# Principal chiral models with non–constant metric*)

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Field equations for generalized principal chiral models with non–constant metric and their possible Lax formulation are considered. Ansatz for Lax operators is taken linear in currents. Results of a complete investigation of models allowing Lax formulation with linear ansatz for Lax operators on solvable 2– and 3–dimensional groups are given; all such models appear to be almost linear. Also models on simple group SU(2) with diagonal metric are considered; it turns out that Lax formulation exists in this case for constant metrics only.

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## 1 Introduction

We have investigated the generalisation of principal chiral models [1] to models with non–constant metric (this idea was suggested by Sochen in [2]) This approach allowed to study also models on non–semisimple groups. We have studied the case of 2– and 3– dimensional groups, both solvable and simple. In this article we provide a brief overview and extension of our results, submitted for publication elsewhere (see [3], [6] and [5]).

Generalised principal chiral models [2] are given by the action

$$I[g] = \int d^2x \eta^{\mu\nu} L_{ab}(g) J^a_\mu J^b_\nu \tag{1}$$

where $G$ is a Lie group, $\mathcal{L}(G)$ its Lie algebra,

$$J_\mu = (g^{-1}\partial_\mu g) \in \mathcal{L}(G), \tag{2}$$

$g : \mathbf{R^2} \to G,\ \mu,\nu \in \{0,1\},\ \eta = diag(1,-1)$, L is a $G$-dependent symmetric nondegenerate bilinear form. We consider the bilinear form $L$ as a metric on the group manifold and the generalization of principal models from ad-invariant Killing form on $\mathcal{L}(G)$ to more general case enables us to introduce the principal models also on non–semisimple groups.

Lie products of elements of the basis of $\mathcal{L}(G)$ define the structure coefficients

$$[t_a, t_b] = c_{ab}{}^c t_c \tag{3}$$

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and in the same basis we define the coordinates of the field $J_\nu$

$$J_\nu = g^{-1}\partial_\nu g = J_\nu^b t_b. \tag{4}$$

Fields automatically satisfy Bianchi identities

$$\partial_\mu J_\nu - \partial_\nu J_\mu + [J_\mu, J_\nu] = 0. \tag{5}$$

Varying the action (1) we obtain the equations of motion for the generalized principal chiral models

$$\partial_\mu J^{\mu,a} + \Gamma^a_{bc} J^b_\mu J^{\mu,c} = 0 \tag{6}$$

where the connection $\Gamma$ is defined by

$$\Gamma^a_{bc} = \frac{1}{2}(L^{-1})^{ad}(c_{db}{}^q L_{qc} + c_{dc}{}^q L_{qb} + U_b L_{cd} + U_c L_{bd} - U_d L_{bc}). \tag{7}$$

The vector fields $U_a$ are defined in the local group coordinates $\theta_i$ as

$$U_a = U_a^i(\theta)\frac{\partial}{\partial\theta_i} \tag{8}$$

where the matrix $U$ is the inverse of the matrix $V$ of vielbein coordinates

$$U_a^i(\theta) = (V^{-1})_a^i(\theta),\ V_i^a = (g^{-1}\frac{\partial g}{\partial\theta_i})^a. \tag{9}$$

Note that the connection (7) is symmetric in the lower indices, $\Gamma^a_{bc} = \Gamma^a_{cb}$.

### 1.1 Lax pairs

The ansatz that we are going to use for the Lax operators $X_0, X_1$ of the generalized principal chiral models is

$$X_0 = \partial_0 + P_{ab}J_0^b t_a + Q_{ab}J_1^b t_a + A_a t_a, \tag{10}$$

$$X_1 = \partial_1 + \tilde{Q}_{ab}J_0^b t_a + \tilde{P}_{ab}J_1^b t_a + B_a t_a, \tag{11}$$

where $P, Q, \tilde{P}, \tilde{Q}$ are four arbitrary constant $dimG \times dimG$ matrices and $A, B$ are two arbitrary constant vectors.

By explicit evaluation of the zero curvature condition

$$[X_0, X_1] = 0, \tag{12}$$

using the equations of motions (6) and Bianchi identities (5) and equating the coefficients of different powers and derivatives of $J^a_\mu$ one finds following necessary and sufficient conditions that the operators $X_0, X_1$ form a Lax pair:

$$\tilde{P} = P, \quad \tilde{Q} = Q, \quad \exists\, Q^{-1} \tag{13}$$

$$(P_{bp}P_{cq} - Q_{bp}Q_{cq})c_{bc}{}^a = P_{ab}c_{pq}{}^b, \tag{14}$$

$$\frac{1}{2}c_{cd}{}^a(P_{cp}Q_{dq} + P_{cq}Q_{dp}) = Q_{ab}\Gamma^b_{pq}, \tag{15}$$

$$c_{cd}{}^a(P_{cp}B_d + A_c Q_{dp}) = 0, \tag{16}$$

$$c_{cd}{}^a(Q_{cp}B_d + A_c P_{dp}) = 0, \tag{17}$$

$$c_{cd}{}^a A_c B_d = 0. \tag{18}$$

Moreover, the previous equations impose a condition on $\Gamma$. Namely, we can express (15) in an equivalent form

$$\frac{1}{2}(Q^{-1})^{ba}c_{cd}{}^{a}(P_{cp}Q_{dq}+P_{cq}Q_{dp})=\Gamma^{b}_{pq} \tag{19}$$

and conclude that **only the generalized principal models with the constant connection $\Gamma$ admit the Lax formulation** (10)–(12) because the left–hand side of the previous equation is constant.

## 2 2–dimensional solvable group

Every non-Abelian two–dimensional connected Lie group is isomorphic to the group of affine transformations of real line. Let us denote it by $Af(1)$. We have used its matrix realisation with the following parametrisation ($\theta_1$, $\theta_2 \in \mathbf{R}$)

$$g(\theta_1,\theta_2)=\begin{pmatrix} \exp(\theta_1) & \theta_2 \\ 0 & 1 \end{pmatrix} \tag{20}$$

There exist for the group $Af(1)$ two classes of metrics allowing Lax pair of the form considered:

1. One class of metrics with Lax formulation, leading to equations of motion

$$\partial_\mu\partial^\mu\theta_1=0 \quad , \quad \partial_\mu\partial^\mu\theta_2+Ke^{\theta_1}\left[\left(\frac{\partial\theta_1}{\partial x_0}\right)^2-\left(\frac{\partial\theta_1}{\partial x_1}\right)^2\right]=0 \tag{21}$$

   The first equation is just the wave equation, its general solution has the well–known form $\theta_1=F(x_0-x_1)+G(x_0+x_1)$. We can then substitute this solution into the second equation and find a linear equation for $\theta_2$.

2. The class of metrics of the form

$$L(\theta_1,\theta_2)=\alpha e^{\theta_1}\begin{pmatrix} \frac{-1+K^2\kappa^2}{\kappa^2} & -K \\ -K & 1 \end{pmatrix} \tag{22}$$

   where $K\in\mathbf{R}$, $\alpha,\kappa\in\mathbf{R}\setminus\{0\}$. Its equations of motion read

$$\begin{aligned} \partial_\nu\partial^\nu\theta_1+\frac{1}{2}\partial_\nu\theta_1\partial^\nu\theta_1-\frac{1}{2}\kappa^2(K^2\partial_\nu\theta_1\partial^\nu\theta_1 & \\ -2Ke^{-\theta_1}\partial_\nu\theta_1\partial^\nu\theta_2+e^{-2\theta_1}\partial_\nu\theta_2\partial^\nu\theta_2) &= 0, \qquad (23)\\ \partial_\nu\partial^\nu\theta_2-Ke^{\theta_1}\partial_\nu\partial^\nu\theta_1 &= \partial_\nu\theta_1\partial^\nu\theta_2 \qquad (24) \end{aligned}$$

   and the Lax pair reads

$$X_0=\begin{pmatrix} \partial_0+\frac{1}{2}Y_0+\lambda, & Y_1-2\lambda \\ 0 & \partial_0 \end{pmatrix},\ X_1=\begin{pmatrix} \partial_1+\frac{1}{2}Y_1+\lambda, & Y_0-2\lambda \\ 0 & \partial_1 \end{pmatrix} \tag{25}$$

   where $Y_0=J^1_0-\kappa J^2_1+K\kappa J^1_1$, $Y_1=J^1_1-\kappa J^2_0+K\kappa J^1_0$ and $J^1_\mu=\partial_\mu\theta_1$, $J^2_\mu=e^{-\theta_1}\partial_\mu\theta_2$, $\lambda$ may be interpreted as a spectral parameter.

In order to get deeper understanding of the model considered we explicitly evaluate corresponding Lax equation (12) and find

$$\partial_0 Y_1 - \partial_1 Y_0 = 0, \qquad \partial_0 Y_0 - \partial_1 Y_1 + \frac{1}{2}(Y_0 Y_0 - Y_1 Y_1) = 0. \tag{26}$$

We may consider the first of these equations (26) a condition for existence of $\phi$ such that

$$Y_\mu = 2\frac{\partial_\mu \phi}{\phi} \tag{27}$$

and the second equation (26) becomes

$$\partial_\mu \partial^\mu \phi = 0. \tag{28}$$

Once one has a solution $\phi$ of the wave equation (28), he may substitute this solution into (27). After explicit calculation of $Y_\mu$ in this model one finds

$$\partial_\mu \theta_1 - \kappa \mathrm{e}^{-\theta_1} \partial_{\bar{\mu}} \theta_2 + K\kappa \partial_{\bar{\mu}} \theta_1 = 2\frac{\partial_\mu \phi}{\phi}, \tag{29}$$

where $\bar{\mu}$ is defined $\bar{1} = 0, \bar{0} = 1$.

After substitution $\mathrm{e}^{\theta_1} = \rho$, $\kappa\theta_2 = W$ we finally obtain a set of linear partial differential equations for $\rho, W$

$$\partial_0 \rho - \partial_1 W + K\kappa \partial_1 \rho = 2\frac{\partial_0 \phi}{\phi}\rho, \ \partial_1 \rho - \partial_0 W + K\kappa \partial_0 \rho = 2\frac{\partial_1 \phi}{\phi}\rho. \tag{30}$$

**We have thus transformed the original nonlinear problem into several steps, each containing linear equations only.** This approach can be used to find some simple solutions of the principal chiral model (23–24), but it is probably impossible to write explicitly $\rho, W$ (and consequently $\theta_1, \theta_2$) for a general solution $\phi$ of the wave equation (27). We also see that we have linearized the equations (23–24) without inverse spectral transform. On the other hand, the Lax pair (25) proved to be useful for guessing the linearizing transformation (27).

## 3 3–dimensional solvable Lie groups

Models on 3-dimensional solvable Lie groups were investigated in [6]. It was shown that most of such groups allow models of the following form only:

$$\begin{aligned} \partial_\mu J^{\mu,A} + 2\Gamma^A_{B3} J^B_\mu J^{\mu,3} + \Gamma^A_{33} J^3_\mu J^{\mu,3} &= 0 \\ \partial_\mu J^{\mu,3} &= 0 \end{aligned}$$

where $J^3_\mu = \partial_\mu \theta_3$, $J^A_\mu$ are linear in $\partial_\mu \theta_B$ and $\theta_B$ (and nonlinear in $\theta_3$); i.e. the equation of motion for $\theta_3$ is just the wave equation $\partial_\mu \partial^\mu \theta_3 = 0$ and $J^1_\mu$, $J^2_\mu$ are linear in $\theta_1, \theta_2$ and their derivatives and consequently the equations of motion for

$\theta_{1,2}$ after substitution of the explicit form of $\theta_3$ turn out to be a system of two coupled linear partial differential equations for unknown $\theta_1, \theta_2$.

The only exceptions, allowing equations of motion of a different form, are 3-dimensional nilpotent group, i.e. Heisenberg group, and centrally extended $Af(1)$ group. These cases were considered separately.

The Heisenberg group leads to models that can be written again in terms of linear equations (although not of the form given above). The case of centrally extended $Af(1)$ group was investigated using computer algebra system and we have also found no intrinsically nonlinear model, i.e. the results are again similar to the previous one.

## 4 Generalized principal chiral models on $SU(2)$

As mentioned in the introduction to the first chapter, chiral models on simple groups were the original ones considered because of nondegeneracy of their Killing form. Results concerning the case of models with such ad-invariant metrics and corresponding inverse scattering method were published firstly in [1]. The generalization by Sochen [2] allowed to consider also the case with nonconstant metric. In the paper [5] one of us has tried to construct such model on $SU(2)$ group for diagonal metric, but has not found any.

In the following we present a simple explanation why there is no such model with diagonal nonconstant metric on $SU(2)$. We use the usual basis of $su(2)$ with the structure coefficients $c^c_{ab} = i\epsilon_{abc}$. As was mentioned in [5], the connection $\Gamma$ in the case of diagonal metric on $SU(2)$ has a following form (no sums over repeated indices):

$$i\Gamma^a_{bc} = \epsilon_{abc}\frac{L_{bb}-L_{cc}}{2L_{aa}},\ \forall a\neq b, a\neq c, c\neq b, \tag{31}$$

$$i\Gamma^a_{bb} = -\frac{U_a L_{bb}}{2L_{aa}},\ \forall a\neq b, \tag{32}$$

$$i\Gamma^a_{ab} = i\Gamma^a_{ba} = \frac{U_b L_{aa}}{2L_{aa}}, \tag{33}$$

If we write explicitly the equations (31) for different choices of indices, we find

$$L_{22} = 2i\Gamma^1_{23}L_{11} + L_{33}, \tag{34}$$

$$L_{33} = 2i\Gamma^2_{31}L_{22} + L_{11}, \qquad L_{11} = 2i\Gamma^3_{12}L_{33} + L_{22}. \tag{35}$$

We eliminate from (35) $L_{22}$ using the equation (34) and find

$$(1-2i\Gamma^1_{23})L_{11} = (2i\Gamma^3_{12}+1)L_{33}, \tag{36}$$

$$(4i\Gamma^1_{23}i\Gamma^2_{31}+1)L_{11} = (1-2i\Gamma^2_{31})L_{33}. \tag{37}$$

Since $\Gamma$s are constant due to (19), we find that $L_{11}$ is a constant multiple of $L_{33}$ (otherwise the nonsingularity of the metric $L$ would require all coefficients in the equations (36–37) be zero, i.e. $\Gamma^1_{23} = -\frac{i}{2}, \Gamma^3_{12} = \frac{i}{2}, \Gamma^2_{31} = -\frac{i}{2}$ and $-4\Gamma^1_{23}\Gamma^2_{31} + 1 = 2 = 0$ leading to a contradiction).

Together with (34) we have found that $L_{11}$ and $L_{22}$ are constant multiples of $L_{33}$. Using the relation

$$\det L = \text{const.} \tag{38}$$

proven in [5] we find $\det L = K(L_{33})^3 = \text{const.}$ (where $K \neq 0$ is a certain constant), i.e. $L_{33}$ = Const.. Therefore, **the only diagonal metrics $L$ admitting the Lax formulation in the form considered (i.e. linear in currents) are the constant ones.** It was shown in [5] that constant diagonal metrics always allow such a Lax pair.

## 5 Conclusion

We have found no interesting, truly nonlinear integrable model on any 2– and 3– dimensional non–semisimple Lie group. We don't know whether it is due to our ansatz (10–11) for Lax operators or whether it is a general property of principal models on non–semisimple groups.

The investigation of principal models on the group $SU(2)$ with diagonal metric and Lax pair linear in currents can be considered complete: only models with constant metric allow Lax pair and all models with constant metric have a Lax pair.

Classification of principal chiral models with nonconstant non-diagonal metric on $SU(2)$ seems to be technically unfeasible in the present time.

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