THANK YOU FOR YOUR ORDER

Thank you for your order from NRC-CISTI's Document Delivery Service. This order is being delivered to you by Infotrieve, the Information Center Specialists. If you need further assistance, please contact NRC-CISTI.

Phone: (613) 998-8544
(800) 668-1222 (toll free)

Fax: (613) 993-7619
Email: info.cisti@nrc.ca

This is not an invoice. / Ceci n'est pas une facture.

CUSTOMER INFORMATION

Ordered By:	NTK PRAHA
Company:	NATIONAL TECHNICAL LIBRARY (CISTI)
Address:	ILL SERVICES
City, St, Zip/PC:	PRAGUE 6, - 16080
Country:	CZE
Phone:	011/420232002419
Fax:	011/420221663459
Email:	ill@stk.cz

ORDER INFORMATION

Infotrieve Order Id:	233710
CISTI Order #:	WS-08068313
CISTI Account #:	FGH42936
Ship Via:	Ariel Email
Delivery Address:	ill@techlib.cz
Cust Order #:	I216263/H58
Ordered:	1/11/2010 11:16 AM
Filled:	1/11/2010 12:40:47 PM

ARTICLE INFORMATION

STD #:	00758434	Type:	Doc Del (Journal Article)
Publisher:	SPRINGER-VERLAG NEW YORK, INC	# Copies:	1
Publication:	Lecture notes in mathematics	Urgency:	Normal
Vol(Iss) Pg:	1037 () p.171-208	Genre:	Article
Date	1983	Total Fee:	$14.00

Article Title: QUANTUM KINEMATICS ON SMOOTH MANIFOLDS

Usage: 1 copy will be used in Default for the following use: "Internal General Business Use"

QUANTUM KINEMATICS ON SMOOTH MANIFOLDS

B. Angermann
H.D. Doebner

Institute for Theoretical Physics A
Technical University of Clausthal
Clausthal, Germany F.R.

J. Tolar

Faculty of Nuclear Science and Physical Engineering
Czech Technical University
Prague, Czechoslovakia

1. Introduction

1.1. Let M be a differentiable manifold (connected and of finite dimension). Consider physical systems with M as configuration space and suppose that, at any time, it is possible to observe the localization of these systems and their motion on M. A mathematical model for this intuitive physical picture depends on the type of the systems. Here we will describe the family Γ_M of non-relativistic quantum mechanical systems on M. We propose mathematical models for the systems in Γ_M, we study the structure of these models and discuss their physical interpretation.

1.2. The abstract framework for a description of any quantum mechanical system S is given by the axiomatic approach (cf. Appendix):

The states of S are in 1-1-correspondence with the positive bounded operators T with unit trace, $\mathrm{Tr}\, T = 1$, in some separable Hilbert space H. All physical observables of S have to be identified with selfadjoint operators $\mathbb{A}$ in H. The probability that a measurement of the observable $\mathbb{A}$ in a state T of S yields a result in the Borel set $\Delta \subset \mathbb{R}$ is given by

$$(1) \qquad \mathrm{Tr}(T \circ \mathbb{E}^{\mathbb{A}}(\Delta)) \quad ,$$

where $\mathbb{E}^{\mathbb{A}}(.)$ denotes the spectral measure of the selfadjoint operator $\mathbb{A}$; T is determined, if the values (1) are known for sufficiently many $\mathbb{A}$ and Δ.

For the application of this scheme to any family Γ of quantum mechanical systems, e.g. to Γ_M, it is necessary

a) to characterize the family by a set $\mathcal{K}$ of mathematically well defined objects which can be identified with observable quantities, which is large enough to describe the physics of Γ,

b) to map $\mathcal{K}$ into the set $SA(H)$ of selfadjoint operators of some Hilbert space H, such that at least some significant properties of $\mathcal{K}$ remain valid on the quantum level.

Such a procedure is called <u>quantization method</u> or <u>quantization</u>; $\mathcal{K}$ serves as a kind of germ of appropriate mathematical models for the systems in Γ.

1.3. The material is organized as follows:

In section 2 the family Γ_M of quantum mechanical systems, localized and moving on M, is characterized by a set $\mathcal{K}_M$ of "kinematical objects" called <u>Borel kinematics on M</u>. $\mathcal{K}_M$ is the germ of mathematical models for the systems in Γ_M and is quantized in section 3 via a mapping of $\mathcal{K}_M$ into the set of selfadjoint operators in some Hilbert space H, such that those properties of $\mathcal{K}_M$ survive, which are on the one hand characteristic for Γ_M and can be used for a rigorous mathematical formulation on the other hand. We end up with the definition of a <u>quantum Borel kinematics on M</u>, which is essentially a collection of abstract position and momentum operators and which represents a possible mathematical model for a system in Γ_M.

The structure of these models is discussed in section 4. We prove that the position operators are unique up to equivalence. For the momentum operators an analogous result is not valid, additional assumptions are necessary. In section 5, we introduce the notion of "differentiability" for those quantum Borel kinematics whose momentum operators are, up to equivalence, differential operators with respect to a (not necessarily trivial) line bundle structure on the point set $M \times \mathbb{C}$ over M. We derive the form of the momentum operators and a characterization of all inequivalent differentiable quantum Borel kinematics in sections 5. and 6 respectively by differential geometric methods; the results are shortly discussed.

2. Borel Kinematics on a Smooth Manifold

In order to characterize the family Γ_M of quantum mechanical systems we introduce a set $\mathcal{K}_M$ of objects representing the localization and motion observables of the systems in Γ_M.

2.1. The intuitive physical picture of a system S being localized on M is that, at any time, there are experiments to observe the "position" of S on M. Hence we assume that a sufficiently large number of regions $B \subset M$ is associated with physical observables - position observables - in the sense that the expectation values of these observables give the probabilities for S being localized in B. A canonical set of regions of M is $\mathcal{B}(M)$, the σ-algebra of Borel sets of M.

2.2. The intuitive physical picture of a localized system S moving on M is that, at any time, there are experiments to observe the (infinitesimal) change of the localization of S on M, i.e. the "momentum" of S on M. The canonical objects characterizing motions on M are infinitesimal generators of one-parameter flows on M, i.e. complete vectorfields on M. We will assume that all $X \in \mathfrak{X}_c(M)$ with $\mathfrak{X}_c(M) = \{ X | X$ is a smooth complete vectorfield on $M \}$ represent physical observables - momentum observables - of S.

2.3. The B's and the X's are related:

The flow of $X \in \mathfrak{X}_c(M)$,

$$\varphi^X : \mathbb{R} \times M \longrightarrow M, \quad (t,m) \longrightarrow \varphi_t^X(m) \quad ,$$

$$(Xf)(m) = \frac{d}{dt}\bigg|_{t=0} (f \circ \varphi_t^X)(m) \quad ,$$

shifts Borel sets via

$$\mathcal{B}(M) \ni B \longrightarrow \varphi_t^X(B) = \{ \varphi_t^X(m) \mid m \in B \} \in \mathcal{B}(M)$$

and models a motion of localization regions of S (flow model).

2.4. The above discussion motivates

Definition 1

For every differentiable manifold M, $\mathcal{K}_M := (\mathcal{B}(M), \mathcal{X}_c(M))$ is called Borel kinematics on M.

We assume now that the characterization of Γ_M through the objects in $\mathcal{B}(M) \cup \mathcal{X}_c(M)$ and the physical observables associated to them is "complete"; internal non-mechanical degrees of freedom, like spin, are not included. We call the systems in Γ_M mechanical. For their physical characterization, $\mathcal{K}_M$ is large enough and will be used as a germ of mathematical models.

2.5. If M is a homogeneous space with respect to the transitive action of a Lie group G (with Lie algebra $\mathfrak{g}$) on M, then there is a natural representation $\tau : \mathfrak{g} \longrightarrow \mathcal{X}_c(M)$ of $\mathfrak{g}$ in $\mathcal{X}_c(M)$. The pair $(\mathcal{B}(M), \tau(\mathfrak{g}))$ is then called a Borel $\mathfrak{g}$-kinematics on M. Borel $\mathfrak{g}$-kinematics are used in a quantization method based on Mackey's systems of imprimitivity ([1],[2],[3]).

3. Quantization of Borel Kinematics

The Borel kinematics $\mathcal{K}_M = (\mathcal{B}(M), \mathcal{X}_c(M))$ is quantized by mapping $\mathcal{B}(M)$ and $\mathcal{X}_c(M)$ into the set $SA(H)$ of selfadjoint operators of some separable Hilbert space H,

(2) $$\mathbb{E} : \mathcal{B}(M) \longrightarrow SA(H)$$

(3) $$\mathbb{P} : \mathfrak{X}_c(M) \longrightarrow SA(H)$$

such that the position observables $\mathbb{E}(B)$ and the momentum observables $\mathbb{P}(X)$ carry some of the characteristic structure of $\mathcal{K}_M$, including an operator version of the flow model.

3.1. <u>Quantization of $\mathcal{B}(M)$</u>

3.1.1. Consider the system $S \in \Gamma_M$ in a state $T \in ST(H) = \{ T | T$ is a positive bounded operator in H, $\mathrm{Tr}\, T = 1 \}$. The mapping (2) is assumed to yield selfadjoint operators such that, for every $B \in \mathcal{B}(M)$, the expectation value of $\mathbb{E}(B)$ in T

(4) $$\mu_T(B) := \mathrm{Tr}(T \circ \mathbb{E}(B)) = \int \mathrm{id}_{\mathbb{R}}\, d\, \mathrm{Tr}(T \circ \mathbb{E}^{\mathbb{E}(B)}(.))$$

(cf. (1)) exists and can be interpreted as the probability to find S, in the state T, localized in B. This forces $\mathbb{E}(B)$ to be a bounded positive operator for every $B \in \mathcal{B}(M)$. It is furthermore plausible to assume, for every $T \in ST(H)$, the map

(5) $$\mu_T : \mathcal{B}(M) \longrightarrow \mathbb{R}^+ , \; B \longrightarrow \mu_T(B)$$

to be a probability measure on $\mathcal{B}(M)$. This leads to the following necessary properties of $\mathbb{E}$:

(6a) $$\|\mathbb{E}(B)\| < \infty , \quad 0 \leqslant \mathbb{E}(B) , \quad B \in \mathcal{B}(M) ,$$

$$\mathbb{E}(\emptyset) = 0 , \quad \mathbb{E}(M) = \mathbb{1} ,$$

$$\mathbb{E}(\bigcup_j B_j) = \operatorname*{s-lim}_{N \longrightarrow \infty} \sum_{j=1}^{N} \mathbb{E}(B_j)$$

for sequences of mutually disjoint Borel sets.

Hence $\mathbb{E}$ should be a (normalized) <u>positive-operator-valued (POV-) measure</u> on $\mathcal{B}(M)$ ([4],[5]).

3.1.2. The set of POV-measures on M is vast. In order to select certain POV-measures as candidates for a map (2) a more detailed description of the measurements of $\mathbb{E}(B)$ is needed: we assume that the possible results are 0, whenever S is not in B, and 1, whenever S is in B, or equivalently, that the spectrum of $\mathbb{E}(B)$ consists of 0 and 1 only, i.e. that $E(B)$ is a projection operator in H. So we have, in addition to (6a),

(6b) $$[\mathbb{E}(B), \mathbb{E}(\tilde{B})] = 0 \quad , \quad B, \tilde{B} \in \mathcal{B}(M)$$

(cf. [4]).
A map (2) with the properties (6) is called a projection-valued (PV-) or spectral measure on $\mathcal{B}(M)$. Thus our discussion leads to the assumption that $\mathcal{B}(M)$ is quantized via a spectral measure on $\mathcal{B}(M)$ (cf. e.g. [6]).

3.1.3. $\mathcal{K}_M$ was assumed to be "complete" (cf. 2.4.), so those projection operators in H, which commute with all $\mathbb{E}(B)$, $B \in \mathcal{B}(M)$, are of some interest. If $\mathbb{F} \in \mathbb{E}(\mathcal{B}(M))$, then obviously $\mathbb{F}$ commutes with $\mathbb{E}$ (cf. (6b)). The reverse is true, if the v.Neumann algebra generated by $\mathbb{E}$ is maximal abelian or, in other words, $\mathbb{E}$ has multiplicity 1. We call $\mathbb{E}$ an elementary spectral measure in this case.

3.1.4. The quantization of $\mathcal{B}(M)$ via a spectral measure $\mathbb{E}$ on $\mathcal{B}(M)$ automatically induces a quantization of the algebra $C^\infty(M,\mathbb{R})$ of real valued smooth functions on M via a map

(7a) $$Q : C^\infty(M,\mathbb{R}) \longrightarrow SA(H)$$

defined by

(7b) $$Q(f) = \int_{\mathbb{R}} id_{\mathbb{R}} \, d(\mathbb{E} \circ f^{-1})(.)$$

$$\text{i.e.} \quad (Q(f)\psi, \psi) = \int_M f \, d(\mathbb{E}(.)\psi, \psi)$$

$$\text{on} \quad \mathfrak{D}(Q(f)) = \{ \psi \in H \mid \int f^2 \, d(\mathbb{E}(.)\psi, \psi) < \infty \}$$

(cf. [4]). The operators $Q(f)$ can be called position operators too.

3.2. Quantization of $\mathcal{X}_c(M)$

3.2.1. Consider a complete vectorfield X on M, with flow φ^X, and quantize $\mathcal{B}(M)$ via a spectral measure $\mathbb{E}$.
For every state $T \in ST(H)$, φ^X induces a shift of the probability measure (5) along X,

$$(8) \qquad \mu_T \longrightarrow \mu_T \circ \varphi^X_{-t}$$

$$(\mu_T \circ \varphi^X_{-t})(B) := \mu_T(\varphi^X_{-t}(B)) \quad , \quad B \in \mathcal{B}(M) \ , \ t \in \mathbb{R} \ .$$

The idea is now to relate this one-parameter shift-group to a continuous one-parameter group of unitary transformations of H,

$$(9a) \qquad \mathbb{V}^X : \mathbb{R} \times H \longrightarrow H \ , \ (t, \psi) \longrightarrow \mathbb{V}^X_t \psi$$

$$\mathbb{V}^X_o = 1 \ , \ \mathbb{V}^X_{t_1} \circ \mathbb{V}^X_{t_2} = \mathbb{V}^X_{t_1+t_2} \quad , \quad t_1, t_2 \in \mathbb{R} \ ,$$

which has the "shift property"

$$(9b) \qquad \mu_T \circ \varphi^X_{-t} = \mu_{T'} \quad , \quad T' = \mathbb{V}^X_t \circ T \circ \mathbb{V}^X_{-t} \quad ,$$

$$\text{for all } \ T \in ST(H) \ , \ t \in \mathbb{R} \ ,$$

i.e. which unitarily implements the shift (8). Exploiting the analogy between φ^X and $\mathbb{V}^X$, we require X to be represented on the quantum level by the selfadjoint infinitesimal generator $\mathbb{P}(X)$ of a unitary shift-group $\mathbb{V}^X$,

$$(10) \qquad \underset{t \to 0}{\text{s-lim}} \ \frac{\mathbb{V}^X_t \psi - \psi}{t} = i \ \mathbb{P}(X) \psi$$

$$\psi \in \mathcal{D}(\mathbb{P}(X)) = \{ \tilde{\psi} \in H \mid \underset{t \to 0}{\text{s-lim}} \ \frac{\mathbb{V}^X_t \tilde{\psi} - \tilde{\psi}}{t} \ \text{exists} \} \quad ,$$

(cf. [7] Ch. VIII).

3.2.2. The map (3) quantizing $\mathcal{X}_c(M)$ should have some further physically motivated properties which are connected with the flow model on M (cf. 2.3.):
Take a pure state $T_\psi \in ST(H)$, $\psi \in H$, $\psi \neq 0$, $(T_\psi \tilde{\psi} = \|\psi\|^{-2} (\tilde{\psi}, \psi) \psi)$, which is localized in $B \in \mathcal{B}(M)$, i.e.

(11) $\mu_{T_\psi}(B) = \mathrm{Tr}(T_\psi \circ \mathbb{E}(B)) = \|\psi\|^{-2}(\mathbb{E}(B)\psi,\psi) = 1$

and a complete vectorfield X on M which vanishes in B,

(12) $$X \restriction B = 0 \quad .$$

Then the flow φ^X of X acts trivially on B and it is resonable to require that $\mathbb{P}(X)$ and $\mathbb{P}(0)$ have the same expectation values in T_ψ :

(13) $$\mathrm{Tr}(T_\psi \circ \mathbb{P}(X)) = \|\psi\|^{-2}(\mathbb{P}(X)\psi,\psi)$$
$$= \|\psi\|^{-2}(\mathbb{P}(0)\psi,\psi) = \mathrm{Tr}(T_\psi \circ \mathbb{P}(0)) \quad ,$$

whenever $\psi \in \mathfrak{D}(\mathbb{P}(0)) \cap \mathfrak{D}(\mathbb{P}(X))$.

We call a map (3) <u>local</u>, if for all $\psi \neq 0$, $B \in \mathcal{B}(M)$, $X \in \mathfrak{X}_c(M)$

(14) $$(11) \wedge (12) \Longrightarrow (13)$$

whenever $\psi \in \mathfrak{D}(P(0)) \cap \mathfrak{D}(P(X))$.

3.2.3. The set of <u>all</u> vectorfields on M is e.g. an infinite-dimensional Lie algebra and a module over $C^\infty(M,\mathbb{R})$, etc.. In $\mathfrak{X}_c(M)$ these structures are spoilt in general, but $\mathfrak{X}_c(M)$ contains Lie subalgebras and vector subspaces. Hence it is plausible to assume that a quantization $\mathbb{P}$ of $\mathfrak{X}_c(M)$ retains the partial Lie algebra structure of $\mathfrak{X}_c(M)$. We shall call a map (3) a <u>partial Lie homomorphism</u> if

(15a) $$\mathbb{P}(X + aY) = \mathbb{P}(X) + a\,\mathbb{P}(Y)$$
for all $X,Y \in \mathfrak{X}_c(M)$, $a \in \mathbb{R}$, whenever $X + aY \in \mathfrak{X}_c(M)$

and

(15b) $$[\mathbb{P}(X),\mathbb{P}(Y)] = i\,\mathbb{P}([X,Y])$$
for all $X,Y \in \mathfrak{X}_c(M)$, whenever $[X,Y] \in \mathfrak{X}_c(M)$,

where the operator identities hold on a domain which is dense in H. A physical motivation for these requirements is not well known. There are examples in which (15b) is not satisfied. Non-linear $\mathbb{P}$ can be constructed but have not been used in physics.

3.3. Quantization of $\mathcal{K}_M$

3.3.1. Summing up we give the definition of a quantization of a Borel kinematics $\mathcal{K}_M$ (cf. [12] for a similar "axiomatization"):

Definition 2

Let M be a differentiable manifold.
A triple $(H,\mathbb{E},\mathbb{P})$ is called a quantum Borel kinematics on M iff

H is a separable Hilbert space,

$\mathbb{E}$ is an elementary spectral measure on $\mathcal{B}(M)$ in H (cf. (2),(6), 3.1.3.) ,

$\mathbb{P} : \mathfrak{X}_c(M) \longrightarrow SA(H)$ is a map with the following properties:

a) for every $X \in \mathfrak{X}_c(M)$, $\mathbb{P}(X)$ is the infinitesimal generator of a unitary one-parameter group of "shifts" along X (cf. (9),(10)) ,

b) $\mathbb{P}$ is local (cf. (14)) ,

c) $\mathbb{P}$ is a partial Lie homomorphism (cf. (15)) ,

and the domain

$$\mathcal{D}^\infty = \bigcap_{\substack{f \in C^\infty(M,\mathbb{R}) \\ X \in \mathfrak{X}_c(M)}} \bigcap_{\substack{\alpha_1,\ldots,\alpha_n \in \mathbb{N} \\ \beta_1,\ldots,\beta_n \in \mathbb{N} \\ n \in \mathbb{N}}} \mathcal{D}(Q(f)^{\alpha_1}\mathbb{P}(X)^{\beta_1}\ldots Q(f)^{\alpha_n}\mathbb{P}(X)^{\beta_n})$$

is dense in H (cf. (7) for the definition of the Q's).

3.3.2. In order to divide quantum Borel kinematics into classes of physically equivalent ones we introduce an equivalence relation by

Definition 3

Two quantum Borel kinematics $(H_j, \mathbb{E}_j, \mathbb{P}_j)$, $j=1,2$, on M are called equivalent iff there exists a unitary map

$$\mathbb{U} : H_1 \longrightarrow H_2$$

such that

$$U\, \mathbb{E}_1(B) U^{-1} = \mathbb{E}_2(B) \qquad \text{for all} \quad B \in \mathcal{B}(M)$$
$$U\, \mathbb{P}_1(X) U^{-1} = \mathbb{P}_2(X) \qquad \text{for all} \quad X \in \mathfrak{X}_c(M) \quad .$$

4. Properties of Quantum Borel Kinematics

There is, up to unitary equivalence, exactly one spectral measure $\mathbb{E}$ on $\mathcal{B}(M)$ which can serve to be part of a quantum Borel kinematics $(H, \mathbb{E}, \mathbb{P})$ on M. The abstract relations between the probability measures (5) and the momentum observables $\mathbb{P}(X)$ yield relations between the operators $\mathbb{E}(B)$, $\mathbb{Q}(f)$, and $\mathbb{P}(X)$; a general form for the shift operators (9) can be derived relative to a "standard" representation of $\mathbb{E}$.

4.1. Uniqueness of the Position Observables

4.1.1. Consider $(H, \mathbb{E}, \mathbb{P})$ on M. Since $\mathbb{E}$ is elementary (cf. Def. 2) the general spectral theorem ([8] § 50) ensures the existence of a σ-finite Borel measure ν on M and of a unitary map

(16a) $$\mathbb{U} : H \longrightarrow L^2(M, \nu)$$

of H onto the Hilbert space of square-ν-integrable complex functions on M, such that

(16b) $$\mathbb{U}\, \mathbb{E}(B) \mathbb{U}^{-1} \psi = \chi_B \psi$$
$$\text{for all} \quad B \in \mathcal{B}(M), \psi \in L^2(M, \nu) \qquad ,$$

where χ_B denotes the indicator function of B. The measure class $[\nu]$ of ν is uniquely determined by $\mathbb{E}$, because

(17) $\mathbb{E}(B) = 0 \iff \chi_B = 0 \iff \nu(B) = 0$, $B \in \mathcal{B}(M)$,

ν must not be trivial, otherwise (5) vanishes for all $T \in ST(H)$.

4.1.2. Let X be a complete vectorfield on M.
Since $\mathbb{P}(X)$ is the infinitesimal generator of a group of shift operators along X, i.e. $\mathbb{V}_t^X = e^{it\mathbb{P}(X)}$ (cf. (9),(10)), we obtain for every $B \in \mathcal{B}(M)$ and every state $T \in ST(H)$, $t \in \mathbb{R}$:

$$Tr(T \circ \mathbb{V}_{-t}^X \circ \mathbb{E}(B) \circ \mathbb{V}_t^X) = Tr(\mathbb{V}_t^X \circ T \circ \mathbb{V}_{-t}^X \circ \mathbb{E}(B)) = (\mu_T \circ \varphi_{-t}^X)(B) \quad (cf.\ (9))$$
$$= Tr(T \circ \mathbb{E}(\varphi_{-t}^X(B))) \quad ,$$

which is equivalent to

(18) $\mathbb{V}_{-t}^X \circ \mathbb{E}(B) \circ \mathbb{V}_t^X = \mathbb{E}(\varphi_{-t}^X(B))$, $B \in \mathcal{B}(M)$, $t \in \mathbb{R}$.

Together with (17) this implies

(19) $\nu(B) = 0 \ (\iff \mathbb{E}(B) = 0 \iff \mathbb{E}(\varphi_{-t}^X(B)) = 0)$

$\iff \nu(\varphi_{-t}^X(B)) = 0$,

$B \in \mathcal{B}(M)$, $t \in \mathbb{R}$.

A Borel measure on M is called flow quasi-invariant if it has the property (19) for all $X \in \mathfrak{X}_c(M)$.
We show that there is only one non-trivial flow quasi-invariant measure class on M. For this we recall that a smooth Borel measure on M is a Borel measure λ on M which, in local coordinates, has a strictly positive density of class C^∞ with respect to the standard Borel measure in R^n, $n = \dim M$:

(20) locally $d\lambda = k^U \cdot dx_1 \ldots dx_n$, $k^U \in C^\infty(U, \mathbb{R}^+)$,

U any coordinate neighbourhood on M ,
(cf. e.g. [9] 16.22).

Theorem 1

Let M be a differentiable manifold.
There exists a non-trivial σ-finite flow quasi-invariant Borel measure

on M. All non-trivial σ-finite flow quasi-invariant Borel measures on M are equivalent, i.e. they have the same null-sets. The (unique) measure class on M, which contains all these Borel measures, also contains all smooth Borel measures on M.

This can be proven following "locally" the proof of the well known result stating that, on $\mathbb{R}^n$, the standard Borel measure is the only translation invariant non-trivial σ-finite Borel measure (cf. e.g. [2] Theorem 8.19 and [10] for a detailed proof).

Together with the result of 4.1.1. we obtain

Theorem 2 (Uniqueness theorem)
Let M be a differentiable manifold and $(H, \mathbb{E}, \mathbb{P})$ be a quantum Borel kinematics on M. Then, up to unitary equivalence, the following holds:

I.

(21) $H = L^2(M, \nu)$, ν being an arbitrary smooth Borel measure on M ,

$\mathbb{E}(B)\psi = \chi_B \psi$ for all $B \in \mathcal{B}(M)$, $\psi \in H$,

II. $\mathbb{E}$ is unique.

Proof:
Part I follows from (16) and from Theorem 1: the measure ν in (16a) is equivalent to every smooth Borel measure on M, and changing ν into an equivalent measure does not affect (16b). Concerning part II we observe that, given $\mathbb{E}$ and $\mathbb{E}'$, both in a representation (21) with measures ν , ν' resp., the map $\mathbb{W} : L^2(M, \nu) \longrightarrow L^2(M, \nu')$, $\psi \longrightarrow (\frac{d\nu}{d\nu'})^{1/2}\psi$ is unitary and transforms $\mathbb{E}$ into $\mathbb{E}'$.

(21) will be called a standard representation in the sequel.

Note that Theorem 2 relates the interpretation of (5) as probability measures for localization in the framework of quantum Borel kinematics to the correspondence between pure states and square-ν-integrable wave-functions with their square moduli being interpreted as probability densities for localization, which is usually assumed in quantum mechanics. This is because in a representation (21) one has for pure states

$T_\psi \in ST(H)$, $\psi \neq 0$,

$$\mu_{T_\psi}(B) = \mathrm{Tr}(T_\psi \circ \mathbb{E}(B)) = \|\psi\|^{-2} (\mathbb{E}(B)\psi, \psi)$$
$$= \|\psi\|^{-2} \int_B |\psi|^2 \, d\nu \quad , \qquad B \in \mathcal{B}(M) \quad .$$

4.1.3. It is easily checked that the position observables $\mathbb{Q}(f)$ which were introduced in 3.1.4., in a standard representation have the form

$$\text{(22)} \qquad \mathbb{Q}(f)\psi = f \cdot \psi$$
$$\psi \in \mathcal{D}(\mathbb{Q}(f)) = \{ \tilde{\psi} \in H \mid f \cdot \tilde{\psi} \in H \} \quad ,$$

for all $f \in C^\infty(M,\mathbb{R})$.
Any common domain for all $\mathbb{Q}(f)$, $f \in C^\infty(M,\mathbb{R})$, has to be contained in the space of square-ν-integrable functions which vanish outside a compact subset of M:

Fix $\psi \in \bigcap_{f \in C^\infty(M,\mathbb{R})} \mathcal{D}(\mathbb{Q}(f))$, then $\int_M f^2 |\psi|^2 \, d\nu < \infty$

for all $f \in C^\infty(M,\mathbb{R})$, and since every real valued continuous function on M can be uniformly approximated by a sequence in $C^\infty(M,\mathbb{R})$, we have

$$\int_M g^2 |\psi|^2 \, d\nu < \infty$$

for all real continuous functions g. This implies $|\psi|^2 \nu$ to have compact support ([9] 13.19.3) or equivalently, that $|\psi|^2$ vanishes outside some compact set because of (20).
Note that this result also applies to the domain $\mathcal{D}^\infty$ (cf. Def. 2); it will be useful later.

4.2. Operator Equations and Shift Operators

4.2.1. In a standard representation (21) an explicit form for the shift groups generated by the momentum observables $\mathbb{P}(X)$ can be obtained starting from (18) and using similar arguments as in the proof of Theorem 9.11 in [2]. We find

(23) $e^{i\ t\ \mathbb{P}(X)}\psi \ = \mathbb{V}_t^X \psi \ = k_t^X \cdot (\varrho_t^X)^{\frac{1}{2}} \cdot (\psi \circ \varphi_{-t}^X)$

$\psi \in H\ ,\ t \in \mathbb{R}\ ,$

for every $X \in \mathfrak{X}_c(M)$;
k_t^X is a measurable complex function on M with modulus 1, and ϱ_t^X denotes the Radon-Nikodym derivative of the shifed measure $\nu \circ \varphi_{-t}^X$ with respect to ν .
For fixed $X \in \mathfrak{X}_c(M)$ the map $t \longrightarrow \mathbb{V}_t^X$ is a unitary representation of the additive group of real numbers (cf. (9)); this property relates the functions k_t^X for different t via a cocycle relation (cf. [2] Ch. XI.5.).
The form (23) of the shift operators is in agreement with the flow model:
If $\psi \in L^2(M,\nu)$ is localized in $B \in \mathfrak{B}(M)$, $|\psi|^2 \restriction (M\setminus B) = 0$, then $\mathbb{V}_t^X \psi$ is localized in $\varphi_t^X B$, $|\mathbb{V}_t^X \psi|^2 \restriction (M \setminus \varphi_t^X(B)) = 0$.

4.2.2. Now the following operator equations are valid for the position and momentum observables:

<u>Lemma:</u>
Let $(H,\mathbb{E},\mathbb{P})$ be a quantum Borel kinematics on M. Then, for all $B \in \mathfrak{B}(M)$, $f \in C^\infty(M,\mathbb{R})$, $X \in \mathfrak{X}_c(M)$, $t \in \mathbb{R}$:

(24) $\mathbb{V}_{-t}^X\ \mathbb{E}(B)\ \mathbb{V}_t^X = \mathbb{E}(\varphi_{-t}^X(B))$

(25) $\mathbb{V}_t^X\ \mathbb{Q}(f)\ \mathbb{V}_{-t}^X = \mathbb{Q}(f \circ \varphi_{-t}^X)$

(26) $[\mathbb{P}(X), \mathbb{Q}(f)] = i\ \mathbb{Q}(Xf)$ on ϑ^∞ ;

(24) and (25) are equivalent.

<u>Proof:</u>
(24) is identical with (18).
(25) follows from (24) and the fact that, according to (7), the spectral measures of the selfadjoint operators $\mathbb{Q}(f)$ and $\mathbb{Q}(f \circ \varphi_{-t}^X)$ are

$$\mathbb{E}^{\mathbb{Q}(f)}(\Delta) = \mathbb{E}(f^{-1}(\Delta))$$

and $$\mathbb{E}^{Q(f\circ\varphi^X_{-t})}(\Delta) = E((f\circ\varphi^X_{-t})^{-1}(\Delta)) = \mathbb{E}(\varphi^X_t(f^{-1}(\Delta))) ,$$

$\Delta \in \mathcal{B}(\mathbb{R})$, respectively.

To prove (25) $\Longrightarrow$ (24) we observe that (25) implies

$$\int_M f\, d\, Tr(T\circ\mathbb{E}(\varphi^X_{-t}(.))) = \int_{\mathbb{R}} id_{\mathbb{R}} d\, Tr(T\circ\mathbb{E}((f\circ\varphi^X_t)^{-1}(.)))$$

$$= Tr(T\circ Q(f\circ\varphi^X_{-t})) = Tr(\mathbb{V}^X_t\circ T\circ\mathbb{V}^X_{-t}\circ Q(f))$$

$$= \int id_{\mathbb{R}}\, d\, Tr(\mathbb{V}^X_t\circ T\circ\mathbb{V}^X_{-t}\circ \mathbb{E}(f^{-1}(.)))$$

$$= \int f\, d\, Tr(T\circ\mathbb{V}^X_{-t}\circ \mathbb{E}(.)\circ\mathbb{V}^X_t)$$

for all bounded $f\in C^\infty(M,\mathbb{R})$, $T\in ST(H)$. This gives

$$\int_M g\, d\, Tr(T\circ\mathbb{E}(\varphi^X_{-t}(.))) = \int_M g\, d\, Tr(T\circ\mathbb{V}^X_{-t}\circ \mathbb{E}(.)\circ\mathbb{V}^X_t)$$

for all bounded continuous functions g on M (see also 4.1.3.), hence

$$Tr(T\circ\mathbb{V}^X_{-t}\circ \mathbb{E}(B)\circ\mathbb{V}^X_t) = Tr(T\circ\mathbb{E}(\varphi^X_{-t}(B)))$$

for all $B\in\mathcal{B}(M)$, $T\in ST(H)$, which is equivalent to (24).

To prove (26) consider $(H,\mathbb{E},\mathbb{P})$ in a standard representation. Let $\psi\in\mathfrak{D}^\infty$. Then $|\psi|^2$ vanishes outside some compact set $K\subset M$ (4.1.3.). For $X\in\mathfrak{X}_c(M)$ and $f\in C^\infty(M,\mathbb{R})$, the function

$$\mathbb{R}\times M\ni(t,m)\longrightarrow F(t,m) = \begin{cases} t^{-1}(f(\varphi^X_{-t}(m)) - f(m)) + (Xf)(m), & t\neq 0\\ 0, & t=0\end{cases}$$

is differentiable, hence $\chi_K F$ is bounded on $[-1,1]\times K$ and one easily obtains

$$\lim_{t \to 0} \| t^{-1} (\mathbb{Q}(f \circ \varphi^X_{-t}) \psi - \mathbb{Q}(f) \psi) + \mathbb{Q}(Xf) \psi \|^2$$

$$= \lim_{t \to 0} \int_K |F(t,m)|^2 \, |\psi(m)|^2 \, d\nu = 0 \quad , \qquad \text{i.e.}$$

$$\frac{d}{dt}\Big|_{t=0} \mathbb{Q}(f \circ \varphi^X_{-t}) \psi = - \mathbb{Q}(Xf) \psi$$

in the norm topology of H.
On the other side,

$$(- \mathbb{Q}(Xf) \psi , \psi) = \lim_{t \to 0} (t^{-1} (\mathbb{Q}(f \circ \varphi^X_{-t}) - \mathbb{Q}(f)) \psi , \psi)$$

$$= \lim_{t \to 0} (t^{-1} (\mathbb{V}^X_t \, \mathbb{Q}(f) \mathbb{V}^X_{-t} - \mathbb{Q}(f)) \psi , \psi)$$

$$= \lim_{t \to 0} \{ (t^{-1} (\mathbb{V}^X_{-t} - \mathbb{1}) \psi , \mathbb{Q}(f) \mathbb{V}^X_{-t} \psi)$$

$$+ (t^{-1} (\mathbb{1} - \mathbb{V}^X_{-t}) \, \mathbb{Q}(f) \psi , \mathbb{V}^X_{-t} \psi) \}$$

$$= (i \, [\mathbb{P}(X) , \mathbb{Q}(f)] \, \psi , \psi) \qquad \text{(cf. (10))} \quad ,$$

and because ψ can vary over the dense domain ϑ^∞ , we get (26).

(24) is some kind of imprimitivity relation for $\mathbb{E}$ and the one-parameter groups of shift operators.
(26) generalizes the Heisenberg commutation relations for position and momentum operators in the case $M = \mathbb{R}^n$.

5. Differentiable Quantum Borel Kinematics and Momentum Observables

It was shown in section 4 that the position observables $\mathbb{E}(B)$ (and $\mathbb{Q}(f)$) of a quantum Borel kinematics on a differentiable manifold M are uniquely determined up to unitary equivalence. A similar result for the momentum observables does not seem to hold in general but is desirable for the construction of explicit quantum Borel kinematics; some further restrictions on the $\mathbb{P}(X)$ are necessary.

A physically motivated postulate is explained and formulated in section 5.1. It yields explicit expressions for momentum operators in section 5.2..

5.1. On Differential Operators and Domains

5.1.1. The best known example of a quantum Borel kinematics is the Schrödinger representation of position and momentum observables for a quantum mechanical system with configuration space $M = \mathbb{R}^n$. By v. Neumann's Theorem ([11]) it can be shown that, for a quantum Borel kinematics on $\mathbb{R}^n$ in a standard representation (21), the maximal invariant domain ϑ^∞ is equal to the set $C_o^\infty(\mathbb{R}^n,\mathbb{C})$ of complex valued, compactly supported smooth functions on $\mathbb{R}^n$ and that the momentum observables are first order differential operators, if (15b) is assumed to hold in its "integrated form"

$$e^{is\mathbb{P}(Y)}\ e^{it\mathbb{P}(X)}\ e^{-is\mathbb{P}(Y)} = e^{it\mathbb{P}(Z_s)} \quad , \quad s,t \in \mathbb{R} \quad ,$$

with Z_s defined by $\varphi^Y_s\ \varphi^X_t\ \varphi^Y_{-s} = \varphi^{Z_s}_t$, $s,t \in \mathbb{R}$.

Since every differentiable manifold M is locally diffeomorphic to $\mathbb{R}^n$, it is plausible to postulate an analogous condition to be satisfied locally in a quantum Borel kinematics.
The idea is to select dense sets of functions in $L^2(M,\nu)$, the elements of which can be interpreted to be smooth compactly supported maps with respect to a differentiable structure D on $M \times \mathbb{C}$, and to assume, that one of these distinguished subspaces of $L^2(M,\nu)$ is contained in ϑ^∞ (cf. Def.2) and is an invariant domain for the $\mathbb{P}(X)$'s; the $\mathbb{P}(X)$'s are then possibly differential operators.

5.1.2. A reasonable method of selection of dense sets of "differentiable" functions in $L^2(M,\nu)$ makes use of the concept of complex line bundles with hermitean metric and is described below:

a) <u>Fibration of $M \times \mathbb{C}$ over M</u>

There is a natural correspondence between complex valued functions

$$s : M \longrightarrow \mathbb{C}$$

and sections

$$\sigma : M \longrightarrow M \times \mathbb{C} \quad ,$$
$$pr_1 \circ \sigma = id_M \quad ,$$

of the fibration

$$\eta_o = (M \times \mathbb{C},\ pr_1, M)$$

with pr_1 being the natural projection of $M \times \mathbb{C}$ onto M,

(27) $$s \longleftrightarrow \sigma$$
$$\sigma(m) = (m, s(m)) \quad , \quad m \in M \quad ,$$

and it is easy to show that

s is Borel-measurable

$\Longleftrightarrow$

σ is Borel-measurable (with respect to the σ-Algebra $\mathcal{B}(M) \otimes \mathcal{B}(\mathbb{C})$ of subsets on $M \times \mathbb{C}$, (cf. [22] Chapt. VII).

b) <u>$L^2(M,\nu)$ as a space of sections</u>

With

$$< , >_o : \bigcup_{m \in M} pr_1^{-1}(m) \times pr_1^{-1}(m) \longrightarrow \mathbb{C}$$
$$((m,z) \ , \ (m,z')) \longrightarrow z \cdot \bar{z}'$$

(27) yields, for every smooth Borel measure ν on M, a natural correspondence between $L^2(M,\nu)$ and the space

$$L^2(\eta_o, <,>_o, \nu) = \{ \sigma : M \longrightarrow M \times \mathbb{C} \mid pr_1 \circ \sigma = id_M \ , \ \sigma \text{ measurable}, \int <\sigma,\sigma>_o d\nu < \infty \}$$

in which, as usual, ν -almost everywhere identical sections are identified. With the scalar product

$$(\sigma, \tau) = \int <\sigma, \tau>_o d\nu \quad ,$$

$L^2(\eta_o, \langle,\rangle_o, \nu)$ is a Hilbert space, isometrically isomorphic to $L^2(M,\nu)$ by (27).

c) Line bundle structures on $M \times \mathbb{C}$ over M

A differentiable structure $D = (\mathcal{T}, \mathfrak{A})$ on $M \times \mathbb{C}$ is a pair consisting of a lcsc Hausdorff topology $\mathcal{T}$ for $M \times \mathbb{C}$ and a maximal C^∞-atlas $\mathfrak{A}$ of charts compatible with $\mathcal{T}$; it turns the point set $M \times \mathbb{C}$ into a differentiable manifold, symbolically $(M \times \mathbb{C}, D)$. Note that the structure of a product manifold on $M \times \mathbb{C}$ inherited from the differentiable manifolds M and $\mathbb{C}$ is always one possible differentiable structure on the point set $M \times \mathbb{C}$. However, there are in general additional ones, as will be shown now:

Theorem 3

Let M be a differentiable manifold and $\langle,\rangle_o$ be defined as above. If $\eta = (E, \pi, M, \mathbb{C})$ is a complex line bundle over M with hermitean metric $\langle,\rangle$, then there exists a differentiable structure $D = (\mathcal{T}, \mathfrak{A})$ on the point set $M \times \mathbb{C}$, such that

$$\eta_o^D = ((M \times \mathbb{C}, D), pr_1, M, \mathbb{C})$$

is a complex line bundle over M with hermitean metric $\langle,\rangle_o$ which is isometrically isomorphic to η; D can be chosen so that the σ-algebra $\mathcal{B}(M \times \mathbb{C}, \mathcal{T})$ generated by $\mathcal{T}$ is equal to the product algebra,

(28) $$\mathcal{B}(M \times \mathbb{C}, \mathcal{T}) = \mathcal{B}(M) \otimes \mathcal{B}(\mathbb{C}) .$$

Proof:
Let $\{U_j, j \in \mathbb{N}\}$ be an open covering of M and let

$$\kappa_j : U_j \times \mathbb{C} \longrightarrow \pi^{-1}(U_j)$$

$$\langle \kappa_j(m,z), \kappa_j(m,z') \rangle = z \cdot \bar{z}'$$

be isometric local trivializations of η, $j \in \mathbb{N}$. Then

$$V_1 = U_1, \; V_j = U_j \setminus \bigcup_{k=1}^{j-1} V_k \quad , \quad j > 1 \quad ,$$

are mutually disjoint Borel sets, which cover M, and the map

$$\Gamma : E \longrightarrow M \times \mathbb{C}$$
$$v \longrightarrow \Gamma v = \varkappa_j^{-1}(v) \quad , \quad \text{if} \quad \pi(v) \in V_j \quad ,$$

is bijective. Hence Γ induces a differentiable structure $D = (\mathcal{T}, \mathfrak{A})$ on the point set $M \times \mathbb{C}$ turning $M \times \mathbb{C}$ into a differentiable manifold $(M \times \mathbb{C}, D)$, diffeomorphic to E via Γ.
Since $pr_1 = \pi \circ \Gamma^{-1}$, pr_1 belongs to $C^\infty((M \times \mathbb{C},D),M)$, and $(M \times \mathbb{C},D)$ is locally trivializable over M with respect to pr_1 via the maps $\Gamma \circ \varkappa_j$, $j \in \mathbb{N}$. Finally $\langle,\rangle_o$ is easily seen to be a differentiable metric on the line bundle $\eta_o^D = ((M \times \mathbb{C},D), pr_1, M, \mathbb{C})$ because Γ maps the fibres of η linearly onto the fibres of η_o^D and is isometric,

$$\langle \Gamma v, \Gamma v \rangle_o = \langle v,v \rangle \quad , \quad v \in E \quad .$$

Concerning the compatibility (28) of D with the natural product Borel structure on $M \times \mathbb{C}$ we observe that on $V_j \times \mathbb{C}$

$$\varkappa_j \circ \Gamma^{-1} = id \quad , \quad j \in \mathbb{N} \quad ,$$

and therefore

$$(\mathcal{B}(M) \otimes \mathcal{B}(\mathbb{C})) \cap (V_j \times \mathbb{C}) = \mathcal{B}(M \times \mathbb{C}, \mathcal{T}) \cap (V_j \times \mathbb{C}) \quad , \quad j \in \mathbb{N} \quad ;$$

hence (28) is obtained, taking into account $V_j \times \mathbb{C} \in \mathcal{B}(M) \otimes \mathcal{B}(\mathbb{C}) \cap \mathcal{B}(M \times \mathbb{C}, \mathcal{T})$, $j \in \mathbb{N}$.

d) Domains of smooth elements of $L^2(M,\nu)$

To every differentiable structure $D = (\mathcal{T}, \mathfrak{A})$ on the point set $M \times \mathbb{C}$ there corresponds a set

$$\vartheta_D = \{ \sigma : M \longrightarrow M \times \mathbb{C} \mid pr_1 \circ \sigma = id_M \, , \, \sigma \in C^\infty(M,(M \times \mathbb{C},D)) \, , \; \sigma \text{ has compact support} \}$$

of compactly supported "D-differentiable" sections of η_o; if the σ-algebra $\mathcal{B}(M \times \mathbb{C}, \mathcal{T})$ of Borel sets generated by $\mathcal{T}$ is larger than the product algebra $\mathcal{B}(M) \otimes \mathcal{B}(\mathbb{C})$ (cf. a), then ϑ_D

is contained in $L^2(\eta_o,\langle,\rangle_o,\nu)$. For those D introduced in Theorem 3 we obtain

Theorem 4

Let M be a differentiable manifold. Suppose $D = (\mathcal{T},\mathfrak{A})$ is a differentiable structure on the point set $M \times \mathbb{C}$ such that

$$\eta_o^D = ((M \times \mathbb{C},D),\ pr_1,M,\mathbb{C})$$

is a complex line bundle with hermitean metric $\langle,\rangle_o$, and that (28) is satisfied.

Then the complex vector space

$$\vartheta_D = Sec_o^\infty(\eta_o^D)$$

of compactly supported differentiable sections of η_o^D is dense in $L^2(\eta_o,\langle,\rangle_o,\nu)$.

For a proof cf. [10].

This theorem shows that there is, in general, a large number of differentiable structures on $M \times \mathbb{C}$ which generate dense sets of "differentiable" functions in $L^2(\eta_o,\langle,\rangle_o,\nu)$ resp. in $L^2(M,\nu)$.

5.1.3. We will now single out those quantum Borel kinematics on M which have the following property of "differentiability":

(29) In a standard representation (21),

$$Sec_o^\infty(\eta_c^D) \subseteq \vartheta^\infty \quad \text{and}$$

$$\mathbb{P}(X)\ Sec_o^\infty(\eta_o^D) \subseteq Sec_o^\infty(\eta_o^D) \quad , \quad X \in \mathfrak{X}_c(M) \quad ,$$

where D is a differentiable structure on $M \times \mathbb{C}$, such that

$$\eta_o^D = ((M \times \mathbb{C},D)\ ,\ pr_1,M,\mathbb{C})$$

is a complex line bundle over M with hermitean metric $\langle,\rangle_o$, and (28) is satisfied.

Definition 4

Let M be a differentiable manifold.

A quantum Borel kinematics on M is called differentiable if it satisfies (29); it is called standard if, in addition, η_o^D is trivializable.

The above discussion shows that there is no essential difference between the description of quantum systems by square integrable "wave functions" and by square integrable "wave sections" of some complex line bundle: functions and sections can be thought of as elements of different dense sets in a common Hilbert space of measurable maps.

In the following section we shall discuss differentiable quantum Borel kinematics only.

5.2. Momentum Operators for Differentiable Quantum Borel Kinematics

5.2.1. Consider a differentiable quantum Borel kinematics $(H, \mathbb{E}, \mathbb{P})$ on M. Then, by definition, there is a complex line bundle $\eta = (E, \pi, M, \mathbb{C})$ over M with hermitean metric $\langle , \rangle$ and a smooth Borel measure ν on M such that (up to equivalence)

(30)
$$H = L^2(\eta, \langle , \rangle, \nu)$$
$$= \{ \psi : M \longrightarrow E \mid \pi \circ \psi = id_M , \ \psi \text{ measurable}, \int \langle \psi, \psi \rangle d\nu < \infty \}$$
$$\mathbb{E}(B)\psi = \chi_B \psi \quad , \quad \psi \in H, \quad B \in \mathcal{B}(M) ,$$
$$Sec_o^\infty(\eta) \subseteq \mathcal{D}^\infty , \quad \mathbb{P}(X)\, Sec_o^\infty(\eta) \subseteq Sec_o^\infty(\eta) \quad , \quad X \in \mathfrak{X}_c(M)$$
$$(\eta = \eta_o^D , \langle , \rangle_o \quad , \quad \text{cf. Thm. 3}) .$$

How do the momentum operators look like?

In order to answer this question, we choose a hermitean linear connection $\tilde{\nabla}$ on η (cf. [24] p 76) and, for every $X \in \mathfrak{X}_c(M)$, we define the linear operator

$$\tilde{\mathbb{P}}(X) : Sec_o^\infty(\eta) \longrightarrow Sec_o^\infty(\eta)$$
$$\tilde{\mathbb{P}}(X)\psi = i\, \tilde{\nabla}_X \psi + \frac{i}{2} (\mathrm{div}_\nu X) \cdot \psi$$

where, locally

(31) $$\operatorname{div}_{\nu}\Big(\sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j}\Big) = \frac{1}{k} \sum_{j=1}^{n} \frac{\partial}{\partial x_j} (a_j k)$$

$$d\nu = k \cdot dx_1 \ldots dx_n \;, \quad n = \dim M \;, \qquad (\text{cf. } (20)) \;.$$

Straightforward calculations show that , for all $X \in \mathfrak{X}_c(M)$, $\tilde{\mathbb{P}}(X)$ is a symmetric operator in H and satisfies

(32) $$[\tilde{\mathbb{P}}(X), \mathbb{Q}(f)] = i\, \mathbb{Q}(Xf) \quad \text{on } \operatorname{Sec}_0^{\infty}(\eta)$$

for all $f \in C^{\infty}(M,\mathbb{R})$ (cf. (22)). Furthermore one has

(33a) $$\tilde{\mathbb{P}}(X + aY) = \tilde{\mathbb{P}}(X) + a\, \tilde{\mathbb{P}}(Y) \quad \text{on } \operatorname{Sec}_0^{\infty}(\eta)$$

for all $X,Y \in \mathfrak{X}_c(M)$, $a \in \mathbb{R}$, whenever $X + aY \in \mathfrak{X}_c(M)$, and

(33b) $$[\tilde{\mathbb{P}}(X), \tilde{\mathbb{P}}(Y)] = i\, \tilde{\mathbb{P}}([X,Y]) + i\, \mathbb{Q}(\tilde{\Omega}(X,Y)) \quad \text{on } \operatorname{Sec}_0^{\infty}(\eta)$$

for all $X,Y \in \mathfrak{X}_c(M)$, whenever $[X,Y] \in \mathfrak{X}_c(M)$, where $\tilde{\Omega}$ is the differential curvature-2-form of $\tilde{\nabla}$ defined by

$$[i\, \tilde{\nabla}_X, i\, \tilde{\nabla}_Y]\psi = -\tilde{\nabla}_{[X,Y]}\psi + i\, \tilde{\Omega}(X,Y)\psi \;.$$

It follows from (22), (26), (30) and (32) that

$$(\tilde{\mathbb{P}}(X) - \mathbb{P}(X))(f \cdot \psi) = f \cdot (\tilde{\mathbb{P}}(X) - \mathbb{P}(X))\psi$$

$$f \in C^{\infty}(M,\mathbb{R}), \quad \psi \in \operatorname{Sec}_0^{\infty}(\eta) \;,$$

hence $\tilde{\mathbb{P}}(X) - \mathbb{P}(X)$ is a symmetric differential operator of order zero on η , i.e. there exists a function $\tilde{\omega}(X) \in C^{\infty}(M,\mathbb{R})$ such that (cf. (22))

(34) $$\mathbb{P}(X) = \tilde{\mathbb{P}}(X) + \mathbb{Q}(\tilde{\omega}(X)) \qquad \text{on } \operatorname{Sec}_0^{\infty}(\eta) \;.$$

Combining (33) with (15) yields for all $X,Y \in \mathfrak{X}_c(M)$, $a \in \mathbb{R}$,

(35a) $$\tilde{\omega}(X + aY) = \tilde{\omega}(X) + a\, \tilde{\omega}(Y)$$

whenever $X + aY \in \mathfrak{X}_c(M)$, and

(35b) $$X\,\tilde{\omega}(Y) - Y\,\tilde{\omega}(X) - \tilde{\omega}([X,Y]) = \tilde{\Omega}(X,Y)$$

whenever $[X,Y] \in \mathfrak{X}_c(M)$.

Consider now a complete vectorfield X on M which vanishes on the open set $U \subset M$. According to the locality (14) of $\mathbb{P}$, we have

$$0 = (\mathbb{P}(0)\psi,\psi) = (\mathbb{P}(X)\psi,\psi)$$
$$= (\tilde{\mathbb{P}}(X) + Q(\tilde{\omega}(X)))\psi,\psi) = (\tilde{\omega}(X)\cdot\psi,\psi)$$

for those $\psi \in Sec_0^\infty(\eta)$ whose support is contained in U. Thus one obtains

(36) $$\operatorname{supp}\tilde{\omega}(X) \subseteq \operatorname{supp} X, \quad X \in \mathfrak{X}_c(M) \quad ,$$

("locality"), and because of this, the map

$$\tilde{\omega} : \mathfrak{X}_c(M) \longrightarrow C^\infty(M,\mathbb{R}) \ , \ X \longrightarrow \tilde{\omega}(X)$$

can be uniquely extended to an $\mathbb{R}$-linear map (also denoted by $\tilde{\omega}$)

(37) $$\tilde{\omega} : \mathfrak{X}(M) \longrightarrow C^\infty(M,\mathbb{R})$$

which satisfies (36) for <u>all</u> vectorfields on M ($\mathfrak{X}(M)$ denotes the set of all vectorfields on M). It is then easy to show that the extension $\tilde{\omega}$ also satisfies (35b) for <u>all</u> vectorfields X,Y on M.

5.2.2. Our postulates for $(H,E,\mathbb{P})$ imply that $\tilde{\omega}$ is the sum of a differential 1-form ω , $d\omega = \tilde{\Omega}$, and a $\mathbb{R}$-linear 1-form proportional to $\operatorname{div}_\nu X$:

Theorem 5

Let M be a differentiable manifold, ν a smooth Borel measure on M, and $\tilde{\Omega}$ a differential 2-form on M.
If $\tilde{\omega} : \mathfrak{X}(M) \longrightarrow C^\infty(M,\mathbb{R})$
is a map with the properties (35)-(36), then there exist a differential 1-form ω on M and a real number c, such that

(38) $$\tilde{\omega}(X) = \omega(X) + c\,\mathrm{div}_\nu X$$

for all $X \in \mathfrak{X}(M)$; ω satisfies

(39) $$(d\omega)(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]) = \tilde{\Omega}(X,Y)$$

for all $X,Y \in \mathfrak{X}(M)$.

Proof:

$\tilde{\omega}$ can be viewed as a linear operator sending the smooth sections of the tangent bundle of M into smooth sections of the trivial real line bundle over M. Due to its locality (36), there exists, for every $m \in M$, a local chart around m in which $\tilde{\omega}$ can be written as a differential operator of finite order, i.e. locally

$$\tilde{\omega}\left(\sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j}\right) = \sum_{|\alpha| < p} \omega_j^\alpha D^\alpha a_j \quad , \qquad j=1,\ldots,n$$

$n = \dim M$, where the α's are multi-indices of length n (Peetre's Theorem, cf. [15] Theorem 6.2).
Inserting this local expression for $\tilde{\omega}$ into (35b) one obtains by comparing coefficients

$$\tilde{\omega}\left(\sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j}\right) = \sum_{j=1}^{n} \left\{ \omega_j^0 \cdot a_j + c \cdot \frac{\partial}{\partial x_j} a_j \right\}$$
$$= \sum_{j=1}^{n} \left(\omega_j^0 - c \cdot \frac{\partial}{\partial x_j} \ln k\right) \cdot a_j + c \cdot \mathrm{div}_\nu \left(\sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j}\right)$$

($d\nu = k \cdot dx_1 \ldots dx_n$, cf. (31)) with a real number c; a further simple calculation shows that c does not depend on the choice of the local chart, hence c is a global constant (M is a connected topological space). Therefore

$$\omega(X) = \tilde{\omega}(X) - c \cdot \mathrm{div}_\nu X$$

defines a real differential 1-form on M, and (38) holds.
Taking into account

(40) $$X\,\mathrm{div}_\nu Y - Y\,\mathrm{div}_\nu X - \mathrm{div}_\nu [X,Y] = 0 \quad ,$$

one easily shows that for every differential 1-form ω

$$(38) \quad \Longrightarrow \quad (39) \quad ;$$

this completes the proof of the theorem.

5.2.3. Applying the above theorem to the map (37) we obtain

$$\tilde{\omega}(X) = \omega(X) + c \cdot \mathrm{div}_\nu X \quad , \qquad X \in \mathfrak{X}(M) \quad ,$$

with a real number c and a differential 1-form ω satisfying

$$d\omega = \tilde{\Omega} \quad .$$

An elementary algebraic calculation shows that, through

$$\nabla_X \psi = \tilde{\nabla}_X \psi - i \cdot \omega(X) \cdot \psi \quad , \quad X \in \mathfrak{X}(M) \; , \; \psi \in \mathrm{Sec}^\infty(\eta) \quad ,$$

a hermitean linear connection ∇ is defined on η , whose curvature-2-form Ω vanishes,

$$\Omega = \tilde{\Omega} - d\omega = 0 \quad .$$

Using (34) we finally obtain for $\mathbb{P}(X)$ on $\mathrm{Sec}_0^\infty(\eta)$:

$$\mathbb{P}(X)\psi = i \nabla_X \psi + (\tfrac{i}{2} + c) \cdot \mathbb{Q}(\mathrm{div}_\nu X) \quad .$$

In the following we shall denote by L a complex line bundle η over M, together with a hermitean metric $\langle , \rangle$ on η and together with a hermitean linear connection ∇ on η with vanishing curvature 2-form, $L = (\eta, \langle , \rangle, \nabla)$.

5.2.4. We are now prepared to describe the general structure of differentiable quantum Borel kinematics:

<u>Theorem 6</u>

Let M be a differentiable manifold.

I. For every triple (ν, L, c) consisting of a smooth Borel measure ν on M, a complex line bundle over M with hermitean metric and hermitean linear connection with vanishing curvature $L = (\eta, \langle,\rangle, \nabla)$, and a real number c,

$$(41) \qquad H = L^2(\eta, \langle,\rangle, \nu) \quad ,$$

$$\mathbb{E}(B)\psi = \chi_B \psi \qquad , \quad B \in \mathcal{B}(M) \quad ,$$

$$\mathbb{P}(X) = \{ (i \cdot \nabla_X + (\tfrac{i}{2}+c)\ \mathfrak{a}(\mathrm{div}_\nu X)) \restriction \mathrm{Sec}_o^\infty(\eta) \}^* \quad ,$$

$$X \in \mathfrak{X}_c(M) \quad ,$$

defines a differentiable quantum Borel kinematics $(H, \mathbb{E}, \mathbb{P})$ on M.

II. Every differentiable quantum Borel kinematics on M is equivalent to one given by (41).

<u>Proof:</u>

I. Let $H, \mathbb{E}$, and $\mathbb{P}$ be given by (41).

H is a separable Hilbert space since M is second countable and locally compact, and $\mathbb{E}$ is an elementary spectral measure ([2] Chapter IX. 2.). Concerning the momentum operators we prove:

a) $\mathbb{P}(X)$ is selfadjoint for every $X \in \mathfrak{X}_c(M)$:
Let $X \in \mathfrak{X}_c(M)$ and $\mathfrak{a}(\mathrm{div}_\nu X)$ be given by (22).
Let $\tilde{X}$ be the unique complete vectorfield on the total space of η which satisfies

$$\pi \circ \varphi_t^{\tilde{X}} = \varphi_t^X \circ \pi \qquad , \quad t \in \mathbb{R} \quad ,$$

$$\nabla_X \psi = \frac{d}{dt}\Big|_{t=0} \varphi_{-t}^{\tilde{X}} \circ \psi \circ \varphi_t^X \quad , \quad \psi \in \mathrm{Sec}_o^\infty(\eta) \quad ,$$

where π is the projection of η ; since ∇ is hermitean, the flow of $\tilde{X}$ is a one-parameter group of linear isometries of η with respect to the metric $\langle,\rangle$ ([18] Chap. III.1.). If we denote by ϱ_t^X the (smooth) Radon-Nikodym derivative of the shifted measure $\nu \circ \varphi_{-t}^X$ with respect to ν then, for every $t \in \mathbb{R}$,

$$(42) \qquad U(t)\psi = \exp(-\ ic\ \ln \varrho_t^X) \cdot (\varrho_t^X)^{\frac{1}{2}} \cdot \varphi_t^{\tilde{X}} \circ \psi \circ \varphi_{-t}^X$$

defines a unitary operator in H, and it is easy to show that $\{U(t),\ t \in \mathbb{R}\}$ is a weakly measurable one-parameter unitary group in H (cf. [7] Chap VIII.4.). Hence there is a unique selfadjoint operator $\mathbb{A}$ in H such that

$$U(t) = e^{i\,t\,\mathbb{A}} \quad , \quad t \in \mathbb{R} \quad ,$$

with

$$i\,\mathbb{A}\,\psi = \lim_{t \to 0} \frac{U(t) - \mathbb{1}}{t}\,\psi$$

whenever this limit exists.

For $\psi \in \mathrm{Sec}_o^\infty(\eta)$,

$$\lim_{t \to 0} \frac{U(t) - \mathbb{1}}{t}\,\psi = i\,\{\, i\nabla_X\psi + (\tfrac{i}{2} + c)\cdot \mathbb{Q}(\mathrm{div}_\nu X)\psi \,\}$$

in the norm topology of H (using $\left.\frac{d}{dt}\right|_{t=0} \rho^X_t = -\,\mathrm{div}_\nu X)$, therefore

$$\mathbb{A} \upharpoonright \mathrm{Sec}_o^\infty(\eta) = \{\, i\,\nabla_X + (\tfrac{i}{2} + c)\cdot \mathbb{Q}(\mathrm{div}_\nu X)\} \upharpoonright \mathrm{Sec}_o^\infty(\eta) \quad .$$

Since all the ingredients for the definition of $U(t)$ are arbitrarily differentiable, $U(t)\cdot \mathrm{Sec}_o^\infty(\eta) \subseteq \mathrm{Sec}_o^\infty(\eta)$, and this proves

$$\mathbb{P}(X) = \{\, \mathbb{A} \upharpoonright \mathrm{Sec}_o^\infty(\eta)\}^{*} = \mathbb{A} \qquad ([7]\ \text{Thm. VIII.11.}).$$

b) The map $X \longrightarrow \mathbb{P}(X)$ is a partial Lie homomorphism:
The required operator identities (15) hold on the dense domain $\mathrm{Sec}_o^\infty(\eta)$, since ∇ has vanishing curvature and (40) holds.

c) The map $X \longrightarrow \mathbb{P}(X)$ is local:
The locality of $\mathbb{P}$ follows from (42) and from the locality of $X \longrightarrow \mathrm{div}_\nu X$.

d) For every $X \in \mathfrak{X}_c(M)$, $\mathbb{P}(X)$ is the infinitesimal generator of a unitary "shift group":
The "shift property" follows from (42).
The set $\mathcal{D}^\infty$ for $H,\mathbb{E}$, and $\mathbb{P}$ is dense in H because it contains the dense set $\mathrm{Sec}_o^\infty(\eta)$.

Finally we show that $(H,\mathbb{E},\mathbb{P})$ is differentiable in the sense of Definition 4:
We observe that there exists a differentiable structure D on $M \times \mathbb{C}$, such that $\eta_o^D = ((M \times \mathbb{C}, D), \mathrm{pr}_1, M, \mathbb{C})$ is a complex line

bundle with hermitean metric $\langle , \rangle_o$ which is isometrically isomorphic to η while (28) is satisfied (Theorem 3). But every isometric isomorphism Γ of η onto η_o^D induces a unitary map

$$U^\Gamma : H \longrightarrow L^2(\eta_o, \langle , \rangle_o, \nu) \simeq L^2(M, \nu)$$

via

$$U^\Gamma \psi = \Gamma \circ \psi \ .$$

Obviously, U^Γ transforms $(H, \mathbb{E}, \mathbb{P})$ into a standard representation (21) and maps $Sec_o^\infty(\eta)$ onto $Sec_o^\infty(\eta_o^D)$.

II. It has already been proven in sections 5.2.1.-5.2.3. that, after applying a suitable unitary transformation, every differentiable quantum Borel kinematics $(H, \mathbb{E}, \mathbb{P})$ can be written with H and $\mathbb{E}$ given by (41) and P satisfying

$$\mathbb{P}(X) \restriction Sec_o^\infty(\eta) = (i \nabla_X + (\tfrac{i}{2} + c) \cdot Q(div_\nu X)) \restriction Sec_o^\infty(\eta)$$

for a suitable complex line bundle L with metric and connection (curvature = 0) and a suitable $c \in \mathbb{R}$. But as shown above, $(\mathbb{P}(X) \restriction Sec_o^\infty(\eta))^* = \mathbb{P}(X)$, and this completes the proof of the second part of this theorem.

6. Parametrization of All Inequivalent Differentiable Quantum Borel Kinematics

It was shown above that for a pair $(H, \mathbb{E})$ given by (21) there are various choices of $\mathbb{P}$ such that $(H, \mathbb{E}, \mathbb{P})$ becomes a differentiable-quantum Borel kinematics on M. More precisely, (41) defines a map which assigns to every triple (ν, L, c) consisting of a smooth Borel measure ν on M, a complex line bundle L over M with hermitean metric and hermitean linear connection with vanishing curvature, and a real number c, a differentiable quantum Borel kinematics on M. Theorem 6 II. states that this map induces a surjective map from the set of these triples onto the set of equivalence classes of differentiable quantum Borel kinematics on M.
In this section we will study the problem how this set of equivalence classes can be parametrized and be described in terms of global topology.

6.1. Equivalent Differentiable Quantum Borel Kinematics

6.1.1. We start with a technical

Lemma

Let $(H,\mathbb{E},\mathbb{P})$ be a differentiable quantum Borel kinematics of type (41) on M. Then

$$\vartheta^{\infty} = \mathrm{Sec}_{o}^{\infty}(\eta)$$

(cf. Definition 2).

Proof:

Obviously $\mathrm{Sec}_{o}^{\infty}(\eta) \subseteq \vartheta^{\infty}$ holds.

In order to prove $\vartheta^{\infty} \subseteq \mathrm{Sec}_{o}^{\infty}(\eta)$ we fix a $\psi \in \vartheta^{\infty}$.

Since $\mathfrak{X}_{c}(M)$ contains all compactly supported vectorfields on M there are, for every $m \in M$ and $K \in \mathbb{N}$, vectorfields $X_1,\ldots,X_n \in \mathfrak{X}_c(M)$, $n = \dim M$, such that

$$\mathbb{D} = \sum_{j=1}^{n} \mathbb{P}(X_j)^{2K}$$

is a differential operator which is elliptic and of order $2K$ at least in a neighbourhood of m. $\mathbb{D}$ is well defined on ϑ^{∞}, hence $\mathbb{D}\psi \in L^2(\eta,<,>,\nu) = H$. But this implies that ψ is at least $(2K - [\frac{n}{2}] - 1)$ - times differentiable in a neighbourhood of m ([16] p.1708 Cor. 4). K and m can be chosen arbitrarily, hence $\psi \in \mathrm{Sec}^{\infty}(\eta)$. The proof is completed by the observation that ψ has to have compact support (cf. 4.1.3.).

6.1.2. Consider now two quantum Borel kinematics $(H_j,\mathbb{E}_j,\mathbb{P}_j)$, $j=1,2$, of type (41) on M characterized by (ν_j,L_j,c_j), $L_j = (\eta_j,<,>_j,\nabla^j)$, $j=1,2$.

a) Suppose $c_1 = c_2 = c$ and that there is an isomorphism Γ of η_1 onto η_2 which is isometric and maps ∇^1 into ∇^2.

Then $(H_1,\mathbb{E}_1,\mathbb{P}_1)$ and $(H_2,\mathbb{E}_2,\mathbb{P}_2)$ are equivalent:

Let ς^{12} be the (smooth) Radon-Nikodym derivative of ν_1 with respect to ν_2. Then

$$H_1 \ni \psi \longrightarrow \mathbb{U}\psi = \exp(-\,ic\,\ln \varrho^{12})\sqrt{\varrho^{12}}\cdot(\Gamma\circ\psi) \in H_2$$

is an isometric isomorphism of Hilbert spaces which obviously transforms the spectral measure $\mathbb{E}_1$ into $\mathbb{E}_2$. Straightforward calculation gives

$$\mathbb{U}\,\mathbb{P}_1(X)\mathbb{U}^{-1}\psi = i\,\nabla^2_X\psi + \{\,i(ic - \tfrac{1}{2})\;X\ln\varrho^{12} + (\tfrac{i}{2}+c)\;\mathrm{div}_{\nu_1}X\,\}\cdot\psi$$

for all $X \in \mathfrak{X}_c(M)$, $\psi \in \mathrm{Sec}^\infty_o(\eta_2)$ $(\mathbb{U}^{-1}\psi \in \mathrm{Sec}^\infty_o(\eta_1)!)$;

it can be derived directly from (20) and (31) that the expression in brackets is equal to $(\frac{i}{2}+c)\;\mathrm{div}_{\nu_2}X$. Therefore we obtain

$$\mathbb{U}\,\mathbb{P}_1(X)\mathbb{U}^{-1}\restriction \mathrm{Sec}^\infty_o(\eta_2) = \mathbb{P}_2(X)\restriction \mathrm{Sec}^\infty_o(\eta_2) \quad ;$$

this is an identity of essentially selfadjoint operators (cf. proof of Theorem 6), thus

$$\mathbb{U}\,\mathbb{P}_1(X)\mathbb{U}^{-1} = \mathbb{P}_2(X) \quad , \quad X \in \mathfrak{X}_c(M) \quad ,$$

and $(H_1,\mathbb{E}_1,\mathbb{P}_1)$ and $(H_2,\mathbb{E}_2,\mathbb{P}_2)$ have been shown to be equivalent.

b) Suppose that $(H_1,\mathbb{E}_1,\mathbb{P}_1)$ and $(H_2,\mathbb{E}_2,\mathbb{P}_2)$ are equivalent. What are the relations between (ν_1,L_1,c_1) and (ν_2,L_2,c_2)?

Let $\mathbb{U}$ be an isometric isomorphism $\mathbb{U}$ of H_1 onto H_2 such that

$$\begin{aligned} (43) \qquad & \mathbb{U}\,\mathbb{E}_1(B)\mathbb{U}^{-1} = \mathbb{E}_2(B) \quad , \quad B \in \mathcal{B}(M) \quad , \\ & \mathbb{U}\,\mathbb{P}_1(X)\mathbb{U}^{-1} = \mathbb{P}_2(X) \quad , \quad X \in \mathfrak{X}_c(M) \quad . \end{aligned}$$

According to our Lemma,

$$\mathbb{U}\,\mathrm{Sec}^\infty_o(\eta_1) = \mathbb{U}\,\vartheta^\infty_1 = \vartheta^\infty_2 = \mathrm{Sec}^\infty_o(\eta_2) \quad ,$$

and, using the position operators $\mathbb{Q}(f)$ (cf. (22)),

$$\mathbb{U}(f\cdot\psi) = \mathbb{U}\,\mathbb{Q}_1(f)\psi = \mathbb{Q}_2(f)\mathbb{U}\psi = f\cdot\mathbb{U}\psi$$

for all $f \in C^\infty(M,\mathbb{R})$, $\psi \in \mathrm{Sec}^\infty_o(\eta_1)$. These properties of $\mathbb{U}$ imply the existence of an isomorphism Γ of η_1 onto η_2 such that

$$(44) \qquad \mathbb{U}\,\psi = \Gamma \circ \psi \qquad , \quad \psi \in \mathrm{Sec}_0^\infty(\eta_1) \quad ,$$

and since $\mathbb{U}$ is isometric, it can be shown that Γ is isometric with respect to the hermitean metrics on η_1 and η_2. Inserting (44) into (43) we obtain

$$i\,\Gamma\circ\nabla_X^1(\Gamma^{-1}\psi) + (\tfrac{i}{2}+c_1)\cdot(\mathrm{div}_{\nu_1}X)\;\psi = \mathbb{U}\;\mathbb{P}_1(X)\mathbb{U}^{-1}\psi$$

$$= \mathbb{P}_2(X)\,\psi = i\;\nabla_X^2\psi + (\tfrac{i}{2}+c_2)(\mathrm{div}_{\nu_2}X)\psi \qquad ;$$

equivalently

$$(\nabla_X^2 - \Gamma\circ\nabla_X^1\circ\Gamma^{-1})\,\psi = i\,\{c_2\cdot\mathrm{div}_{\nu_2}X - c_1\cdot\mathrm{div}_{\nu_1}X\}\,\psi$$

for all $X \in \mathfrak{X}_c(M)$, $\psi \in \mathrm{Sec}_0^\infty(\eta_2)$. For fixed ψ , the l.h.s. of the last equation is linear over $C_0^\infty(M,\mathbb{R})$ in X, whereas the r.h.s. has this property if and only if $c_1 = c_2 = c$ (cf. (31)). Therefore we arrive at

$$c_1 = c_2 = c$$

and
$$\nabla_X^2 = \Gamma\circ\nabla_X^1\circ\Gamma^{-1} \qquad , \qquad X \in \mathfrak{X}_c(M) \quad .$$

Summarizing we have

<u>Theorem 7</u> (Equivalence theorem)
Let M be a differentiable manifold.
Two differentiable quantum Borel kinematics $(H_j, \mathbb{E}_j, \mathbb{P}_j)$ of type (41) on M, characterized by (ν_j, L_j, c_j) , $j=1,2$, are equivalent if and only if $c_1 = c_2$ and if there is an isometric isomorphism of L_1 onto L_2, which transforms the connections into each other.

6.2. The Set of Equivalence Classes of Differentiable Quantum Borel Kinematics

Combining our Theorems 6 and 7 with a result of Kostant on hermitean line bundles with flat connection we obtain the final result.

Theorem 8

Let M be a differentiable manifold and let ν be a smooth Borel measure on M.

I. For every pair (L,c) consisting of a complex line bundle with hermitean metric and hermitean linear connection with vanishing curvature, $L = (\eta, <,>, \nabla)$, and a real number c, the formulas (41) define a differentiable quantum Borel kinematics on M.

II. Every differentiable quantum Borel kinematics on M is equivalent to one of type (41).

III. The set of equivalence classes of differentiable quantum Borel kinematics on M can be mapped bijectively onto the cartesian product

$$\pi_1(M)^* \times \mathbb{R} \quad ,$$

where $\pi_1(M)$ denotes the fundamental group of M and $\pi_1(M)^*$ its group of characters.

For the proof see Theorems 6,7, and [23].

6.3. Standard Quantum Borel Kinematics

In Definition 4 we introduced the notion of standard quantum Borel kinematics on a manifold M for those quantum Borel kinematics which, up to equivalence, satisfy

$$H = L^2(M, \nu)$$

$$\mathbb{E}(B)\psi = \chi_B \psi \quad , \quad B \in \mathcal{B}(M),$$

$$C_0^\infty(M,\mathbb{C}) \subseteq \mathcal{D}^\infty$$

$$\mathbb{P}(X) C_0^\infty(M,\mathbb{C}) \subseteq C_0^\infty(M,\mathbb{C}) \quad , \quad X \in \mathcal{X}_c(M).$$

Concerning the explicit form of $\mathbb{P}(X)$ we obtain from the discussion in section 5.2. as a necessary and sufficient condition for

$$\mathbb{P} : \mathcal{X}_c(M) \longrightarrow SA(H)$$

to constitute, together with H and $\mathbb{E}$, a standard quantum Borel kinematics on M:

$$\mathbb{P}(X) = \{ (i\ (X - i\,\omega(X)) + (\tfrac{i}{2} + c)\ Q(\mathrm{div}_\nu X)) \upharpoonright C_0^\infty(M,\mathbb{C}) \}^* = \mathbb{P}^{(\omega,c)}(X) \quad , \quad X \in \mathfrak{X}_c(M) \quad ,$$

with a closed (real) differential 1-form ω and a real number c.

Two such quantum kinematics $(H,\mathbb{E},\mathbb{P}^{(\omega,c)})$ and $(H,\mathbb{E},\mathbb{P}^{(\tilde\omega,\tilde c)})$ are equivalent, if and only if

$$c = \tilde c \text{ and } \omega - \tilde\omega \text{ is logarithmically exact, i.e. } (\omega - \tilde\omega)(X) = i\ h^{-1}\ Xh$$

with a function $h \in C^\infty(M,S^1)$, $S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$.
Hence the equivalence classes of standard quantum Borel kinematics on M are in one-to-one correspondence with the elements of

$$\left({}^{Z_1(M)} \big/ {}_{L_1(M)} \right) \times \mathbb{R} \quad ,$$

where $Z_1(M)$ denotes the abelian group of closed differential 1-forms on M and $L_1(M)$ its subgroup of logarithmically exact 1-forms on M.

If fundamental group $\pi_1(M)$ is finitely generated, $Z_1(M)/L_1(M)$ can be shown to be isomorphic to

$$\bigotimes_{j=1}^{b_1} U(1) \simeq \pi_1(M)^* \big/ (\Gamma\pi_1(M))^*$$

where b_1 denotes the first Betti number of M and $\Gamma\pi_1(M)$ is the subgroup of $\pi_1(M)$ generated by all its elements of finite order (cf. [17]).

In [13] and [14] quantized versions of Borel kinematics were discussed which, in this approach, turn out to be standard quantum Borel kinematics (with c = 0).

6.4. Quantum Borel Kinematics and Homology

Concluding the technical discussion we add that, according to the Hurewicz theorem, the parametrization of classes of equivalent differentiable quantum Borel kinematics can also be given in terms of (Čech) cohomology theory:

One has

$$\pi_1(M)^* \simeq H_1(M,Z)^* \simeq H^1(M,U(1))$$

for the general case (cf. Thm. 8) and

$$\pi_1(M)^* \Big/ (\Gamma\pi_1(M))^* \simeq (BH)_1(M,Z)^*$$

for the "standard" case (with finitely generated $\pi_1(M)$), where $(BH)_1(M,Z)$ denotes the subgroup of $H_1(M,Z)$ consisting of all (non-trivial) elements of infinite order and 0.

This result shows clearly that differentiable quantum Borel kinematics depend on the global structure of the underlying configuration space M. Because these are mathematical models for physical systems which can be tested by local experiments in a laboratory, these local experiments can "feel" the global physical space similarly as one can "hear the shape of a drum" (Mark Kac).

APPENDIX

Abstract Quantum Mechanics

1. Let S be a physical system.
It is generally assumed that, at least in principle, the set of all elementary propositions of the form

(1) "S has the property A" $\hat{=}$ "A"

can be embedded into an orthocomplemented σ-lattice $\mathcal{L}_S$, and that every state of S, i.e. every result P of a series of physical manipulations on S, gives rise to a unique function

(2a) $$p : \mathcal{L}_S \longrightarrow [0,1] = \{ r \in \mathbb{R} \mid 0 \leqslant r \leqslant 1 \} \quad ,$$

such that $p(A)$ can be interpreted as the probability for obtaing the outcome (1) in an experiment in which the interaction with the measuring apparatus forces S either to show the property A or not, after S has been brought into the state P.
The functions (2a) are assumed to have the properties

(2b) $$p(\mathbb{1}) = 1$$

$$p(\bigvee_{j\in\mathbb{N}} A_j) = \sum_{j\in\mathbb{N}} p(A_j)$$

for sequences $(A_j) \subset \mathcal{L}_S$ with

$$A_j \leqslant A_k^{\perp} \quad , \quad j \neq k \quad ,$$

and they are called state functions or shortly <u>states of $\mathcal{L}_S$</u> (resp. of S), here "$\leqslant$" denotes the partial ordering relation on $\mathcal{L}_S$ and "$\vee$" the induced supremum-operation, $\mathbb{1}$ is the unit element of $\mathcal{L}_S$, and $A^{\perp}$ denotes the orthocomplement of A (cf. [19]).

In this context one defines an <u>observable of S</u> to be a set

(3a) $$e = \{ e(\Delta) \mid \Delta \in \mathcal{B}(\mathbb{R}) \} \subset \mathcal{L}_S$$

of properties (resp. propositions, cf. (1)), which satisfy

(3b) $$e(\mathbb{R}) = 1$$

$$e(\bigcup_{j\in\mathbb{N}} \Delta_j) = \sum_{j\in\mathbb{N}} e(\Delta_j)$$

$$e(\mathbb{R}\setminus\Delta) = e(\Delta)^{\perp} \quad ,$$

($\mathcal{B}(\mathbb{R})$ denotes the σ-algebra of Borel subsets of $\mathbb{R}$), and $e(\Delta)$ is interpreted to represent the property < the observable e has a value in Δ > (resp. the associated proposition, cf. (1)).

Cf. [2],[20],[21].

2. A physical system S is called a quantum mechanical system, if its "logic", i.e. its lattice $\mathcal{L}_S$ can ce chosen to be isomorphic to the orthocomplemented σ-lattice $\mathcal{L}(H)$ of projection operators of a separable complex Hilbert space H. The partial ordering in such a lattice is given by the usual ordering relation for bounded operators, the orthocomplement $\mathbb{E}^{\perp}$ of a projection $\mathbb{E}$ is defined to be the projection $\mathbb{1} - \mathbb{E}$.

Theorem

Let H be a complex separable Hilbert space, dim $H \geq 3$.

I. For every state p of $\mathcal{L}(H)$ (cf. (2)) there is a unique bounded positive operator T with trace equal to one such that

(4) $$p(\mathbb{E}) = \mathrm{Tr}(T \cdot \mathbb{E}) \quad \text{for all } \mathbb{E} \in \mathcal{L}(H) .$$

Every bounded positive operator T with trace equal to one defines a state of $\mathcal{L}(H)$ via (4).

II. Every observable e of $\mathcal{L}(H)$ (cf. (3)) is equal to the spectral measure of a unique selfadjoint operator in H, and vice versa.

Proof: [2].

REFERENCES

[1] G.W. Mackey, Quantum mechanics and induced representations, Benjamin, New York 1968

[2] V.S. Varadarajan, Geometry of quantum theory Vols. I,II, Van Nostrand, Princeton 1968

[3] H.D. Doebner, J. Tolar, Quantum mechanics on homogeneous spaces, J.Math.Phys. 16, 1975, pp 975-984

[4] S.K. Berberian, Notes on spectral theory, Van Nostrand, Princeton 1966

[5] S.T. Ali, G.G. Emch, Fuzzy observables in quantum mechanics, J.Math.Phys. 15, 1974, 176-182

[6] A.S. Wightman, On the localizability of quantum mechanical systems, Rev.Mod.Phys. 34, 1962, 845-872

[7] M. Reed, B. Simon, Methods of modern mathematical physics, Vol.I, Academic Press, New York 1972

[8] P.R. Halmos, Introduction to Hilbert space and the theory of spectral multiplicity, Chelsea Publ.Comp., New York 1957

[9] J. Dieudonné, Foundations of modern analysis, Academic Press, New York 1960

[10] B. Angermann, Über Quantisierungen lokalisierter Systeme - Physikalisch interpretierbare mathematische Modelle, Ph.D.Thesis, Clausthal 1983

[11] J.v.Neumann, Die Eindeutigkeit der Schrödinger'schen Operatoren, Math.Ann. 104, 1931, 570-578

[12] B. Angermann, H.D. Doebner, Homotopy groups and the quantization of localizable systems, Physica 114A, 1982, 433-439

[13] H.D. Doebner, J. Tolar, On global properties of quantum systems, in: Symmetries in science, Plenum Press, New York 1980

[14] I.E. Segal, Quantization of non-linear systems, J.Math.Phys. 1, 1960, 468-488

[15] D.W. Kahn, Introduction to global analysis, Academic Press, New York 1980

[16] N. Dunford, J.T. Schwartz, Linear operators, Vol.II, Interscience, New York 1957

[17] R.S. Palais, Logarithmically exact differential forms, Proc.Amer.Math.Soc. 12, 1961, 50-52

[18] S. Kobayashi, K. Nomizu, Foundations of differential geometry, Vol.I, Interscience-Wiley, New York 1963

[19] G. Birkhoff, Lattice theory, Amer.Math.Soc.Publ. XXV, 1967

[20] G. Birkhoff, J.v.Neumann, On the logic of quantum mechanics, Ann. of Math. 37, 1936, 823-843

[21] J.M. Jauch, Foundations of quantum mechanics, Addison Wesley, London 1973

[22] P.R. Halmos, Measure theory, Van Nostrand, Princeton 1968

[23] B. Kostant, Quantization and unitary representations, Springer Lecture Notes in Mathematics 170, 1970, 86-208

[24] R.O. Wells, Differential analysis on complex manifolds, Springer, New York 1973.