# CONSTRUCTION OF QUANTUM DOUBLES FROM SOLUTIONS OF YANG-BAXTER SYSTEM

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The construction of quantum doubles based on the results of Vladimirov's article ( Mod. Phys. Lett. A 8, 1315 (1993) ) was considered. A.A. Vladimirov claimed that to every triple of invertible matrices satisfying certain Yang-Baxter-like system should be possible to associate a quantum double. The most interesting conclusion of the current research is that the corresponding quantum double is completely determined by one matrix of the triple, the other matrices are irrelevant. Several examples of quantum doubles were obtained.

## 1 Introduction

We have studied the possibilities and problems of the construction of Drinfeld's quantum doubles using a method published by A.A. Vladimirov in [5]. The original motivation for this research came from Vladimirov's article and from successes in classifying the solutions of different Yang-Baxter systems, namely from work of L. Hlavaty [2].

The original idea of this research was to find firstly solutions of the Yang-Baxter system (1) in dimension 2 and then to use them in construction of corresponding quantum doubles. We have found and published last year ( together with L. Hlavaty ) the complete solution of given Yang-Baxter system ( see [1] ). In the present work we have at first tried to use those solutions in order to construct associated quantum doubles. As we will show later, this approach had to be modified and the current work is surprisingly almost independent on the previous one.

## 2 Preliminaries

We use the following definition of quantum double:

**Definition 2.1** *Let $H$ be a finite-dimensional Hopf algebra. The* quantum double *$D(H)$ in form containing $H, H^{*op} \equiv H^o$ is a quasitriangular Hopf algebra generated by these as sub-Hopf algebras with the cross multiplication relations and the quasitriangular structure*

$$\langle h_{(1)}, a_{(1)}\rangle h_{(2)} a_{(2)} = a_{(1)} h_{(1)} \langle h_{(2)}, a_{(2)}\rangle,\ \mathcal{R} = \sum_j e^j \otimes e_j,$$

*where $h \in H, a \in H^o$, $e_j$ is a basis of $H$ and $e^j$ is the dual basis of $H^o$, i.e. $\langle e_j, e^k\rangle = \delta_j^k$. It can be realised on the vector space $H \otimes H^*$ ( $= H \otimes H^o$ as a vector* *space ) with the multiplication*

$$(h \otimes a).(g \otimes b) = \langle g_{(1)}, a_{(1)} \rangle h.g_{(2)} \otimes b.a_{(2)} \langle Sg_{(3)}, a_{(3)} \rangle$$

*and tensor product coproduct, unit and counit.*
**Generalization:** *It is possible to replace $H$ and $H^o$ by an antidual pair of Hopf algebras, not necessary of finite dimension.*

All structures are considered over the field of complex numbers $\mathcal{C}$. The unit element is denoted by $\mathbf{1}$, the $n \times n$ unit matrix by $\mathbf{1}_n$.

## 3 Vladimirov's Method for Obtaining Quantum Doubles

In this section we explain a method for obtaining a quantum double from any set of matrices satisfying the system of Yang-Baxter-like equations. In [5] was proposed that to every constant solution $\{W, X, Z\}$ of the Yang-Baxter system

$$\begin{array}{ll} [W,W,W] = 0 & [Z,Z,Z] = 0 \\ [W,X,X] = 0 & [X,X,Z] = 0, \end{array} \tag{1}$$

where $[A,B,C] = A_{12}B_{13}C_{23} - C_{23}B_{13}A_{12}$, it is possible to associate a quantum double. Actually, as we will show later, there are some problems applying this procedure.

In order to construct the associated quantum double one should find the following structures:

1. **Quantum matrix bialgebras $A_W$, $A_Z$ ( $W_{12}U_1U_2 = U_2U_1W_{12}$, $Z_{12}V_1V_2 = V_2V_1Z_{12}$ ) with antidual pairing $\langle U_1, V_2 \rangle = X_{12}$**
Equations $[W,X,X] = 0$, $[X,X,Z] = 0$ guarantee that the pairing can be correctly extended on the whole bialgebras $A_W$, $A_Z$.

2. **Null biideals of the pairing and factor them out, finding the bialgebras $\tilde{A}_W, \tilde{A}_Z$.**
It is known from theory of bialgebras that such factorization is always possible, but there is no known general algorithm how to actually perform the factorization. Usually it is possible to find some elements of low order lying in the null biideal using computer algebra systems and then try to prove that they generate the whole null biideal.

3. **Antipodes in $\tilde{A}_W, \tilde{A}_Z$.**
The existence of antipodes is not generally guaranteed but in some interesting cases the antipodes exist. They can be deduced e.g. from so-called weak antipode $\langle SU_1, V_2 \rangle = X_{12}^{-1}$.[1])

[1]) The problem of the existence of antipodes can be solved in some cases by enlarging both $A_W$ and $A_Z$ by "inverse formal matrix $\bar{U}$" ( and "$\bar{V}$" ) satisfying $\bar{U}U = U\bar{U} = \mathbf{1}_n$ and $\bar{V}V = V\bar{V} = \mathbf{1}_n$ and also factoring out relations like $W_{12}\bar{U}_2\bar{U}_1 = \bar{U}_1\bar{U}_2W_{12}$ etc. The antipodes may then be defined $SU = \bar{U}$, $S\bar{U}$ is a bit more complicated. More detailed description of this method is given e.g. in [6].

4. **Structure of quantum double $D(\tilde{A}_W)$,**
   i.e. evaluation of the cross-product. This step is just substitution of concrete Hopf algebra structures into Definition 2.1 given above.

5. **Dual bases and consequently the $\mathcal{R}$-matrix**
   Again there is no known general algorithm for doing this; also the existence of dual bases is in general not guaranteed ( when the Hopf algebras $\tilde{A}_W, \tilde{A}_Z$ have infinite dimension, in finite dimensional case it is almost trivial to find a pair of dual bases ).

## 4 Modification of Original Method

In the previous section we have explained Vladimirov's method for obtaining quantum doubles. In this section we propose its modification. Following theorem is probably the most interesting result of this paper.

**Theorem 4.1** *Let $D(\tilde{A}_W)$ be a quantum double obtained by Vladimirov's construction from the triple $W, X, Z$. Then its structure doesn't depend on matrices $W, Z$ at all, it is completely determined by the matrix $X$ itself.*

**Proof:** Let us assume that $X$ is a $n^2 \times n^2$ matrix and let us define an antidual pairing of tensor algebras ( with coproduct $\Delta(U) = U \dot{\otimes} U$, $\Delta(V) = V \dot{\otimes} V$ and counit $\epsilon(U) = \mathbf{1}_n, \epsilon(V) = \mathbf{1}_n$ extended as algebra maps )[2] $T(U) \times T(V) \to k$ by defining the pairing between generators $\langle u^i_j, v^k_l \rangle = X^{i,k}_{j,l}$ ( where $U = (u^i_j), V = (v^i_j)$ ) and requiring the properties of antidual pairing. Let $W, Z$ be arbitrary matrices satisfying $[W, X, X] = 0, [X, X, Z] = 0$. Then from [5] we know that the pairing is consistent with the structure of $A_W$ and $A_Z$, i.e. with factoring out the elements $W^{ab}_{ij} u^i_c u^j_d - u^b_j u^a_i W^{ij}_{cd}$ and $Z^{ab}_{ij} v^i_c v^j_d - v^b_j v^a_i Z^{ij}_{cd}$, so those elements lie in the corresponding null biideals of the pairing of tensor algebras. Since during the construction of quantum double we factor out the null biideals of the pairing, any information about matrices $W, Z$ disappears, the bialgebras $\tilde{A}_W$ and $\tilde{A}_Z$ are the same for given pair $W, Z$ and for $W' = P,\ Z' = P$ ( $P$ being the permutation matrix $P^{i,j}_{k,l} = \delta^i_l \delta^j_k$. It satisfies $[P, X, X] = [X, X, P] = 0$ for any $X$ ). The conclusion is that $\tilde{A}_W$ and $\tilde{A}_Z$ are the same for any pair $W, Z$ satisfying $[W, X, X] = 0, [X, X, Z] = 0$. Completing the construction of the quantum double ( and remembering that the antipodes on $\tilde{A}_W$ and $\tilde{A}_Z$ are unique if they exist, so they cannot depend on $W$ or $Z$ ) we prove the theorem.

## 5 Examples

In the present section we illustrate the scheme described in the previous section on examples. Since the matrix $X$ is the only one important of the triple $W, X, Z$, the matrices $W$,$Z$ are not specified. Also the notation is modified accordingly, we write

[2]) Such bialgebras can be imagined as quantum matrix bialgebras given by permutation matrix.

$\mathrm{U}_X$ instead of $\tilde{A}_W$ and $\mathrm{V}_X$ instead of $\tilde{A}_Z$. Since commutative quantum doubles are not very interesting, we are presenting only noncommutative ones.

### 5.1 Infinite dimensional quantum double

We will study the following matrix $X$:

$$X = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2}$$

Factorization of null biideals in this case leads to bialgebras $\mathrm{U}_X$ and $\mathrm{V}_X$:

- $\mathrm{U}_X = \mathrm{span}\{\mathbf{1}, u_1^1, (u_1^2)^j, u_1^1(u_1^2)^j \,|\, j \in \mathcal{N}\}$ with the multiplication generated by $u_1^1 u_1^1 = \mathbf{1}, u_1^1 u_1^2 = -u_1^2 u_1^1$ and the coproduct $\Delta(u_1^1) = u_1^1 \otimes u_1^1$, $\Delta(u_1^2) = u_1^2 \otimes u_1^1 + u_1^1 \otimes u_1^2$. This bialgebra is evidently cocommutative, but not commutative.
- $\mathrm{V}_X = \mathrm{span}\{\mathbf{1}, v_1^1, (v_2^1)^j, v_1^1(v_2^1)^j \,|\, j \in \mathcal{N}\}$ with the multiplication generated by $v_1^1 v_1^1 = \mathbf{1}, v_1^1 v_2^1 = v_2^1 v_1^1$ and with the coproduct $\Delta(v_1^1) = v_1^1 \otimes v_1^1$, $\Delta(v_2^1) = v_1^1 \otimes v_2^1 + v_2^1 \otimes \mathbf{1}$. This bialgebra is commutative, but not cocommutative.

Now we can deduce the antipodes:

- $Su_1^1 = u_1^1$ and $Su_1^2 = u_1^2$
- $Sv_1^1 = v_1^1$ and $Sv_2^1 = -v_1^1 v_2^1$.

The quantum double $D(U_X)$ is generated by $U_X$ and $V_X$ as subbialgebras with the cross product generated by the relations

$$\begin{aligned} v_1^1 u_1^1 &= u_1^1 v_1^1, & v_1^1 u_1^2 &= u_1^2 v_1^1, \\ v_2^1 u_1^1 &= -u_1^1 v_2^1, & v_2^1 u_1^2 &= u_1^1 - u_1^2 v_2^1 - u_1^1 v_1^1. \end{aligned}$$

It remains to find a pair of dual bases. In order to do this, we firstly evaluate the pairing between the powers of generators.

- $\langle (u_1^2)^j, (v_2^1)^k \rangle = k!(-1)^{\frac{k(k-1)}{2}} \delta_{jk}$
- $\langle u_1^1(u_1^2)^j, (v_2^1)^k \rangle = (-1)^k \langle u_1^1(u_1^2)^j, (v_2^1)^k \rangle = k!(-1)^{\frac{k(k-1)}{2}+k} \delta_{jk}$,
- $\langle (u_1^2)^j, v_1^1(v_2^1)^k \rangle = (-1)^k \langle u_1^1(u_1^2)^j, (v_2^1)^k \rangle = k!(-1)^{\frac{k(k-1)}{2}+k} \delta_{jk}$,
- $\langle u_1^1(u_1^2)^j, v_1^1(v_2^1)^k \rangle = (-1)^{2k-1} \langle u_1^1(u_1^2)^j, (v_2^1)^k \rangle = -k!(-1)^{\frac{k(k-1)}{2}} \delta_{jk}$.

The pairings given above allow to write a pair of dual bases in the form

$$e_{k1} = (u_1^2)^k \qquad e^{k1} = \frac{1}{2k!}(-1)^{\frac{k(k-1)}{2}}((v_2^1)^k + (-1)^k v_1^1(v_2^1)^k)$$

$$e_{k2} = u_1^1(u_1^2)^k \qquad e^{k2} = \frac{1}{2k!}(-1)^{\frac{k(k-1)}{2}+k}((v_2^1)^k - (-1)^k v_1^1(v_2^1)^k).$$

Finally, we can write the canonical $\mathcal{R}$-matrix in the form of formal power series:

$$\mathcal{R} = \sum_{k=0}^{\infty} \frac{1}{2k!}(-1)^{\frac{k(k-1)}{2}} \big((v_2^1)^k \otimes (u_1^2)^k \quad + \quad (-1)^k v_1^1 (v_2^1)^k \otimes (u_1^2)^k + \\ +(-1)^k (v_2^1)^k \otimes u_1^1 (u_1^2)^k \quad - \quad v_1^1 (v_2^1)^k \otimes u_1^1 (u_1^2)^k\big)\,. \tag{3}$$

### 5.2 Finite dimensional quantum double

There is a natural question whether the method explained above can be used also to find some noncommutative finite dimensional quantum doubles. The answer is positive, as we show in the present subsection.

Let us study the matrix $X$:

$$X = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4}$$

The bialgebras $\mathrm{U}_X$, $\mathrm{V}_X$ in this case are

- $\mathrm{U}_X = \mathrm{span}\{\mathbf{1}, u_1^1, u_1^2, u_1^1 u_1^2\}$ with the multiplication $u_1^1 u_1^1 = \mathbf{1}$, $u_1^2 u_1^2 = 0$, $u_1^2 u_1^1 = -u_1^1 u_1^2$ and the coproduct $\Delta(u_1^1) = u_1^1 \otimes u_1^1$, $\Delta(u_1^2) = \mathbf{1} \otimes u_1^2 + u_1^2 \otimes u_1^1$.
- $\mathrm{V}_X = \mathrm{span}\{\mathbf{1}, v_1^1, v_2^1, v_1^1 v_2^1\}$ with the multiplication $v_1^1 v_1^1 = \mathbf{1}$, $v_2^1 v_2^1 = 0$, $v_2^1 v_1^1 = -v_1^1 v_2^1$ and with the coproduct $\Delta(v_1^1) = v_1^1 \otimes v_1^1$, $\Delta(v_2^1) = v_1^1 \otimes v_2^1 + v_2^1 \otimes \mathbf{1}$.

Given bialgebras are Hopf algebras with the antipodes :

- $Su_1^1 = u_1^1$, $Su_1^2 = u_1^1 u_1^2$
- $Sv_1^1 = v_1^1$, $Sv_2^1 = -v_1^1 v_2^1$.

It is now possible to construct the quantum double $D(U_X)$ containing the $\mathrm{U}_X$ and $\mathrm{V}_X$ as subbialgebras with the cross product given by relations

$$\begin{array}{ccc} v_1^1 u_1^1 = u_1^1 v_1^1 & , & v_1^1 u_1^2 = -u_1^2 v_1^1 \\ v_2^1 u_1^1 = -u_1^1 v_2^1 & , & v_2^1 u_1^2 = u_1^2 v_1^1 + u_1^1 + v_1^1 . \end{array}$$

In this finite dimensional case it is easy to find a pair of dual bases and express the canonical $\mathcal{R}$-matrix of the quantum double

$$\mathcal{R} = \frac{1}{2}\big(\mathbf{1} \otimes \mathbf{1} + v_1^1 \otimes \mathbf{1} \quad + \quad \mathbf{1} \otimes u_1^1 - v_1^1 \otimes u_1^1 \\ + v_2^1 \otimes u_1^2 - v_1^1 v_2^1 \otimes u_1^2 \quad - \quad v_2^1 \otimes u_1^1 u_1^2 - v_1^1 v_2^1 \otimes u_1^1 u_1^2\big)\,. \tag{5}$$

**Remark:** The Hopf algebra $\mathrm{U}_X$ obtained in this example is not new, it has been already published in the literature, namely [4] ( and [3] ).

## 6 Conclusions

As we have mentioned earlier, the original motivation of this work was to use the solutions of the Yang-Baxter system found earlier ( see [1] ) in order to construct some new quantum doubles and corresponding universal $\mathcal{R}$-matrices. As we have shown in the present work, this idea had to be modified, since in the original method the matrices $W$,$Z$ were not important, the associated quantum double depends only on the matrix $X$.

During the study of different examples, the main difficulties of the given method for finding quantum doubles became more clear. It seems that the problem of the existence of antipodes is rather a technical one, it often can be overcome by modification of the matrix $X$. The most difficult problem is probably the suitable choice of the matrix $X$. As we have shown in examples, a change of one sign in the matrix $X$ may turn the resulting quantum double from finite-dimensional into infinite-dimensional. Therefore the resulting quantum double is very sensitive to changes in the matrix $X$ and it is very difficult to predict its properties without actually finding the corresponding null biideals.

On the other hand, matrices $X$ and

$$\tilde{X} = T_1 S_2 X S_2^{-1} T_1^{-1}$$

( where $T, S$ are arbitrary invertible $2\times 2$ matrices ) may look very different, but the corresponding bialgebras $\mathrm{U}_X$,$\mathrm{V}_X$ and $\mathrm{U}_{\tilde{X}}$,$\mathrm{V}_{\tilde{X}}$ ( and appropriate quantum doubles $D(\mathrm{U}_X)$ and $D(\mathrm{U}_{\tilde{X}})$ ) are isomorphic. Consequently, the structure of the set of quantum doubles that can be obtained using the given method remains hidden.

Nevertheless, it is surely possible to find new interesting examples of quantum doubles using given method and some well-known examples of quantum doubles can be reobtained this way.

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