Random processes

Winter term 2021/22, FJFI ČVUT

3. Exercise

1. ($M|M|3$) Let us assume that the customers, which want to pay in a shop, are modeled by a Poisson process, i.e. that the times between two new customers coming to the cash desk are independent random variables with exponential distribution with the expected value 10 seconds. Customers are served by three cash desks and if these are fully occupied, a queue is formed which is common for all the cash desks. The average time needed to serve a customer at the cash desk is 20 seconds (again, exponentially distributed, independent). Model the number of customers in the system (at the cash desks and in the queue) by a Markov chain X with continuous time.
   - 1. Show, that there is a stationary distribution of this system.
   - 2. What is the average number of customers just served in the stationary distribution?
   - 3. What is the average number of customers just waiting in the queue in the stationary distribution?
   - 4. What is the probability, that a new customer does not have to wait in the queue?
2. ($M|M|\infty$) Let us assume that a web page is visited by new visitors according to a Poisson process, i.e., that the times between two new visitors are independent and exponentially distributed with expected value $1/\lambda$ for $\lambda > 0$. Next assume that the time, which the visitors spend on this web page, is exponentially distributed random variable with expected value $1/\mu$ for $\mu > 0$ and that these times are also independent. Finally assume that there is no upper limit on the number of visitors of this web page. How many percent of time does this web page in a long-term horizon has exactly $k \in \mathbb{N}_0$ visitors?
3. Let $W(t) = (W^1(t), W^2(t))$ be a two-dimensional Wiener process, i.e., $W^1(t), W^2(t)$ are two independent Wiener processes. Find (for $R > 0$) the probability $\mathbb{P}(\|W(t)\|_2 < R)$.
4. Using the random number generator, find one instance of $W_0, W_1, \ldots, W_{100}$. For this instance then produce "zoom" (or refinement) $W_0, W_{1/2}, W_1, \ldots, W_{99,5}, W_{100}$ and afterwards $W_0, W_{1/4}, W_{1/2}, W_{3/4}, W_1, \ldots, W_{100}$.
5. Try to generate the paths of Wiener bridge in this way (and their refinements) and of fractional Wiener process.
6. Find the Karhunen-Loèv expansion of the Wiener bridge.
7. Use the Karhunen-Loèv decomposition to generate the paths of a) Brownian motion on $[0, 1]$, b) Brownian bridge on $[0, 1]$, c) Ornstein-Uhlenbeck process on $[0, 1]$.