subspaces

$$\mathfrak{n} = \mathfrak{m}_K \dotplus \mathfrak{m}_{K-1} \dotplus \ldots \dotplus \mathfrak{m}_1$$

where

$$\mathfrak{n}^j = \mathfrak{m}_j \dotplus \mathfrak{n}^{j+1},\ \mathfrak{m}_j \subset [\mathfrak{m}_{j-1}, \mathfrak{m}_1],\ \mathrm{ad}(\mathfrak{p})\mathfrak{m}_j \subset \mathfrak{m}_j.$$

such that $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$ is a faithful representation of $\mathfrak{p}$ on $\mathfrak{m}_1$.

Main result II

Theorem

For $j = 2, \ldots, K$ the representation $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_j}$ of $\mathfrak{p}$ on the subspace $\mathfrak{m}_j$ can be decomposed into some subset of irreducible components of the tensor representation $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_{j-1}} \otimes \mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$.

If any of the subspaces $\mathfrak{m}_j$ is one–dimensional then $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{n}}$ must contain a copy of the trivial representation corresponding to the subspace $\mathfrak{m}_j$. When $j < K$, the representation of $\mathfrak{p}$ on $\mathfrak{m}_{j+1}$ can be decomposed into a sum of irreducible representations, each of which is equivalent to an irreducible representation contained in the decomposition of $\mathfrak{m}_1$.

In particular, when $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$ is irreducible and $\dim \mathfrak{m}_j = 1$, $1 < j < K$ then the representation $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_{j+1}}$ on $\mathfrak{m}_{j+1}$ is equivalent to $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$.

Main result II

Theorem

For $j = 2, \ldots, K$ the representation $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_j}$ of $\mathfrak{p}$ on the subspace $\mathfrak{m}_j$ can be decomposed into some subset of irreducible components of the tensor representation $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_{j-1}} \otimes \mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$.

If any of the subspaces $\mathfrak{m}_j$ is one–dimensional then $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{n}}$ must contain a copy of the trivial representation corresponding to the subspace $\mathfrak{m}_j$. When $j < K$, the representation of $\mathfrak{p}$ on $\mathfrak{m}_{j+1}$ can be decomposed into a sum of irreducible representations, each of which is equivalent to an irreducible representation contained in the decomposition of $\mathfrak{m}_1$.

In particular, when $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$ is irreducible and $\dim \mathfrak{m}_j = 1$, $1 < j < K$ then the representation $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_{j+1}}$ on $\mathfrak{m}_{j+1}$ is equivalent to $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$.

Main result II

Theorem

For $j = 2, \ldots, K$ the representation $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_j}$ of $\mathfrak{p}$ on the subspace $\mathfrak{m}_j$ can be decomposed into some subset of irreducible components of the tensor representation $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_{j-1}} \otimes \mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$.

If any of the subspaces $\mathfrak{m}_j$ is one–dimensional then $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{n}}$ must contain a copy of the trivial representation corresponding to the subspace $\mathfrak{m}_j$. When $j < K$, the representation of $\mathfrak{p}$ on $\mathfrak{m}_{j+1}$ can be decomposed into a sum of irreducible representations, each of which is equivalent to an irreducible representation contained in the decomposition of $\mathfrak{m}_1$.

In particular, when $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$ is irreducible and $\dim \mathfrak{m}_j = 1$, $1 < j < K$ then the representation $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_{j+1}}$ on $\mathfrak{m}_{j+1}$ is equivalent to $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$.

Why certain nilpotent algebras are not in Turkowski's lists?

Let us consider the 6–dimensional nilpotent algebra

$\mathbf{A_{6,15}}$: $[e_1, e_2] = e_3 + e_5,\ [e_1, e_3] = e_4,\ [e_1, e_4] = e_6, [e_2, e_5] = e_6.$

It has an incomplete flag of characteristic ideals

$$0 \subset \mathfrak{n}^4 \subset \mathfrak{n}^3 \subset \mathfrak{n}^2 \subset \mathfrak{z}_3 \subset \mathfrak{n}$$

in which only a 5–dimensional ideal is missing. Therefore, if any Levi decomposable algebra with the radical $\mathbf{A_{6,15}}$ exists then we have a decomposition of the subspaces $\mathfrak{m}_i$ into irreducible representations as follows

$$\mathfrak{m}_1 = \mathbf{2}_1 \dot{+} \mathbf{1}_1,\ \mathfrak{m}_2 = \mathbf{1}_2,\ \mathfrak{m}_3 = \mathbf{1}_3,\ \mathfrak{m}_4 = \mathbf{1}_4.$$

The representation space $\mathbf{1}_1$ coincides with the 1–dimensional subspace $\mathfrak{m}_1 \cap \mathfrak{z}_3$.

Why certain nilpotent algebras are not in Turkowski's lists?

Let us consider the 6–dimensional nilpotent algebra

$\mathbf{A}_{6,15}$: $[e_1, e_2] = e_3 + e_5,\ [e_1, e_3] = e_4,\ [e_1, e_4] = e_6, [e_2, e_5] = e_6.$

It has an incomplete flag of characteristic ideals

$$0 \subset \mathfrak{n}^4 \subset \mathfrak{n}^3 \subset \mathfrak{n}^2 \subset \mathfrak{z}_3 \subset \mathfrak{n}$$

in which only a 5–dimensional ideal is missing. Therefore, if any Levi decomposable algebra with the radical $\mathbf{A}_{6,15}$ exists then we have a decomposition of the subspaces $\mathfrak{m}_i$ into irreducible representations as follows

$$\mathfrak{m}_1 = \mathbf{2}_1 \dot{+} \mathbf{1}_1,\ \mathfrak{m}_2 = \mathbf{1}_2,\ \mathfrak{m}_3 = \mathbf{1}_3,\ \mathfrak{m}_4 = \mathbf{1}_4.$$

The representation space $\mathbf{1}_1$ coincides with the 1–dimensional subspace $\mathfrak{m}_1 \cap \mathfrak{z}_3$.

Why certain nilpotent algebras are not in Turkowski's lists?

Let us consider the 6–dimensional nilpotent algebra

$\mathbf{A_{6,15}}$: $[e_1, e_2] = e_3 + e_5,\ [e_1, e_3] = e_4,\ [e_1, e_4] = e_6, [e_2, e_5] = e_6.$

It has an incomplete flag of characteristic ideals

$$0 \subset \mathfrak{n}^4 \subset \mathfrak{n}^3 \subset \mathfrak{n}^2 \subset \mathfrak{z}_3 \subset \mathfrak{n}$$

in which only a 5–dimensional ideal is missing. Therefore, if any Levi decomposable algebra with the radical $\mathbf{A_{6,15}}$ exists then we have a decomposition of the subspaces $\mathfrak{m}_i$ into irreducible representations as follows

$$\mathfrak{m}_1 = \mathbf{2}_1 \dot{+} \mathbf{1}_1,\ \mathfrak{m}_2 = \mathbf{1}_2,\ \mathfrak{m}_3 = \mathbf{1}_3,\ \mathfrak{m}_4 = \mathbf{1}_4.$$

The representation space $\mathbf{1}_1$ coincides with the 1–dimensional subspace $\mathfrak{m}_1 \cap \mathfrak{z}_3$.

By Theorem 2 we have

$$\mathfrak{m}_3 = [\mathbf{1}_1, \mathfrak{m}_2] = [\mathfrak{m}_1 \cap \mathfrak{z}_3, \mathfrak{m}_2], \; \mathfrak{m}_4 = [\mathbf{1}_1, \mathfrak{m}_3] = [\mathfrak{m}_1 \cap \mathfrak{z}_3, \mathfrak{m}_3]. \tag{9}$$

At the same time, $\mathfrak{z}_3$ of the algebra $\mathbf{A_{6,15}}$ is Abelian and contains both $\mathfrak{m}_2$ and $\mathfrak{m}_3$ which leads to a contradiction with Eq. (9); namely, $[\mathfrak{m}_1 \cap \mathfrak{z}_3, \mathfrak{m}_3] = 0$. Therefore, no Levi extension of $\mathbf{A_{6,15}}$ exists. The same argument can be applied also to the algebra $\mathbf{A_{6,17}}$.

By Theorem 2 we have

$$\mathfrak{m}_3 = [\mathbf{1}_1, \mathfrak{m}_2] = [\mathfrak{m}_1 \cap \mathfrak{z}_3, \mathfrak{m}_2], \ \mathfrak{m}_4 = [\mathbf{1}_1, \mathfrak{m}_3] = [\mathfrak{m}_1 \cap \mathfrak{z}_3, \mathfrak{m}_3]. \tag{9}$$

At the same time, $\mathfrak{z}_3$ of the algebra $\mathbf{A_{6,15}}$ is Abelian and contains both $\mathfrak{m}_2$ and $\mathfrak{m}_3$ which leads to a contradiction with Eq. (9); namely, $[\mathfrak{m}_1 \cap \mathfrak{z}_3, \mathfrak{m}_3] = 0$. Therefore, no Levi extension of $\mathbf{A_{6,15}}$ exists. The same argument can be applied also to the algebra $\mathbf{A_{6,17}}$.

By Theorem 2 we have

$$\mathfrak{m}_3 = [\mathbf{1}_1, \mathfrak{m}_2] = [\mathfrak{m}_1 \cap \mathfrak{z}_3, \mathfrak{m}_2], \ \mathfrak{m}_4 = [\mathbf{1}_1, \mathfrak{m}_3] = [\mathfrak{m}_1 \cap \mathfrak{z}_3, \mathfrak{m}_3]. \tag{9}$$

At the same time, $\mathfrak{z}_3$ of the algebra $\mathbf{A_{6,15}}$ is Abelian and contains both $\mathfrak{m}_2$ and $\mathfrak{m}_3$ which leads to a contradiction with Eq. (9); namely, $[\mathfrak{m}_1 \cap \mathfrak{z}_3, \mathfrak{m}_3] = 0$. Therefore, no Levi extension of $\mathbf{A_{6,15}}$ exists. The same argument can be applied also to the algebra $\mathbf{A_{6,17}}$.

Second example

Consider $\mathbf{A_{6,13}}$

$$[e_1, e_2] = e_5,\ [e_1, e_3] = e_4,\ [e_1, e_4] = e_6,\ [e_2, e_5] = e_6. \qquad (10)$$

We have

$$\mathfrak{n}^2 = \mathrm{span}\{e_4, e_5, e_6\}, \mathfrak{n}^3 = \mathrm{span}\{e_6\}, \mathfrak{cent}(\mathfrak{n}^2) = \mathrm{span}\{e_3, e_4, e_5, e_6\}.$$

By dimensional analysis alone we have the following structure of irreducible representations of hypothetical $\mathfrak{p}$:

$$\mathfrak{m}_1 = \mathbf{2}_1 \dot{+} \mathbf{1}_1,\ \mathfrak{m}_3 = \mathbf{1}_3$$

and two options for $\mathfrak{m}_2$: either $\mathfrak{m}_2 = \mathbf{2}_2$ or $\mathfrak{m}_2 = \mathbf{1} \dot{+} \mathbf{1}$.

Second example

Consider $\mathbf{A}_{6,13}$

$$[e_1, e_2] = e_5, \ [e_1, e_3] = e_4, \ [e_1, e_4] = e_6, \ [e_2, e_5] = e_6. \quad (10)$$

We have

$$\mathfrak{n}^2 = \mathrm{span}\{e_4, e_5, e_6\}, \mathfrak{n}^3 = \mathrm{span}\{e_6\}, \mathfrak{cent}(\mathfrak{n}^2) = \mathrm{span}\{e_3, e_4, e_5, e_6\}.$$

By dimensional analysis alone we have the following structure of irreducible representations of hypothetical $\mathfrak{p}$:

$$\mathfrak{m}_1 = \mathbf{2}_1 \dotplus \mathbf{1}_1, \ \mathfrak{m}_3 = \mathbf{1}_3$$

and two options for $\mathfrak{m}_2$: either $\mathfrak{m}_2 = \mathbf{2}_2$ or $\mathfrak{m}_2 = \mathbf{1} \dotplus \mathbf{1}$.

Second example

Consider $\mathbf{A}_{6,13}$

$$[e_1, e_2] = e_5,\ [e_1, e_3] = e_4,\ [e_1, e_4] = e_6,\ [e_2, e_5] = e_6. \quad (10)$$

We have

$$\mathfrak{n}^2 = \mathrm{span}\{e_4, e_5, e_6\}, \mathfrak{n}^3 = \mathrm{span}\{e_6\}, \mathfrak{cent}(\mathfrak{n}^2) = \mathrm{span}\{e_3, e_4, e_5, e_6\}.$$

By dimensional analysis alone we have the following structure of irreducible representations of hypothetical $\mathfrak{p}$:

$$\mathfrak{m}_1 = \mathbf{2}_1 \dotplus \mathbf{1}_1,\ \mathfrak{m}_3 = \mathbf{1}_3$$

and two options for $\mathfrak{m}_2$: either $\mathfrak{m}_2 = \mathbf{2}_2$ or $\mathfrak{m}_2 = \mathbf{1} \dotplus \mathbf{1}$.

Out of the two, $\mathfrak{m}_2 = \mathbf{1} \dot{+} \mathbf{1}$ cannot be found in the antisymmetrized tensor product of $\mathbf{2_1} \dot{+} \mathbf{1_1}$ with itself; therefore, it must be $\mathfrak{m}_2 = \mathbf{2_2} = [\mathbf{2_1}, \mathbf{1_1}]$. On the other hand, from the Lie brackets (10) we have

$$[\mathbf{2_1}, \mathbf{1_1}] \subset [\mathfrak{n}, \mathfrak{cent}(\mathfrak{n}^2)] = \operatorname{span}\{e_4, e_6\}$$

which splits into $\mathfrak{m}_3$ and a 1–dimensional subspace of $\mathfrak{m}_2$. Therefore, $[\mathbf{2_1}, \mathbf{1_1}]$ must be simultaneously a trivial representation and a 2–dimensional irreducible representation, a contradiction showing that no Levi extension of $\mathbf{A}_{6,13}$ exists.

Out of the two, $\mathfrak{m}_2 = \mathbf{1} \dot{+} \mathbf{1}$ cannot be found in the antisymmetrized tensor product of $\mathbf{2_1} \dot{+} \mathbf{1_1}$ with itself; therefore, it must be $\mathfrak{m}_2 = \mathbf{2_2} = [\mathbf{2_1}, \mathbf{1_1}]$. On the other hand, from the Lie brackets (10) we have

$$[\mathbf{2_1}, \mathbf{1_1}] \subset [\mathfrak{n}, \mathfrak{cent}(\mathfrak{n}^2)] = \mathrm{span}\{e_4, e_6\}$$

which splits into $\mathfrak{m}_3$ and a 1–dimensional subspace of $\mathfrak{m}_2$. Therefore, $[\mathbf{2_1}, \mathbf{1_1}]$ must be simultaneously a trivial representation and a 2–dimensional irreducible representation, a contradiction showing that no Levi extension of $\mathbf{A_{6,13}}$ exists.

In a similar fashion, most of 6–dimensional nilpotent algebras missing in Turkowski's list can be shown to possess no Levi extension by dimensional arguments only.

Only in two cases $\mathbf{A_{6,14}}$

$$[e_1, e_3] = e_4,\ [e_1, e_4] = e_6,\ [e_2, e_3] = e_5,\ [e_2, e_5] = \epsilon e_6,\ \epsilon^2 = 1$$

and $\mathbf{A_{6,18}}$

$$[e_1, e_2] = e_3,\ [e_1, e_3] = e_4,\ [e_1, e_4] = e_6,\ [e_2, e_3] = e_5,\ [e_2, e_4] = e_6$$

we had to resort to at least partial explicit construction of derivations of $\mathfrak{n}$ in order to demonstrate that they cannot have a Levi extension.

In a similar fashion, most of 6–dimensional nilpotent algebras missing in Turkowski's list can be shown to possess no Levi extension by dimensional arguments only.

Only in two cases $\mathbf{A_{6,14}}$

$$[e_1, e_3] = e_4,\ [e_1, e_4] = e_6,\ [e_2, e_3] = e_5,\ [e_2, e_5] = \epsilon e_6,\ \epsilon^2 = 1$$

and $\mathbf{A_{6,18}}$

$$[e_1, e_2] = e_3,\ [e_1, e_3] = e_4,\ [e_1, e_4] = e_6,\ [e_2, e_3] = e_5,\ [e_2, e_4] = e_6$$

we had to resort to at least partial explicit construction of derivations of $\mathfrak{n}$ in order to demonstrate that they cannot have a Levi extension.

Nonnilpotent radicals

Let us assume that $\mathfrak{r} = \mathfrak{n} \dot{+} \mathfrak{q}$, $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{q}} = 0$ as is always possible to achieve by the theorem of Turkowski.

We have

$$\mathrm{ad}(p)[x, y] = [\mathrm{ad}(p)x, y] + [x, \mathrm{ad}(p)y] = 0$$

for any $x, y \in \mathfrak{q}$ and $p \in \mathfrak{p}$, i.e. the subspace $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{n}$ must be a representation space of the trivial representation (if $[\mathfrak{q}, \mathfrak{q}]$ is nonvanishing). Furthermore, for any $z \in \mathfrak{n}$, $y \in \mathfrak{q}$, $p \in \mathfrak{p}$ we have

$$\mathrm{ad}(p)[y, z] = [y, \mathrm{ad}(p)z]$$

i.e. $\mathrm{ad}(y)$, $y \in \mathfrak{q}$ maps any representation subspace $V \subset \mathfrak{n}$ of an irreducible representation of $\mathfrak{p}$ either to a representation space of an equivalent representation (including V itself) or to zero.

Nonnilpotent radicals

Let us assume that $\mathfrak{r} = \mathfrak{n} \dot{+} \mathfrak{q}$, $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{q}} = 0$ as is always possible to achieve by the theorem of Turkowski.
We have

$$\mathrm{ad}(p)[x, y] = [\mathrm{ad}(p)x, y] + [x, \mathrm{ad}(p)y] = 0$$

for any $x, y \in \mathfrak{q}$ and $p \in \mathfrak{p}$, i.e. the subspace $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{n}$ must be a representation space of the trivial representation (if $[\mathfrak{q}, \mathfrak{q}]$ is nonvanishing). Furthermore, for any $z \in \mathfrak{n}$, $y \in \mathfrak{q}$, $p \in \mathfrak{p}$ we have

$$\mathrm{ad}(p)[y, z] = [y, \mathrm{ad}(p)z]$$

i.e. $\mathrm{ad}(y)$, $y \in \mathfrak{q}$ maps any representation subspace $V \subset \mathfrak{n}$ of an irreducible representation of $\mathfrak{p}$ either to a representation space of an equivalent representation (including V itself) or to zero.

Nonnilpotent radicals

Let us assume that $\mathfrak{r} = \mathfrak{n} \dot{+} \mathfrak{q}$, $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{q}} = 0$ as is always possible to achieve by the theorem of Turkowski.
We have

$$\mathrm{ad}(p)[x, y] = [\mathrm{ad}(p)x, y] + [x, \mathrm{ad}(p)y] = 0$$

for any $x, y \in \mathfrak{q}$ and $p \in \mathfrak{p}$, i.e. the subspace $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{n}$ must be a representation space of the trivial representation (if $[\mathfrak{q}, \mathfrak{q}]$ is nonvanishing). Furthermore, for any $z \in \mathfrak{n}$, $y \in \mathfrak{q}$, $p \in \mathfrak{p}$ we have

$$\mathrm{ad}(p)[y, z] = [y, \mathrm{ad}(p)z]$$

i.e. $\mathrm{ad}(y)$, $y \in \mathfrak{q}$ maps any representation subspace $V \subset \mathfrak{n}$ of an irreducible representation of $\mathfrak{p}$ either to a representation space of an equivalent representation (including V itself) or to zero.

Another restriction: $[\mathfrak{p}, \mathfrak{q}] \subset \mathfrak{n}$ means for the corresponding derivations acting on $\mathfrak{n}$ that

$$[\mathrm{ad}(p)|_{\mathfrak{n}}, \mathrm{ad}(x)|_{\mathfrak{n}}] \in \mathfrak{Inn}(\mathfrak{n}), \; \forall p \in \mathfrak{p}, x \in \mathfrak{q}.$$

This in turn implies that the $\mathfrak{m}_1\mathfrak{m}_1$–blocks of $\mathrm{ad}(p)|_{\mathfrak{n}}$ and $\mathrm{ad}(x)|_{\mathfrak{n}}$ commute.

Theorem on nonnilpotent radicals

Theorem

Let $\mathfrak{g}$ be a Levi decomposable Lie algebra which cannot be decomposed into a direct sum of ideals, $\mathfrak{p}$ its Levi factor, $\mathfrak{r}$ its radical, $\mathfrak{n}$ its nilradical. Let $\mathfrak{n} = \sum_{k=1}^{K} \mathfrak{m}_k$ be the decomposition (6) of the nilradical $\mathfrak{n}$. Then for any $p \in \mathfrak{p}$ and $x \in \mathfrak{r}$ the submatrices $(\mathrm{ad}(p))_{\mathfrak{m}_1\mathfrak{m}_1}$ and $(\mathrm{ad}(x))_{\mathfrak{m}_1\mathfrak{m}_1}$ of $\mathrm{ad}(p)|_{\mathfrak{n}}$ and $\mathrm{ad}(x)|_{\mathfrak{n}}$, respectively, commute

$$[(\mathrm{ad}(p))_{\mathfrak{m}_1\mathfrak{m}_1}, (\mathrm{ad}(x))_{\mathfrak{m}_1\mathfrak{m}_1}] = 0.$$

In particular, if the restriction of $\mathrm{ad}(\mathfrak{p})$ to $\mathfrak{m}_1$ is irreducible and $\mathfrak{g}$ is an algebra over $\mathbb{C}$ then $\dim \mathfrak{r} - \dim \mathfrak{n} \leq 1$. When equality holds then the $\mathfrak{m}_1\mathfrak{m}_1$–block of $\mathrm{ad}(f_1)$ ($f_1 \in \mathfrak{r} \setminus \mathfrak{n}$) is a nonvanishing multiple of the unit operator.

Theorem on nonnilpotent radicals

Theorem

Let $\mathfrak{g}$ *be a Levi decomposable Lie algebra which cannot be decomposed into a direct sum of ideals,* $\mathfrak{p}$ *its Levi factor,* $\mathfrak{r}$ *its radical,* $\mathfrak{n}$ *its nilradical. Let* $\mathfrak{n} = \sum_{k=1}^{K} \mathfrak{m}_k$ *be the decomposition (6) of the nilradical* $\mathfrak{n}$*. Then for any* $p \in \mathfrak{p}$ *and* $x \in \mathfrak{r}$ *the submatrices* $(\mathrm{ad}(p))_{\mathfrak{m}_1\mathfrak{m}_1}$ *and* $(\mathrm{ad}(x))_{\mathfrak{m}_1\mathfrak{m}_1}$ *of* $\mathrm{ad}(p)|_{\mathfrak{n}}$ *and* $\mathrm{ad}(x)|_{\mathfrak{n}}$*, respectively, commute*

$$[(\mathrm{ad}(p))_{\mathfrak{m}_1\mathfrak{m}_1}, (\mathrm{ad}(x))_{\mathfrak{m}_1\mathfrak{m}_1}] = 0.$$

In particular, if the restriction of $\mathrm{ad}(\mathfrak{p})$ *to* $\mathfrak{m}_1$ *is irreducible and* $\mathfrak{g}$ *is an algebra over* $\mathbb{C}$ *then* $\dim \mathfrak{r} - \dim \mathfrak{n} \leq 1$*. When equality holds then the* $\mathfrak{m}_1\mathfrak{m}_1$*–block of* $\mathrm{ad}(f_1)$ $(f_1 \in \mathfrak{r} \setminus \mathfrak{n})$ *is a nonvanishing multiple of the unit operator.*

Theorem on nonnilpotent radicals

Theorem

Let $\mathfrak{g}$ be a Levi decomposable Lie algebra which cannot be decomposed into a direct sum of ideals, $\mathfrak{p}$ its Levi factor, $\mathfrak{r}$ its radical, $\mathfrak{n}$ its nilradical. Let $\mathfrak{n} = \sum_{k=1}^{K} \mathfrak{m}_k$ be the decomposition (6) of the nilradical $\mathfrak{n}$. Then for any $p \in \mathfrak{p}$ and $x \in \mathfrak{r}$ the submatrices $(\mathrm{ad}(p))_{\mathfrak{m}_1\mathfrak{m}_1}$ and $(\mathrm{ad}(x))_{\mathfrak{m}_1\mathfrak{m}_1}$ of $\mathrm{ad}(p)|_{\mathfrak{n}}$ and $\mathrm{ad}(x)|_{\mathfrak{n}}$, respectively, commute

$$[(\mathrm{ad}(p))_{\mathfrak{m}_1\mathfrak{m}_1}, (\mathrm{ad}(x))_{\mathfrak{m}_1\mathfrak{m}_1}] = 0.$$

In particular, if the restriction of $\mathrm{ad}(\mathfrak{p})$ to $\mathfrak{m}_1$ is irreducible and $\mathfrak{g}$ is an algebra over $\mathbb{C}$ then $\dim \mathfrak{r} - \dim \mathfrak{n} \leq 1$. When equality holds then the $\mathfrak{m}_1\mathfrak{m}_1$–block of $\mathrm{ad}(f_1)$ ($f_1 \in \mathfrak{r} \setminus \mathfrak{n}$) is a nonvanishing multiple of the unit operator.

Theorem on nonnilpotent radicals

Theorem

Let $\mathfrak{g}$ be a Levi decomposable Lie algebra which cannot be decomposed into a direct sum of ideals, $\mathfrak{p}$ its Levi factor, $\mathfrak{r}$ its radical, $\mathfrak{n}$ its nilradical. Let $\mathfrak{n} = \sum_{k=1}^{K} \mathfrak{m}_k$ be the decomposition (6) of the nilradical $\mathfrak{n}$. Then for any $p \in \mathfrak{p}$ and $x \in \mathfrak{r}$ the submatrices $(\mathrm{ad}(p))_{\mathfrak{m}_1\mathfrak{m}_1}$ and $(\mathrm{ad}(x))_{\mathfrak{m}_1\mathfrak{m}_1}$ of $\mathrm{ad}(p)|_{\mathfrak{n}}$ and $\mathrm{ad}(x)|_{\mathfrak{n}}$, respectively, commute

$$[(\mathrm{ad}(p))_{\mathfrak{m}_1\mathfrak{m}_1}, (\mathrm{ad}(x))_{\mathfrak{m}_1\mathfrak{m}_1}] = 0.$$

In particular, if the restriction of $\mathrm{ad}(\mathfrak{p})$ to $\mathfrak{m}_1$ is irreducible and $\mathfrak{g}$ is an algebra over $\mathbb{C}$ then $\dim \mathfrak{r} - \dim \mathfrak{n} \leq 1$. When equality holds then the $\mathfrak{m}_1\mathfrak{m}_1$–block of $\mathrm{ad}(f_1)$ ($f_1 \in \mathfrak{r} \setminus \mathfrak{n}$) is a nonvanishing multiple of the unit operator.

The proof of the statements when $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$ is irreducible is a direct consequence of Schur's lemma. $\square$

Theorem 4 can be used in explaining the particular values of parameters of solvable radicals $\mathfrak{r}$ allowing Levi extension. E.g. in the algebra $g_{6,54}$ in Mubarakzyanov's classification of solvable algebras there are two parameters whereas its Levi extension $L^{p}_{9,49}$ has only one. The reason is that in order to have $(\mathrm{ad}(f_1))_{\mathfrak{m}_1\mathfrak{m}_1}$ commuting with $(\mathrm{ad}(\mathfrak{p}))_{\mathfrak{m}_1\mathfrak{m}_1}$ one of the parameters must be equal to 1. For the same reason the four parameters in the algebra $\mathcal{N}^{\alpha\beta\gamma\delta}_{6,1}$ are reduced to the values $\gamma = \alpha \equiv p, \beta = \delta \equiv q$ in the Levi extension $L^{p,q}_{9,28}$. And similarly for other parametric families in Turkowski's classifications.

The proof of the statements when $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$ is irreducible is a direct consequence of Schur's lemma. □

Theorem 4 can be used in explaining the particular values of parameters of solvable radicals $\mathfrak{r}$ allowing Levi extension. E.g. in the algebra $g_{6,54}$ in Mubarakzyanov's classification of solvable algebras there are two parameters whereas its Levi extension $L^{p}_{9,49}$ has only one. The reason is that in order to have $(\mathrm{ad}(f_1))_{\mathfrak{m}_1\mathfrak{m}_1}$ commuting with $(\mathrm{ad}(\mathfrak{p}))_{\mathfrak{m}_1\mathfrak{m}_1}$ one of the parameters must be equal to 1. For the same reason the four parameters in the algebra $\mathcal{N}^{\alpha\beta\gamma\delta}_{6,1}$ are reduced to the values $\gamma = \alpha \equiv p, \beta = \delta \equiv q$ in the Levi extension $L^{p,q}_{9,28}$. And similarly for other parametric families in Turkowski's classifications.

The proof of the statements when $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$ is irreducible is a direct consequence of Schur's lemma. □

Theorem 4 can be used in explaining the particular values of parameters of solvable radicals $\mathfrak{r}$ allowing Levi extension. E.g. in the algebra $g_{6,54}$ in Mubarakzyanov's classification of solvable algebras there are two parameters whereas its Levi extension $L^{p}_{9,49}$ has only one. The reason is that in order to have $(\mathrm{ad}(f_1))_{\mathfrak{m}_1\mathfrak{m}_1}$ commuting with $(\mathrm{ad}(\mathfrak{p}))_{\mathfrak{m}_1\mathfrak{m}_1}$ one of the parameters must be equal to 1. For the same reason the four parameters in the algebra $\mathcal{N}^{\alpha\beta\gamma\delta}_{6,1}$ are reduced to the values $\gamma = \alpha \equiv p, \beta = \delta \equiv q$ in the Levi extension $L^{p,q}_{9,28}$. And similarly for other parametric families in Turkowski's classifications.

The proof of the statements when $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$ is irreducible is a direct consequence of Schur's lemma. $\square$

Theorem 4 can be used in explaining the particular values of parameters of solvable radicals $\mathfrak{r}$ allowing Levi extension. E.g. in the algebra $g_{6,54}$ in Mubarakzyanov's classification of solvable algebras there are two parameters whereas its Levi extension $L^{p}_{9,49}$ has only one. The reason is that in order to have $(\mathrm{ad}(f_1))_{\mathfrak{m}_1\mathfrak{m}_1}$ commuting with $(\mathrm{ad}(\mathfrak{p}))_{\mathfrak{m}_1\mathfrak{m}_1}$ one of the parameters must be equal to 1. For the same reason the four parameters in the algebra $\mathcal{N}^{\alpha\beta\gamma\delta}_{6,1}$ are reduced to the values $\gamma = \alpha \equiv p, \beta = \delta \equiv q$ in the Levi extension $L^{p,q}_{9,28}$. And similarly for other parametric families in Turkowski's classifications.

The proof of the statements when $\mathrm{ad}(\mathfrak{p})|_{\mathfrak{m}_1}$ is irreducible is a direct consequence of Schur's lemma. $\square$

Theorem 4 can be used in explaining the particular values of parameters of solvable radicals $\mathfrak{r}$ allowing Levi extension. E.g. in the algebra $g_{6,54}$ in Mubarakzyanov's classification of solvable algebras there are two parameters whereas its Levi extension $L^{p}_{9,49}$ has only one. The reason is that in order to have $(\mathrm{ad}(f_1))_{\mathfrak{m}_1\mathfrak{m}_1}$ commuting with $(\mathrm{ad}(\mathfrak{p}))_{\mathfrak{m}_1\mathfrak{m}_1}$ one of the parameters must be equal to 1. For the same reason the four parameters in the algebra $\mathcal{N}^{\alpha\beta\gamma\delta}_{6,1}$ are reduced to the values $\gamma = \alpha \equiv p, \beta = \delta \equiv q$ in the Levi extension $L^{p,q}_{9,28}$. And similarly for other parametric families in Turkowski's classifications.

Conclusions

- We have investigated the structure of Levi decomposable algebras. We have formulated several general properties that the nilradical of any Levi decomposable algebra must necessarily satisfy.
- We were able to explain the absence of a Levi extension for all but two 6–dimensional indecomposable nilpotent algebras by abstract, mostly dimensional, considerations, without an explicit construction of derivations.
- The results and methods used in this section can be applied to Levi extensions of arbitrary dimension.
- Especially, if a particular choice of the Levi factor and the nilradical is desired by some application our results can be easily used to estimate whether such a Levi extension may exist.

Conclusions

- We have investigated the structure of Levi decomposable algebras. We have formulated several general properties that the nilradical of any Levi decomposable algebra must necessarily satisfy.
- We were able to explain the absence of a Levi extension for all but two 6–dimensional indecomposable nilpotent algebras by abstract, mostly dimensional, considerations, without an explicit construction of derivations.
- The results and methods used in this section can be applied to Levi extensions of arbitrary dimension.
- Especially, if a particular choice of the Levi factor and the nilradical is desired by some application our results can be easily used to estimate whether such a Levi extension may exist.

Conclusions

- We have investigated the structure of Levi decomposable algebras. We have formulated several general properties that the nilradical of any Levi decomposable algebra must necessarily satisfy.
- We were able to explain the absence of a Levi extension for all but two 6–dimensional indecomposable nilpotent algebras by abstract, mostly dimensional, considerations, without an explicit construction of derivations.
- The results and methods used in this section can be applied to Levi extensions of arbitrary dimension.
- Especially, if a particular choice of the Levi factor and the nilradical is desired by some application our results can be easily used to estimate whether such a Levi extension may exist.

Conclusions

- We have investigated the structure of Levi decomposable algebras. We have formulated several general properties that the nilradical of any Levi decomposable algebra must necessarily satisfy.
- We were able to explain the absence of a Levi extension for all but two 6–dimensional indecomposable nilpotent algebras by abstract, mostly dimensional, considerations, without an explicit construction of derivations.
- The results and methods used in this section can be applied to Levi extensions of arbitrary dimension.
- Especially, if a particular choice of the Levi factor and the nilradical is desired by some application our results can be easily used to estimate whether such a Levi extension may exist.

Conclusions II

- One particular immediate consequence: no nilpotent algebra $\mathfrak{n}$ such that
  1. $\dim \mathfrak{n} - \dim \mathfrak{n}^2 = 2$ and
  2. $j \in \mathbb{N}$ exists such that $\dim \mathfrak{n}^j - \dim \mathfrak{n}^{j+1} = \dim \mathfrak{n}^{j+1} - \dim \mathfrak{n}^{j+2} = 1$

  can be a nilradical of a Levi decomposable algebra (generalization of a known result for nilpotent algebras of maximal degree of nilpotency).
- It remains an open problem to find some structurally interesting series of nilradicals in arbitrary dimension allowing the classification of its nontrivial Levi extensions other than $\mathfrak{n}$ being Abelian or Heisenberg.

Conclusions II

- One particular immediate consequence: no nilpotent algebra $\mathfrak{n}$ such that
  1. $\dim \mathfrak{n} - \dim \mathfrak{n}^2 = 2$ and
  2. $j \in \mathbb{N}$ exists such that $\dim \mathfrak{n}^j - \dim \mathfrak{n}^{j+1} = \dim \mathfrak{n}^{j+1} - \dim \mathfrak{n}^{j+2} = 1$

  can be a nilradical of a Levi decomposable algebra (generalization of a known result for nilpotent algebras of maximal degree of nilpotency).
- It remains an open problem to find some structurally interesting series of nilradicals in arbitrary dimension allowing the classification of its nontrivial Levi extensions other than $\mathfrak{n}$ being Abelian or Heisenberg.

Quantum Theory and Symmetries 7
Prague
August 7–13, 2011
PŘÍSTA IŠTĚ
Advisory Committee:
N. Arkani-Hamed (IAS)
N. Berkovits (Sao Paulo)
E. Corrigan (Durham)
M. Douglas (SUNY)
L. D. Faddeev (St. Petersburg)
D. Freedman (MIT)
J. P. Gazeau (Paris 7)
H. Grosse (Wien)
P. Hořava (Berkeley)
E. A. Ivanov (Dubna)
R. Kerner (Paris 6)
A. Molev (Sydney)
J. Patera (Montreal)
O. Ragnisco (Roma)
M. Rocek (SUNY)
F. Smirnov (LPTHE)
D. Sorokin (Padova)
A. Strominger (Harvard)
G. 't Hooft (Utrecht)
M. Vasiliev (Moscow)
F. Wilczek (MIT)
P. Winternitz (Montreal)
E. Witten (IAS)
S. L. Woronowicz (Warszaw)
Local Organizing Committee:
Č. Burdík (chairman, Prague)
M. Schnabl (vice-chairman, Prague)
T. Erler (Prague), D. Francia (Prague)
L. Hlavatý (Prague), O. Navrátil (Prague)
S. Pošta (Prague), L. Šnobl (Prague)
R. von Unge (Brno), M. Znojil (Řež)
Topics include
• Integrable systems
• String theory
• Quantum groups
• Noncommutative geometry
• Supersymmetry
• Foundations of quantum mechanics
• Modern mathematical methods
www.km.fjfi.cvut.cz/qts7
Goslar 1999
Krakow 2001
Cincinnati 2003
Prague 2011
Varna 2005
Valladolid 2007
Lexington 2009

Thank you for your attention