Random processes

Winter term 2022/23, FJFI ČVUT

Exercises 1

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1. Let $X_1, \dots, X_n$ be random variables. Show that the matrix

$$C = (\mathrm{Cov}(X_j, X_k))_{j,k=1}^n = (\mathbb{E}(X_j - \mathbb{E}X_j)(X_k - \mathbb{E}X_k))_{j,k=1}^n$$

   is positive semidefinite.

2. Let $X_1, X_2, \dots$ be independent random variables. Decide, if the variables

   a) $Y_n = \max(X_1, \dots, X_n)$
   b) $Z_n = X_1 + \dots + X_n$
   c) $W_n = X_1 \cdot \ldots \cdot X_n$

   are Markov processes.

3. Let $X_1, X_2, \dots$ be independent random variables with $\mathbb{P}(X_i = +1) = p$ and $\mathbb{P}(X_i = -1) = q = 1 - p$ and put $S_n = X_1 + \dots + X_n$. Show that

   a) $\mathbb{E}X_i = p - q$, $\mathbb{E}S_n = n(p - q)$, $\mathrm{Var}[X_n] = 4pq$, $\mathrm{Var}[S_n] = 4npq$,
   b) $\mathbb{P}(S_{2n+1} = 2k) = \mathbb{P}(S_{2n} = 2k + 1) = 0$ for $k \in \mathbb{Z}$ and $n \in \mathbb{N}$,
   c) $\mathbb{P}(S_n = k) = 0$ for $|k| > n$,
   d) $\mathbb{P}(S_{2n} = 2k) = \binom{2n}{n+k} p^{n+k} q^{n-k}$, $\mathbb{P}(S_{2n+1} = 2k + 1) = \binom{2n+1}{n+k+1} p^{n+k+1} q^{n-k}$ for $|k| \le n$.

4. Calculate the expected value and the autocovariance function for the following processes.

   a) Let φ be a random uniformly distributed vector on the unit circle in $\mathbb{R}^2$ and $X_t = t \cdot \varphi, t \ge 0$;
   b) S_n is the random walk with parameters $p, q = 1 - p$;
   c) $\theta \in \mathbb{R}$ is fixed, $Y, Z \sim \mathcal{N}(0, 1)$ and $X(t) = Y \cos(\theta t) + Z \sin(\theta t)$;
   d) Y is uniformly distributed random variable on $(0, 1)$ and $X_t = te^Y$;
   e) Let $Y_n, n \in \mathbb{Z}$ be independent random variables with zero mean and variance σ^2, $m \in \mathbb{N}$ is a fixed positive integer and

$$X_n = \frac{1}{2m+1} \sum_{l=n-m}^{n+m} Z_l;$$

   f) $N = (N_t)_{t \ge 0}$ is the Poisson process;
   g) $W = (W_t)_{t \ge 0}$ is the Wiener process;
   h) $B_t = W_t - tW_1, t \in [0, 1]$ is the Brownian bridge.