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DIPLOMA THESIS

Quantum Doubles and Universal $\mathcal{R}$-matrices

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I declare that I have written my Diploma Thesis independently using the cited literature.

Introduction

During last two decades the theory of Hopf algebras has been continuously developing. Although basic properties of Hopf algebras were studied already in 1960's (see for example [8]), the most important impulse for the study of Hopf algebras came from the quantum inverse scattering method in early 1980's. This approach lead to notions of quasitriangular Hopf algebras, quantum universal enveloping algebras and quantum matrix groups and also connected Hopf algebras with the Yang-Baxter equation. (The most remarkable articles from this period include [1] and [2].) One of the most significant results of this epoch was Drinfeld's quantum double published in [1]. The quantum double (in one of several equivalent definitions) is basically a tensor product $A \otimes A^*$ (as coalgebras) of an arbitrary Hopf algebra A with its dual coalgebra A^* (and with multiplication on A^* twisted). Defining a special multiplication on $A \otimes A^*$ one obtains a new Hopf algebra $D(A)$. This new Hopf algebra $D(A)$ possesses an important feature: it has a so-called universal $\mathcal{R}$-matrix, i.e. it is quasitriangular. Different examples of quantum doubles were found during subsequent years and also the connection of quantum doubles to quantum universal enveloping algebras (due to works of Majid all quantum enveloping algebras like $U_q(sl_2)$ can be written as factor bialgebras of some quantum doubles - see e.g. [5]).

In the presented Diploma Thesis I have studied the possibilities and problems of the construction of Drinfeld's quantum doubles using a method published by A.A. Vladimirov in [9]. The original motivation for this research came from Vladimirov's article and from successes in classifying the solutions of different Yang-Baxter systems, namely from work of L. Hlavaty [4]. Vladimirov's result was roughly speaking a statement that to every triple of invertible $n^2 \times n^2$ matrices (where $n \in \mathcal{N}$) W, X, Z satisfying Yang-Baxter-like equations

$$\begin{array}{rcl} [W,W,W]=0 & , & [Z,Z,Z]=0 \\ [W,X,X]=0 & , & [X,X,Z]=0 \end{array}$$

($[A,B,C]$ being so-called Yang-Baxter commutator, $[A,B,C] = A_{12}B_{13}C_{23}-$

$C_{23}B_{13}A_{12}$) it should be possible to construct an associated quantum double.

The original idea of my research was to find firstly the solutions of this Yang-Baxter system in dimension 2 and then to use them in the construction of corresponding quantum doubles. In my pre-Diploma work finished last year we found and published (together with L. Hlavaty) the complete solution of the given Yang-Baxter system (see [3]). In the present work I have at first tried to use those solutions in order to construct associated quantum doubles. As I will show later, this approach had to be modified and the current work is surprisingly almost independent on the previous one.

The whole text is divided into four chapters. In the first chapter I briefly recall the theory of Hopf algebras, quasitriangular Hopf algebras ("quantum groups") and quantum doubles. In the second chapter I explain in detail Vladimirov's method for the construction of quantum doubles. Also some difficulties and problems of this method are described there. In the third chapter I present possible modifications of this method. In the fourth chapter I show several examples of the method explained in the previous chapters and the resulting quantum doubles, some of them seem not to be published until now.

Chapter 1

Fundamentals of Theory of Hopf Algebras and Quantum Doubles

In this chapter I recall most important facts from the theory of Hopf algebras. At the end of this chapter, the quantum double as a special construction of quasitriangular Hopf algebras and its possible generalisations are considered. All facts are given in the form of definitions, theorems and remarks. Proofs are given usually in order to introduce some useful techniques or to clarify the meaning of the definitions and of the notation. Most of the proofs in this chapter are modified, usually more detailed versions of proofs given in [5]. Some proofs of theorems and remarks are omitted and can be found in the literature, e.g. [5]. Notation is based on Majid's book [5].

1.1 Basic definitions and notation

In following definitions I use notation:

- k denotes a field.
- $(A, +, k)$ denotes a vector space over field k.
- If not specified, all algebraic structures are understood to be over field k.
- A^* denotes vector space dual to vector space A, i.e. space of linear maps $A \to k$. Action of elements $\phi \in A^*$ on $a \in A$ is written $\phi(a) = \langle \phi, a \rangle$. Vector spaces A and A^{**} are identified in finite-dimensional cases.

- All maps are understood to be linear.
- *id* denotes identical map.
- Dot indicating multiplication is often omitted.
- τ denotes the twist map, i.e. $\tau : A \otimes A \to A \otimes A$ such that $\tau(a \otimes b) = b \otimes a,\ \forall a, b \in A$.
- $\mathbf{1}$ denotes the unit matrix (in suitable dimension) or the unit element in an algebra according to the context. To prevent misunderstanding I often denote unit matrix with subscript indicating its rank (e.g. $\mathbf{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.). The unit elements in algebras are always without indices.

Definition 1.1.1 *A* **formal linear span** $\mathcal{L}[M]$ *of a given set* M *is a vector space (over field* k *) of functions* $\phi : M \to k \wedge (M - \mathrm{Ker}\phi)$ *is finite, with pointwise operations* $(\phi + \psi)(m) = \phi(m) + \psi(m),\ \forall \phi, \psi \in \mathcal{L}[M], m \in M$ *and* $(\alpha\phi)(m) = \alpha\phi(m),\ \forall \phi \in \mathcal{L}[M], m \in M, \alpha \in k$.

Remark: Any element $\phi \in \mathcal{L}[M]$ can be written $\phi = \sum_{x \in M} \phi(x)\psi_x$ where ψ_x denotes function $\psi_x(y) = \delta_{xy}$.

Definition 1.1.2 *A* **tensor product of two vector spaces** V, W *is a factor space* $\mathcal{L}[V \times W]/\mathcal{Z}$*, where* $\mathcal{Z}$ *is a linear subspace generated by elements* $\psi_{(v,w_1+w_2)} - \psi_{(v,w_1)} - \psi_{(v,w_2)}, \psi_{(v_1+v_2,w)} - \psi_{(v_1,w)} - \psi_{(v_2,w)}, \psi_{(\alpha v,w)} - \alpha\psi_{(v,w)}, \psi_{(v,\alpha w)} - \alpha\psi_{(v,w)}, \forall v, v_1, v_2 \in V, w, w_1, w_2 \in W, \alpha \in k$. *After factorization I denote* $v \otimes w = [\psi_{(v,w)}]$.

Definition 1.1.3 *Let* A, B, C, D *be vector spaces and* ϕ, ψ *be maps* $\phi : A \to B$, $\psi : C \to D$. *A* **tensor product of maps** $\phi \otimes \psi$ *is a map* $A \otimes C \to B \otimes D$ *defined on elements* $a \in A, c \in C : (\phi \otimes \psi)(a \otimes c) = \phi(a) \otimes \psi(c)$ *and extended linearly.*

Definition 1.1.4 *An* **(associative) algebra (with unit)** $(A, +, ., \eta, k)$ *(or abbreviated* $(A, ., \eta)$*) is a vector space (A,+,k) for which there are linear maps* $. : A \otimes A \to A$ *and* $\eta : k \to A$ *such that* $a.(b.c) = (a.b).c,\ \forall a, b, c, \in A$ *(the associativity axiom) and* $a.\eta(\alpha) = \eta(\alpha).a = \alpha a\ \forall \alpha \in k, a \in A$ *(the existence of a unit) (see Fig.1.1).*

Remark: This can be reformulated in an usual way defining the unit element of A $\mathbf{1} = \eta(1)$. Vice versa defining $\eta(\alpha) = \alpha\mathbf{1}$ one obtains the definition given above.

$$\begin{tikzcd}
A\otimes A\otimes A \arrow[r, "(id\otimes .)"] \arrow[d, "(.\otimes id)"'] & A\otimes A \arrow[d, "."] \\
A\otimes A \arrow[r, "."] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes A \arrow[dr, "."] & \\
k\otimes A \arrow[u, "\eta\otimes id"] \arrow[r, equal] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes A \arrow[dr, "."] & \\
A\otimes k \arrow[u, "id\otimes\eta"] \arrow[r, equal] & A
\end{tikzcd}$$

Figure 1.1: Algebra axioms

$$\begin{tikzcd}
A\otimes A\otimes A & A\otimes A \arrow[l, "(id\otimes\Delta)"'] \\
A\otimes A \arrow[u, "(\Delta\otimes id)"] & A \arrow[l, "\Delta"'] \arrow[u, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes A \arrow[d, "\epsilon\otimes id"'] & \\
k\otimes A \arrow[r, equal] & A \arrow[ul, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes A \arrow[d, "id\otimes\epsilon"'] & \\
A\otimes k \arrow[r, equal] & A \arrow[ul, "\Delta"']
\end{tikzcd}$$

Figure 1.2: Coalgebra axioms

Definition 1.1.5 *Let $(A, ._1, \eta_1)$ and $(B, ._2, \eta_2)$ be algebras. An* **algebra map** *$\phi : A \to B$ is a linear map satisfying $\phi(a)._2\phi(b) = \phi(a._1 b)$, $\forall a, b, \in A$ and $\phi(\eta_1(\alpha)) = \eta_2(\alpha)$, $\forall \alpha \in k$.*

Remark: Let A, B be two algebras. Then I may define their **tensor product** $A \otimes B$ as the tensor product of vector spaces with the multiplication : $(a^1 \otimes b^1).(a^2 \otimes b^2) = (a^1.a^2 \otimes b^1.b^2)$, extended linearly (i.e. $\cdot_{A\otimes B} = (\cdot_A \otimes \cdot_B) \circ (id \otimes \tau \otimes id)$) and with the unit $\eta_{A\otimes B} = \eta_A \otimes \eta_B$.

Definition 1.1.6 *An algebra $(A, ., \eta)$ is said to be* **commutative** *if for any pair $a, b \in A$ is $a.b = b.a$.*

Definition 1.1.7 *Let V be a vector space. A* **tensor algebra** $(T(V), +, .)$ *is the noncommutative algebra defined on the vector space $T(V) = \sum_{k=0}^{\infty} V^{\otimes k}$ $= k \oplus V \oplus V \otimes V \oplus V \otimes V \otimes V \ldots$, where the addition $+$ is linear extension of addition on every $V \otimes V \otimes V \ldots \otimes V$ and the algebra multiplication is linear extension of maps $\cdot_{(m,n)} : V^{\otimes m} \otimes V^{\otimes n} \to V^{\otimes(m+n)} : a._{(m,n)}b = a \otimes b$.*

Definition 1.1.8 *A* **representation of an algebra** *A is a pair (ϕ, V) where V is a vector space and $\phi : A \otimes V \to V$ such that $\phi(ab \otimes u) = \phi(a \otimes \phi(b \otimes u))$, $\forall a, b \in A, u \in V$ and $\phi(\eta(\alpha) \otimes u) = \alpha u$, $\forall \alpha \in k, u \in V$. Also V is called a left A-module.*

Definition 1.1.9 *A* **(coassociative) coalgebra (with counit)** (A, Δ, ϵ) *is a vector space $(A, +, k)$ equipped with linear operations $\Delta : A \to A \otimes A$ and $\epsilon : A \to k$ such that $(id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta$ (the coassociativity axiom) and $(\epsilon \otimes id) \circ \Delta = (id \otimes \epsilon) \circ \Delta = id$ (the existence of a counit) (see Fig.1.2).*

Notation: $\Delta(h) \in A \otimes A$ can be written $\Delta(h) = \sum_i h^i_{(1)} \otimes h^i_{(2)}$, where $h^i_{(1)}, h^i_{(2)} \in A$. It is convenient to formally leave out index i and $\sum$ and write $\Delta(h) = h_{(1)} \otimes h_{(2)}$. In this notation the coassociativity axiom implies $h_{(1)(1)} \otimes h_{(1)(2)} \otimes h_{(2)} = h_{(1)} \otimes h_{(2)(1)} \otimes h_{(2)(2)}$. This allows a bit misleading ($h_{(2)}$ in expression involving also $h_{(3)}$ is not equal to usual $h_{(2)}$) but useful notation $h_{(1)} \otimes h_{(2)} \otimes h_{(3)} = h_{(1)(1)} \otimes h_{(1)(2)} \otimes h_{(2)} = h_{(1)} \otimes h_{(2)(1)} \otimes h_{(2)(2)}$ and similarly in expression involving more Δs. This notation will be used later in proofs.

Remark: Let (A, Δ, ϵ) be a coalgebra. On the vector space $(A^*, +, k)$ dual to the coalgebra (A, Δ, ϵ) it is possible to construct an algebra defining $. : A^* \times A^* \to A^* : \langle \phi.\psi, a\rangle = \langle \phi \otimes \psi, \Delta(a)\rangle$ and $\eta : k \to A^* : \langle \eta(\alpha), a\rangle = \alpha\epsilon(a)$. Vice versa, for a finite-dimensional algebra A it is possible to define coalgebra structure on A^*, i.e. notions of algebra and coalgebra are dual.

Definition 1.1.10 *Let $(A, \Delta_1, \epsilon_1)$ and $(B, \Delta_2, \epsilon_2)$ be coalgebras. A* **coalgebra map** *$\psi : A \to B$ is a linear map satisfying $\Delta_2 \circ \psi = (\psi \otimes \psi)\Delta_1$ and $\epsilon_1 = \epsilon_2 \circ \psi$.*

Remark: Let A, B be two coalgebras. Then I may define their **tensor product** $A \otimes B$ with following operations: $\Delta(a \otimes b) = a_{(1)} \otimes b_{(1)} \otimes a_{(2)} \otimes b_{(2)}$, $\epsilon(a \otimes b) = \epsilon(a)\epsilon(b)$, $\forall a \in A,\ b \in B$, extended linearly (i.e. $\Delta_{A\otimes B} = (id \otimes \tau \otimes id) \circ (\Delta_A \otimes \Delta_B) \wedge \epsilon_{a\otimes B} = \epsilon_A \otimes \epsilon_B$).

Definition 1.1.11 *A coalgebra (A, Δ, ϵ) is said to be* **cocommutative** *if $\tau \circ \Delta = \Delta$.*

1.2 Hopf algebras

Definition 1.2.1 *A* **bialgebra** *$(A, ., \eta, \Delta, \epsilon)$ is simultaneously an algebra $(A, ., \eta)$ and a coalgebra (A, Δ, ϵ) such that Δ, ϵ are algebra maps (or equivalently $., \eta$ are coalgebra maps).*

Definition 1.2.2 *A bialgebra $(H, ., \eta, \Delta, \epsilon)$ is called a* **Hopf algebra** *if there exists a linear map $S : H \to H$ called an antipode such that $. \circ (id \otimes S) \circ \Delta = . \circ (S \otimes id) \circ \Delta = \eta \circ \epsilon$ (see Fig. 1.3).*

Remark: For given bialgebra exists at most one antipode S, i.e. the antipode is unique.

Proof: Let me suppose that for given bialgebra $(A, ., \eta, \Delta, \epsilon)$ two antipodes S_1, S_2 exist. Then for an arbitrary element $h \in A$ holds :

$$\begin{array}{ccccc} A \otimes A & & \xrightarrow{id \otimes S,\, S \otimes id} & & A \otimes A \\ \uparrow \scriptstyle{\Delta} & & & & \downarrow \scriptstyle{.} \\ A & \xrightarrow{\epsilon} & k & \xrightarrow{\eta} & A \end{array}$$

Figure 1.3: Properties of antipode

$S_1 h = S_1((id \otimes \epsilon) \circ \Delta) h = S_1(h_{(1)} \epsilon(h_{(2)})) = S_1(h_{(1)}) \epsilon(h_{(2)}) = S_1(h_{(1)}).(\eta \circ \epsilon(h_{(2)})) = S_1(h_{(1)}).(h_{(2)(1)}.S_2 h_{(2)(2)}) = S_1(h_{(1)(1)}).h_{(1)(2)}.S_2(h_{(2)}) = \epsilon(h_1) S_2(h_{(2)}) = S_2 h$, i.e. $S_1 = S_2$.

Remark: Antipode S obeys $. \circ (S \otimes S) = S \circ . \circ \tau$, $S \circ \eta = \eta$, $(S \otimes S) \circ \Delta = \tau \circ \Delta \circ S$, $\epsilon \circ S = \epsilon$. (see e.g. [5])

Example 1.2.1 *Let G be a finite group, e be its unit. Then $k(G) = \{\phi : G \to k\}$ with following operations $(\phi + \psi)(u) = \phi(u) + \psi(u)$, $(\lambda \phi)(u) = \lambda \phi(u)$, $(\phi.\psi)(u) = \phi(u).\psi(u)$, $\eta(\lambda)(u) = \lambda$, $(\Delta(\phi))(u, v) = \phi(u.v)$, $\epsilon(\phi) = \phi(e)$, $(S\phi)(u) = \phi(u^{-1})$, $\forall u, v \in G, \phi, \psi \in k(G), \lambda \in k$ is a commutative Hopf algebra.*

Example 1.2.2 *Let G be a finite group, e be its unit. Let $kG = \mathcal{L}[G]$, i.e. any $a \in kG$ can be written $a = \sum_{u \in G} a_u \psi_u \equiv \sum_{u \in G} a_u u$. Then kG with operations $a.b = \sum_{u,v \in G} a_u b_v (u.v)$, $\eta(\lambda) = \lambda.e$, $\Delta(a) = \sum_{u \in G} a_u u \otimes u$, $\epsilon(a) = \sum_{u \in G} a_u$, $S(a) = \sum_{u \in G} a_u u^{-1}$, where $a = \sum_{u \in G} a_u u, b = \sum_{u \in G} b_u u \in kG, \lambda \in k$, is a cocommutative Hopf algebra.*

Example 1.2.3 *Let $\mathcal{G}$ be a Lie algebra. Then the factor algebra $U(\mathcal{G}) = T(\mathcal{G})\mathrm{mod}([\xi, \zeta] = \xi\zeta - \zeta\xi)$ with the coproduct $\Delta(\xi) = \xi \otimes 1 + 1 \otimes \xi$, the counit $\epsilon(\xi) = 0, \epsilon(1) = 1$ and the antipode $S\xi = -\xi$ (all maps are defined on elements of $\xi \in \mathcal{G}$ and extended as algebra maps (antialgebra map in case of antipode) on the tensor algebra) is a cocommutative Hopf algebra called an* **universal enveloping algebra** *of Lie algebra $\mathcal{G}$.*

Remark: Let A, B be two bialgebras (Hopf algebras). Then I may define their **tensor product** $A \otimes B$, which is simultaneously the tensor product of algebras and of coalgebras (It can be proved that it is a bialgebra). In the case of Hopf algebras I define $S(a \otimes b) = S(a) \otimes S(b)$ extended linearly.

Remark: Let H be a Hopf algebra. Then I may define a bialgebra H^{op} (H^{cop}) by changing multiplication $a._{H^{op}} b = b._H a$, i.e. $._{H^{op}} = ._H \circ \tau$ (comultiplication $\Delta_{H^{cop}} = (\tau \circ \Delta_H)$) and leaving other operations unchanged. Given bialgebras are Hopf algebras if and only if antipode in H S_H is invertible. Then their antipode maps are equal to S_H^{-1}.

Definition 1.2.3 *Two Hopf algebras (or bialgebras - then the condition on antipodes should be left out) A, H are* **paired** *if there exists a map $\langle\ ,\ \rangle : A \otimes H \to k$ such that $\langle \phi\psi, h\rangle = \langle \phi \otimes \psi, \Delta(h)\rangle$, $\langle \Delta(\phi), h \otimes g\rangle = \langle \phi, hg\rangle$, $\langle \eta(\alpha), h\rangle = \alpha\epsilon(h)$, $\alpha\epsilon(\phi) = \langle \phi, \eta(\alpha)\rangle$, $\langle S\phi, h\rangle = \langle \phi, Sh\rangle$ $\forall \phi, \psi \in A, g, h \in H, \alpha \in k$. They are* **strictly dual pair** *if the pairing is nondegenerate (i.e. if $\langle \phi, h\rangle = 0$ $\forall h \in H$ implies $\phi = 0$ and $\langle \phi, h\rangle = 0$ $\forall \phi \in A$ implies $h = 0$).*

Motivation: Let H be a finite-dimensional Hopf algebra. Then using relations given above ($\phi, \psi \in H^*, g, h \in H$) one can define Hopf algebra structure on the vector space H^*, i.e. dual Hopf algebra.

Theorem 1.2.1 *A pairing between bialgebras (or Hopf algebras) A, B can always be made nondegenerate by factoring out biideals generated by elements that pair as zero with all elements of the other algebra (or Hopf algebra).*

Proof: Firstly I define algebra maps $j_1 : A \to B^* : j_1(\phi) = \langle \phi, \ \rangle$, $j_2 : B \to A^* : j_2(b) = \langle\ , b\rangle$ and find their kernels J_1, J_2 (from linear algebra is known that they are also ideals in algebras A, B). Then I construct factor algebras A/J_1,B/J_2. Since for any $b \in J_2$ $0 = \langle \phi\psi, b\rangle = \langle \phi \otimes \psi, \Delta(b)\rangle = \langle \phi, b_{(1)}\rangle\langle \psi, b_{(2)}\rangle$ $\forall \phi, \psi$, I conclude[1] $\Delta(J_2) \subset B \otimes J_2 + J_2 \otimes B$ and similarly $\Delta(J_1) \subset A \otimes J_1 + J_1 \otimes A$. Consequently Δ is well defined on A/J_1,B/J_2 and I have factor bialgebras. For Hopf algebras, if $b \in J_2$ then $\langle \phi, Sb\rangle = \langle S\phi, b\rangle = 0$ i.e. $SJ_2 \subset J_2$ and similarly $SJ_2 \subset J_2$, so I can correctly quotient also antipode S and I have dual Hopf algebras A/J_1,B/J_2.

Remark: Let A be a coalgebra and J be a linear subspace of A such that $\Delta(J) \subset A \otimes J + J \otimes A$. Then $(A \otimes A)/(A \otimes J + J \otimes A) \subset A/J \otimes A/J$ and I may correctly define the comultiplication Δ on A/J.

Definition 1.2.4 *Two Hopf algebras (or bialgebras - then the condition on antipodes should be left out) A, H are* **antidual pair** *if there exists a map*

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[1]**Proof:** Let me decompose $B = J_2 \oplus I$ where $\forall x \in I \setminus \{0\}$ $\exists \phi : \langle \phi, x\rangle \neq 0$. Then $B \otimes B = (J_2 + I) \otimes (J_2 + I) = J_2 \otimes J_2 + J_2 \otimes I + I \otimes J_2 + I \otimes I \subset B \otimes J_2 + J_2 \otimes B + I \otimes I$. It remains to derive that $\forall x \in I \otimes I$ relation $0 = \langle \phi \otimes \psi, x\rangle$ $\forall \phi, \psi \in A$ implies $x = 0$. This can be done expressing $x = \sum_i c_i \otimes d_i$ where $d_i \in I$ are linearly independent and $0 \neq c_i \in I$ (it is clear that any nonzero element of $I \otimes I$ can be expressed in this way). Fixing index i and finding $\phi \in A$ such that $\langle \phi, c_i\rangle \neq 0$ I find for any $\psi \in A$

$$0 = \langle \psi, d_i + \sum_{k \neq i} \frac{\langle \phi, c_k\rangle}{\langle \phi, c_i\rangle} d_k\rangle$$

i.e. $d_i + \sum_{k\neq i} \frac{\langle \phi, c_k\rangle}{\langle \phi, c_i\rangle} d_k \in J_2 \cap I = 0$, i.e. I have found a contradiction with the linear independency of d_i. The proof is finished.

$$
\begin{array}{ccccc}
A & \xrightarrow{\Delta} & A \otimes A = A \otimes A \otimes k & \xrightarrow{\tau \otimes R} & A \otimes A \otimes A \otimes A \\
\downarrow {\scriptstyle \Delta} & & & & \downarrow {\scriptstyle id \otimes \tau \otimes id} \\
A \otimes A = k \otimes A \otimes A & & & & A \otimes A \otimes A \otimes A \\
\downarrow {\scriptstyle \mathcal{R} \otimes id} & & & & \downarrow {\scriptstyle \cdot \otimes \cdot} \\
A \otimes A \otimes A \otimes A & \xrightarrow{id \otimes \tau \otimes id} & A \otimes A \otimes A \otimes A & \xrightarrow{\cdot \otimes \cdot} & A \otimes A
\end{array}
$$

$$
\begin{array}{ccc}
k & \xrightarrow{R} & A \otimes A \\
\downarrow {\scriptstyle \mathcal{R} \otimes \mathcal{R}} & & \downarrow {\scriptstyle id \otimes \Delta} \\
A \otimes A \otimes A \otimes A & & A \otimes A \otimes A \\
\downarrow {\scriptstyle id \otimes \tau \otimes id} & & \downarrow {\scriptstyle id} \\
A \otimes A \otimes A \otimes A & \xrightarrow{\cdot \otimes id \otimes id} & A \otimes A \otimes A
\end{array}
\qquad
\begin{array}{ccc}
k & \xrightarrow{R} & A \otimes A \\
\downarrow {\scriptstyle \mathcal{R} \otimes \mathcal{R}} & & \downarrow {\scriptstyle \Delta \otimes id} \\
A \otimes A \otimes A \otimes A & & A \otimes A \otimes A \\
\downarrow {\scriptstyle id \otimes \tau \otimes id} & & \downarrow {\scriptstyle id} \\
A \otimes A \otimes A \otimes A & \xrightarrow{id \otimes id \otimes \cdot} & A \otimes A \otimes A
\end{array}
$$

Figure 1.4: Axioms of quasitriangular Hopf algebra

$\langle\ ,\ \rangle : A \otimes H \to k$ *such that* $\langle \phi\psi, h\rangle = \langle \phi \otimes \psi, \Delta(h)\rangle$, $\langle \Delta(\phi), h \otimes g\rangle = \langle \phi, gh\rangle$, $\langle \eta(\alpha), h\rangle = \alpha\epsilon(h)$, $\alpha\epsilon(\phi) = \langle \phi, \eta(\alpha)\rangle$, $\langle S\phi, h\rangle = \langle \phi, S^{-1}h\rangle$ $\forall \phi, \psi \in A, g, h \in H, \alpha \in k$. *They are* **strictly antidual pair** *if the pairing is non-degenerate.*

Remark: An analog of previous theorem is valid also for antidual pair of bialgebras (Hopf algebras).

1.3 Quasitriangular Hopf algebras

Notation: Let $R = \sum_l R_l^{(1)} \otimes R_l^{(2)} \in A \otimes A$, $n \in \mathcal{N}$. Then $R_{ij} \in A^{\otimes n}$ is defined $R_{ij} = \sum_l \mathbf{1} \otimes \mathbf{1} \otimes \ldots \otimes R_l^{(1)} \otimes \ldots \otimes R_l^{(2)} \otimes \ldots \otimes \mathbf{1}$ where $R_l^{(1)}$ is on the i-th position and $R_l^{(2)}$ is on the j-th position.

Definition 1.3.1 *A pair* $H, \mathcal{R}$ *where* H *is a Hopf algebra and* $\mathcal{R} \in H \otimes H, \exists \mathcal{R}^{-1}$ *is called a* **quasitriangular Hopf algebra** *(or a* **quantum group** *) if and only if* $\mathcal{R}$ *satisfies* $(\Delta \otimes id)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}$, $(id \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}$ *and* $(\tau \circ \Delta)h = \mathcal{R}(\Delta(h))\mathcal{R}^{-1}$, $\forall h \in H$. *This definition can be also reformulated using commutative diagrams and* $\mathcal{R} : k \to H : \mathcal{R}(\alpha) = \alpha\mathcal{R}$ *(see Fig. 1.4).*

Remark: It is also possible to define a **quasitriangular bialgebra**. Its definition is analogous to the one given above, the only difference is that H is a bialgebra, not a Hopf algebra. Also the following theorems are valid for quasitriangular bialgebras except the statements involving antipodes.

Remark: $\mathcal{R}$ is not unique, there may exist several $\mathcal{R}$s for given H. For example if H is cocommutative then $\mathcal{R} = \mathbf{1} \otimes \mathbf{1}$ satisfies conditions given in the definition above but for special H also exists another $\mathcal{R}$ satisfying given conditions. (see Section 4.1)

Theorem 1.3.1 *Let $(H, \mathcal{R})$ be a quasitriangular Hopf algebra. Then $(\epsilon \otimes id)\mathcal{R} = (id \otimes \epsilon)\mathcal{R} = \mathbf{1}$, $(S \otimes id)\mathcal{R} = \mathcal{R}^{-1}, (id \otimes S)\mathcal{R}^{-1} = \mathcal{R}$.*

Proof: $(\epsilon \otimes id \otimes id)(\Delta \otimes id)\mathcal{R} = (\epsilon \otimes id \otimes id)\mathcal{R}_{13}\mathcal{R}_{23} = \sum(\epsilon(\mathcal{R}^{(1)})\mathbf{1} \otimes \mathcal{R}^{(2)}) \sum(\mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}) = \sum(\epsilon(\mathcal{R}^{(1)})\mathbf{1} \otimes \mathcal{R}^{(2)})\mathcal{R} \;\wedge\; (\epsilon \otimes id \otimes id)(\Delta \otimes id)\mathcal{R} = (((\epsilon \otimes id)\Delta) \otimes id)\mathcal{R} = (id \otimes id)\mathcal{R} \Rightarrow (\epsilon \otimes id)\mathcal{R} = \mathbf{1}$, $(id \otimes \epsilon)\mathcal{R} = \mathbf{1}$ can be proved in a similar way. $(S \otimes id)$: I evaluate $\sum\sum \mathcal{R}^{(1)}_{(1)} S\mathcal{R}^{(1)}_{(2)} \otimes \mathcal{R}^{(2)}$ in two ways: $\sum\sum \mathcal{R}^{(1)}_{(1)} S\mathcal{R}^{(1)}_{(2)} \otimes \mathcal{R}^{(2)} = (.(id \otimes S)\Delta\mathcal{R}^{(1)}) \otimes \mathcal{R}^{(2)} = (\eta(\epsilon(\mathcal{R}^{(1)}))) \otimes \mathcal{R}^{(2)} = (\eta \otimes id)(\epsilon \otimes id)\mathcal{R} = \mathbf{1} \;\wedge \sum\sum \mathcal{R}^{(1)}_{(1)} S\mathcal{R}^{(1)}_{(2)} \otimes \mathcal{R}^{(2)} = (. \otimes id)(id \otimes S \otimes id)\mathcal{R}_{13}\mathcal{R}_{23} = (. \otimes id)(\mathcal{R}_{13}((S \otimes id)\mathcal{R})_{23}) = \mathcal{R}(S \otimes id)\mathcal{R}$. Consequently $(S \otimes id)\mathcal{R} = \mathcal{R}^{-1}$. The last part of the proof is analogous.

Theorem 1.3.2 *Let $(H, \mathcal{R})$ be a quasitriangular Hopf algebra. Then $\mathcal{R}$ satisfies abstract quantum Yang-Baxter equation $\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$.*

Proof: $(id \otimes (\tau \circ \Delta))\mathcal{R}$ can be evaluated in two different ways: $(id \otimes (\tau \circ \Delta))\mathcal{R} = (id \otimes \tau) \circ (id \otimes \Delta)\mathcal{R} = (id \otimes \tau)\mathcal{R}_{13}\mathcal{R}_{12} = \mathcal{R}_{12}\mathcal{R}_{13} \;\wedge\; (id \otimes (\tau \circ \Delta))\mathcal{R} = \mathcal{R}_{23}(id \otimes \Delta)\mathcal{R}\mathcal{R}_{23}^{-1} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}\mathcal{R}_{23}^{-1}$, together giving $\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$.

Motivation: Now I want to define a dual quasitriangular Hopf algebra. This can be done by "inverting arrows" in the commutative diagrams Fig. 1.4. The only problem is the existence of an inverse element to the map $\tilde{\mathcal{R}} : A \otimes A \to k$. This can be solved by defining a convolution algebra.

Definition 1.3.2 *Let C be a coalgebra, B be an algebra. I define an algebra structure (called* **convolution algebra** *) on the vector space* $\mathrm{Hom}(C, B)$ *of maps $C \to B$:*

$$(\phi.\psi)(c) = \phi(c_{(1)})._B\psi(c_{(2)}),\;\; \mathbf{1}(c) = \eta_B \circ \epsilon(c),\;\; \forall c \in C, \forall \phi, \psi \in \mathrm{Hom}(C, B).$$

Remark: The definition given above allows to define an inverse element (if it exists) to the map $\tilde{\mathcal{R}} : A \otimes A \to k$: $\tilde{\mathcal{R}}^{-1} : A \otimes A \to k$ and $\forall a, b \in A$

$$\tilde{\mathcal{R}}^{-1}(a_{(1)} \otimes b_{(1)})\tilde{\mathcal{R}}(a_{(2)} \otimes b_{(2)}) = \epsilon(a)\epsilon(b) = \tilde{\mathcal{R}}(a_{(1)} \otimes b_{(1)})\tilde{\mathcal{R}}^{-1}(a_{(2)} \otimes b_{(2)}).$$

Such inverse element is unique (if it exists) (*Proof:* Let there are two inverse elements $\tilde{\mathcal{R}}_1^{-1}$ and $\tilde{\mathcal{R}}_2^{-1}$. Then $\tilde{\mathcal{R}}_1^{-1}(a \otimes b) = \tilde{\mathcal{R}}_1^{-1}(a_{(1)} \otimes b_{(1)})\epsilon(a_{(2)})\epsilon(b_{(2)}) = \tilde{\mathcal{R}}_1^{-1}(a_{(1)} \otimes b_{(1)})\tilde{\mathcal{R}}(a_{(2)} \otimes b_{(2)})\tilde{\mathcal{R}}_2^{-1}(a_{(3)} \otimes b_{(3)}) = \epsilon(a_{(1)})\epsilon(b_{(1)})\tilde{\mathcal{R}}_2^{-1}(a_{(2)} \otimes b_{(2)}) = \tilde{\mathcal{R}}_2^{-1}(a \otimes b)$, i.e. $\tilde{\mathcal{R}}_1^{-1} = \tilde{\mathcal{R}}_2^{-1}$.)

$$\begin{array}{ccccc}
A & \xleftarrow{\cdot} & A \otimes A = A \otimes A \otimes k & \xleftarrow{\tau \otimes \tilde{R}} & A \otimes A \otimes A \otimes A \\
\uparrow {\scriptstyle \cdot} & & & & \uparrow {\scriptstyle id \otimes \tau \otimes id} \\
A \otimes A = k \otimes A \otimes A & & & & A \otimes A \otimes A \otimes A \\
\uparrow {\scriptstyle \tilde{\mathcal{R}} \otimes id} & & & & \uparrow {\scriptstyle \Delta \otimes \Delta} \\
A \otimes A \otimes A \otimes A & \xleftarrow{id \otimes \tau \otimes id} & A \otimes A \otimes A \otimes A & \xleftarrow{\Delta \otimes \Delta} & A \otimes A
\end{array}$$

$$\begin{array}{ccc}
k & \xleftarrow{\tilde{R}} & A \otimes A \\
\uparrow {\scriptstyle \tilde{\mathcal{R}} \otimes \tilde{\mathcal{R}}} & & \uparrow {\scriptstyle id \otimes \cdot} \\
A \otimes A \otimes A \otimes A & & A \otimes A \otimes A \\
\uparrow {\scriptstyle id \otimes \tau \otimes id} & & \uparrow {\scriptstyle id} \\
A \otimes A \otimes A \otimes A & \xleftarrow{\Delta \otimes id \otimes id} & A \otimes A \otimes A
\end{array}
\qquad
\begin{array}{ccc}
k & \xleftarrow{\tilde{R}} & A \otimes A \\
\uparrow {\scriptstyle \tilde{\mathcal{R}} \otimes \tilde{\mathcal{R}}} & & \uparrow {\scriptstyle \cdot \otimes id} \\
A \otimes A \otimes A \otimes A & & A \otimes A \otimes A \\
\uparrow {\scriptstyle id \otimes \tau \otimes id} & & \uparrow {\scriptstyle id} \\
A \otimes A \otimes A \otimes A & \xleftarrow{id \otimes id \otimes \Delta} & A \otimes A \otimes A
\end{array}$$

Figure 1.5: Axioms of dual quasitriangular Hopf algebra

Definition 1.3.3 *A pair* $A, \tilde{\mathcal{R}}$ *where* A *is a Hopf algebra and* $\tilde{\mathcal{R}} : A \otimes A \to k, \exists \tilde{\mathcal{R}}^{-1}$ *is called a* **dual quasitriangular Hopf algebra** *if and only if* $\tilde{\mathcal{R}}$ *satisfies*

$$\tilde{\mathcal{R}}(ab \otimes c) = \tilde{\mathcal{R}}(a \otimes c_{(1)})\tilde{\mathcal{R}}(b \otimes c_{(2)}),\ \forall a, b, c \in A$$

$$\tilde{\mathcal{R}}(a \otimes bc) = \tilde{\mathcal{R}}(a_{(1)} \otimes c)\tilde{\mathcal{R}}(a_{(2)} \otimes b),\ \forall a, b, c \in A$$

$$b_{(1)}a_{(1)}\tilde{\mathcal{R}}(a_{(2)} \otimes b_{(2)}) = \tilde{\mathcal{R}}(a_{(1)} \otimes b_{(1)})a_{(2)}b_{(2)},\ \forall a, b \in A$$

Remark: Given definition of dual quasitriangular algebra seems to be quite obscure, but it can also be rewritten in the form of commutative diagrams, see Fig. 1.5. Also, $\tilde{\mathcal{R}}$ will be later (in following chapters) denoted simply by $\mathcal{R}$.

Remark: Again, it is also possible to define a **dual quasitriangular bialgebra**. Its definition is similar to the one given above, the only difference is that A is a bialgebra, not a Hopf algebra. Also the following theorems are valid for quasitriangular bialgebras except the statements involving antipodes.

Theorem 1.3.3 *Let* $(A, \tilde{\mathcal{R}})$ *be a dual quasitriangular Hopf algebra. Then for any* $a, b \in A$ *is* $\tilde{\mathcal{R}}(a \otimes \mathbf{1}) = \tilde{\mathcal{R}}(\mathbf{1} \otimes a) = \epsilon(a), \tilde{\mathcal{R}}(Sa \otimes b) = \tilde{\mathcal{R}}^{-1}(a \otimes b), \tilde{\mathcal{R}}^{-1}(a \otimes Sb) = \tilde{\mathcal{R}}(a \otimes b), \tilde{\mathcal{R}}(Sa \otimes Sb) = \tilde{\mathcal{R}}(a \otimes b)$.

Proof: $\tilde{\mathcal{R}}(a \otimes \mathbf{1}) = \tilde{\mathcal{R}}(\epsilon(a_{(1)})a_{(2)} \otimes \mathbf{1}) = (\tilde{\mathcal{R}}^{-1}(a_{(1)(1)} \otimes \mathbf{1})\tilde{\mathcal{R}}(a_{(1)(2)} \otimes \mathbf{1}))\tilde{\mathcal{R}}(a_{(2)} \otimes \mathbf{1}) = \tilde{\mathcal{R}}^{-1}(a_{(1)} \otimes \mathbf{1})(\tilde{\mathcal{R}}(a_{(2)(1)} \otimes \mathbf{1})\tilde{\mathcal{R}}(a_{(2)(2)} \otimes \mathbf{1})) = \tilde{\mathcal{R}}^{-1}(a_{(1)} \otimes \mathbf{1})\tilde{\mathcal{R}}(a_{(2)} \otimes \mathbf{1}) = \epsilon(a), \forall a \in A$, the second equality is analogous. $\tilde{\mathcal{R}}(Sa_{(1)} \otimes b_{(1)})\tilde{\mathcal{R}}(a_{(2)} \otimes b_{(2)}) = \tilde{\mathcal{R}}((Sa_{(1)})a_{(2)} \otimes b) = \epsilon(a)\tilde{\mathcal{R}}(\mathbf{1} \otimes b) = \epsilon(a)\epsilon(b), \forall a, b \in A$ and it is also possible to prove $\tilde{\mathcal{R}}(a_{(1)} \otimes b_{(1)})\tilde{\mathcal{R}}(Sa_{(2)} \otimes b_{(2)}) = \epsilon(a)\epsilon(b)$, so $\tilde{\mathcal{R}}(Sa \otimes b) = \tilde{\mathcal{R}}^{-1}(a \otimes b)$. Similarly, $\tilde{\mathcal{R}}(Sa \otimes Sb)$ is convolution-inverse to $\tilde{\mathcal{R}}^{-1}$, so $\tilde{\mathcal{R}}(Sa \otimes Sb) = \tilde{\mathcal{R}}(a \otimes b)$.

Theorem 1.3.4 *Let $(A, \tilde{\mathcal{R}})$ be a dual quasitriangular Hopf algebra. Then $\tilde{\mathcal{R}}(a_{(1)} \otimes b_{(1)})\tilde{\mathcal{R}}(a_{(2)} \otimes c_{(1)})\tilde{\mathcal{R}}(b_{(2)} \otimes c_{(2)}) = \tilde{\mathcal{R}}(b_{(1)} \otimes c_{(1)})\tilde{\mathcal{R}}(a_{(1)} \otimes c_{(2)})\tilde{\mathcal{R}}(a_{(2)} \otimes b_{(2)})$ for any $a, b, c \in A$.*

Proof: $\tilde{\mathcal{R}}(a_{(1)} \otimes b_{(1)})\tilde{\mathcal{R}}(a_{(2)} \otimes c_{(1)})\tilde{\mathcal{R}}(b_{(2)} \otimes c_{(2)}) = \tilde{\mathcal{R}}(a \otimes c_{(1)}b_{(1)})\tilde{\mathcal{R}}(b_{(2)} \otimes c_{(2)}) = \tilde{\mathcal{R}}(a \otimes b_{(2)}c_{(2)})\tilde{\mathcal{R}}(b_{(1)} \otimes c_{(1)}) = \tilde{\mathcal{R}}(b_{(1)} \otimes c_{(1)})\tilde{\mathcal{R}}(a_{(1)} \otimes c_{(2)})\tilde{\mathcal{R}}(a_{(2)} \otimes b_{(2)})$

Theorem 1.3.5 *Let $(A, \tilde{\mathcal{R}})$ be a dual quasitriangular Hopf algebra. Let me define*

$$v(a) = \tilde{\mathcal{R}}(a_{(1)} \otimes Sa_{(2)}),\ v^{-1}(a) = \tilde{\mathcal{R}}(S^2a_{(1)} \otimes a_{(2)}).$$

Then $a_{(1)}v(a_{(2)}) = v(a_{(1)})S^2a_{(2)}$ and v^{-1} is the inverse of v in the convolution algebra $\mathrm{Hom}(A, k)$. *Similarly, definitions*

$$u(a) = \tilde{\mathcal{R}}(a_{(2)} \otimes Sa_{(1)}),\ u^{-1}(a) = \tilde{\mathcal{R}}(S^2a_{(2)} \otimes a_{(1)}).$$

imply that $u(a_{(1)})a_{(2)} = S^2a_{(1)}u(a_{(2)})$ and u^{-1} is the inverse of u in the convolution algebra $\mathrm{Hom}(A, k)$. *The antipode of A is bijective.*

Proof: First step in the proof is the proof of equality $a_{(1)}v(a_{(2)}) = v(a_{(1)})S^2a_{(2)}$. I remind that $\tilde{\mathcal{R}}, v$ etc. are k-valued maps, so their positions in the products are arbitrary. Using $\epsilon(b) = Sb_{(1)}.b_{(2)}$ and Definition 1.3.3 I evaluate: $a_{(1)}v(a_{(2)}) = a_{(1)}\tilde{\mathcal{R}}(a_{(2)} \otimes Sa_{(3)}) = a_{(1)}\tilde{\mathcal{R}}(a_{(2)} \otimes \epsilon((Sa_{(3)})_{(1)})(Sa_{(3)})_{(2)}) = (S(Sa_{(3)})_{(1)})(Sa_{(3)})_{(2)}$ $a_{(1)}\tilde{\mathcal{R}}(a_{(2)} \otimes (Sa_{(3)})_{(3)}) = (S(Sa_{(3)})_{(1)})a_{(2)}(Sa_{(3)})_{(3)}\tilde{\mathcal{R}}(a_{(1)} \otimes (Sa_{(3)})_{(2)}) = \tilde{\mathcal{R}}(a_{(1)} \otimes Sa_{(4)})S^2a_{(5)}a_{(2)}Sa_{(3)} = \tilde{\mathcal{R}}(a_{(1)} \otimes Sa_{(2)})S^2a_{(3)} = v(a_{(1)})S^2a_{(2)}$. (I have used $S(b_{(1)}) \otimes S(b_{(2)}) \otimes S(b_{(3)}) = (Sb)_{(3)} \otimes (Sb)_{(2)} \otimes (Sb)_{(1)}$.)

In order to prove that v and v^{-1} are mutually inverse elements of convolution algebra I firstly evaluate auxiliary expression: $\tilde{\mathcal{R}}(a_{(1)} \otimes a_{(3)})v(a_{(2)}) = \tilde{\mathcal{R}}(a_{(1)} \otimes a_{(4)})\tilde{\mathcal{R}}(a_{(2)} \otimes Sa_{(3)}) = \tilde{\mathcal{R}}(a_{(1)} \otimes (Sa_{(2)}a_{(3)})) = \epsilon(a_{(2)})\tilde{\mathcal{R}}(a_{(1)} \otimes \mathbf{1}) = \epsilon(a)$. Now I am able to evaluate $\epsilon(a) = \tilde{\mathcal{R}}(a_{(1)} \otimes a_{(3)})v(a_{(2)}) = v(a_{(1)})\tilde{\mathcal{R}}(S^2a_{(2)} \otimes a_{(3)}) = v(a_{(1)})v^{-1}(a_{(2)})$ and $\epsilon(a) = \tilde{\mathcal{R}}(a_{(1)} \otimes a_{(3)})v(a_{(2)}) = v(a_{(2)})\tilde{\mathcal{R}}(S^2a_{(1)} \otimes S^2a_{(3)}) = \tilde{\mathcal{R}}(S^2a_{(1)} \otimes a_{(2)})v(a_{(3)}) = v^{-1}(a_{(1)})v(a_{(2)})$ (I have used $\tilde{\mathcal{R}}(a \otimes b) = \tilde{\mathcal{R}}(Sa \otimes Sb)$.) giving the property that v and v^{-1} are mutually inverse. Proofs for u's are similar.

Finally I prove that S is bijective. I define $S^{-1}(a) = v(a_{(1)})Sa_{(2)}v^{-1}(a_{(3)})$ and check whether it is really an inverse of S: $S(S^{-1}(a)) = S(v(a_{(1)})Sa_{(2)}v^{-1}(a_{(3)})) = v(a_{(1)})S^2a_{(2)}v^{-1}(a_{(3)}) = a_{(1)}v(a_{(2)})v^{-1}(a_{(3)}) = a_{(1)}\epsilon(a_{(2)}) = a$ and $S^{-1}(S(a)) = v((Sa)_{(1)})S(Sa)_{(2)}v^{-1}((Sa)_{(3)}) = v(Sa_{(3)})S^2a_{(2)}v^{-1}(Sa_{(1)}) = u^{-1}(a_{(1)})S^2a_{(2)}$ $u(a_{(3)}) = a$ (because $v(Sa) = \tilde{\mathcal{R}}((Sa)_{(1)} \otimes S(Sa)_{(2)}) = \tilde{\mathcal{R}}(Sa_{(2)} \otimes S^2a_{(1)}) = \tilde{\mathcal{R}}(a_{(2)} \otimes Sa_{(1)}) = u(a)$). Since both S and S^{-1} are defined on the whole A, S is bijective. This **finishes the proof.**

1.4 Quantum matrices

Definition 1.4.1 *Let R be a $n^2 \times n^2$ matrix. Then I may define a* **quantum matrix bialgebra** A_R *in the following way: I suppose that there is a n^2-dimensional vector space V generated by linearly independent elements t^i_j. I rewrite generators as a formal matrix $T : T_{(i,j)} = t^i_j$. Secondly, I create the tensor algebra $T(V)$ defined earlier and factor out an ideal generated by relations $R_{12}T_1T_2 = T_2T_1R_{12}$ i.e. $R^{ij}_{kl}t^k_p t^l_q = t^j_k t^i_l R^{lk}_{pq}$ and I obtain A_R as an algebra. Thirdly I define comultiplication as an algebraic extension of the map $\Delta : V \to V \otimes V : \Delta(t^i_j) = t^i_k \otimes t^k_j$, i.e. $\Delta(T) = T \otimes T$ (in tensor products of formal matrices also matrix multiplication is assumed) and counit $\epsilon(t^i_j) = \delta^i_j$, i.e. $\epsilon(T) = \mathbf{1}_2$. It is possible to prove that structure just defined is really a bialgebra, so-called quantum matrix algebra A_R.*

Remark: If R is an invertible matrix fulfilling the Yang-Baxter equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$, then the quantum matrix bialgebra A_R is dual quasitriangular, the $\tilde{\mathcal{R}}$ is given by relations $\tilde{\mathcal{R}}(T \otimes \mathbf{1}) = \mathbf{1}_2 = \tilde{\mathcal{R}}(\mathbf{1} \otimes T)$ and $\tilde{\mathcal{R}}(T_1 \otimes T_2) = R_{12}$ extended by the relations from Definition 1.3.3. (Proof is given in ref. [5].)

1.5 Quantum doubles

Motivation: Following definition of a quantum double gives a canonical way how to construct a quasitriangular Hopf algebra from a strictly (anti)dual pair of Hopf algebras.

Definition 1.5.1 *Let H be a finite-dimensional Hopf algebra.* **The quantum double** $D(H)$ *in a form containing $H, H^{*op} \equiv H^o$ is a quasitriangular Hopf algebra generated by these as sub-Hopf algebras with the cross multiplication relations and the quasitriangular structure*

$$\langle h_{(1)}, a_{(1)}\rangle h_{(2)}a_{(2)} = a_{(1)}h_{(1)}\langle h_{(2)}, a_{(2)}\rangle, \ \mathcal{R} = \sum_j e^j \otimes e_j, \tag{1.1}$$

where $h \in H, a \in H^o$, e_j is a basis of H and e^j is a dual basis of H^o, i.e. $\langle e_j, e^k\rangle = \delta^k_j$.

Remark: The definition implies that $D(H) \cong H \otimes H^o \equiv H \otimes H^*$ as a vector space with canonical embedding of $H, H^o : h = h \otimes \mathbf{1}$, $a = \mathbf{1} \otimes a$. (It can be proved in a following way: Since $D(H)$ is generated by H, H^o, it consists (before factorization given by cross-multiplication rules) of linear combinations of finite chains like $a^1h^1a^2h^2 \dots$ and $h^1a^1h^2a^2 \dots$ where $h^1, h^2, \dots \in H$,

$a^1, a^2, \ldots \in H^o$. Using cross-multiplication rules rewritten in the form $ah = \epsilon(a_{(2)})\epsilon(h_{(2)})a_{(1)}h_{(1)} = a_{(1)}h_{(1)}\langle h_{(2)}Sh_{(3)}, a_{(2)}\rangle = a_{(1)}h_{(1)}\langle h_{(2)}, a_{(2)}\rangle\langle Sh_{(3)}, a_{(3)}\rangle$ $= \langle h_{(1)}, a_{(1)}\rangle h_{(2)}a_{(2)}\langle Sh_{(3)}, a_{(3)}\rangle$ allows to write any ah as a linear combination of elements like $\tilde{h}\tilde{a}$ and vice versa, i.e. it is possible to change the ordering in the chains and collapse any chain into a linear combination of elements in the form $h.a$. A vector space of such elements can be realised as $H \otimes H^o$.)

Remark: The definition of quantum double given above is slightly different from the one given by Drinfeld in [1] (the Drinfeld's quantum double was built above $H \otimes (H^*)^{cop}$ instead of $H \otimes (H^*)^{op}$), but it can be proved that resulting quantum doubles are isomorphic.

Remark: Definition given above can be extended on an infinite-dimensional case replacing H, H^{*op} with some strictly antidual pair of Hopf algebras H, H^o and giving sense to recipe on construction of $\mathcal{R}$ element. (In infinite-dimensional case it is a formal infinite linear combination.)

Theorem 1.5.1 *Let H be a finite-dimensional Hopf algebra. Then the quantum double $D(H)$ can be realised on the vector space $H \otimes H^*$ ($= H \otimes H^o$ as a vector space) with multiplication*

$$(h \otimes a).(g \otimes b) = \langle g_{(1)}, a_{(1)}\rangle h.g_{(2)} \otimes b.a_{(2)}\langle Sg_{(3)}, a_{(3)}\rangle \tag{1.2}$$

and tensor product comultiplication, unit and counit. (Operations on the right-hand side of the definition of product are in H and H^* and consequently already well-defined.)

Remark: Given theorem can be partly extended to infinite-dimensional case. Those extensions will be commented during the proof.

Proof: During the first part of the proof I will show that given structure is a bialgebra and fulfils the relation on cross-multiplication given in the definition of $D(H)$. To be able to prove it, I need only the fact that H,H^o are antidual pair, the dimension and the strict duality are not needed. The only problem is the notation, in case when there is no $H^* = (H^o)^{op}$ (i.e. when antipode on H^o is not invertible). Then in all expressions should be subexpressions like $b.a$ (where multiplication . is in H^*) replaced by $a._{H^o}b$ etc.

At first it should be proved that given multiplication defines an algebra, i.e. the associativity condition (and $(h \otimes a).(\mathbf{1} \otimes \mathbf{1}) = (\mathbf{1} \otimes \mathbf{1}).(h \otimes a) = (h \otimes a)$, this is a trivial consequence of the properties of pairing, namely $\langle h, \mathbf{1}\rangle = \epsilon(h)$ etc.) :
$((h \otimes a).(g \otimes b)).(f \otimes c) = (\langle g_{(1)}, a_{(1)}\rangle h.g_{(2)} \otimes b.a_{(2)}\langle Sg_{(3)}, a_{(3)}\rangle).(f \otimes c) =$
$\langle g_{(1)}, a_{(1)}\rangle\langle Sg_{(3)}, a_{(3)}\rangle h.g_{(2)}.f_{(2)} \otimes c.(ba_{(2)})_{(2)}\langle f_{(1)}, (ba_{(2)})_{(1)}\rangle\langle Sf_{(3)}, (ba_{(2)})_{(3)}\rangle =$
$\langle g_{(1)}, a_{(1)}\rangle\langle Sg_{(3)}, a_{(5)}\rangle h.g_{(2)}.f_{(2)} \otimes c.b_{(2)}a_{(3)}\langle f_{(1)}, b_{(1)}a_{(2)}\rangle\langle Sf_{(3)}, b_{(3)}a_{(4)}\rangle =$
$\langle g_{(1)}, a_{(1)}\rangle\langle Sg_{(3)}, a_{(5)}\rangle h.g_{(2)}.f_{(3)} \otimes c.b_{(2)}a_{(3)}\langle f_{(1)}, b_{(1)}\rangle\langle f_{(2)}, a_{(2)}\rangle\langle Sf_{(4)}, b_{(3)}a_{(4)}\rangle =$

$\langle g_{(1)}, a_{(1)}\rangle\langle Sg_{(3)}, a_{(5)}\rangle h.g_{(2)}.f_{(3)} \otimes c.b_{(2)}a_{(3)}\langle f_{(1)}, b_{(1)}\rangle\langle f_{(2)}, a_{(2)}\rangle\langle Sf_{(4)}, a_{(4)}\rangle$ $\langle Sf_{(5)}, b_{(3)}\rangle = \langle g_{(1)}f_{(2)}, a_{(1)}\rangle\langle S(f_{(4)})S(g_{(3)}), a_{(3)}\rangle hg_{(2)}f_{(3)} \otimes cb_{(2)}a_{(2)}\langle f_{(1)}, b_{(1)}\rangle$ $\langle Sf_{(5)}, b_{(3)}\rangle = \langle (gf_{(2)})_{(1)}, a_{(1)}\rangle\langle S(gf_{(2)})_{(3)}, a_{(3)}\rangle h(gf_{(2)})_{(2)} \otimes (cb_{(2)})a_{(2)}\langle f_{(1)}, b_{(1)}\rangle$ $\langle Sf_{(3)}, b_{(3)}\rangle = (h \otimes a)((g \otimes b)(f \otimes c))$ using the coassociativity, properties of bialgebra, of pairing and of antipode.

Next, it should be verified whether algebra and coalgebra structures are compatible, i.e. whether comultiplication and counit are algebra maps. I'll prove $\Delta((h \otimes a).(g \otimes b)) = \Delta(h \otimes a)\Delta(g \otimes b)$ only, other proofs are similar (and simpler). $\Delta(h\otimes a)\Delta(g\otimes b) = ((h_{(1)}\otimes a_{(1)})\otimes(h_{(2)}\otimes a_{(2)})).((g_{(1)}\otimes b_{(1)})\otimes(g_{(2)}\otimes b_{(2)})) = ((h_{(1)} \otimes a_{(1)}).(g_{(1)} \otimes b_{(1)})) \otimes ((h_{(2)} \otimes a_{(2)}).(g_{(2)} \otimes b_{(2)})) = \langle g_{(1)}, a_{(1)}\rangle h_{(1)}g_{(2)} \otimes b_{(1)}a_{(2)}\langle Sg_{(3)}, a_{(3)}\rangle \otimes \langle g_{(4)}, a_{(4)}\rangle h_{(2)}g_{(5)} \otimes b_{(2)}a_{(5)}\langle Sg_{(6)}, a_{(6)}\rangle = \langle g_{(1)}, a_{(1)}\rangle h_{(1)}g_{(2)} \otimes b_{(1)}a_{(2)}\langle Sg_{(3)}g_{(4)}, a_{(3)}\rangle \otimes h_{(2)}g_{(5)} \otimes b_{(2)}a_{(4)}\langle Sg_{(6)}, a_{(5)}\rangle = \langle g_{(1)}, a_{(1)}\rangle h_{(1)}g_{(2)} \otimes b_{(1)}a_{(2)} \epsilon(g_{(3)})\epsilon(a_{(3)})\otimes h_{(2)}g_{(4)}\otimes b_{(2)}a_{(4)}\langle Sg_{(5)}, a_{(5)}\rangle = \langle g_{(1)}, a_{(1)}\rangle h_{(1)}g_{(2)}\otimes b_{(1)}a_{(2)}\otimes h_{(2)}g_{(3)}\otimes b_{(2)}a_{(3)}\langle Sg_{(4)}, a_{(4)}\rangle = \Delta(\langle g_{(1)}, a_{(1)}\rangle hg_{(2)} \otimes ba_{(2)}\langle Sg_{(3)}, a_{(3)}\rangle) = \Delta((h \otimes a).(g \otimes b))$, again using the coassociativity, properties of pairing, of bialgebra and of antipode. To conclude, I have shown that structure defined above is a bialgebra.

Now, I'll prove that $\langle h_{(1)}, a_{(1)}\rangle h_{(2)}a_{(2)} = a_{(1)}h_{(1)}\langle h_{(2)}, a_{(2)}\rangle$, where the canonical embeddings of H, H^* are $h \equiv h \otimes \mathbf{1}, a \equiv \mathbf{1} \otimes a$:
$(\mathbf{1} \otimes a_{(1)})(h_{(1)} \otimes \mathbf{1})\langle h_{(2)}, a_{(2)}\rangle = \langle h_{(1)}, a_{(1)}\rangle h_{(2)} \otimes a_{(2)}\langle Sh_{(3)}, a_{(3)}\rangle\langle h_{(4)}, a_{(4)}\rangle =$
$\langle h_{(1)}, a_{(1)}\rangle h_{(2)} \otimes a_{(2)}\langle Sh_{(3)}h_{(4)}, a_{(3)}\rangle = \langle h_{(1)}, a_{(1)}\rangle h_{(2)} \otimes a_{(2)}\epsilon(h_{(3)})\langle \mathbf{1}, a_{(3)}\rangle =$
$\langle h_{(1)}, a_{(1)}\rangle h_{(2)} \otimes a_{(2)}\epsilon(h_{(3)})\epsilon(a_{(3)}) = \langle h_{(1)}, a_{(1)}\rangle h_{(2)} \otimes a_{(2)}$.

In the second part of the proof I'll prove the existence of the antipode in the bialgebra constructed above. This is also valid in infinite-dimensional case. About notation in this case: the antipode in H^o is denoted by S^{-1} as an inverse of antipode in $H^* = (H^o)^{op}$. In infinite-dimensional case this notation is a bit misleading, since there may be no Hopf algebra H^* at all, then S^{-1} is simply the antipode on H^o. The antipode of the bialgebra just constructed is given from properties of $S(h \otimes a) = S((h \otimes \mathbf{1})(\mathbf{1} \otimes a)) = S(\mathbf{1} \otimes a).S(h \otimes \mathbf{1}) = (\mathbf{1} \otimes S^{-1}a).(Sh \otimes \mathbf{1}) = \langle h_{(1)}, S^{-1}a_{(1)}\rangle Sh_{(2)} \otimes S^{-1}a_{(2)}\langle SSh_{(3)}, S^{-1}a_{(3)}\rangle = \langle h_{(1)}, S^{-1}a_{(1)}\rangle Sh_{(2)} \otimes S^{-1}a_{(2)}\langle Sh_{(3)}, a_{(3)}\rangle$. (Last equation holds if at least of the antipodes S_H, $S^{-1}_{H^o}$ is invertible. This is true e.g. if both H^o and H^* are Hopf algebras.) In order to prove that given relations define an antipode, one evaluates: $S(h_{(1)} \otimes a_{(1)}).(h_{(2)} \otimes a_{(2)}) = (\mathbf{1} \otimes S^{-1}a_{(1)}).(Sh_{(1)} \otimes \mathbf{1}).(h_{(2)} \otimes \mathbf{1}).(\mathbf{1} \otimes a_{(2)}) = (\mathbf{1} \otimes S^{-1}a_{(1)}).(Sh_{(1)}.h_{(2)} \otimes \mathbf{1}).(\mathbf{1} \otimes a_{(2)}) = \epsilon(h)(\mathbf{1} \otimes S^{-1}a_{(1)})(\mathbf{1} \otimes a_{(2)}) = \epsilon(h)\epsilon(a) = \epsilon(h \otimes a)$ and similarly $(h_{(1)} \otimes a_{(1)})S(h_{(2)} \otimes a_{(2)}) = \epsilon(h \otimes a)$.

In the third, last part of the proof I'll deal with the element $\mathcal{R} = \mathbf{1}\otimes e^j\otimes e_j\otimes\mathbf{1} \in (H \otimes H^*)^{\otimes 2}$. In this case important features (in addition to those given above) are strictly antidual pairing of H, H^o and existence of a basis in H and a dual basis in H^* (all those features are automatic in finite-dimensional case). In infinite-dimensional case one also needs some recipe how to sum infinite linear combination. I should prove that $\mathcal{R}$ satisfies following four conditions $(\Delta\otimes id)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}$, $(id \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}$, $\mathcal{R}\Delta(h \otimes a) = (\tau \circ \Delta)(h \otimes a)\mathcal{R}$, $\forall h \in H, a \in H^*$ and $\exists\mathcal{R}^{-1}$. During the proofs I often use decomposition of a vector in a basis

$a = \langle e_j, a\rangle e^j$, $h = \langle h, e^j\rangle e_j$, $a_{(1)} \otimes a_{(2)} = \Delta(a) = \Delta(\langle e_j, a\rangle e^j) = \langle e_j, a\rangle e^j_{(1)} \otimes e^j_{(2)}$ etc. (The proof follows from $a = \alpha_j e^j \Rightarrow \langle e_k, a\rangle = \alpha_j \delta^j_k$, since now summation convention is assumed.) Also I often need to pair some expression in a form of tensor product with a general element from the dual space in one component. This will be denoted by pairing with index giving the number of the component, e.g. $\langle h, a \otimes b\rangle_1 = \langle h, a\rangle b$ etc.

At first I'll prove $(\Delta \otimes id)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}$. Both left and right-hand sides can be evaluated $(\Delta \otimes id)\mathcal{R} = \mathbf{1} \otimes e^j_{(1)} \otimes \mathbf{1} \otimes e^j_{(2)} \otimes e_j \otimes \mathbf{1}$ and $\mathcal{R}_{13}\mathcal{R}_{23} = (\mathbf{1} \otimes e^j \otimes \mathbf{1} \otimes \mathbf{1} \otimes e_j \otimes \mathbf{1}).(\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes e^k \otimes e_k \otimes \mathbf{1}) = \mathbf{1} \otimes e^j \otimes \mathbf{1} \otimes e^k \otimes e_j e_k \otimes \mathbf{1}$. In order to prove the equality of right-hand sides I'll pair it in the 5th component with general element $a \in H^*$ and verify equality of results. (The equality $\langle h, a\rangle = \langle g, a\rangle$ valid for any $a \in H^*$ implies $h = g$ from the nondegeneracy of the pairing.): $\langle \mathbf{1} \otimes e^j_{(1)} \otimes \mathbf{1} \otimes e^j_{(2)} \otimes e_j \otimes \mathbf{1}, a\rangle_5 = \mathbf{1} \otimes e^j_{(1)} \otimes \mathbf{1} \otimes e^j_{(2)} \otimes \mathbf{1}\langle e_j, a\rangle = \mathbf{1} \otimes a_{(1)} \otimes \mathbf{1} \otimes a_{(2)} \otimes \mathbf{1} = \mathbf{1} \otimes e^j \otimes \mathbf{1} \otimes e^k \otimes \mathbf{1}\langle e_j, a_{(1)}\rangle\langle e_k, a_{(2)}\rangle = \mathbf{1} \otimes e^j \otimes \mathbf{1} \otimes e^k \otimes \mathbf{1}\langle e_j e_k, a\rangle = \langle \mathbf{1} \otimes e^j \otimes \mathbf{1} \otimes e^k \otimes e_j e_k \otimes \mathbf{1}, a\rangle_5$. The proof of the second condition is similar.

Next I'll prove that $\mathcal{R}$ satisfies $\mathcal{R}\Delta(h \otimes a) = (\tau \circ \Delta)(h \otimes a)\mathcal{R}$, $\forall h \in H, a \in H^*$. The left-hand side can be rewritten $\mathcal{R}\Delta(h \otimes a) = (\mathbf{1} \otimes e^j \otimes e_j \otimes \mathbf{1}).(h_{(1)} \otimes a_{(1)} \otimes h_{(2)} \otimes a_{(2)}) = \langle h_{(1)}, e^j_{(1)}\rangle\langle Sh_{(3)}, e^j_{(3)}\rangle h_{(2)} \otimes a_{(1)} e^j_{(2)} \otimes e_j h_{(4)} \otimes a_{(2)}$ and the right-hand side $(\tau \circ \Delta)(h \otimes a)\mathcal{R} = ((h_{(2)} \otimes a_{(2)}) \otimes (h_{(1)} \otimes a_{(1)})).\mathbf{1} \otimes e^j \otimes e_j \otimes \mathbf{1} = \langle e_{j(1)}, a_{(1)}\rangle\langle Se_{j(3)}, a_{(3)}\rangle h_{(2)} \otimes e^j a_{(4)} \otimes h_{(1)} e_{j(2)} \otimes a_{(2)}$. In the next step I'll pair given expressions with a general element $g \in H$ in the 2nd component and prove that they are equal. Left-hand side gives $\langle h_{(1)}, e^j_{(1)}\rangle\langle Sh_{(3)}, e^j_{(3)}\rangle\langle g, h_{(2)} \otimes a_{(1)} e^j_{(2)} \otimes e_j h_{(4)} \otimes a_{(2)}\rangle_2 = \langle h_{(1)}, e^j_{(1)}\rangle\langle Sh_{(3)}, e^j_{(3)}\rangle\langle g, a_{(1)} e^j_{(2)}\rangle h_{(2)} \otimes e_j h_{(4)} \otimes a_{(2)} = \langle h_{(1)}, e^j_{(1)}\rangle\langle Sh_{(3)}, e^j_{(3)}\rangle\langle g_{(1)}, a_{(1)}\rangle\langle g_{(2)}, e^j_{(2)}\rangle h_{(2)} \otimes e_j h_{(4)} \otimes a_{(2)} = \langle g_{(1)}, a_{(1)}\rangle h_{(2)} \otimes h_{(1)} g_{(2)} Sh_{(3)} h_{(4)} \otimes a_{(2)} = \langle g_{(1)}, a_{(1)}\rangle h_{(2)} \otimes h_{(1)} g_{(2)} \otimes a_{(2)}$ and the right-hand side gives $\langle g, \langle e_{j(1)}, a_{(1)}\rangle\langle Se_{j(3)}, a_{(3)}\rangle h_{(2)} \otimes e^j a_{(4)} \otimes h_{(1)} e_{j(2)} \otimes a_{(2)}\rangle_2 = \langle e_{j(1)}, a_{(1)}\rangle\langle Se_{j(3)}, a_{(3)}\rangle \langle g_{(1)}, e^j\rangle\langle g_{(2)}, a_{(4)}\rangle h_{(2)} \otimes h_{(1)} e_{j(2)} \otimes a_{(2)} = \langle g_{(1)}, e^j\rangle\langle e_{j(1)}, a_{(1)}\rangle\langle Se_{j(3)} g_{(2)}, a_{(3)}\rangle h_{(2)} \otimes h_{(1)} e_{j(2)} \otimes a_{(2)} = \langle g_{(1)}, a_{(1)}\rangle\langle Sg_{(3)} g_{(4)}, a_{(3)}\rangle h_{(2)} \otimes h_{(1)} g_{(2)} \otimes a_{(2)} = \langle g_{(1)}, a_{(1)}\rangle h_{(2)} \otimes h_{(1)} g_{(2)} \otimes a_{(2)}$ giving the identity wanted.

Finally I'll prove the invertibility of $\mathcal{R}$. First I should guess how $\mathcal{R}^{-1}$ looks like and then I'll prove that it is really an inverse. Guess (based on some properties of quasitriangular Hopf algebras) is $\mathcal{R}^{-1} = \mathbf{1} \otimes S^{-1}e^j \otimes e_j \otimes \mathbf{1}$. The first identity $\mathcal{R}\mathcal{R}^{-1} = \mathbf{1}$ then looks like $(\mathbf{1} \otimes e^j \otimes e_j \otimes \mathbf{1}).(\mathbf{1} \otimes S^{-1}e^j \otimes e_j \otimes \mathbf{1}) = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}$. This again is proved by pairing with a general element $a \in H^o$ on the third position, i.e. $\langle(\mathbf{1} \otimes e^j \otimes e_j \otimes \mathbf{1}).(\mathbf{1} \otimes S^{-1}e^k \otimes e_k \otimes \mathbf{1}), a\rangle_3 = \langle \mathbf{1} \otimes S^{-1}e^k e^j \otimes e_j e_k \otimes \mathbf{1}, a\rangle_3 = \langle e^j, a_{(1)}\rangle\langle e^k, a_{(2)}\rangle \mathbf{1} \otimes S^{-1}e_k e_j \otimes \mathbf{1} = \mathbf{1} \otimes S^{-1}a_{(2)} a_{(1)} \otimes \mathbf{1} = \epsilon(a)\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} = \langle \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}, a\rangle_3$. Proof of the second identity $\mathcal{R}^{-1}\mathcal{R} = \mathbf{1}$ is analogous. This **finishes the proof of the theorem.**

Remark: The structure similar to the quantum double can be also build on a pair of strictly (anti)dual bialgebras if the pairing is convolution-invertible

(then this convolution inverse $\langle\, , \rangle^{-1}$ to $\langle\, , \rangle$ replaces $\langle S\, , \rangle$). The resulting structure is a bialgebra with cross-multiplication like (1.2), but may not be quasitriangular (the resulting $\mathcal{R}$-matrix may not be invertible). Even more general structure (called a **generalised quantum double**) can be built on a pair of antidual bialgebras (not necessarily strictly antidual) if the pairing is convolution-invertible. The resulting bialgebra has a cross multiplication of quantum double type, but may not possess a $\mathcal{R}$-matrix (canonical $\mathcal{R}$-matrix $\mathcal{R} = \sum_i e^i \otimes e_i$ may not satisfy the properties of $\mathcal{R}$-matrix because all proofs of the fact that $\mathcal{R} = \sum_i e^i \otimes e_i$ satisfies the conditions given in Definition 1.3.1 are based on the property that H and H^o form a *strictly* antidual pair of bialgebras).

1.6 Double cross product

Motivation: Quantum doubles (and also generalised quantum doubles) defined in the previous section can be also seen from a more general point of view, namely they represent special examples of double cross product bialgebras. In the present section I briefly explain the notion of double cross product bialgebra (based on [6]) and the most important result of the theory of double cross product bialgebras that in fact in certain sense any bialgebra built on a tensor product of bialgebras as a vector spaces is a double cross product bialgebra. On the other hand, since double cross product bialgebras in general don't possess $\mathcal{R}$-matrix and also all structures resulting from Vladimirov's method discussed later are at least generalised quantum doubles, I'll not use the notion of double cross product bialgebra in the remaining chapters, this section is included just to complete the discussion of possible cross multiplication on tensor products of bialgebras similar to quantum doubles.

Definition 1.6.1 *Let B,A be bialgebras, $\triangleleft : B \otimes A \to B$ be a right action (i.e. $\triangleleft \circ (id \otimes .) = \triangleleft \circ (\triangleleft \otimes id)$, $\triangleleft \circ (id \otimes \eta) = id \otimes \eta$) of algebra A on B, $\triangleright : B \otimes A \to A$ be a left action of algebra B on A (i.e. $\triangleright \circ (. \otimes id) = \triangleright \circ (id \otimes \triangleright)$, $\triangleright \circ (\eta \otimes id) = \eta \otimes id$). Let $\triangleleft$ and $\triangleright$ be also coalgebra maps. Maps $\triangleleft$ and $\triangleright$ form a* **matched pair** *if and only if*

$$(bd) \triangleleft a = (b \triangleleft (d_{(1)} \triangleright a_{(1)}))(d_{(2)} \triangleleft a_{(2)}),\ \mathbf{1} \triangleleft a = \epsilon(a)\mathbf{1}$$

$$b \triangleright (ac) = (b_{(1)} \triangleright a_{(1)})((b_{(2)} \triangleleft a_{(2)}) \triangleright c),\ b \triangleright \mathbf{1} = \epsilon(b)\mathbf{1}$$

$$b_{(1)} \triangleleft a_{(1)} \otimes b_{(2)} \triangleright a_{(2)} = b_{(2)} \triangleleft a_{(2)} \otimes b_{(1)} \triangleright a_{(1)}$$

for all possible $a, c \in A$, $b, d \in B$.

Definition 1.6.2 *Let A,B be bialgebras,* $\triangleleft : B \otimes A \to B$ *and* $\triangleright : B \otimes A \to A$ *be a matched pair. A* **double cross product bialgebra** $A \bowtie B$ *is a bialgebra defined on* $A \otimes B$ *with tensor product comultiplication, unit and counit and with the multiplication defined by*

$$(a \otimes b)(c \otimes d) = a(b_{(1)} \triangleright c_{(1)}) \otimes (b_{(2)} \triangleleft c_{(2)})d,\ \forall a, c \in A,\ b, d \in B$$

and extended linearly.

Remark: It can be proved that Definition 1.6.2 is correct, i.e. that $A \bowtie B$ is really a bialgebra. The proof goes in a way very similar to the first part of the proof of Theorem 1.5.1 using the properties of $\triangleleft$,$\triangleright$ instead of the properties of the pairing.

Theorem 1.6.1 *Let* X, A, B *be bialgebras such that exist injective bialgebra maps* $i : A \to X$ *and* $j : B \to X$ *and* $.\circ(i \otimes j) : A \otimes B \to X$ *is an isomorphism of vector spaces. Then* A, B *are a matched pair and* $X \cong A \bowtie B$*.*

Proof: Firstly let me define a linear map $\Psi : B \otimes A \to A \otimes B$ by implicit definition $. \circ (j \otimes i) = . \circ (i \otimes j) \circ \Psi$. Because $. \circ (i \otimes j)$ is an isomorphism, i.e. invertible, Ψ is well-defined. Next I express some of the properties of Ψ: $. \circ (i \otimes j) \circ \Psi(bc \otimes a) = j(b)j(c)i(a) = j(b)(. \circ (i \otimes j)) \circ \Psi(c \otimes a) = . \circ (j \otimes .) \circ (id \otimes i \otimes j) \circ (id \otimes \Psi)(b \otimes c \otimes a) = . \circ (id \otimes .) \circ (j \otimes i \otimes j) \circ (id \otimes \Psi)(b \otimes c \otimes a) = . \circ (. \otimes id) \circ (j \otimes i \otimes j) \circ (id \otimes \Psi)(b \otimes c \otimes a) = . \circ ((. \circ (i \otimes j) \circ \Psi) \otimes j) \otimes (id \otimes \Psi)(b \otimes c \otimes a) = . \circ ((. \circ (i \otimes j)) \otimes j)\Psi_{12} \circ \Psi_{23}(b \otimes c \otimes a) = . \circ (i \otimes j) \circ (id \otimes .) \circ \Psi_{12} \circ \Psi_{23}(b \otimes c \otimes a)$ ($\Psi_{12} = \Psi \otimes id$, $\Psi_{23} = id \otimes \Psi$, during the computation the associativity of the product has been used). Together with evaluating other expressions in the same way I obtain:

$$\Psi \circ (. \otimes id) = (id \otimes .) \circ \Psi_{12} \circ \Psi_{23},\ \Psi(\mathbf{1} \otimes a) = a \otimes \mathbf{1}$$

$$\Psi \circ (id \otimes .) = (. \otimes id) \circ \Psi_{23} \circ \Psi_{12},\ \Psi(b \otimes \mathbf{1}) = \mathbf{1} \otimes b.$$

Now it is possible to define actions

$$\triangleleft = (\epsilon \otimes id) \circ \Psi,\ \triangleright = (id \otimes \epsilon) \circ \Psi$$

and to check their properties.

Firstly I verify that they are actions, i.e. $\triangleright \circ (. \otimes id) = \triangleright \circ (id \otimes \triangleright)$ (and other conditions on actions, their proofs are similar): $\triangleright \circ (. \otimes id) = (id \otimes \epsilon) \circ \Psi \circ (. \otimes id) = (id \otimes \epsilon)(id \otimes .) \circ \Psi_{12} \circ \Psi_{23} = (id \otimes \epsilon \otimes \epsilon) \circ \Psi_{12} \circ \Psi_{23} = (id \otimes \epsilon) \circ \Psi \circ (id \otimes id \otimes \epsilon) \circ \Psi_{23} = (id \otimes \epsilon) \circ \Psi \circ (id \otimes \triangleright) = \triangleright \circ (id \otimes \triangleright)$. Similar evaluation gives relations $\triangleleft \circ (. \otimes id) = (\epsilon \otimes id) \circ \Psi \circ (. \otimes id) = (\epsilon \otimes id) \circ (id \otimes .) \circ \Psi_{12} \circ \Psi_{23} = . \circ (\epsilon \otimes id \otimes id) \circ \Psi_{12} \circ \Psi_{23} = . \circ (\triangleleft \otimes id) \circ (id \otimes \Psi)$, i.e. (together with other similar evaluations):

$$(bd) \triangleleft a = . \circ (\triangleleft \otimes id) \circ (b \otimes \Psi(d \otimes a)),\ \mathbf{1} \triangleleft a = \epsilon(a)\mathbf{1}$$

$$b \triangleright (ac) = . \circ (id \otimes \triangleright) \circ (\Psi(b \otimes a) \otimes c), \;\; b \triangleright \mathbf{1} = \epsilon(b)\mathbf{1}.$$

Because $i, j, .$ are coalgebra maps, also $. \circ (i \otimes j)$, its inverse and $. \circ (j \otimes i)$ are coalgebra maps. As a consequence Ψ is a coalgebra map, i.e.

$$\Delta_{A \otimes B} \circ \Psi = (\Psi \otimes \Psi) \circ \Delta_{B \otimes A}. \tag{1.3}$$

Applying different tensor products of id's and ϵ's to this equality I obtain $\Psi = (id \otimes \epsilon \otimes \epsilon \otimes id)\Delta_{A \otimes B} \circ \Psi = (id \otimes \epsilon \otimes \epsilon \otimes id)(\Psi \otimes \Psi) \circ \Delta_{B \otimes A} = (\triangleright \otimes \triangleleft) \circ \Delta_{B \otimes A}$, i.e.

$$\Psi(b \otimes a) = b_{(1)} \triangleright a_{(1)} \otimes b_{(2)} \triangleleft a_{(2)}.$$

Substituting this expression into the results of the previous paragraph I obtain the first two rows of conditions on matched pair in Definition 1.6.1.

In a similar way, applying $(\epsilon \otimes id \otimes id \otimes \epsilon)$ on the expression (1.3) I obtain

$$\tau \circ \Psi(b \otimes a) = b_{(1)} \triangleleft a_{(1)} \otimes b_{(2)} \triangleright a_{(2)}$$

and consequently the last condition in Definition 1.6.1 ($b_{(1)} \triangleleft a_{(1)} \otimes b_{(2)} \triangleright a_{(2)} = \tau \circ (b_{(1)} \triangleright a_{(1)} \otimes b_{(2)} \triangleleft a_{(2)}) = h_{(2)} \triangleleft a_{(2)} \otimes b_{(1)} \triangleright a_{(1)}$).

Application of $\epsilon \otimes id \otimes \epsilon \otimes id$ in front of the expression (1.3) gives

$$(\triangleleft \otimes \triangleleft) \circ \Delta_{B \otimes A} = (\epsilon \otimes id \otimes \epsilon \otimes id) \circ \Delta_{A \otimes B} \circ \Psi = \Delta_{A \otimes B} \circ (\epsilon \otimes id) \circ \Psi = \Delta_{A \otimes B} \circ \triangleleft,$$

i.e. $\triangleleft$ is a coalgebra map (for correctness it is of course also possible to check the relations involving ϵ). Similarly, application of $id \otimes \epsilon \otimes id \otimes \epsilon$ in front of the expression (1.3) shows that $\triangleright$ is a coalgebra map.

To sum up, the actions $\triangleleft$ and $\triangleright$ form a matched pair. Consequently it is possible to construct $A \bowtie B$ and to check that $. \circ (i \otimes j)$ is an isomorphism of bialgebras $A \bowtie B$ and X (as shown earlier, $. \circ (i \otimes j)$ is an isomorphism of coalgebras, in order to verify that it preserves multiplication, I again evaluate $j(b)i(a) = . \circ (i \otimes j) \circ \Psi(b \otimes a) = i(b_{(1)} \triangleright a_{(1)}) j(b_{(2)} \triangleleft a_{(2)})$ using the properties of Ψ and compare it to evaluation in $A \bowtie B$ $(\mathbf{1} \otimes b)(a \otimes \mathbf{1}) = (b_{(1)} \triangleright a_{(1)}) \otimes (b_{(2)} \triangleleft a_{(2)})$. From comparison of those expressions and from the fact that both A, B are immersed in both $A \bowtie B$ and X as bialgebras I conclude that $. \circ (i \otimes j)$ preserves the multiplication. It is possible by trivial computation to verify that $. \circ (i \otimes j)$ also preserves the unit map). The conclusion is that $. \circ (i \otimes j)$ is a bialgebra map between $A \bowtie B$ and X. **This finishes the proof**.

Remark: Let $D(H)$ be a quantum double, $D(H) = H \otimes H^o$ as a coalgebra. Then assignment $A = H$, $B = H^o$ in the previous theorem makes possible to express the quantum double as a double cross product bialgebra $H \bowtie H^o$. In this case maps i, j are just immersions $i(h) = h \otimes \mathbf{1}$, $j(a) = \mathbf{1} \otimes a$. The map Ψ and the actions can be deduced from the expression (1.2), i.e. $j(a)i(h) =$

$\langle h_{(1)}, a_{(1)}\rangle i(h_{(2)}) j(a_{(2)}) \langle h_{(3)}, a_{(3)}\rangle^{-1}$. This implies $\Psi(a \otimes h) = \langle h_{(1)}, a_{(1)}\rangle h_{(2)} \otimes a_{(2)} \langle h_{(3)}, a_{(3)}\rangle^{-1}$ and

$$
\begin{aligned}
a \triangleright h &= \langle h_{(1)}, a_{(1)}\rangle h_{(2)} \langle h_{(3)}, a_{(2)}\rangle^{-1} && (1.4) \\
a \triangleleft h &= \langle h_{(1)}, a_{(1)}\rangle a_{(2)} \langle h_{(2)}, a_{(3)}\rangle^{-1}. && (1.5)
\end{aligned}
$$

To verify that $D(H) = H \bowtie H^o$ I evaluate $(\mathbf{1} \otimes a)(h \otimes \mathbf{1}) = (a_{(1)} \triangleright h_{(1)}) \otimes (a_{(2)} \triangleleft h_{(2)}) = \langle h_{(1)}, a_{(1)}\rangle h_{(2)} \langle h_{(3)}, a_{(2)}\rangle^{-1} \otimes \langle h_{(4)}, a_{(3)}\rangle a_{(4)} \langle h_{(5)}, a_{(5)}\rangle^{-1} = \langle h_{(1)}, a_{(1)}\rangle h_{(2)} \otimes a_{(2)} \langle h_{(3)}, a_{(3)}\rangle^{-1}$ and from the the fact that H, H^o are subalgebras of $D(H)$ I conclude that the product in $H \bowtie H^o$ is identical with the product in $D(H)$. The same is of course true also for generalised quantum doubles.

Chapter 2

Vladimirov's Method for Obtaining Quantum Doubles

In this chapter I explain a method for obtaining the quantum double from the set of matrices satisfying the system of Yang-Baxter-like equations. The method was proposed by A.A. Vladimirov in [9]. In this article Vladimirov claims that to every constant solution of the Yang-Baxter system

$$\begin{aligned} W_{12}W_{13}W_{23} &= W_{23}W_{13}W_{12} \qquad Z_{12}Z_{13}Z_{23} = Z_{23}Z_{13}Z_{12} \\ W_{12}X_{13}X_{23} &= X_{23}X_{13}W_{12} \qquad X_{12}X_{13}Z_{23} = Z_{23}X_{13}X_{12} \end{aligned} \tag{2.1}$$

it is possible to associate a quantum double. Actually, as I'll show later, there are some problems applying this procedure.

In the first section I explain the construction of two strictly antidual bialgebras and the origin of equations (2.1) (this section is in fact a reformulated version of [9], the other sections of this chapter contain mostly my new results). In the second section I study the consequences of the symmetries of the system (2.1). In the third section I study the possibility of turning those bialgebras into Hopf algebras, i.e. introducing antipodes. In the last section I try to complete the procedure of constructing the quantum double, i.e. finding the $\mathcal{R}$-matrix. In the next chapter I'll explain possible generalisations and modifications of the Vladimirov's method.

2.1 Antidual pair of quantum matrix bialgebras

Let me begin with two quantum matrix bialgebras A_W, A_Z defined by two matrices W, Z (of the same dimension $n^2 \times n^2$). The first two equations in (2.1) follow from usual requirements on matrices W, Z, ensuring e.g. the existence of dual quasitriangular structure. I assume that generators of A_W are written as a formal matrix $U_{(i,j)} = u^i_j$ and the generators of A_Z as a formal matrix $V_{(i,j)} = v^i_j$. Now I try to introduce a antidual pairing between those bialgebras. I introduce new matrix X to simplify the notation: $X^{i,j}_{k,l} = \langle u^i_k, v^j_l \rangle$, i.e. $X_{12} = \langle U_1, V_2 \rangle$ (I assume the notation based on tensor product of matrices, i.e. $X^{i,k}_{j,l} = X^{n(i-1)+k}_{n(j-1)+l}$). Once I know the matrix X, I know in fact pairing of any two elements $u \in A_W, v \in A_Z$ from the properties of antidual pairing, but there is a question whether any matrix X can define a pairing. The answer is negative and I'll show that it leads to the second pair of equations in (2.1).

Firstly I write the explicit form of pairing of two elements $u = u^{i_1}_{j_1} u^{i_2}_{j_2} \dots u^{i_n}_{j_n}$ and $v = v^{k_1}_{l_1} v^{k_2}_{l_2} \dots v^{k_m}_{l_m}$: $\langle u^{i_1}_{j_1} u^{i_2}_{j_2} \dots u^{i_n}_{j_n}, v^{k_1}_{l_1} v^{k_2}_{l_2} \dots v^{k_m}_{l_m} \rangle = X^{i_1,k_m}_{p_{1,1},q_{1,1}} \dots X^{i_n q_{n-1,1}}_{p_{1,n} l_m}$ $X^{p_{1,1},k_{m-1}}_{p_{2,1},q_{1,2}} \dots X^{p_{m-1,1},k_1}_{j_1,q_{1,m}} \dots X^{p_{m-1,n},q_{n-1,m}}_{j_n,l_1}$, where summation over repeated indices (i.e. p's and q's) is assumed. In the matrix form it can be written $\langle U_{i_1} U_{i_2} \dots U_{i_n}, V_{j_1} V_{j_2} \dots V_{j_m} \rangle = X_{i_1 j_m} \dots X_{i_n j_m} X_{i_1 j_{m-1}} \dots X_{i_1 j_1} \dots X_{i_n j_1}$ (provided $i_k \neq j_l \ \forall k \leq n, l \leq m$). Proof is simple, using the properties of antidual pairing and the fact that comultiplication is an algebra map.

Using expressions given above I can check when the pairing is compatible with factoring out relations $W_{12} U_1 U_2 = U_2 U_1 W_{12}$ and $Z_{12} V_1 V_2 = V_2 V_1 Z_{12}$. For example, $\langle W_{12} U_1 U_2 - U_2 U_1 W_{12}, V_3 \rangle = W_{12} \langle U_1, V_3 \rangle \langle U_2, V_3 \rangle - \langle U_2, V_3 \rangle \langle U_1, V_3 \rangle W_{12} = W_{12} X_{13} X_{23} - X_{23} X_{13} W_{12}$ leads to the condition $W_{12} X_{13} X_{23} = X_{23} X_{13} W_{12}$. Similarly, from $Z_{12} V_1 V_2 = V_2 V_1 Z_{12}$ I obtain the necessary condition $X_{12} X_{13} Z_{23} = Z_{23} X_{13} X_{12}$, i.e. I have explained the role of the second pair of equations in (2.1). It remains to be proved that the compatibility of the pairing and the algebra structure doesn't impose other conditions. This can be done evaluating $\langle W_{12} U_1 U_2 - U_2 U_1 W_{12}, V_3 \dots V_m \rangle = W_{12} X_{1m} X_{2m} X_{1(m-1)} X_{2(m-1)} \dots X_{13} X_{23} - X_{2m} X_{1m} \dots X_{23} X_{13} W_{12} = W_{12} X_{1m}$ $X_{2m} \dots X_{13} X_{23} - X_{2m} X_{1m} \dots W_{12} X_{13} X_{23} = \dots = (W_{12} X_{1m} X_{2m} - X_{2m} X_{1m} W_{12})$ $X_{1(m-1)} X_{2(m-1)} \dots X_{13} X_{23} = 0$, since $W_{12} X_{13} X_{23} = X_{23} X_{13} W_{12}$ implies $W_{12} X_{1m} X_{2m} = X_{2m} X_{1m} W_{12}$ for any $m \in \mathcal{N}$. The properties of antidual pairing imply that the pairing is zero also for any element in the form $u = u^{i_1}_{j_1} \dots u^{i_n}_{j_n} (W^{ij}_{kl} u^k_p u^l_q - u^j_k u^i_l W^{lk}_{pq}) u^{i_{n+1}}_{j_{n+1}} \dots u^{i_m}_{j_m}$, so the pairing and algebra structure on A_W are compatible provided $W_{12} X_{13} X_{23} = X_{23} X_{13} W_{12}$. Similar reasoning can be applied to the A_Z and one obtains that also $X_{12} X_{13} Z_{23} = Z_{23} X_{13} X_{12}$ is the only condition of the correctness of the pairing, i.e. I can conclude by the theorem

Theorem 2.1.1 *Let W, Z be $n^2 \times n^2$ matrices, $n \in \mathcal{N}$, let A_W and A_Z be corresponding quantum matrix bialgebras. Let $\langle\, , \rangle$ be an antidual pair-*

ing of tensor algebras (with the coproduct and the counit as in the quantum matrix bialgebra case) $T(\mathcal{L}\{\mathbf{1}, u^i_j,\ i,j \in \{1,\dots,n\}\}) \times T(\mathcal{L}\{\mathbf{1}, v^i_j,\ i,j \in \{1,\dots,n\}\}) \to k$ *and* X *be a matrix given by relations* $X^{i,j}_{k,l} = \langle u^i_k, v^k_l \rangle$. *Then the map* $\langle\, , \rangle$ *defines an antidual pairing of bialgebras* A_W *and* A_Z *if and only if* X *satisfies*

$$[W, X, X] = 0 \qquad [X, X, Z] = 0. \tag{2.2}$$

Proof: follows from the evaluation given above.

Since now I have a pair of antidual bialgebras, I would like to convert it into a strictly antidual pair (since the quantum double is *ex definitione* built on the tensor product of strictly antidual Hopf algebras). This can be always formally done by factoring out null biideals (remember that Theorem 1.2.1 is valid also for antidual pairing) and bialgebras obtained this way I denote $\tilde{A}_W$,$\tilde{A}_Z$, but in concrete cases it can be almost impossible to perform the factorization, since there is no known efficient general algorithm for finding the null biideals. (Also it can be quite difficult to prove that the pairing obtained is nondegenerate.) The only method that seems to work in the moment is to find linear combinations of generators and of their small powers (say up to the second powers of generators)[1] lying in the null biideal and then try to prove that after factoring out those elements (and their products) one obtains a nondegenerate pairing. Both steps of this procedure can be very demanding and obviously this method is not usable in some cases when it requires high powers of generators.

[1] This evaluation was done usually with help of the software for symbolic manipulations, namely Maple V Release 5. Using this program it was possible to find all elements of the form $u = \alpha_0 \mathbf{1} + \sum_{i,j=1}^2 \alpha_{i,j} u^i_j + \sum_{i,j,k,l=1}^2 \alpha_{i,j,k,l} u^i_j u^k_l$ such that their pairing with any element of the form $v_1 v_2 v_3$ (where $v_1, v_2, v_3 \in \{\mathbf{1}, v^i_j | i,j = 1,2\}$) is zero. (It was not possible to use higher powers of generators because of limitations of both the available hardware and the software used.) Now using the knowledge of those elements u of A_W that may lie in the null biideal it was possible to try to prove that those elements are really elements of the null biideal. The similar procedure was of course applied to the elements of A_Z. Also this scheme allowed to discard in the very beginning matrices X that would lead to only small null biideals, i.e. to complicated bialgebras $\tilde{A}_W$ and $\tilde{A}_Z$ that would be technically impossible to study in detail. Similarly some X matrices that didn't allow existence of antipodes could be discarded in this moment (more detailed explanation of this is given in Theorem 4.4.3).

2.2 Symmetries of the Yang-Baxter system and of the generated bialgebras

The Yang-Baxter system (2.1) has a lot of symmetries. Many of them have been used in classification of its solutions in [3]. It is therefore reasonable to find the relation between two solutions of (2.1) connected by those symmetries and between the bialgebras generated by those solutions.

The most important symmetry used during the solving of (2.1) was the following:

$$\begin{aligned} W \to \tilde{W} &= (T \otimes T)W(T \otimes T)^{-1} \\ X \to \tilde{X} &= (T \otimes S)X(T \otimes S)^{-1} \\ Z \to \tilde{Z} &= (S \otimes S)Z(S \otimes S)^{-1} \end{aligned} \tag{2.3}$$

where T, S are arbitrary invertible 2×2 matrices. The pair of bialgebras $A_{\tilde{W}}$, $A_{\tilde{Z}}$ generated by $\tilde{W}, \tilde{X}, \tilde{Z}$ can be written

$$\begin{aligned} \tilde{W}_{12}\tilde{U}_1\tilde{U}_2 &= \tilde{U}_2\tilde{U}_1\tilde{W}_{12} \\ \tilde{Z}_{12}\tilde{V}_1\tilde{V}_2 &= \tilde{V}_2\tilde{V}_1\tilde{Z}_{12} \\ \langle \tilde{U}_1, \tilde{V}_2 \rangle &= \tilde{X}_{12} \\ \Delta(\tilde{U}) = \tilde{U} \otimes \tilde{U} & \qquad \Delta(\tilde{V}) = \tilde{V} \otimes \tilde{V} \end{aligned}$$

Substituting the expressions for $\tilde{W}, \tilde{X}, \tilde{Z}$ one obtains:

$$\begin{aligned} T_1T_2W_{12}T_1^{-1}T_2^{-1}\tilde{U}_1\tilde{U}_2 &= \tilde{U}_2\tilde{U}_1T_1T_2W_{12}T_1^{-1}T_2^{-1} \\ S_1S_2Z_{12}S_1^{-1}S_2^{-1}\tilde{V}_1\tilde{V}_2 &= \tilde{V}_2\tilde{V}_1S_1S_2Z_{12}S_1^{-1}S_2^{-1} \\ \langle \tilde{U}_1, \tilde{V}_2 \rangle &= T_1S_2X_{12}T_1^{-1}S_2^{-1} \end{aligned}$$

Finally substituting $U = T^{-1}\tilde{U}T$, $V = S^{-1}\tilde{V}S$ and multiplying given equations by Ts and Ss from left and right one reconstructs the relations valid in the pair of A_W and A_Z, namely

$$\begin{aligned} W_{12}U_1U_2 &= U_2U_1W_{12} \\ Z_{12}V_1V_2 &= V_2V_1Z_{12} \\ \langle U_1, V_2 \rangle &= X_{12} \end{aligned}$$

It remains to be proved that the algebra map generated $\tilde{U} \to U = T^{-1}\tilde{U}T$, $\tilde{V} \to V = S^{-1}\tilde{V}S$ is also a coalgebra map, i.e. $\Delta(U) = \Delta(T^{-1}\tilde{U}T) = T^{-1}\Delta(\tilde{U})T = T^{-1}\tilde{U} \otimes \tilde{U}T = T^{-1}\tilde{U}T \otimes T^{-1}\tilde{U}T = U \otimes U$ and similarly $\Delta(V) = V \otimes V$ (remember that the matrix multiplication is implicitly assumed in the tensor products of formal matrices). **To conclude, I have proved that the maps defined above are bialgebra maps, i.e. pairs of bialgebras A_W, A_Z and $A_{\tilde{W}}$, $A_{\tilde{Z}}$ are isomorphic.**

The second important symmetry of the system (2.1) is

$$W \to \tilde{W} = \lambda W,\ X \to \tilde{X} = \mu X,\ Z \to \tilde{Z} = \nu Z \tag{2.4}$$

where λ, μ, ν are nonzero complex numbers. The multiplication by λ, ν doesn't change the bialgebras A_W, A_Z at all, because both W and Z appear only in the relations of type $W_{12}U_1U_2 = U_2U_1W_{12}$. **The multiplication of X by μ is not a symmetry of generated bialgebras**, namely, as I show later on simple examples, the null biideals and the existence of antipodes depend on the choice of μ. Requirement of the existence of antipodes can be used to fix (up to the multiplication by any root of unity) the number μ. (If antipodes exist in both bialgebras for some value of μ.)

Another symmetry of the system (2.1) is

$$W \to \tilde{W} = W^T,\ X \to \tilde{X} = X^T,\ Z \to \tilde{Z} = Z^T \tag{2.5}$$

where A^T is the matrix transposed to A. The new algebras A_{W^T} and A_{Z^T} are given by relations

$$W^T_{12}\tilde{U}_1\tilde{U}_2 = \tilde{U}_2\tilde{U}_1W^T_{12} \qquad Z^T_{12}\tilde{V}_1\tilde{V}_2 = \tilde{V}_2\tilde{V}_1Z^T_{12}.$$

and the pairing is given $\langle \tilde{U}_1, \tilde{V}_2\rangle = X^T_{12}$. Defining $u^i_j = \tilde{u}^j_i$ i.e. $U = \tilde{U}^T$ and $V = \tilde{V}^T$ one recovers the original algebra structure of A_W and A_Z and the original pairing between generators. The coalgebra structure is not the same, since $\Delta(u^i_j) = \Delta(\tilde{u}^j_i) = \tilde{u}^j_k \otimes \tilde{u}^k_i = u^k_j \otimes u^i_k = \tau(u^i_k \otimes u^k_j)$. **The conclusion is that the given symmetry transformation (2.5) of the system (2.1) leads to the transition from A_W and A_Z to $A^{cop}_{W^T}$ and $A^{cop}_{Z^T}$.**

Next symmetry of the system (2.1) is

$$W \to \tilde{W} = Z^\pm,\ X \to \tilde{X} = X^+,\ Z \to \tilde{Z} = W^\pm \tag{2.6}$$

where $A^+ = PAP$ and $A^- = A^{-1}$. Firstly I find that $A_{W^\pm}$ is given by relations $W^\pm_{12}U_1U_2 = U_2U_1W^\pm_{12}$. Those relations can be rewritten in the form $U_1U_2W_{12} = W_{12}U_2U_1$ (Proof follows in case of W^+ from $PU_1P = U_2, PU_2P = U_1$, in case of W^- it is just the multiplication of the original expression by W_{12} from left and right.), i.e. $A_{W^\pm} = A^{op}_W$. Consequently the transformation (2.6) leads to a new pair of bialgebras A^{op}_Z and A^{op}_W with the pairing $\widetilde{\langle \tilde{U}_1, \tilde{V}_2\rangle} = X^+_{12}$. It is natural to identify the generators $\tilde{u}^i_j = v^i_j$

and $\tilde{v}^i_j = u^i_j$. The pairing is then connected with the original one by relation $\tilde{\langle}\tilde{U}_{i_1}\tilde{U}_{i_2}\ldots\tilde{U}_{i_n},\tilde{V}_{j_1}\ldots\tilde{V}_{j_m}\tilde{\rangle} = X^+_{i_1j_m}\ldots X^+_{i_nj_m}X^+_{i_1j_{m-1}}\ldots X^+_{i_1j_1}\ldots X^+_{i_nj_1} = X_{j_mi_1}\ldots X_{j_mi_n}X_{j_{m-1}i_1}\ldots X_{j_1i_1}\ldots X_{j_1i_n} = X_{j_mi_1}\ldots X_{j_mi_n}X_{j_{m-1}i_1}\ldots X_{j_1i_1}\ldots X_{j_1i_n} = X_{j_mi_1}\ldots X_{j_1i_1}\ldots X_{j_1i_n} = \langle U_{j_m}\ldots U_{j_1}, V_{i_n}\ldots V_{i_1}\rangle$. This relation implies that also the appropriate null biideals can be found by reversing the order of multiplication of generators in the original ones and substituting $\tilde{u}^i_j = v^i_j$ and $\tilde{v}^i_j = u^i_j$.

The other symmetries of (2.1) used in [3] don't seem to lead to the symmetries of the generated bialgebras.

2.3 Existence of antipodes

Until now, I have considered only bialgebras. For the construction of the quantum double I need Hopf algebras, i.e. antipodes in both bialgebras. The idea how to introduce the antipode is to define a map $\tilde{S} : A_W \rightarrow A^*_Z$ by relation $\tilde{S}_{(u^i_j)}(v^k_l) = (X^{-1})^{ik}_{jl}$ and extending the action of $\tilde{S}_{(u^i_j)}$ on the whole A_Z by requiring the properties that would be satisfied if S was an antipode in A_W and $\tilde{S}$ was a map $\tilde{S}_u = \langle Su, \ \rangle$ such as $\tilde{S}_{(u^1u^2)}(v) = \tilde{S}_{(u^1)}(v_{(2)}).\tilde{S}_{(u^2)}(v_{(1)}), \tilde{S}_{(u)}(v^1v^2) = \tilde{S}_{u_{(1)}}(v^1).\tilde{S}_{u_{(2)}}(v^2)$ etc. ($\tilde{S}$ is called a **weak antipode**.) In the explicit matrix form the weak antipode can be written $\tilde{S}_{(U_{i_1}U_{i_2}\ldots U_{i_n})}(V_{j_1}V_{j_2}\ldots V_{j_m}) = X^{-1}_{i_nj_1}\ldots X^{-1}_{i_1j_1}X^{-1}_{i_nj_2}\ldots X^{-1}_{i_1j_m}$. After factoring out the null biideals any element of $\tilde{A}_W$ is fully determined by its pairing to any element of $\tilde{A}_Z$, so hopefully map $\tilde{S}$ may define a map $S : \tilde{A}_W \rightarrow \tilde{A}_W$ with all properties of the antipode. Unfortunately, this is in general not the case, because there may be no element in $\tilde{A}_W$ corresponding to such pairing, and the procedure as given in [9] can be used only in special cases.

In this paragraph, I show how can be proved that $\tilde{S}$ is well-defined on A_W. Firstly, it should be shown that $\tilde{S}$ is consistent with relation $W_{12}U_1U_2 = U_2U_1W_{12}$. The pairing with generators gives $\tilde{S}_{(W_{12}U_1U_2-U_2U_1W_{12})}(V_3) = W_{12}\tilde{S}_{(U_2)}(V_3)\tilde{S}_{(U_1)}(V_3) - \tilde{S}_{(U_1)}(V_3)\tilde{S}_{(U_2)}(V_3)W_{12} = W_{12}X^{-1}_{23}X^{-1}_{13} - X^{-1}_{13}X^{-1}_{23}W_{12} = X^{-1}_{13}X^{-1}_{23}(X_{23}X_{13}W_{12} - W_{12}X_{13}X_{23})X^{-1}_{23}X^{-1}_{13} = 0$ and this result can be again extended on the whole null biideal and on the pairing with any element of A_Z. Next it should be proved that the map $\tilde{S}$ is consistent with the usual condition on the antipode $u_{(1)}.Su_{(2)} = Su_{(1)}.u_{(2)} = \epsilon(u)$, i.e. $\langle u_{(1)}, v_{(1)}\rangle\tilde{S}_{(u_{(2)})}(v_{(2)}) = \tilde{S}_{(u_{(1)})}(v_{(1)})\langle u_{(2)}, v_{(2)}\rangle = \epsilon(u)\langle 1, v\rangle = \epsilon(u)\epsilon(v)$. I evaluate it in the matrix form: $\tilde{S}_{(U_{i_1}U_{i_2}\ldots U_{i_n})}(V_{j_1}V_{j_2}\ldots V_{j_m})\langle U_{i_1}U_{i_2}\ldots U_{i_n}, V_{j_1}V_{j_2}\ldots V_{j_m}\rangle = X^{-1}_{i_nj_1}\ldots X^{-1}_{i_1j_1}X^{-1}_{i_nj_2}\ldots X^{-1}_{i_1j_m}X_{i_1j_m}\ldots X_{i_nj_m}X_{i_1j_{m-1}}\ldots X_{i_1j_1}\ldots X_{i_nj_1} = X^{-1}_{i_nj_1}\ldots X^{-1}_{i_1j_1}X^{-1}_{i_nj_2}\ldots X^{-1}_{i_2j_m}X_{i_2j_m}\ldots X_{i_nj_m}X_{i_1j_{m-1}}\ldots X_{i_1j_1}\ldots X_{i_nj_1} = \ldots = \mathbf{1} \quad \wedge \quad \mathbf{1} = \epsilon(U_{i_1}U_{i_2}\ldots U_{i_n})\epsilon(V_{j_1}V_{j_2}\ldots V_{j_m})$ and similarly from the other side. In fact I have just proved that $\tilde{S}_u(v) = \langle u, v\rangle^{-1}$, i.e. if the matrix X is invertible, then the convolution inverse of the pairing exists (and it is possible to immediately construct a generalized quantum double. On the other hand it usually doesn't possess a $\mathcal{R}$-matrix, as I have mentioned earlier, so I won't go this way.) Finally, I should show that the map $\tilde{S}$ is consistent with factoring out null biideals. Regrettably, I'm not able to prove this consistency and I even don't have an idea whether it holds or not.

Similar procedure to the one given above can be applied to A_Z, defining $\langle U_1, S^{-1}(V_2)\rangle = X_{12}^{-1}$. In this case arises one additional problem. Even if there exists a map $S^{-1} : \tilde{A}_Z \to \tilde{A}_Z$ (which is not guaranteed as I have shown above) it may not be invertible.

To summarize, the existence of antipodes on $\tilde{A}_W$ and $\tilde{A}_Z$ is not generally guaranteed but in some interesting cases the antipodes exist.

Remark: The problem of the existence of antipodes can be solved in some cases by enlarging both A_W and A_Z by "inverse formal matrix $\bar{u}^i_j$" (and "$\bar{v}^i_j$") satisfying $\bar{u}^i_k u^k_j = u^i_k \bar{u}^k_j = \delta^i_j$ and $\bar{v}^i_k v^k_j = v^i_k \bar{v}^k_j = \delta^i_j$ and also factoring out relations like $R_{12}\bar{U}_2\bar{U}_1 = \bar{U}_1\bar{U}_2 R_{12}$ etc. The antipodes may then be defined ($Su^i_j = \bar{u}^i_j$, $S\bar{u}^i_j$ is a bit more complicated). More detailed description of this way of introducing the antipodes will be given in next chapter.

2.4 Dual bases and the construction of $\mathcal{R}$-matrix

In the case when there are antipodes in both bialgebras, I may continue with the construction of the quantum double. As I have shown in the first chapter (see Section 1.5), it is now possible to construct the Hopf algebra $D(\tilde{A}_W)$. The only remaining problem (in the infinite-dimensional case) is whether $D(\tilde{A}_W)$ is really a quasitriangular Hopf algebra, i.e. the existence of $\mathcal{R}$-matrix. To be able to construct the canonical $\mathcal{R}$-matrix of quantum double I need a pair of dual bases of $\tilde{A}_W$ and $\tilde{A}_Z$. The existence of such pair is evident if $\tilde{A}_W$ is finite-dimensional[2] but remains an open question otherwise. Even the notion of basis in infinite-dimensional case is not unique. I use the following definition:

Definition 2.4.1 *Let V be a vector space and J be some set of indices (for example $\mathcal{N}$ or some subset of $\mathcal{N}$). An ordered set $\{e_j\}_{j\in J}$ is a* **basis of**

[2] It is known from linear algebra that to any basis of finite dimensional vector space V exists a dual basis of its dual space V^*. I remind that $H^o = H^*$ as vector spaces.

the vector space V *(also called a* **Hammel basis** *) if and only if any element $v \in V$ can be written as a finite linear combination of elements e_j and simultaneously the only finite linear combination of elements e_j equal to* 0 *is the trivial one.*

Using the definition given above I can investigate the following problem: Let U,V be a strictly antidual pair of bialgebras, $\{e_j\}$ a basis of U, $\{e^j\}$ a dual basis of V. Let $\{f_j\}$ be another basis of U. I suppose that both index sets are infinite, countable. (This is the usual case encountered in study of $\tilde{A}_W$ and $\tilde{A}_Z$.) **Does the dual basis of** V $\{f^j\}$ **exist ?** I shall prove that in general the answer is negative.

Because $f_j \in U$, $\forall j$ and $\{e_j\}$ is a basis, it is possible to express $f_j = \sum_k \alpha_{jk} e_k$ where for given j only finite number of coefficients α_{jk} are nonzero (see the first condition in Definition 2.4.1). Similarly it is possible to express $e_j = \sum_k \beta_{jk} f_k$ (because also $\{f_j\}$ is a basis). Simple evaluation $e_j = \sum_k \beta_{jk} f_k = \sum_k \sum_l \beta_{jk}(\alpha_{kl} e_l) = \sum_k \sum_l (\beta_{jk}\alpha_{kl}) e_l$ and the second condition in Definition 2.4.1 gives $\sum_k \beta_{jk}\alpha_{kl} = \delta_{jl} = \sum_k \alpha_{jk}\beta_{kl}$, i.e. $A = (\alpha_{jk})$ and $B = (\beta_{jk})$ are mutually inverse matrices. Finally I define $f^j = \sum_k \beta_{kj} e^k$ and verify $\langle f_j, f^k \rangle = \langle \sum_l \alpha_{jl} e_l, \sum_m \beta_{mk} e^m \rangle = \sum_l \sum_m \alpha_{jl}\beta_{mk} \langle e_l, e^m \rangle = \sum_l \sum_m \alpha_{jl}\beta_{mk}\delta_l^m = \delta_j^k$. Regrettably, the sum $f^j = \sum_k \beta_{kj} e^k$ may not be finite (a simple example: the transformation $f_j = e_j + e_1$ implies $\forall j \neq 1 :\ f^j = e^j$, $f^1 = \frac{1}{2}(e^1 - \sum_{j \neq 1} e^j)$, i.e. f^1 is not well defined), i.e. the existence of a pair of dual bases doesn't guarantee the existence of a dual basis of V to arbitrary basis of U. It remains to be investigated whether it is possible to given pair of bases that are not dual (e.g. some system of generators and their powers and products) to find a linear transformation that will transform given bases into new (unknown) ones that would be mutually dual.

An obvious way one should try out in order to construct a pair of dual bases is to use some kind of modified Gramm-Schmidt orthogonalisation. During this inductive process one begins with two arbitrary bases $\{e_j \in U\}_{j \in \mathcal{N}}$ and $\{\tilde{e}^j \in V\}_{j \in \mathcal{N}}$ and assumes that he already knows $\{f_j\}_{j \in \{1,\ldots,k\}}$ and $\{f^j\}_{j \in \{1,\ldots,k\}}$ such that $\langle f_i, f^j \rangle = \delta_i^j$ and $\mathcal{L}\{f_j,\, j \in \{1,\ldots,k\}\} = \mathcal{L}\{e_j,\, j \in \{1,\ldots,k\}\}$, $\mathcal{L}\{f^j,\, j \in \{1,\ldots,k\}\} = \mathcal{L}\{\tilde{e}^j,\, j \in \{1,\ldots,k\}\}$. Then one defines $f_{k+1} = e_{k+1} - \sum_{j=1}^k \langle e^{k+1}, f^j \rangle f_j$ and $f^{k+1} = \tilde{e}^{k+1} - \sum_{j=1}^k \langle f_j, \tilde{e}^{k+1} \rangle f^j$. New elements satisfy $\langle f_{k+1}, f^j \rangle = \langle f_j, f^{k+1} \rangle = 0\ \forall j < k+1$ and also the condition on linear spans is satisfied for $k \to k+1$. It remains to find whether is possible by some normalisation of f_{k+1} and f^{k+1} satisfy $\langle f_{k+1}, f^{k+1} \rangle = 1$. Evaluation gives $\langle f_{k+1}, f^{k+1} \rangle = \langle e_{k+1}, \tilde{e}^{k+1} \rangle - \sum_{j=1}^k \langle e_{k+1}, f^j \rangle \langle f_j, \tilde{e}^{k+1} \rangle$. However, this expression can be zero and then there is no possible normalisation giving $\langle f_{k+1}, f^{k+1} \rangle = 1$. (This obstacle is in original Gramm-Schmidt orthogonalisation solved by requesting the original quadratic form F (scalar product etc.) be strictly positive, i.e. $F(x,x) > 0 \; \forall x \neq 0$. In the investigated case I have a map $U \times V \to k$, so there is no evident solution of the problem except some identification of elements of $x \in U$ and $\tilde{x} \in V$ (for example identifying e_j and $\tilde{e}^j$) and requesting $\langle x, \tilde{x} \rangle \neq 0 \; \forall x \neq 0$. However, this condition is usually not satisfied in cases considered, the only known properties of pairing (neglecting properties that have no sense on vector space level which is the only important for bases) are bilinearity and nondegeneracy.) In some cases it may be possible to solve the problem using some suitable reordering of the elements of bases $\{e_j \in U\}_{j \in \mathcal{N}}$ and $\{\tilde{e}^j \in V\}_{j \in \mathcal{N}}$ but generally the construction of dual bases using modification of Gramm-Schmidt orthogonalisation seems not to work. Also I don't know any other method that would seem to be usable for the construction of a pair of dual bases.

The conclusion of this section is that the search for a pair of dual bases of infinite dimensional bialgebras may be (and usually is) very problematic. It is difficult to find it and in the case when it doesn't exist to prove its nonexistence. (I don't know any general sufficient or necessary condition for existence of a pair of dual bases of infinite dimensional bialgebras.)

Remark: There are some $\mathcal{R}$-matrices published in literature (e.g. [10]) which have been found using some kind of exponential ansatz. Regrettably, I have not found any explanation of this procedure (usually the whole discussion of this point in articles consists of stating "using exponential ansatz was found" and publishing the resulting $\mathcal{R}$-matrix without any details and comments) and I don't know exactly what is meant by "exponential ansatz" in this particular application. Nevertheless I have used something that could be called exponential ansatz in the following way: Sometimes it is handy to replace some of the generators $u^j_k = u$ by formal exponential $u = \exp(h)$ and study the properties of h. The pairing of h with the elements from the bialgebra $\tilde{A}_Z$ might be simpler and it might be possible to find a pair of dual bases and consequently the canonical $\mathcal{R}$-matrix.

Finally, if a pair of dual bases is found (and antipodes exist etc.), it is possible to construct the $\mathcal{R}$-matrix, at least as a formal infinite sum (such formal $\mathcal{R}$-matrix can have well defined meaning e.g. in suitable representation of $D(U)$).

Chapter 3

Generalisations and Modifications of Vladimirov's Method

In the previous chapter I have explained in detail Vladimirov's method for obtaining quantum doubles. In this chapter I propose some generalisations and modifications. In the first section I explain that the result of Vladimirov's method doesn't depend on the whole triple W, X, Z, it depends only on X. In the second section I study the possibility of introducing antipodes in A_W and A_Z using so-called FRT method based on enlarging both A_W and A_Z.

3.1 Simplification of Vladimirov's method

The most important result of this section is the following theorem.

Theorem 3.1.1 *Let $D(\tilde{A}_W)$ be a quantum double obtained by Vladimirov's construction from the triple W, X, Z. Then its structure doesn't depend on matrices W, Z at all, it is completely determined by the matrix X itself.*

Proof: The proof is very simple, based on the previous proof of compatibility of pairing and the algebra structures (see Theorem 2.1.1) and reversing its reasoning. Let me assume that X is a $n^2 \times n^2$ matrix and let me define an antidual pairing of tensor algebras (with coproduct $\Delta(U) = U \otimes U$, $\Delta(V) = V \otimes V$ and counit $\epsilon(U) = \mathbf{1}_n, \epsilon(V) = \mathbf{1}_n$)[1] $T(\mathcal{L}\{\mathbf{1}, u^i_j, \; i, j \in \{1, \ldots, n\}\}) \times T(\mathcal{L}\{\mathbf{1}, v^i_j, \; i, j \in \{1, \ldots, n\}\}) \to k$ by defining the pairing between generators

[1]Such bialgebras can be imagined as quantum matrix bialgebras given by permutation matrix.

$\langle u^i_j, v^k_l \rangle = X^{i,k}_{j,l}$ and requiring the properties of antidual pairing. Let W, Z be arbitrary matrices satisfying $[W, X, X] = 0, [X, X, Z] = 0$. Then from the proof of Theorem 2.1.1 I know that the pairing is consistent with the structure of A_W and A_Z, i.e. with factoring out the elements $W^{ab}_{ij} u^i_c u^j_d - u^b_j u^a_i W^{ij}_{cd}$ and $Z^{ab}_{ij} v^i_c v^j_d - v^b_j v^a_i Z^{ij}_{cd}$, so those elements lie in the corresponding null biideals of the pairing of tensor algebras. Since during the construction of quantum double I factor out the null biideals of the pairing, any information about matrices W, Z disappears, the bialgebras $\tilde{A}_W$ and $\tilde{A}_Z$ are the same for given pair W, Z and for $W' = P$, $Z' = P$ (P being the permutation matrix $P^{i,j}_{k,l} = \delta^i_l \delta^j_k$. It satisfies $[P, X, X] = [X, X, P] = 0$ for any X). The conclusion is that $\tilde{A}_W$ and $\tilde{A}_Z$ are the same for any pair W, Z satisfying $[W, X, X] = 0, [X, X, Z] = 0$. Completing the construction of the quantum double (and remembering that the antipodes on $\tilde{A}_W$ and $\tilde{A}_Z$ are unique if they exist, so they cannot depend on W or Z) **I prove the theorem**.

Theorem 3.1.1 has the following consequence: the work done in [3] shows to be less important for the construction of the quantum doubles using Vladimirov's method than before, since the construction can be *in principle* based on arbitrary matrix X of suitable dimension starting from tensor algebras with coalgebra structure instead of quantum matrix bialgebras (i.e. assuming $W = Z = P$).

Of course, the problems explained in the previous chapter (finding the null biideals, the existence of antipodes and of dual bases) remain but Theorem 3.1.1 can be used also in solving these problems.The first obstacle that one encounters during the construction of quantum double is factoring out the null biideals. It is usually difficult to find elements lying in them and to prove that one has found the whole null biideal. There is a possible method of finding some quadratic elements lying in the null biideal of $T(\mathcal{L}\{\mathbf{1}, u^i_j,\ i, j \in \{1, \ldots, n\}\})$ by finding any matrix W satisfying $[W, X, X] = 0$ and then constructing the elements $W^{ab}_{ij} u^i_c u^j_d - u^b_j u^a_i W^{ij}_{cd}$. They are then surely the elements of the null biideal. Similar technique can be applied to find the null biideal of $T(\mathcal{L}\{\mathbf{1}, v^i_j,\ i, j \in \{1, \ldots, n\}\})$. Also the matrices W,Z could give some hints on the possible form of antipodes in $\tilde{A}_W$ and $\tilde{A}_Z$.

The conclusion of this section is that the construction of the quantum double can be based on just one matrix X. On the other hand the knowledge of matrices W,Z satisfying $[W, X, X] = 0, [X, X, Z] = 0$ can help to solve some of the technical problems during the construction.

3.2 Introducing antipodes using FRT method

In this section I explain a different way of introducing antipodes in $\tilde{A}_W$ and $\tilde{A}_Z$. This method was proposed by Vladimirov in [10] (for a special case $W = X = Z$, but it can be generalised for almost arbitrary invertible triple solving (2.1)) and is based on introducing antipodes already in A_W and A_Z. The main idea of this procedure (known as FRT method, published firstly in [2]) is to enlarge both A_W and A_Z by formal matrices SU and SV with suitable relations, i.e. the new bialgebras are factor algebras of $T(\mathcal{L}\{\mathbf{1}, u^i_j, Su^i_j\})$ ($T(\mathcal{L}\{\mathbf{1}, v^i_j, Sv^i_j\})$). As I'll show a bit later, for W, Z solving the Yang-Baxter equation and being invertible and so-called second invertible such new bialgebras are Hopf algebras (i.e. $S(St^i_j)$ can be expressed using products of elements $\mathbf{1}, t^k_l$ and St^k_l, it is not needed to add another set of elements $S(St^i_j)$). The foundation of this method can be formulated in following theorem.

Theorem 3.2.1 *Let A_R be a quantum matrix bialgebra (generated by a formal matrix T) with a dual quasitriangular structure $\mathcal{R}$ (i.e. R is invertible and satisfies $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ and $\mathcal{R}$ is given by $\mathcal{R}(T_1 \otimes T_2) = R_{12}$, $\mathcal{R}(T \otimes \mathbf{1}) = \mathbf{1}_2$ and $\mathcal{R}(\mathbf{1} \otimes T) = \mathbf{1}_2$). Let $\tilde{A}_R$ be some factor bialgebra of A_R such that the antipode on $\tilde{A}_R$ exists and the dual quasitriangular structure is compatible with the factorization. Then*

$$\tilde{R} = \mathcal{R}(T_1 \otimes ST_2)$$

satisfies $\tilde{R}^{ib}_{aj} R^{ak}_{lb} = \delta^i_l \delta^k_j = R^{ib}_{aj} \tilde{R}^{ak}_{lb}$. If I define a matrix $v^i_j = \tilde{R}^{ia}_{aj}$, then (of course matrix multiplication is assumed)

$$S^2 T = v^{-1} T v$$

Proof: Firstly I evaluate $\tilde{R}^{ib}_{aj} = \mathcal{R}(t^i_a \otimes St^b_j)$ in the product $\tilde{R}^{ib}_{aj} R^{ak}_{lb} = \mathcal{R}(t^i_a \otimes St^b_j)\mathcal{R}(t^a_l \otimes t^k_b) = \mathcal{R}(t^i_l \otimes t^k_b St^b_j) = \delta^k_j \mathcal{R}(t^i_b \otimes \mathbf{1}) = \delta^k_j \delta^i_l$ and similarly from the other side.

Next, writing $v^i_j = \tilde{R}^{ia}_{aj} = \mathcal{R}(t^i_a \otimes St^a_j)$ I find that $v^i_j = v(t^i_j)$ where the map v is given in Theorem 1.3.5, and using the same theorem I find $S^2 T = v^{-1} T v$. This **finishes the proof**.

The previous theorem gives a possible way of introducing the antipode on arbitrary A_R, where R is a biinvertible (i.e. $\tilde{R}$ and R^{-1} exist) solution of the Yang-Baxter equation. Let me add a new set of generators to A_R, namely formal matrix ST and require relations $T(ST) = \mathbf{1}_2$, $(ST)T = \mathbf{1}_2$, $R_{12}ST_2ST_1 = ST_1ST_2R_{12}$, $\Delta(ST) = \tau(ST \otimes ST)$ ($\Delta(St^i_j) = St^k_j \otimes St^i_k$) and

$\epsilon(ST) = \mathbf{1}_2$ and denote the new bialgebra GL_R. Now it is possible to define the antipode on GL_R setting $S(T) = ST$ and $S(ST) = v^{-1}Tv$ and extend it as an antihomomorphism of bialgebras. Then from the proof of the previous theorem it is possible to conclude that the antipode map just defined is really an antipode.

In the next step I should find pairing between new Hopf algebras GL_W and GL_Z. The obvious choice is $\langle U_1, V_2\rangle = X_{12}$, but it remains a question how to extend it on elements involving antipodes. The logical way can be deduced from the antiduality and from $U(SU) = \mathbf{1}_2$: $\mathbf{1}_4 = \langle(\mathbf{1}_2)_1, V_2\rangle = \langle U_1(SU_1), V_2\rangle = \langle U_1, V_2\rangle\langle SU_1, V_2\rangle = X_{12}\langle SU_1, V_2\rangle$ implies $\langle SU_1, V_2\rangle = X_{12}^{-1}$. Similarly $\delta^i_j\delta^k_m = \langle u^i_j, v^k_l Sv^l_m\rangle = \langle u^i_n, Sv^l_m\rangle\langle u^n_j, v^k_l\rangle = \langle u^i_n, Sv^l_m\rangle X^{nk}_{jl}$ indicating that matrix $Y^{ik}_{jl} = \langle u^i_j, Sv^k_l\rangle$ is the second inverse of X, i.e. there is a new condition on X, it should be biinvertible - this excludes from my considerations e.g. $X = P$. Finally $\delta^i_j\delta^l_n = \langle Su^i_j, \mathbf{1}\rangle\delta^l_n = \langle Su^i_j, v^l{}_m Sv^m_n\rangle = \langle Su^i_k, v^l_m\rangle\langle Su^k_j, Sv^m_n\rangle = (X^{-1})^{il}_{km}\langle Su^k_j, Sv^m_n\rangle$ gives $\langle SU_1, SV_2\rangle = X_{12}$. Given the pairing between generators, I can extend it consistently on the whole Hopf algebras, so **it is possible to define an antidual pairing of GL_W and GL_Z provided X is biinvertible.**

Now I can factor out null biideals and find a nondegenerate pairing between quotients $\tilde{GL}_W$ and $\tilde{GL}_Z$ provided X. From Section 1.2 I know that $\tilde{GL}_W$ and $\tilde{GL}_Z$ are Hopf algebras, so I can continue with the construction of the quantum double in a usual way.

It remains a question whether Hopf algebras $\tilde{GL}_W$ and $\tilde{GL}_Z$ coincide with $\tilde{A}_W$ and $\tilde{A}_Z$ when antipodes on $\tilde{A}_W$ and $\tilde{A}_Z$ exist. This is of course the case in some examples but generally the equality might not hold because of the following reason: During the construction of $\tilde{A}_W$ I factor out elements that pair with any element from A_Z to zero. On the other hand during construction of $\tilde{GL}_W$ I factor out elements that pair to zero not only with elements of A_Z but also with their antipodes (and other elements of GL_Z). Nothing guarantees that such pairing must be zero for all elements of A_W that pair to zero with elements of A_Z. Consequently, using FRT approach one may obtain a quantum double different from the one obtained in the original procedure. Also using FRT approach the result might depend on matrices W,Z the analog of Theorem 3.1.1 is not valid in this case.

To conclude, the method explained in this section can be used under additional conditions on original matrices W,X,Z to introduce antipodes and make the construction of the quantum double possible. On the other hand, in many cases the problem of antipodes can be solved in the context of the original method by proper normalization of X or by choosing suitable X (e.g. instead of $X = a\mathbf{1}_4, a^n \neq 1 \forall n \in \mathcal{N}$ that doesn't allow antipodes (see next chapter) one may use $X = diag(a, a^{-1}, a^{-1}, a)$ that allows existence of inverse elements in both bialgebras $\tilde{A}_W$ and $\tilde{A}_Z$ and the resulting $\tilde{A}_W$ and $\tilde{A}_Z$ contain the original bialgebras of $X = a\mathbf{1}_4$ as sub-bialgebras). Also introducing the antipodes using FRT approach can make factorization of null biideals in concrete cases more difficult (one has to check more relations, also the final proof of the nondegeneracy of the resulting pairing might be more difficult).

Chapter 4

Examples

In the present chapter I illustrate the scheme described in the previous chapter on different examples. I use the modification of the original method without FRT approach (i.e. I use the approach based on Section 3.1). Since the X-matrix is the only one important of the triple W, X, Z, I study the consequences of different X-matrices in the following sections, the matrices W,Z are not specified. Also the notation is modified accordingly, I write U_X instead of $\tilde{A}_W$ and V_X instead of $\tilde{A}_Z$. Instead of A_W I consider simply $A_P \equiv T(U)$ with matrix quantum group coalgebra structure and similarly instead of A_Z I consider $A_P \equiv T(V)$ with matrix quantum group coalgebra structure. The field k is taken to be the field of complex numbers $\mathcal{C}$.

4.1 Diagonal X-matrix

In this section I show several examples based on one of so-called generic X matrices (see [3]), namely a diagonal matrix.

4.1.1 General properties of diagonal X-matrix

Let me consider X being an arbitrary diagonal matrix,

$$X = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}.$$

I study some general properties of bialgebras generated this way. It is easy to see that both U_X and V_X have at most two generators each since u^1_2, u^2_1 and v^1_2, v^2_1 lie in null biideals. (Proof is straightforward: $\langle u^1_2, v^{k_1}_{l_1} v^{k_2}_{l_2} \dots v^{k_m}_{l_m} \rangle =$

$\langle u^1_{j_1}, v^{k_m}_{l_m}\rangle\langle u^{j_1}_{j_2}, v^{k_{m-1}}_{l_{m-1}}\rangle \dots \langle u^{j_{m-1}}_2, v^{k_1}_{l_1}\rangle = 0$, because the first term is nonzero only if $j_1 = 1$, similarly then the second term only if $j_2 = 1$ and so on and finally I obtain that the whole expression can be nonzero only if j_{m-1} is simultaneously 1 and 2. Proof that the other elements are in the corresponding null biideals is analogous.) The first important task is to find some simple sufficient conditions on the existence of antipodes. It can be guaranteed (in U_X) e.g. if one generator $u^1_1 \equiv u^{-1}$ is the inverse element to the second one $u^2_2 = u$, i.e. $u.u^{-1} = u^{-1}.u = \mathbf{1}$. Then any element of U_X can be written in the form $x = \sum_{i\in\mathcal{Z}} \alpha_i u^i$ and the antipode map is just $Sx = \sum_{i\in\mathcal{Z}} \alpha_i u^{-i}$. The verification is simple since $x_{(1)}Sx_{(2)} = \sum_{i\in\mathcal{Z}} \alpha_i u^i S u^i = \sum_{i\in\mathcal{Z}} \alpha_{-i} u^i u^{-i} = \sum_{i\in\mathcal{Z}} \alpha_i \mathbf{1} = \epsilon(x)$ and similarly the second condition on the antipode. It remains to find conditions on a, b, c, d implying $u^2_2.u^1_1 = u^1_1.u^2_2 = \mathbf{1}$. Such conditions can be found evaluating $\langle u^1_1u^2_2, v^1_1\rangle = \langle u^1_1, v^1_1\rangle\langle u^2_2, v^1_1\rangle = a.c$ (I have used $\Delta(u^i) = u^i \otimes u^i$ implied by $u^1_2 = u^2_1 = 0$.) and $\langle u^1_1u^2_2, v^2_2\rangle = \langle u^1_1, v^2_2\rangle\langle u^2_2, v^2_2\rangle = b.d$ and comparing the results with $\langle \mathbf{1}, v^1_1\rangle = 1$ and $\langle \mathbf{1}, v^2_2\rangle = 1$. This comparison gives conditions $ac = 1, bd = 1$. Evaluating the general expression $\langle u^1_1u^2_2, v^{i_1}_{i_1} \dots v^{i_k}_{i_k}\rangle = \langle u^1_1u^2_2, v^{i_k}_{i_k}\rangle \dots \langle u^1_1u^2_2, v^{i_1}_{i_1}\rangle = 1 = \langle \mathbf{1}, v^{i_1}_{i_1} \dots v^{i_k}_{i_k}\rangle$ (I have used $\Delta(u^1_1u^2_2) = \Delta(u^1_1)\Delta(u^2_2) = (u^1_1 \otimes u^1_1).(u^2_2 \otimes u^2_2) = u^1_1u^2_2 \otimes u^1_1u^2_2$) I find that the given conditions $ac = 1, bd = 1$ really imply $(u^1_1)^{-1} = u^2_2$. Similar examination of V_X gives conditions $ab = 1, cd = 1$ implying $v^1_1v^2_2 = v^2_2v^1_1 = \mathbf{1}$ and the existence of the antipode on V_X. **To sum up, conditions $ab = cd = ac = bd = 1$ (i.e. $a = d = b^{-1} = c^{-1}$) imply the existence of antipodes in both bialgebras.**

Next interesting point is the study of commutativity and cocommutativity of bialgebras U_X and V_X (I again assume X being an arbitrary diagonal matrix given above) .

Theorem 4.1.1 *Let $\langle u^i_j, v^k_l\rangle = X^{ik}_{jl}$ be an antidual pairing of two matrix bialgebras A_W and A_Z defined by a diagonal matrix X. After factoring out the null biideals, bialgebras U_X and V_X are both commutative and cocommutative.*

Proof: is straightforward (again proofs are given in U_X, in V_X are analogous)

- Commutativity: $\langle u^1_1u^2_2, v^{i_1}_{i_1} \dots v^{i_k}_{i_k}\rangle = \langle u^1_1u^2_2, v^{i_k}_{i_k}\rangle \dots \langle u^1_1u^2_2, v^{i_1}_{i_1}\rangle = \langle u^1_1, v^{i_k}_{i_k}\rangle\langle u^2_2, v^{i_k}_{i_k}\rangle \dots \langle u^1_1, v^{i_1}_{i_1}\rangle\langle u^2_2, v^{i_1}_{i_1}\rangle = \langle u^2_2u^1_1, v^{i_k}_{i_k}\rangle \dots \langle u^2_2u^1_1, v^{i_1}_{i_1}\rangle = \langle u^2_2u^1_1, v^{i_1}_{i_1} \dots v^{i_k}_{i_k}\rangle$ (I have used $\Delta(v^{i_k}_{i_k}) = v^{i_k}_{i_k} \otimes v^{i_k}_{i_k}$ after factoring out the null-biideals.) The fact that the pairing is nondegenerate **finishes the proof**.

- Cocommutativity: $\Delta(u^{i_1}_{i_1}\dots u^{i_k}_{i_k}) = \Delta(u^{i_1}_{i_1})\dots\Delta(u^{i_k}_{i_k}) = (u^{i_1}_{i_1}\otimes u^{i_1}_{i_1})\dots(u^{i_k}_{i_k}\otimes u^{i_k}_{i_k}) = (u^{i_1}_{i_1}\dots u^{i_k}_{i_k})\otimes(u^{i_1}_{i_1}\dots u^{i_k}_{i_k}) = (\tau\circ\Delta)(u^{i_1}_{i_1}\dots u^{i_k}_{i_k})$ and the linearity of the coproduct **finishes the proof**.

The consequence of this Theorem is that also the quantum double constructed over such pair of bialgebras is a cocommutative bialgebra (and therefore not very interesting). The whole construction can only generate some interesting $\mathcal{R}$-matrices different from $\mathcal{R} = \mathbf{1}$ (which is implied by the cocommutativity). (From Theorem 4.4.2 follows that the quantum double is also commutative.)

4.1.2 Special diagonal X-matrix

In this subsection I study the consequences of the X-matrix having the form

$$X = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, \tag{4.1}$$

where $a, b \in \mathcal{C}$ are nonzero numbers. The pairing given by this X matrix is degenerate. As is shown in the general example of diagonal matrix, I should factor out biideals generated by $u^1_2, u^2_1, u^1_1u^2_2 - u^2_2u^1_1$ and $v^1_2, v^2_1, v^1_1v^2_2 - v^2_2v^1_1$. Other, less obvious element lying in the null biideal of $T(U)$ is $u^1_1 - u^2_2$. Proof follows from $\langle u^1_1, v^k_l\rangle = \langle u^2_2, v^k_l\rangle$ and from the fact that after factoring out u^1_2, u^2_1 is $\Delta(u^1_1) = u^1_1 \otimes u^1_1$, i.e. $\langle u^1_1 - u^2_2, v^{k_1}_{l_1}v^{k_2}_{l_2}\dots v^{k_m}_{l_m}\rangle = \langle u^1_{j_1}, v^{k_m}_{l_m}\rangle\langle u^{j_1}_{j_2}, v^{k_{m-1}}_{l_{m-1}}\rangle\dots\langle u^{j_{m-1}}_1, v^{k_1}_{l_1}\rangle - \langle u^2_{j_1}, v^{k_m}_{l_m}\rangle\langle u^{j_1}_{j_2}, v^{k_{m-1}}_{l_{m-1}}\rangle\dots\langle u^{j_{m-1}}_2, v^{k_1}_{l_1}\rangle = \langle u^1_1, v^{k_m}_{l_m}\rangle\langle u^1_1, v^{k_{m-1}}_{l_{m-1}}\rangle\dots\langle u^1_1, v^{k_1}_{l_1}\rangle - \langle u^2_2, v^{k_m}_{l_m}\rangle\langle u^2_2, v^{k_{m-1}}_{l_{m-1}}\rangle\dots\langle u^2_2, v^{k_1}_{l_1}\rangle = 0$.
Whether the pairing after performing given factorization (i.e. the factorization generic for diagonal X-matrix plus $u^1_1 = u^2_2$) is nondegenerate depends on values of a, b. In generic case (i.e. when a, b and their products are not roots of unity) it really is. In the following text I examine special cases.

1. $X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

   First of all I study the most trivial example. In this case, both U_X and V_X have at most one generator $u = u^1_1$ and $v = v^1_1$. Studying the pairing more carefully one finds that for any $n \in \mathcal{N}$ $\langle u - \mathbf{1}, v^n\rangle =$

$\langle \Delta^{(n)}(u) - \Delta^{(n)}(\mathbf{1}), v \otimes v \otimes \ldots \otimes v \rangle = (\langle u, v \rangle)^n - 1 = 1^n - 1 = 0$, i.e. $u = \mathbf{1}$ and similarly $v = \mathbf{1}$; in this case the factorization has **trivialized the resulting quantum double to** $D(\mathrm{U}_X) = \mathrm{U}_X \otimes \mathrm{V}_X = \mathcal{C}$.

2. $X = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}$, where a is not a root of unity

This example is a small variation of the previous. Again in this case both U_X and V_X have at most one generator $u = u_1^1$ and $v = v_1^1$. Actually if I assume that a is not a root of unity then

$$\mathrm{U}_X = \mathcal{L}\{\mathbf{1}, u^n, n \in \mathcal{N}\}, \qquad \mathrm{V}_X = \mathcal{L}\{\mathbf{1}, v^n, n \in \mathcal{N}\}.$$

Proof of this statement can be done supposing there exists for any $n \in N$ $x = \alpha_0 \mathbf{1} + \sum_{i=1}^{n} \alpha_n u^n$ such that $\langle x, v^k \rangle = 0, \forall k \in \mathcal{N}$. Then $0 = \langle x, v^k \rangle = \sum_{i=0}^{n} \alpha_i a^{i.k}$ is an infinite set of homogenous linear equations. Taking into account only first $n+1$ of them and finding the determinant of the matrix A, $A_{(i,j)} = a^{i.j}$ of this set of equations gives (see Vandermond's determinant in linear algebra) $\det(A) = \prod_{i<j;i,j=0}^{n} (a^j - a^i) \neq 0$ (since A is not a root of unity). Consequently, given set of equations has only one, trivial solution $\alpha_i = 0, i \in \mathcal{N} \cup 0$ and the pairing is nondegenerate.

Since now I have an antidual pair of bialgebras, I try to find antipodes in U_X and V_X. I obtain from the construction given before that the antipode should be given $\langle Su, v \rangle = \langle u, S^{-1} v \rangle = \frac{1}{a}$ implying $\langle Su, v^k \rangle = \left(\frac{1}{a}\right)^k$. Supposing $Su = \alpha_0 \mathbf{1} + \sum_{i=1}^{n} \alpha_n u^n$ for some $n \in \mathcal{N}$ I find an infinite set of equations on coefficients α_i:

$$\sum_{i=0}^{n} \alpha_i a^{(i+1)k} = 1, \ k \in \mathcal{N} \cup 0 \tag{4.2}$$

Regrettably, this set of equations (in α_i) hasn't got a solution for any $n \in \mathcal{N}$. (**Proof:** Suppose that there is $n \in \mathcal{N}$ such that given equations have a solution. Then from first $n+1$ equations it is clear that such solution for given n is unique (again the determinant of the system is equal to $\prod_{i<j;i,j=1}^{n+1} (a^j - a^i) \neq 0$). I formally add α_{n+1} into $Su = \alpha_0 \mathbf{1} + \sum_{i=1}^{n+1} \alpha_n u^n$ and try to solve (4.2) for $n \to n+1$. Again I obtain that solution is unique and using Cramer's method for finding the solution of a system of linear equations I find $\alpha_{n+1} = -(\prod_{i<j;i,j=0}^{n+1} (a^j - a^i))/(\prod_{i<j;i,j=1}^{n+2} (a^j - a^i)) \neq 0$ (using Vandermond's

determinant, the numerator is obtained by replacing the last column of the matrix of the system by column consisting only of 1s and swapping the columns). Simultaneously from the uniqueness of the solution and from the initial assumption (that there is a solution in the form $Su = \alpha_0 \mathbf{1} + \sum_{i=1}^{n} \alpha_n u^n$) I get $\alpha_{n+1} = 0$ and I have found a contradiction, i.e. there isn't an element Su in U_X. (I remind that the tensor algebra (and consequently all its factor algebras) consists of all *finite* "powers" of generators.) Similar statement is also valid for V_X. **The conclusion is that in this case there are no antipodes in** U_X **and** V_X **and the construction of the quantum double is not possible.**

3. $X = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}$, $a^n = 1$, $a \neq 1$

When a is a root of unity then the results are completely different. Let me consider the simpliest nontrivial case $a = -1$. This implies $u^2 = \mathbf{1}$, $v^2 = \mathbf{1}$ leading to

$$\mathrm{U}_X = \mathcal{L}\{\mathbf{1}, u\}, \qquad \mathrm{V}_X = \mathcal{L}\{\mathbf{1}, v\}.$$

The antipodes can be identified with identity maps $Su = u$, $Sv = v$ ($u.Su = u^2 = \mathbf{1} = \epsilon(u)$ etc.). Consequently I can construct the quantum double $D(\mathrm{U}_X)$ which is coincident with coalgebra $\mathrm{U}_X \otimes \mathrm{V}_X$ and the cross-multiplication is defined from (1.2) as

$$v.u = \langle u, v\rangle u.v\langle u, v\rangle = u.v$$

and I obtain a Hopf algebra (the antipode is just the identity map) $D(\mathrm{U}_X) = \mathcal{L}\{\mathbf{1}, u, v, u.v\}$. The $\mathcal{R}$-matrix is then given by the canonical way $\mathcal{R} = \sum_j e^j \otimes e_j$ where $\{e_j\}$ and $\{e^j\}$ are dual basis of U_X and V_X. In this concrete case I choose $e_1 = \mathbf{1}$, $e_2 = u$ and find (from $\langle e_j, e^k\rangle = \delta_j^k$) that $e^1 = (\mathbf{1} + v)/2$ and $e^2 = (1 - v)/2$. Resulting nontrivial $\mathcal{R}$-matrix is

$$\mathcal{R} = \frac{1}{2}(\mathbf{1} + v \otimes \mathbf{1} + \mathbf{1} \otimes u - v \otimes u) \tag{4.3}$$

By explicit evaluation can be checked that I have really found an $\mathcal{R}$-matrix.

This result can be generalised as follows. Let $a^n = 1 \wedge a^k \neq 1 \forall k < n,\ k \in \mathcal{N}$. Then

$$\mathrm{U}_X = \mathcal{L}\{1, u, u^2, \ldots, u^{n-1}\}, \quad \mathrm{V}_X = \mathcal{L}\{1, v, v^2, \ldots, v^{n-1}\},$$

i.e. $\dim(\tilde{U}) = n$ and $\dim(\tilde{V}) = n$. The antipodes can be defined $Su = u^{n-1}$, $Sv = v^{n-1}$ and extended as antialgebra maps. The quantum double is

$$D(\mathrm{U}_X) = \mathcal{L}\{\mathbf{1}, u, \ldots, u^{n-1}, v, \ldots, v^{n-1}, uv, \ldots, u^{n-1}v^{n-1}\},$$

$\dim(D(\mathrm{U}_X)) = n^2$. The cross multiplication is generated by

$$v.u = \langle u, v\rangle u.v \langle u^{n-1}, v\rangle = a^n u.v = u.v,$$

i.e. $D(\mathrm{U}_X)$ is commutative (and cocommutative, since the comultiplication is a tensor product comultiplication of cocommutative comultiplications). The dual pair of bases can be chosen for example $e_j = u^j$, $j \in \{0, 1, \ldots, n-1\}$ and

$$e^k = \frac{1}{n}\sum_{l=0}^{n-1} a^{k(n-l)} v^l.$$

(**Proof:** $\langle e_j, e^k\rangle = \frac{1}{n}\sum_{l=0}^{n-1} a^{k(n-l)}a^{jl} = \frac{1}{n}\sum_{l=0}^{n-1} a^{kn}a^{l(j-k)} = \sum_{l=0}^{n-1}\frac{a^{l(j-k)}}{n}$. If $j = k$ then $\langle e_j, e^j\rangle = \sum_{l=0}^{n-1}\frac{1}{n} = 1$, otherwise $\langle e_j, e^k\rangle = \sum_{l=0}^{n-1}\frac{a^{l(j-k)}}{n} = 0$ since $\frac{1}{n}\sum_{l=0}^{n-1} a^{l(j-k)}.(a^{j-k} - 1) = a^{(j-k)n} - 1 = 1 - 1 = 0$ and $(a^{j-k} - 1) \neq 0$.) The resulting $\mathcal{R}$-matrix is

$$\mathcal{R} = \frac{1}{n}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1} a^{k(n-l)} v^l \otimes u^k. \tag{4.4}$$

4.2 Another generic X-matrix

In this section I study another generic (in the sense of [3]) X-matrix, namely

$$X = \begin{pmatrix} a & 0 & 0 & 0 \\ c & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & d & b \end{pmatrix}. \tag{4.5}$$

Factoring out the null biideals of the pairing given by this X-matrix evidently gives $u_2^1 = 0, u_1^2 = 0$ and $v_2^1 = 0$. Remaining generators u_1^1, u_2^2 and v_1^1, v_2^2 are group-like (i.e. $\Delta(u_1^1) = u_1^1 \otimes u_1^1$ etc.), the coproduct of v_1^2 is $\Delta(v_1^2) = v_1^2 \otimes v_1^1 + v_2^2 \otimes v_1^2$.

Studying the pairing more carefully one obtains that u_1^1 and u_2^2 commute ($\langle u_1^1 u_2^2, v_j^j\rangle = \langle u_1^1, v_j^j\rangle\langle u_2^2, v_j^j\rangle = a.b = \langle u_2^2 u_1^1, v_j^j\rangle$, $j = 1, 2$ and $\langle u_1^1 u_2^2, v_1^2\rangle =$

$\langle u_1^1 \otimes u_2^2, v_1^2 \otimes v_1^1 + v_2^2 \otimes v_1^2 \rangle = c.b + a.d = \ldots = \langle u_2^2 u_1^1, v_1^2 \rangle$. General equality $\langle u_1^1 u_2^2, v_{j_1}^{i_1} \ldots v_{j_k}^{i_k} \rangle = \langle u_2^2 u_1^1, v_{j_1}^{i_1} \ldots v_{j_k}^{i_k} \rangle$ then follows from the fact that $u_1^1 u_2^2$ (and $u_2^2 u_1^1$, of course) is group-like, i.e. the general expression can be rewritten as a product of expressions evaluated above.) Another consequence of factoring out the null biideals is $v_1^1 = v_2^2$ in V_X. (The proof follows from $\langle (u_2^2)^k (u_1^1)^l, v_1^1 \rangle = \langle u_2^2, v_1^1 \rangle^k \langle u_1^1, v_1^1 \rangle^l = b^k a^l = \langle u_2^2, v_2^2 \rangle^k \langle u_1^1, v_2^2 \rangle^l = \langle (u_2^2)^k (u_1^1)^l, v_2^2 \rangle$.) I also find $v_1^1 v_1^2 = v_1^2 v_1^1$ (The proof follows from the fact that u_1^1, u_2^2 (and also their products and powers) are group-like: $\langle u_{i_1}^{i_1} \ldots u_{i_k}^{i_k}, v_1^1 v_1^2 \rangle = \langle \Delta(u_{i_1}^{i_1} \ldots u_{i_k}^{i_k}), v_1^2 \otimes v_1^1 \rangle = \langle u_{i_1}^{i_1} \ldots u_{i_k}^{i_k}, v_1^2 \rangle \langle u_{i_1}^{i_1} \ldots u_{i_k}^{i_k}, v_1^1 \rangle = \langle \Delta(u_{i_1}^{i_1} \ldots u_{i_k}^{i_k}), v_1^2 \otimes v_1^1 \rangle = \langle u_{i_1}^{i_1} \ldots u_{i_k}^{i_k}, v_1^2 v_1^1 \rangle$).

The conclusion is that $u_2^1 = u_1^2 = 0,\ v_2^1 = 0,\ v_1^1 = v_2^2$ **and both** U_X **and** V_X **are commutative and cocommutative.**

4.2.1 Simple examples of the X-matrix (4.5)

The first example is $a = b = 1,\ c = d = \lambda$. Those relations lead to U_X generated as a tensor algebra by elements $\mathbf{1}, u_1^1$ and V_X generated by $\mathbf{1}, v_1^2$. Regrettably there is no antipode in U_X (**Proof:** Suppose there is $m \in \mathcal{N}$ such that an antipode is given by $S u_1^1 = \sum_{k=0}^m \alpha_k (u_1^1)^k$. This leads immediately to contradiction since $u_1^1.S u_1^1 = \sum_{k=0}^m \alpha_k (u_1^1)^{k+1}$ cannot be equal to $\epsilon(u_1^1) = 1$ for any α_k.), so **this case is not interesting**.

The second example is $a = b = c = 1,\ d = -1$. In this case I obtain that U_X is generated as a tensor algebra by elements $\mathbf{1}, u_1^1, u_2^2$ factorised by relations $u_1^1 u_2^2 = \mathbf{1},\ u_2^2 u_1^1 = \mathbf{1}$, i.e. $(u_1^1)^{-1} = u_2^2$ and V_X is generated by $\mathbf{1}, v_1^2$. (The nondegeneracy of the pairing can be proved supposing there exists $v = \sum_{k=0}^m \alpha_k (v_1^2)^k$ lying in the null biideal, i.e. $\forall j \in \mathcal{Z}\ \langle (u_1^1)^j, \sum_{k=0}^m \alpha_k (v_1^2)^k \rangle = \sum_{k=0}^m \alpha_k (j)^k = 0$ (I have used $\Delta^j(v_1^2) = v_1^2 \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1} + \mathbf{1} \otimes v_1^2 \otimes \ldots \otimes \mathbf{1} + \ldots + \mathbf{1} \otimes \ldots \otimes \mathbf{1} \otimes v_1^2$). Using similar argumentation as in the proof of nondegeneracy of pairing in the case of diagonal X-matrix based on the Vandermond's determinant I find that $\alpha_k = 0,\ \forall k$. (It suffices to consider the first $n{+}1$ equations with prime j's.) Similarly in the U_X I obtain for element in the null biideal $u = \sum_{k=-m}^m \alpha_k (u_1^1)^k$ equations of type $\sum_{k=-m}^m \alpha_k (k)^j = 0$ and again I use the properties of Vandermond's determinant to conclude that the pairing is really nondegenerate.) There are antipode maps in this case, namely $S u_1^1 = u_2^2,\ S u_2^2 = u_1^1$ and $S v_1^2 = -v_1^2$. It is therefore possible to construct $D(\mathrm{U}_X)$. To be able to construct the $\mathcal{R}$-matrix I need a pair of dual bases of U_X and V_X. I try to construct a basis of U_X dual to basis $e^j = (v_1^2)^j$ of V_X. I express $e_k = \sum_{m=-n_k}^{n_k} \alpha_{km} (u_1^1)^m$ where $n_k \in \mathcal{N}$. It is possible to find $e_0 = \mathbf{1}$ from $\langle \sum_{m=-n_0}^{n_0} \alpha_{0m} (u_1^1)^m, e^j \rangle = \delta_0^j$, but the condition on e_1 gives $\langle \sum_{m=-n_1}^{n_1} \alpha_{1m} (u_1^1)^m, e^j \rangle = \sum_{m=-n_1}^{n_1} \alpha_{km} m^j = \delta_1^j,\ \forall j \in \mathcal{N}$ and from the knowledge of Vandermond's determinant is again possible to prove that this infinite system of equations hasn't got any solution for arbitrary finite n_1. It remains a question whether is possible to find some linear transformation of the basis $\{e^j\}$ that would allow the existence of a dual basis.

Because I am not able to construct a pair of dual bases in a rigorous way, I try the following approach based on the properties of the exponential function $\exp(a) = \sum_{i=0}^{\infty} \frac{a^i}{i!}$: I formally write $u_1^1 = \exp(h),\ u_2^2 = \exp(-h)$ and study the properties of h implied by the properties of u_1^1 and u_2^2. The exponential is not a finite linear combination of elements a^i, on the other hand it converges in many of usual topologies for a large set of elements a. It is reasonable to expect that expressions like $\exp(h)$ have good meaning in similar cases as the infinite linear combination in the definition of the canonical $\mathcal{R}$-matrix of the quantum double has (e.g. suitable representation). The properties of h should be as follows: $\Delta(h) = h \otimes \mathbf{1} + \mathbf{1} \otimes h$ in order to satisfy $\exp(h) \otimes \exp(h) = \Delta(\exp(h)) = (\sum_{i=0}^{\infty} \frac{\Delta(h)^i}{i!}) = \exp(\Delta(h)) = \exp(h \otimes \mathbf{1} + \mathbf{1} \otimes h) = \exp(h \otimes \mathbf{1}) \exp(\mathbf{1} \otimes h) = (\exp(h) \otimes \mathbf{1})(\mathbf{1} \otimes \exp(h)) = \exp(h) \otimes \exp(h))$, the antipode on h should be $Sh = -h$ (in order to satisfy $\exp(-h) = Su = S\exp(h) = \sum_i (Sh)^i/i! = \sum_i (-h)^i/i! = \exp(-h)$) and the pairing should be $\langle h^j, v^k\rangle = k!\delta_{jk}$ (this follows from the requirement $\langle \exp(jh), v^k\rangle = \langle u^j, v^k\rangle = j^k$ i.e. $\langle \exp(jh), v^k\rangle = \sum_i j^i/i! \langle h^i, v^k\rangle = j^k k!/k! = j^k$). Now it is possible to define a pair of dual "bases" of U_X and V_X $e_j = h^j,\ e^j = v^j,\ j \in \mathcal{N} \cup 0$ (of course $e_j = h^j$ isn't strictly speaking a base of U_X, it is a base of a new algebra A generated by 1 and h as a tensor algebra. If it is possible to give a well-defined meaning to the exponential, it is also possible to identify A and U_X.) and to construct the $\mathcal{R}$-matrix

$$\mathcal{R} = \sum_{j=0}^{\infty} e^j \otimes e_j = \sum_{j=0}^{\infty} \frac{1}{j!} v^j \otimes h^j = \exp(v \otimes h). \tag{4.6}$$

4.3 Example of less trivial X-matrix leading to almost trivial results

In the previous sections I have studied the examples based on X-matrices that have had a lot of zero elements and consequently lead to very "hard" factorization, almost all interesting properties (non-commutativity, non-cocommutativity) have been lost. In this example I investigate properties of an X-matrix

$$X = \alpha \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}. \tag{4.7}$$

This X-matrix has no elements equal to zero, so one may think for the first moment that it may lead to more interesting bialgebras. As I show, this assumption is wrong. Also in this case the resulting quantum double is a commutative and cocommutative bialgebra, for a special choice of parameter α it is even finite-dimensional.

Factoring out the appropriate null-biideals for general α I obtain relations $u^1_1 = u^2_2$, $u^1_2 = u^2_1$, $v^1_1 = v^2_2$, $v^1_2 = v^2_1, u^1_1 u^1_2 = 0, u^1_2 u^1_1 = 0, v^1_1 v^1_2 = 0, u^1_2 v^1_1 = 0$. This factorization implies both commutativity and cocommutativity ($\Delta(u^1_1) = u^1_1 \otimes u^1_1 + u^1_2 \otimes u^1_2, \Delta(u^1_2) = u^1_2 \otimes u^1_1 + u^1_1 \otimes u^1_2$ and similarly for v's). The proof of the relations given above can be done e.g. by induction. Firstly I check by explicit evaluation its validity for the pairing with the generators, i.e.$\langle u^1_1, v^i_j\rangle = \langle u^2_2, v^i_j\rangle$, $\langle u^1_2, v^i_j\rangle = \langle u^2_1, v^i_j\rangle$, $\langle u^1_1 u^1_2, v^i_j\rangle = \langle u^1_1, v^i_k\rangle\langle u^1_2, v^k_j\rangle = \sum_{k=1,2} 1\langle u^1_2, v^k_j\rangle = 1 - 1 = 0$, $\langle u^1_2 u^1_1, v^i_j\rangle = 0$. In the second step I assume that for $k \leq n$ and $\forall v^{i_1}_{j_1}, v^{i_2}_{j_2} \ldots v^{i_k}_{j_k}$ is $\langle u^1_1 - u^2_2, v^{i_1}_{j_1} v^{i_2}_{j_2} \ldots v^{i_k}_{j_k}\rangle = 0$ and similarly the other pairings like $\langle u^1_1 u^1_2, v^{i_1}_{j_1} v^{i_2}_{j_2} \ldots v^{i_k}_{j_k}\rangle = 0$ etc. Now I prove that those pairings are equal to zero also for $k = n+1$ and any $v^{i_1}_{j_1} v^{i_2}_{j_2} \ldots v^{i_k}_{j_k}$. I evaluate using the assumed equalities $\langle u^1_1, v^{i_1}_{j_1} v^{i_2}_{j_2} \ldots v^{i_{n+1}}_{j_{n+1}}\rangle = \langle \Delta(u^1_1), v^{i_{n+1}}_{j_{n+1}} \otimes v^{i_1}_{j_1} v^{i_2}_{j_2} \ldots v^{i_n}_{j_n}\rangle = \langle u^1_1 \otimes u^1_1 + u^1_2 \otimes u^2_1, v^{i_{n+1}}_{j_{n+1}} \otimes v^{i_1}_{j_1} v^{i_2}_{j_2} \ldots v^{i_n}_{j_n}\rangle = \langle u^1_1, v^{i_{n+1}}_{j_{n+1}}\rangle\langle u^1_1, v^{i_1}_{j_1} v^{i_2}_{j_2} \ldots v^{i_n}_{j_n}\rangle + \langle u^1_2, v^{i_{n+1}}_{j_{n+1}}\rangle\langle u^2_1, v^{i_1}_{j_1} v^{i_2}_{j_2} \ldots v^{i_n}_{j_n}\rangle = \langle u^2_2, v^{i_{n+1}}_{j_{n+1}}\rangle \langle u^2_2, v^{i_1}_{j_1} v^{i_2}_{j_2} \ldots v^{i_n}_{j_n}\rangle + \langle u^2_1, v^{i_{n+1}}_{j_{n+1}}\rangle\langle u^1_2, v^{i_1}_{j_1} v^{i_2}_{j_2} \ldots v^{i_n}_{j_n}\rangle = \ldots = \langle u^2_2, v^{i_1}_{j_1} v^{i_2}_{j_2} \ldots v^{i_{n+1}}_{j_{n+1}}\rangle$. Similarly $\langle u^1_2, v^{i_1}_{j_1} v^{i_2}_{j_2} \ldots v^{i_{n+1}}_{j_{n+1}}\rangle = \langle \Delta(u^1_2), v^{i_{n+1}}_{j_{n+1}} \otimes v^{i_1}_{j_1} v^{i_2}_{j_2} \ldots v^{i_n}_{j_n}\rangle = \langle u^1_2 \otimes u^2_2 + u^1_1 \otimes u^1_2, v^{i_{n+1}}_{j_{n+1}} \otimes v^{i_1}_{j_1} v^{i_2}_{j_2} \ldots v^{i_n}_{j_n}\rangle = \langle u^1_2, v^{i_{n+1}}_{j_{n+1}}\rangle\langle u^2_2, v^{i_1}_{j_1} v^{i_2}_{j_2} \ldots v^{i_n}_{j_n}\rangle + \langle u^1_1, v^{i_{n+1}}_{j_{n+1}}\rangle\langle u^1_2, v^{i_1}_{j_1} v^{i_2}_{j_2} \ldots v^{i_n}_{j_n}\rangle = \langle u^2_1, v^{i_{n+1}}_{j_{n+1}}\rangle\langle u^1_1, v^{i_1}_{j_1} v^{i_2}_{j_2} \ldots v^{i_n}_{j_n}\rangle + \langle u^2_2, v^{i_{n+1}}_{j_{n+1}}\rangle\langle u^2_1, v^{i_1}_{j_1} v^{i_2}_{j_2} \ldots v^{i_n}_{j_n}\rangle = \ldots = \langle u^2_1, v^{i_1}_{j_1} v^{i_2}_{j_2} \ldots v^{i_{n+1}}_{j_{n+1}}\rangle$, i.e. I have just shown by induction that factoring out the null biideals leads to relations $u^1_1 = u^2_2, u^1_2 = u^2_1$ (and similarly can be proved $v^1_1 = v^2_2, v^1_2 = v^2_1$). The remaining pairings can be evaluated for $k = n+1$ in a similar way (using the statements $u^1_1 = u^2_2, u^1_2 = u^2_1$ just proved), e.g. $\langle u^1_1 u^1_2, v^{i_1}_{j_1} v^{i_2}_{j_2} \ldots v^{i_{n+1}}_{j_{n+1}}\rangle = \langle \Delta(u^1_1 u^1_2), v^{i_{n+1}}_{j_{n+1}} \otimes v^{i_1}_{j_1} v^{i_2}_{j_2} \ldots v^{i_n}_{j_n}\rangle = \langle u^1_1 u^1_2 \otimes u^1_1 u^1_1 + u^1_2 u^1_2 \otimes u^1_2 u^1_1 + u^1_1 u^1_1 \otimes u^1_1 u^1_2 + u^1_2 u^1_1 \otimes u^1_2 u^1_2, v^{i_{n+1}}_{j_{n+1}} \otimes v^{i_1}_{j_1} v^{i_2}_{j_2} \ldots v^{i_n}_{j_n}\rangle = 0$ (using linearity and the assumed equality $\langle u^1_1 u^1_2, v^{i_1}_{j_1} v^{i_2}_{j_2} \ldots v^{i_k}_{j_k}\rangle = 0$, $\forall k \leq n$). The remaining relations in U_X and V_X can be proved in a similar way. The relations just proved imply the commutativity (and also cocommutativity) of both U_X and V_X. This leads according Theorem 4.4.2 to a commutative and cocommutative quantum double (or at least a commutative and cocommutative generalised quantum double, if the antipodes don't exist for given value of α), i.e. not very interesting case.

In the following I won't study the general value of the scaling parameter α (for generic values of α the antipodes don't exist, since in general the null-biideals don't contain any linear combination of generators and its powers containing unit, so $x.y \neq \mathbf{1}$ except $x = \mathbf{1} \wedge y = \mathbf{1}$ and it is not possible to fulfill $(u^1_1)_{(1)} S(u^1_1)_{(2)} = \mathbf{1}$.), I restrict myself to a special value of parameter $\alpha = \frac{1}{2}$. This assumption leads to further factorizations $\mathbf{1} = u^1_1 + u^1_2$, $\mathbf{1} = v^1_1 + v^1_2$ (again can be proved by induction), i.e. together with relations $u^1_1 u^1_2 = 0$ etc. I find that $u^1_1 u^1_1 = u^1_1$ (and $v^1_1 v^1_1 = v^1_1$).

The resulting bialgebras U_X and V_X are therefore 2-dimensional,

- $\mathrm{U}_X = \mathcal{L}\{\mathbf{1}, u^1_1\}$, with the multiplication $u^1_1 u^1_1 = u^1_1$ and the coproduct $\Delta(u^1_1) = 2u^1_1 \otimes u^1_1 + \mathbf{1} \otimes \mathbf{1} - \mathbf{1} \otimes u^1_1 - u^1_1 \otimes \mathbf{1}$,
- $\mathrm{V}_X = \mathcal{L}\{\mathbf{1}, v^1_1\}$, with the multiplication $v^1_1 v^1_1 = v^1_1$ and the coproduct $\Delta(v^1_1) = 2v^1_1 \otimes v^1_1 + \mathbf{1} \otimes \mathbf{1} - \mathbf{1} \otimes v^1_1 - v^1_1 \otimes \mathbf{1}$.

The antipodes can be identified with the identity maps. A pair of dual bases can be chosen e.g. $e_0 = \mathbf{1}, e^0 = \mathbf{1}, e_1 = \sqrt{2}(\mathbf{1} - u^1_1), e^1 = \sqrt{2}(v^1_1 - \mathbf{1})$.

The quantum double $D(\mathrm{U}_X) = \mathcal{L}\{\mathbf{1}, u^1_1, v^1_1, u^1_1 v^1_1\}$ then has got an $\mathcal{R}$-matrix

$$\mathcal{R} = 2v \otimes \mathbf{1} + 2.\mathbf{1} \otimes u - 2v \otimes u - \mathbf{1} \otimes \mathbf{1}. \tag{4.8}$$

4.4 Conclusions based on previous examples

Some of the conclusions of the results of previous examples can be formulated in the following theorems:

Theorem 4.4.1 U_X *is a commutative bialgebra if and only if* V_X *is a cocommutative bialgebra. Similarly,* U_X *is a cocommutative bialgebra if and only if* V_X *is a commutative bialgebra.*

Proof: I prove only the first equivalence, the proof of the second one is completely analogous. The proof follows from evaluation $\langle u.\tilde{u}, v\rangle = \langle u \otimes \tilde{u}, v_{(1)} \otimes v_{(2)}\rangle$. If the bialgebra U_X is commutative then $\langle u \otimes \tilde{u}, \Delta(v)\rangle = \langle u \otimes \tilde{u}, v_{(1)} \otimes v_{(2)}\rangle = \langle u.\tilde{u}, v\rangle = \langle \tilde{u}.u, v\rangle = \langle u \otimes \tilde{u}, v_{(2)} \otimes v_{(1)}\rangle = \langle u \otimes \tilde{u}, (\tau \circ \Delta)(v)\rangle$, $\forall u, \tilde{u} \in \mathrm{U}_X, v \in \mathrm{V}_X$, together with the definition of the tensor product of vector spaces (using the fact that any element of the tensor product can be written as a linear combination of elements in the form $u \otimes \tilde{u}$) and the non-degeneracy of the pairing between U_X and V_X I have proved the first implication ($\Rightarrow$). On the other hand, if the bialgebra

V_X is cocommutative then $\langle u.\tilde{u}, v\rangle = \langle u \otimes \tilde{u}, v_{(1)} \otimes v_{(2)}\rangle = \langle u \otimes \tilde{u}, v_{(2)} \otimes v_{(1)}\rangle = \langle \tilde{u} \otimes u, v_{(1)} \otimes v_{(2)}\rangle = \langle \tilde{u}.u, v\rangle$, $\forall u, \tilde{u} \in \mathrm{U}_X, v \in \mathrm{V}_X$, the non-degeneracy of the pairing between U_X and V_X **finishes the proof of the second implication ($\Leftarrow$).**

Theorem 4.4.2 *Let* U_X *and* V_X *be cocommutative Hopf algebras. Then the resulting quantum double* $D(\mathrm{U}_X)$ *is a cocommutative and commutative Hopf algebra.*

Proof: The first statement follows directly from the definition of the quantum double, namely from $\Delta_{D(\mathrm{U}_X)} = (id \otimes \tau \otimes id) \circ \Delta_{\mathrm{U}_X} \otimes \Delta_{\mathrm{V}_X}$. The second statement is the consequence of the commutativity of the subbialgebras U_X and V_X and of the cocommutativity. The cocommutativity implies $u_{(1)} \otimes u_{(2)} \otimes u_{(3)} = u_{(2)} \otimes u_{(1)} \otimes u_{(3)}$ $\forall u \in \mathrm{U}_X$ (and similarly for v's from V_X) and those equalities imply the commutativity of the cross product $vu = \langle u_{(1)}, v_{(1)}\rangle u_{(2)} v_{(2)} \langle u_{(3)}, v_{(3)}\rangle^{-1} = u_{(1)} v_{(1)} \langle u_{(2)}, v_{(2)}\rangle \langle u_{(3)}, v_{(3)}\rangle^{-1} = u_{(1)} v_{(1)} \epsilon(u_{(2)}) \epsilon(v_{(2)}) = uv$, $\forall u \in \mathrm{U}_X, v \in \mathrm{V}_X$.

Another conclusion based on the previous examples is a simple but useful necessary condition for the existence of antipodes. As I have already mentioned in the last example, in order to have antipodes in U_X and V_X the null biideals must contain some elements that involve the algebra unit in their expansion into the bases of $T(U)$ and $T(V)$ composed of the powers of generators. More precisely:

Theorem 4.4.3 *Let* I *be the null biideal of the pairing* $\langle,\rangle$ *in* $T(U)$. *The necessary condition for the existence of an antipode in* $U_X = T(U)/I$ *is* $I \not\subset \mathcal{U} \oplus \mathcal{U} \otimes \mathcal{U} \oplus \mathcal{U} \otimes \mathcal{U} \otimes \mathcal{U} \oplus \ldots$, *where* $\mathcal{U} = \mathcal{L}[u^i_j, i, j = 1 \ldots n]$.

Proof: Let me assume that the antipode S in $T(U)$ exists and $I \subset \mathcal{U} \oplus \mathcal{U} \otimes \mathcal{U} \oplus \mathcal{U} \otimes \mathcal{U} \otimes \mathcal{U} \oplus \ldots$. Since I know $\epsilon(u^1_1) = 1, \Delta(u^1_1) = \sum_i u^1_i \otimes u^i_1$, after factorization using the definition of antipode ($[u] = u + I$) I find:

$$
\begin{aligned}
. \circ (S \otimes id) \circ \Delta([u^1_1]) &= \sum_i (S[u^1_i]).[u^i_1] \\
&= \epsilon([u^1_1])[\mathbf{1}] = [\mathbf{1}]
\end{aligned}
$$

Denoting x_i representants of $S[u^1_i]$ I find that in $T(U)$ must hold for some $a \in I$

$$
\sum_i x_i.u^i_1 + a = \mathbf{1}.
$$

From the definition of tensor algebra I find that $\sum_i x_i.u^i_1 \in \mathcal{U} \oplus \mathcal{U} \otimes \mathcal{U} \oplus \mathcal{U} \otimes \mathcal{U} \otimes \mathcal{U} \oplus \ldots$, together with the initial assumption I have derived that the left-hand side lies in $\mathcal{U} \oplus \mathcal{U} \otimes \mathcal{U} \oplus \mathcal{U} \otimes \mathcal{U} \otimes \mathcal{U} \oplus \ldots$ and simultaneously the right-hand side $\mathbf{1} \notin \mathcal{U} \oplus \mathcal{U} \otimes \mathcal{U} \oplus \mathcal{U} \otimes \mathcal{U} \otimes \mathcal{U} \oplus \ldots$. **This contradiction finishes the proof.**

The theorem just proved is particularly useful in finding the suitable overall multiplicative factor standing in front of the matrix X. It has of course an analog applicable to $T(V)$ and V_X.

4.5 Noncommutative quantum double

In this section I give an example of a noncommutative (and noncocommutative) quantum double obtained by the method of Section 3.1. I will study the following X-matrix:

$$X = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.9}$$

Corresponding factorization of null biideals gives relations $u^1_2 = 0$, $u^2_2 = u^1_1$, $u^1_1 u^1_1 = \mathbf{1}$, $u^1_1 u^2_1 = -u^2_1 u^1_1$ in U_X and $v^2_1 = 0$, $v^2_2 = \mathbf{1}$, $v^1_1 v^1_1 = \mathbf{1}$, $v^1_1 v^1_2 = v^1_2 v^1_1$ in V_X. (The proofs are quite simple; they can be done by induction as in the previous examples, I show only one example of the more difficult ones: $\langle u^1_1 u^2_1, v\rangle = -\langle u^2_1 u^1_1, v\rangle$, $\forall v$ can be evaluated in the first step for $v = v^i_j$: $\langle u^1_1 u^2_1, v^i_j\rangle = \langle u^1_1 \otimes u^2_1, \Delta(v^i_j)\rangle = \langle u^1_1, v^i_k\rangle\langle u^2_1, v^k_j\rangle = -\langle u^2_1, v^i_k\rangle\langle u^1_1, v^k_j\rangle = -\langle u^2_1 \otimes u^1_1, \Delta(v^i_j)\rangle = -\langle u^2_1 u^1_1, v^i_j\rangle$ by explicit evaluation. ($i = j \Rightarrow 0 = 0$, $i = 1, j = 2 \Rightarrow (-1).1 = -1.1$) In the second step I assume that for given n and for arbitrary $k \leq n$ holds $\langle u^1_1 u^2_1, v^{i_1}_{j_1} \ldots v^{i_k}_{j_k}\rangle = -\langle u^2_1 u^1_1, v^{i_1}_{j_1} \ldots v^{i_k}_{j_k}\rangle$ and I try to prove the same statement for $k = n+1$: $\langle u^1_1 u^2_1, v^{i_1}_{j_1} \ldots v^{i_{n+1}}_{j_{n+1}}\rangle = \langle \Delta(u^1_1 u^2_1), v^{i_{n+1}}_{j_{n+1}} \otimes v^{i_1}_{j_1} \ldots v^{i_n}_{j_n}\rangle = \langle u^1_1 u^2_1 \otimes u^1_1 u^1_1 + u^1_1 u^2_2 \otimes u^1_1 u^2_1, v^{i_{n+1}}_{j_{n+1}} \otimes v^{i_1}_{j_1} \ldots v^{i_n}_{j_n}\rangle = \langle u^1_1 u^2_1, v^{i_{n+1}}_{j_{n+1}}\rangle\langle u^1_1 u^1_1, v^{i_1}_{j_1} \ldots v^{i_n}_{j_n}\rangle + \langle u^1_1 u^2_2, v^{i_{n+1}}_{j_{n+1}}\rangle\langle u^1_1 u^2_1, v^{i_1}_{j_1} \ldots v^{i_n}_{j_n}\rangle = -\langle u^2_1 u^1_1, v^{i_{n+1}}_{j_{n+1}}\rangle \langle u^1_1 u^1_1, v^{i_1}_{j_1} \ldots v^{i_n}_{j_n}\rangle - \langle u^2_2 u^1_1, v^{i_{n+1}}_{j_{n+1}}\rangle\langle u^2_1 u^1_1, v^{i_1}_{j_1} \ldots v^{i_n}_{j_n}\rangle = \ldots = -\langle u^2_1 u^1_1, v^{i_1}_{j_1} \ldots v^{i_{n+1}}_{j_{n+1}}\rangle$ ending the proof by induction (I have already used $u^1_1 = u^2_2$ and $\Delta(u^1_1) = u^1_1 \otimes u^1_1$ in order to shorten the evaluation, $\Delta(u^1_1) = u^1_1 \otimes u^1_1$ follows from $\Delta(u^1_1) = u^1_1 \otimes u^1_1 + u^1_2 \otimes u^2_1$ and $u^1_2 = 0$. Proofs of $u^1_2 = 0$ and $u^1_1 = u^2_2$ are simpler then the one given and do not depend on this one, in fact they are the same as in Subsection 4.1.1)).

Now it is possible to rewrite both U_X and V_X in a more readable form:

- $\mathrm{U}_X = \mathcal{L}\{\mathbf{1}, u^1_1, (u^2_1)^j, u^1_1(u^2_1)^j \,|\, j \in \mathcal{N}\}$ with the multiplication generated by $u^1_1 u^1_1 = \mathbf{1}, u^1_1 u^2_1 = -u^2_1 u^1_1$ and the comultiplication $\Delta(u^1_1) =$

$u_1^1 \otimes u_1^1$, $\Delta(u_1^2) = u_1^2 \otimes u_1^1 + u_1^1 \otimes u_1^2$. This bialgebra is evidently cocommutative, but not commutative.

- $\mathrm{V}_X = \mathcal{L}\{\mathbf{1}, v_1^1, (v_2^1)^j, v_1^1(u_2^1)^j \,|\, j \in \mathcal{N}\}$ with the multiplication generated by $v_1^1 v_1^1 = \mathbf{1}, v_1^1 v_2^1 = v_2^1 v_1^1$ and with the comultiplication $\Delta(v_1^1) = v_1^1 \otimes v_1^1$, $\Delta(v_2^1) = v_1^1 \otimes v_2^1 + v_2^1 \otimes \mathbf{1}$. This bialgebra is commutative, but not cocommutative.

Now I can deduce the antipodes:

- $Su_1^1 = u_1^1$ and $Su_1^2 = u_1^2$
- $Sv_1^1 = v_1^1$ and $Sv_2^1 = -v_1^1 v_2^1$

and check the relations like $.\circ(S \otimes id)\Delta(v_2^1) = Sv_2^1 + (Sv_1^1)v_2^1 = -v_1^1 v_2^1 + v_1^1 v_2^1 = 0$ etc.

The quantum double $D(U_X)$ is generated by U_X and V_X as subbialgebras with the cross product generated by the relations

$$\begin{aligned} v_1^1 u_1^1 &= u_1^1 v_1^1, & v_1^1 u_1^2 &= u_1^2 v_1^1, \\ v_2^1 u_1^1 &= -u_1^1 v_2^1, & v_2^1 u_1^2 &= u_1^1 - u_1^2 v_2^1 - u_1^1 v_1^1. \end{aligned}$$

It remains to find a pair of dual bases. In order to do this, I should first evaluate the pairing between the powers of generators.

It seems that the most important one is $\langle (u_1^2)^j, (v_2^1)^k \rangle$, $i, j \in \mathcal{N} \cup 0$, the others can be deduced from this one (as I'll show a bit later). The first step in the evaluation follows from the definition of the antidual pairing: $\langle (u_1^2)^j, (v_2^1)^k \rangle = \langle \Delta^k((u_1^2)^j), v_2^1 \otimes v_2^1 \ldots \otimes v_2^1 \rangle = \langle (\Delta^k(u_1^2))^j, v_2^1 \otimes v_2^1 \ldots \otimes v_2^1 \rangle = \langle (\sum_{i=1}^k u_1^1 \otimes \ldots \otimes u_1^2 \otimes \ldots \otimes u_1^1)^j, v_2^1 \otimes v_2^1 \ldots \otimes v_2^1 \rangle$, (using $\Delta^k(u_1^2) = \sum_{i=1}^k u_1^1 \otimes \ldots \otimes u_1^2 \otimes \ldots \otimes u_1^1$ where u_1^2 is on the i-th position). In the next step I expand the power $\langle (\sum_{i=1}^k u_1^1 \otimes \ldots \otimes u_1^2 \otimes \ldots \otimes u_1^1)^j, v_2^1 \otimes v_2^1 \otimes \ldots \otimes v_2^1 \rangle = \langle \sum_{i_1=1}^k \ldots \sum_{i_j=1}^k (u_1^1 u_1^1 \ldots u_1^1) \otimes \ldots \otimes (u_1^1)^\alpha u_1^2 (u_1^1)^\beta \otimes \ldots \otimes (u_1^1)^\gamma v_1^2 (u_1^1)^\epsilon \otimes \ldots \otimes (u_1^1 u_1^1 \ldots u_1^1), v_2^1 \otimes v_2^1 \otimes \ldots \otimes v_2^1 \rangle$, where on the right hand side in each summand is a tensor product of products containing mostly u_1^1 and for all $l \leq j$ on the i_l-th position in the l-th tensor product is u_1^2. Now I can state that for $j \neq k$ is $\langle (u_1^2)^j, (v_2^1)^k \rangle = 0$ (Proof: Let me assume $j < k$. Then at least one factor in the tensor product in every summand is equal to $(u_1^1)^j$ and the pairing of such summand with $v_2^1 \otimes v_2^1 \ldots \otimes v_2^1$ is zero, because $\langle (u_1^1)^j, v_2^1 \rangle = \langle u_1^1, v_2^1 \rangle \vee \langle \mathbf{1}, v_2^1 \rangle = 0$. Similarly $j > k$ by expressing the pairing using $\langle a.b, c \rangle = \langle a \otimes b, \Delta(c) \rangle$). It remains to find the pairing $\langle (u_1^2)^k, (v_2^1)^k \rangle$. It is easy to realize by the same reasoning as above that in order to give nonzero contribution to the pairing the sequence $\pi = (i_1, \ldots, i_k)$ must be a permutation. Evaluating e.g. the summands for identical permutation and for τ_{12} I find

$u_1^1u_1^2(u_1^1)^{k-2}\otimes u_1^2(u_1^1)^{k-1}\otimes(u_1^1)^2u_1^2(u_1^1)^{k-3}\otimes\ldots(u_1^1)^{k-1}u_1^2=(-u_1^2u_1^1)(u_1^1)^{k-2}\otimes(-u_1^1u_1^2)(u_1^1)^{k-2}\otimes(u_1^1)^2u_1^2(u_1^1)^{k-3}\otimes\ldots(u_1^1)^{k-1}u_1^2=u_1^2(u_1^1)^{k-1}\otimes u_1^1u_1^2(u_1^1)^{k-2}\otimes(u_1^1)^2u_1^2(u_1^1)^{k-3}\otimes\ldots(u_1^1)^{k-1}u_1^2$, i.e the summands are equal. The same is true for any two permutations differing by transposition of neighbouring elements and consequently for any two permutations. The conclusion is $\langle(u_1^2)^k,(v_2^1)^k\rangle=k!\langle u_1^2(u_1^1)^{k-1}\otimes u_1^1u_1^2(u_1^1)^{k-2}\otimes\ldots\otimes(u_1^1)^{k-1}u_1^2,v_2^1\otimes v_2^1\otimes\ldots\otimes v_2^1\rangle=k!(-1)^{\sum_{i=1}^k(i-1)}\langle(u_1^1)^{k-1}u_1^2\otimes(u_1^1)^{k-1}u_1^2\otimes\ldots\otimes(u_1^1)^{k-1}u_1^2,v_2^1\otimes v_2^1\otimes\ldots\otimes v_2^1\rangle=k!(-1)^{\frac{k(k-1)}{2}}(-1)^{k(k-1)}=k!(-1)^{\frac{k(k-1)}{2}}$.

To sum up,

$$\langle(u_1^2)^j,(v_2^1)^k\rangle=k!(-1)^{\frac{k(k-1)}{2}}\delta_{jk}.$$

The other pairings can be expressed using the previous one:

- $\langle u_1^1(u_1^2)^j,(v_2^1)^k\rangle=(-1)^k\langle u_1^1(u_1^2)^j,(v_2^1)^k\rangle=k!(-1)^{\frac{k(k-1)}{2}+k}\delta_{jk}$,
- $\langle(u_1^2)^j,v_1^1(v_2^1)^k\rangle=(-1)^k\langle u_1^1(u_1^2)^j,(v_2^1)^k\rangle=k!(-1)^{\frac{k(k-1)}{2}+k}\delta_{jk}$,
- $\langle u_1^1(u_1^2)^j,v_1^1(v_2^1)^k\rangle=(-1)^{2k-1}\langle u_1^1(u_1^2)^j,(v_2^1)^k\rangle=-k!(-1)^{\frac{k(k-1)}{2}}\delta_{jk}$.

Proofs can be done by explicit evaluation, e.g. $\langle u_1^1(u_1^2)^j,(v_2^1)^k\rangle=\langle u_1^1\otimes(u_1^2)^j,(\Delta(v_2^1))^k\rangle=\langle u_1^1\otimes(u_1^2)^j,(v_1^1\otimes v_2^1+v_2^1\otimes\mathbf{1})^k\rangle=\langle u_1^1\otimes(u_1^2)^j,(v_1^1\otimes v_2^1)^k\rangle+0=\langle u_1^1,(v_1^1)^k\rangle\langle(u_1^2)^j,(v_2^1)^k\rangle=(-1)^k\langle(u_1^2)^j,(v_2^1)^k\rangle$ etc.

The pairings given above allow me to write a pair of dual bases in the form

$$e_{k1}=(u_1^2)^k\qquad e^{k1}=\frac{1}{2k!}(-1)^{\frac{k(k-1)}{2}}((v_2^1)^k+(-1)^kv_1^1(v_2^1)^k)$$

$$e_{k2}=u_1^1(u_1^2)^k\qquad e^{k2}=\frac{1}{2k!}(-1)^{\frac{k(k-1)}{2}+k}((v_2^1)^k-(-1)^kv_1^1(v_2^1)^k).$$

Finally, I can write the canonical $\mathcal{R}$-matrix in the form of formal power series:

$$\mathcal{R}=\sum_{k=0}^{\infty}\frac{1}{2k!}(-1)^{\frac{k(k-1)}{2}}\Big((v_2^1)^k\otimes(u_1^2)^k\quad+\quad(-1)^kv_1^1(v_2^1)^k\otimes(u_1^2)^k+ +(-1)^k(v_2^1)^k\otimes u_1^1(u_1^2)^k\quad-\quad v_1^1(v_2^1)^k\otimes u_1^1(u_1^2)^k\Big).\qquad(4.10)$$

4.6 Finite dimensional noncommutative quantum doubles

In the previous sections I have presented several examples of quantum doubles obtained using modified Vladimirov's method. Except the most trivial commutative examples, all quantum doubles obtained until now are infinite dimensional. There is a natural question whether this method can be used also to find some noncommutative finite dimensional quantum doubles. The answer is positive, as I show in the present section.

Let me study the X-matrix:

$$X = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.11}$$

This X-matrix implies following identifications after the factorization of the null biideals:

- In $T(U)$: $u^1_2 = 0$, $u^2_2 = \mathbf{1}$, $u^1_1 u^1_1 = \mathbf{1}$, $u^2_1 u^1_1 = -u^1_1 u^2_1$, $u^2_1 u^2_1 = 0$.
- In $T(V)$: $v^2_1 = 0$, $v^2_2 = \mathbf{1}$, $v^1_1 v^1_1 = \mathbf{1}$, $v^1_2 v^1_1 = -v^1_1 v^1_2$, $v^1_2 v^1_2 = 0$.

Proofs are analogs of proofs in the previous example, the only new relation is $(u^2_1)^2 = 0$ (and its counterpart $(v^1_2)^2 = 0$). It can be proved by induction: by explicit evaluation I find $\langle u^2_1 u^2_1, v^i_j \rangle = 0$. Then I assume that for given $n \in \mathcal{N}$ and for any $k \leq n$ is $\langle u^2_1 u^2_1, v^{i_1}_{j_1} \dots v^{i_k}_{j_k} \rangle = 0$ and prove $\langle u^2_1 u^2_1, v^{i_1}_{j_1} \dots v^{i_n}_{j_n} v^{i_{n+1}}_{j_{n+1}} \rangle = 0$ evaluating $\langle u^2_1 u^2_1, v^{i_1}_{j_1} \dots v^{i_n}_{j_n} v^{i_{n+1}}_{j_{n+1}} \rangle = \langle \Delta(u^2_1 u^2_1), v^{i_{n+1}}_{j_{n+1}} \otimes v^{i_1}_{j_1} \dots v^{i_n}_{j_n} \rangle = \langle (u^2_1)^2 \otimes (u^1_1)^2 + u^2_1 u^2_2 \otimes u^1_1 u^2_1 + u^2_2 u^2_1 \otimes u^2_1 u^1_1 + (u^2_2)^2 \otimes (u^2_1)^2, v^{i_{n+1}}_{j_{n+1}} \otimes v^{i_1}_{j_1} \dots v^{i_n}_{j_n} \rangle = 0 + \langle u^2_1 u^2_2 \otimes u^1_1 u^2_1 + u^2_2 u^2_1 \otimes u^2_1 u^1_1, v^{i_{n+1}}_{j_{n+1}} \otimes v^{i_1}_{j_1} \dots v^{i_n}_{j_n} \rangle = \langle u^2_1 \otimes u^1_1 u^2_1 + u^2_1 \otimes u^2_1 u^1_1, v^{i_{n+1}}_{j_{n+1}} \otimes v^{i_1}_{j_1} \dots v^{i_n}_{j_n} \rangle = \langle u^2_1 \otimes (u^1_1 u^2_1 + u^2_1 u^1_1), v^{i_{n+1}}_{j_{n+1}} \otimes v^{i_1}_{j_1} \dots v^{i_n}_{j_n} \rangle = 0$ using the initial assumption (twice) and the equalities $u^2_2 = \mathbf{1}$ and $u^2_1 u^1_1 + u^1_1 u^2_1 = 0$ proved earlier.

To sum up, the bialgebras U_X, V_X in this case are

- $\mathrm{U}_X = \mathcal{L}\{\mathbf{1}, u^1_1, u^2_1, u^1_1 u^2_1\}$ with the multiplication $u^1_1 u^1_1 = \mathbf{1}$, $u^2_1 u^2_1 = 0$, $u^2_1 u^1_1 = -u^1_1 u^2_1$ and the coproduct $\Delta(u^1_1) = u^1_1 \otimes u^1_1$, $\Delta(u^2_1) = \mathbf{1} \otimes u^2_1 + u^2_1 \otimes u^1_1$.
- $\mathrm{V}_X = \mathcal{L}\{\mathbf{1}, v^1_1, v^1_2, v^1_1 v^1_2\}$ with the multiplication $v^1_1 v^1_1 = \mathbf{1}$, $v^1_2 v^1_2 = 0$, $v^1_2 v^1_1 = -v^1_1 v^1_2$ and with the coproduct $\Delta(v^1_1) = v^1_1 \otimes v^1_1$, $\Delta(v^1_2) = v^1_1 \otimes v^1_2 + v^1_2 \otimes \mathbf{1}$.

Given bialgebras are Hopf algebras with the antipodes :

- $Su^1_1 = u^1_1$, $Su^2_1 = u^1_1 u^2_1$
- $Sv^1_1 = v^1_1$, $Sv^1_2 = -v^1_1 v^1_2$.

(It is easy to prove explicitly that given maps define antipodes.)

It is now possible to construct the quantum double $D(U_X)$ containing the U_X and V_X as subbialgebras with the cross product given by relations

$$\begin{array}{rcl} v_1^1 u_1^1 = u_1^1 v_1^1 & , & v_1^1 u_1^2 = -u_1^2 v_1^1 \\ v_2^1 u_1^1 = -u_1^1 v_2^1 & , & v_2^1 u_1^2 = u_1^2 v_1^1 + u_1^1 + v_1^1. \end{array}$$

A pair of dual bases can written in the following way

$$\begin{array}{ll} e_0 = \mathbf{1} & e^0 = \frac{1}{2}(\mathbf{1} + v_1^1) \\ e_1 = u_1^1 & e^1 = \frac{1}{2}(\mathbf{1} - v_1^1) \\ e_2 = u_1^2 & e^2 = \frac{1}{2}(v_2^1 - v_1^1 v_2^1) \\ e_3 = u_1^1 u_1^2 & e^3 = -\frac{1}{2}(v_2^1 + v_1^1 v_2^1), \end{array}$$

and the canonical $\mathcal{R}$-matrix of the quantum double is

$$\begin{aligned} \mathcal{R} = \frac{1}{2}\Big((\mathbf{1} + v_1^1) \otimes \mathbf{1} \quad &+ \quad (\mathbf{1} - v_1^1) \otimes u_1^1 + \\ +(v_2^1 - v_1^1 v_2^1) \otimes u_1^2 \quad &- \quad (v_2^1 + v_1^1 v_2^1) \otimes u_1^1 u_1^2\Big) = \\ = \frac{1}{2}\Big(\mathbf{1} \otimes \mathbf{1} + v_1^1 \otimes \mathbf{1} \quad &+ \quad \mathbf{1} \otimes u_1^1 - v_1^1 \otimes u_1^1 \\ +v_2^1 \otimes u_1^2 - v_1^1 v_2^1 \otimes u_1^2 \quad &- \quad v_2^1 \otimes u_1^1 u_1^2 - v_1^1 v_2^1 \otimes u_1^1 u_1^2\Big) \end{aligned} \tag{4.12}$$

Remark: The Hopf algebra U_X obtained in this example is not new, it has been already published in the literature, namely [7] (and [5]). Also the quantum double $D(\mathrm{U}_X)$ was mentioned in the same article (although expressed in a different, hopefully equivalent way), i.e. the example given above shows that some well-known examples of quantum doubles (and their factor bialgebras, like $sl_q(2)$ - see [9]) can be derived using the generalised Vladimirov's method.

It is also possible to construct more complicated finite-dimensional quantum doubles in a very similar way, using only slightly different X-matrix. For example, let me consider

$$X = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.13}$$

Then the corresponding strictly antidual Hopf algebras U_X and V_X are given by

- $\mathrm{U}_X = \mathcal{L}\{\mathbf{1}, u_1^1, u_2^2, u_1^1u_2^2, u_1^2, u_1^2u_1^1, u_1^2u_2^2, u_1^2u_1^1u_2^2\}$ with the multiplication $u_1^1u_1^1 = \mathbf{1}$, $u_2^2u_2^2 = \mathbf{1}$, $u_1^2u_1^2 = 0$, $u_2^2u_1^1 = u_1^1u_2^2$, $u_1^2u_1^1 = u_1^1u_1^2$, $u_1^2u_2^2 = -u_2^2u_1^2$, the coproduct $\Delta(u_1^1) = u_1^1 \otimes u_1^1$, $\Delta(u_2^2) = u_2^2 \otimes u_2^2$, $\Delta(u_1^2) = u_2^2 \otimes u_1^2 + u_1^2 \otimes u_1^1$ and the antipode $Su_1^1 = u_1^1$, $Su_2^2 = u_2^2$, $Su_1^2 = -u_1^2u_1^1u_2^2$.
- $\mathrm{V}_X = \mathcal{L}\{\mathbf{1}, v_1^1, v_2^2, v_2^1, v_1^1v_2^2, v_2^1v_1^1, v_2^1v_2^2, v_2^1v_1^1v_2^2\}$ with the multiplication $v_1^1v_1^1 = \mathbf{1}$, $v_2^2v_2^2 = \mathbf{1}$, $v_2^1v_2^1 = 0$, $v_2^2v_1^1 = v_1^1v_2^2$, $v_2^1v_1^1 = v_1^1v_2^1$, $v_2^1v_2^2 = -v_2^2v_2^1$ and with the coproduct $\Delta(v_1^1) = v_1^1 \otimes v_1^1$, $\Delta(v_1^1) = v_2^2 \otimes v_2^2$, $\Delta(v_2^1) = v_1^1 \otimes v_2^1 + v_2^1 \otimes v_2^2$ and the antipode $Sv_1^1 = v_1^1$, $Sv_2^2 = v_2^2$, $Sv_2^1 = v_2^1v_1^1v_2^2$.

(i.e. U_X is a tensor product of U_X from the previous example with $\mathcal{Z}_2$ as an algebra, but not as a bialgebra.)

Now it is possible to finish the construction of Hopf algebra structure of the quantum double $D(\mathrm{U}_X)$ expressing the cross multiplication from the formula (1.2):

$$\begin{array}{rcl}
v_1^1u_1^1 = u_1^1v_1^1 & , & v_2^2u_1^1 = u_1^1v_2^2 \\
v_1^1u_2^2 = u_2^2v_1^1 & , & v_2^2u_2^2 = u_2^2v_2^2 \\
v_2^1u_1^1 = u_1^1v_2^1 & , & v_2^1u_2^2 = -u_2^2v_2^1 \\
v_1^1u_1^2 = u_1^2v_1^1 & , & v_2^2u_1^2 = -u_1^2v_2^2 \\
v_2^1u_1^2 = u_1^2v_2^1 & - & u_1^1v_2^2 - u_2^2v_1^1.
\end{array}$$

Finally I find a pair of dual bases:

$$\begin{array}{rl}
e_0 = \mathbf{1} & e^0 = \frac{1}{4}(\mathbf{1} + v_1^1 + v_2^2 + v_1^1v_2^2) \\
e_1 = u_1^1 & e^1 = \frac{1}{4}(\mathbf{1} - v_1^1 - v_2^2 + v_1^1v_2^2) \\
e_2 = u_2^2 & e^2 = \frac{1}{4}(\mathbf{1} - v_1^1 + v_2^2 - v_1^1v_2^2) \\
e_3 = u_1^1u_2^2 & e^3 = \frac{1}{4}(\mathbf{1} + v_1^1 - v_2^2 - v_1^1v_2^2) \\
e_4 = u_1^2 & e^4 = \frac{1}{4}v_2^1(\mathbf{1} - v_1^1 + v_2^2 - v_1^1v_2^2) \\
e_5 = u_1^2u_1^1 & e^5 = \frac{1}{4}v_2^1(-\mathbf{1} - v_1^1 + v_2^2 + v_1^1v_2^2) \\
e_6 = u_1^2u_2^2 & e^6 = \frac{1}{4}v_2^1(\mathbf{1} + v_1^1 + v_2^2 + v_1^1v_2^2) \\
e_7 = u_1^2u_1^1u_2^2 & e^7 = \frac{1}{4}v_2^1(-\mathbf{1} + v_1^1 + v_2^2 - v_1^1v_2^2)
\end{array}$$

and it is possible to write the canonical $\mathcal{R}$-matrix substituting into $\mathcal{R} = \sum_i e^i \otimes e_i$. (Since the final expression is rather complicated, I don't present it here.)

Conclusion

As I have mentioned earlier, the original motivation of this work was to use the solutions of the Yang-Baxter system found earlier (in my pre-Diploma work, see [3]) in order to construct some new quantum doubles and corresponding universal $\mathcal{R}$-matrices. As I have shown in the present work, this idea had to be modified, since in the original method the matrices W,Z were not important, the associated quantum double depends only on the matrix X. Consequently the previous work [3] can serve only as a kind of clue for choosing X-matrices that allow larger number of W and Z matrices and thus "big" factorization leading to uncomplicated structure of quantum double. Such X-matrices were investigated in Sections 4.1, 4.2 and 4.3. Regrettably those X-matrices lead to almost trivial commutative quantum doubles. The remaining, nontrivial examples were found using "educated guess" based on the knowledge of well-known $sl_q(2)$ R-matrix:

$$X = R = \begin{pmatrix} \sqrt{q} & 0 & 0 & 0 \\ 0 & \sqrt{q^{-1}} & 0 & 0 \\ 0 & \sqrt{q} - \sqrt{q^{-3}} & \sqrt{q^{-1}} & 0 \\ 0 & 0 & 0 & \sqrt{q} \end{pmatrix}$$

During the study of different examples, the main difficulties of the given method for finding quantum doubles became more clear. It seems that the problem of the existence of antipodes is rather a technical one, it often can be overcome by a suitable choice of matrix X. Also it is usually possible to find the null biideals using the method sketched above, i.e. to find some simple elements lying in the null biideal and then to prove that those elements generate the whole null biideal (although I have encountered some cases when this method fails or when the null biideals seem to be very complicated and consequently it is technically impossible to study the corresponding quantum doubles).

The most difficult problem is probably the suitable choice of the matrix X. As I have shown in the last examples, a change of one sign in the matrix X can turn the resulting quantum double from finite-dimensional to infinite-dimensional. Therefore the resulting quantum double is very sensitive to changes in the matrix X and it is very difficult to predict its properties without actually finding the corresponding null biideals. On the other hand, matrices X and

$$\tilde{X} = T_1 S_2 X S_2^{-1} T_1^{-1}$$

(where T, S are arbitrary invertible 2×2 matrices) may look very different, but the corresponding bialgebras U_X,V_X and $\mathrm{U}_{\tilde{X}}$,$\mathrm{V}_{\tilde{X}}$ (and appropriate quantum doubles $D(\mathrm{U}_X)$ and $D(\mathrm{U}_{\tilde{X}})$) are isomorphic. Consequently, the structure of the set of quantum doubles that can be obtained using the given method remains hidden. In order to find it, it would seem possible to create a set of classes with respect to symmetry given above and then study possible quantum doubles for X-matrices from given class. But due to the sensitivity of the method it appears to be very difficult to study in general quantum doubles given by X-matrices involving several parameters and the question of full classification of possible quantum doubles obtained using Vladimirov's method and its modifications remains (and probably will remain in near future) unanswered.

Nevertheless, it is surely possible to find new interesting examples of quantum doubles using Vladimirov's approach and some well-known examples of quantum doubles can be reobtained in this way.

Finally, I would like to briefly summarize which parts of the presented text contain my results and which parts are providing necessary background and are not original. The whole first chapter is based on the study of the literature, especially Majid's book [5]. Almost all relevant material in this chapter is taken from this book, although some proofs are modified and some details and comments are added. The second chapter contains material that was inspired by Vladimirov's article [9], especially Section 2.1. On the other hand, Sections 2.2 and 2.4 are original, containing new results. The third chapter again contains my own work (Section 3.1) together with text based on the literature (namely [10] and [5]) in Section 3.2. The fourth chapter is completely new, up to my best knowledge only one of the resulting quantum doubles (namely the one in Section 4.6) has been published in [7] (in a different representation).

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