Lower bounds
for numerical integration and approximation

Foundations of Computational Mathematics 2023

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joint work (mainly) with
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Outline

- Introduction, notation, problem setting
- Classical tools:
  - Radius of information
  - Gelfand numbers
  - Entropy numbers
  - Bump functions
  - Tractability
- Selected results:
  - Carl's inequality
  - Erich's problem
  - Schur technique
  - On bump functions
- Closing remarks

Introduction

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- Numerical approximation: f is a function on $\Omega \subset \mathbb{R}^d$; approximate f in some sense from a limited information (n function values, n linear functionals)

Notation

The class of all possible input objects: F

- $F = K$ - a (convex?) subset of $\mathbb{R}^d$
- A (bounded? compact?) set of functions
- Set of matrices, signals, ...

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- Numerical integration: $T(f) = \mathrm{INT}(f) = \int_\Omega f(x)dx$
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The quality of approximation: Y - measures the distance between $T(f)$ and our guess

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E. Novak and H. Woźniakowski, Tractability of multivariate problems
3 volumes - EMS Tracts in Mathematics

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- $n \geq 1$ fixed - the amount of information we can use
- **Information map:** $N : F \to \mathbb{R}^n$
  For example: $N_n(f) = (f(x_1), \ldots, f(x_n))$ - *standard information*
  $N_n(f) = (L_1(f), \ldots, L_n(f))$ - *linear information*

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- **Recovery algorithm:** $A_n(f) = (\varphi_n \circ N_n)(f) = \varphi_n(N_n(f))$

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- **Worst-case error of A_n:**

$$\mathrm{err}(A_n) = \sup_{f \in F} \|T(f) - A_n(f)\|_Y$$

- **Worst-case error of T:**

$$\mathrm{err}_n(T) = \inf_{A_n} \mathrm{err}(A_n)$$

E. Novak and H. Woźniakowski, Tractability of multivariate problems
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Example: Numerical integration on $[0,1]^d$

Classical task of approximation theory:
For some function f on $[0,1]^d$, approximate

$$T(f) = \mathrm{INT}(f) = \int_{[0,1]^d} f(x)dx$$

if you know only the function values of f at finitely many points.

Quadrature formulas: $\sum_{j=1}^{n} c_j f(x_j), \qquad c_j \in \mathbb{R}, \quad x_j \in [0,1]^d$

Information/Recovery map:

$$N_n(f) = (f(x_1), \ldots, f(x_n)), \varphi_n(y) = \sum_{j=1}^{n} c_j y_j$$

$$A_n(f) = (\varphi_n \circ N_n)(f) = \sum_{j=1}^{n} c_j f(x_j)$$

Classical results: $d = 1$: rectangle rule, trapezoidal rule, Simpson's rule, Newton-Cotes rule(s), ...
d large: Important to avoid *curse of dimension*: Lattices, (Quasi-) Monte Carlo methods, sparse grids

Classical tools: Radius of information

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If $N_n(f) = N_n(g)$, then $A_n(f) = A_n(g)$: algorithms can not distinguish between objects described by the same information

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Radius of information N_n:

$$\begin{aligned}\operatorname{rad}(N_n) &= \sup_{y \in N_n(F)} \operatorname{rad}(\{f : N_n(f) = y\}) = \inf_{\varphi_n} \operatorname{err}(\varphi_n \circ N_n) \\ &= \inf_{\varphi_n} \sup_{f \in F} \| T(f) - \varphi_n(N_n(f)) \|\end{aligned}$$

Classical tools: Gelfand numbers

Gelfand numbers

Let F be the unit ball of a linear space X and
let $N_n(f) = (L_1(f), \ldots, L_n(f))$ be a linear information map.

The set $N_n^{-1}(0) = \{f \in X : N_n(f) = 0\}$ is a linear subspace of X and

$$\operatorname{rad}(N_n) \geq \operatorname{rad}(N_n^{-1}(0) \cup F) \geq c_{n+1}(T)$$

where

$$c_n(T) := \inf_{\substack{M \subset X \\ \operatorname{codim}(M) < n}} \sup_{\substack{x \in M \\ \|x\|_X \leq 1}} \|Tx\|_Y$$

A. Pinkus, *n*-Widths in Approximation Theory, Springer, 1985

G. Lorentz, M. von Golitschek, Y. Makovoz, Constructive Approximation: Advanced Problems, Springer, 1996

Classical tools: Entropy numbers

Entropy numbers

$$e_n(T) := e_n(T(B_X), Y)$$
$$= \inf \left\{ \varepsilon > 0 : \exists y_1, \ldots, y_{2^{n-1}} : T(B_X) \subset \bigcup_{j=1}^{2^{n-1}} (y_j + \varepsilon B_Y) \right\}$$

A. Pietsch, Operator ideals, 1980
B. Carl, I. Stephani, Entropy, compactness and the approximation of operators, 1990
G. Pisier, The volume of convex bodies and Banach space geometry, Cambridge, 1989
B. Carl, Entropy numbers, s-numbers, and eigenvalue problems, J. Funct. Anal. 1981

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Carl's inequality: Let $\alpha > 0$. Then there is $\gamma_\alpha > 0$ such that for all Banach spaces X and Y and all bounded linear $T : X \to Y$

$$\sup_{1\le k\le n} k^\alpha e_k(T) \le \gamma_\alpha \sup_{1\le k\le n} k^\alpha c_k(T).$$

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Classical tools: Fooling and bump functions

Fooling functions:

For given information map N_n, construct $f \in F$ with $N_n(f) = 0$ and large $T(f)$ (radius of the zero information)

N. S. Bakhvalov, On the approximate calculation of multiple integrals, J. Complexity 2015; translation of N. S. Bakhvalov, 1959, in Russian

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Bump functions: A way how to construct the fooling function

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Bump functions: A way how to construct the fooling function
If $N_n(f) = (f(x_1), \ldots, f(x_n))$ for some points $x_1, \ldots, x_n \in \Omega$, consider a compactly supported φ and define

$$f(x) = \sum_{j=1}^{m} \varphi(x - z_j)$$

such that

- $\varphi(x - z_j)$ have disjoint supports
- these supports avoid all $x_1, \ldots, x_n$ and
- $f \in F$

N. S. Bakhvalov, On the approximate calculation of multiple integrals, J. Complexity 2015; translation of N. S. Bakhvalov, 1959, in Russian

Curse of dimension

Many classical problems suffer from exponential dependence of the results on d!

Example: Approximation of smooth functions
Let $\mathcal{F}_d := \{f : [0,1]^d \to \mathbb{R}, \|D^\alpha f\|_\infty \le 1, \alpha \in \mathbb{N}_0^d\}$
Smoothness does not help!
Infinitely differentiable functions on $\Omega = [0,1]^d$:

Theorem: Initial error is the same as error of uniform approximation for $n \le 2^{\lfloor d/2 \rfloor} - 1$

...curse of dimension!

...the number of sampling points must grow exponentially in d

E. Novak, H. Woźniakowski, Approximation of infinitely differentiable multivariate functions is intractable, J. Complexity 2009

Lower bound in compressed sensing

Case study: Compressed Sensing (CS)

- $F =$ unit ball of ℓ_p^d with $0 < p < 1$ (compressible vectors)
- Linear information: $N_n(x) = y = Ax \in \mathbb{R}^n$
- Arbitrary non-linear recovery map $\Delta : \mathbb{R}^n \to \mathbb{R}^d$
- CS: Random information $A_n \in \mathbb{R}^{n\times d}$, basis pursuit Δ_n:

$$\inf_{A,\Delta} \sup_{\|x\|_p \le 1} \|x - \Delta(Ax)\|_2 \le \sup_{\|x\|_p \le 1} \|x - \Delta_n(A_n x)\|_2 \lesssim \min\left\{1, \frac{1+\log(d/n)}{n}\right\}^{1/p-1/2}$$

D. Donoho, Compressed sensing, IEEE Trans. Inform. Theory 2006

E.J. Candès, J. Romberg, and T. Tao, IEEE Trans. Inform. Theory 2006

S. Foucart, A. Pajor, H. Rauhut, and T. Ullrich, J. Compl. 2010

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- Lower bound directly and through Carl's inequality!

D. Donoho, Compressed sensing, IEEE Trans. Inform. Theory 2006

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Bounds on entropy numbers I

Let $0 < p, q \leq \infty$ and let $n \in \mathbb{N}$.

a) If $0 < p \leq q \leq \infty$ then for all $k \in \mathbb{N}$ it holds

$$e_k(id : \ell_p^n \to \ell_q^n) \sim \begin{cases} 1 & \text{if} \quad 1 \leq k \leq \log_2 n, \\ \left(\frac{\log_2(1 + n/k)}{k}\right)^{\frac{1}{p}-\frac{1}{q}} & \text{if} \quad \log_2 n \leq k \leq n, \\ 2^{-\frac{k-1}{n}} n^{\frac{1}{q}-\frac{1}{p}} & \text{if} \quad n \leq k. \end{cases}$$

b) If $0 < q \leq p \leq \infty$ then for all $k \in \mathbb{N}$ it holds

$$e_k(id : \ell_p^n \to \ell_q^n) \sim 2^{-\frac{k-1}{n}} n^{\frac{1}{q}-\frac{1}{p}}.$$

C. Schütt, J. Approx. Theory 1984, D.E. Edmunds H. Triebel, Cambridge, 1996
T. Kühn, J. Approx. Theory 2001, O. Guédon, A.E. Litvak, Springer, 2000
M. Kossaczka, J.V., Entropy numbers of finite-dimensional embeddings, Exp. Math. 2020

Bounds on entropy numbers II

Entropy (and Gelfand) numbers are well studied for many different operators, like:

Diagonal operators:
$D_\sigma : (x_1, x_2, \dots) \to (\sigma_1 x_1, \sigma_2 x_2, \dots)$, estimate $e_n(D_\sigma : \ell_p \to \ell_q)$

Embeddings of function spaces:
$e_n(id : X \to Y)$, where X and Y are different (Sobolev, Besov, Triebel-Lizorkin, dominating mixed smoothness) function spaces

Embeddings of Schatten classes:
$e_n(id : \mathcal{S}_p^N \to \mathcal{S}_q^N)$, where $\mathcal{S}_p^N$ is the Schatten class of $N \times N$ matrices

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A. Pietsch, Operator ideals, 1980

H. König, Eigenvalue distribution of compact operators 1986

B. Carl, Entropy numbers, s-numbers, and eigenvalue problems, J. Funct. Anal. 1981

D. E. Edmunds and H. Triebel, Function spaces, entropy numbers, differential operators 1996

A. Hinrichs, J. Prochno, J.V.: J. Funct. Anal. 2017, Math. Ann. 2021

(In)tractability of integration of very smooth functions

- H_1: a 3-dimensional space with the orthonormal basis

$$e_1(x) = 1, \quad e_2(x) = \cos(2\pi x), \quad e_3(x) = \sin(2\pi x), \quad x \in [0,1]$$

- Kernel, $d = 1$:

$$\begin{aligned} K_x(y) &= 1 + \cos(2\pi x)\cos(2\pi y) + \sin(2\pi x)\sin(2\pi y) \\ &= 1 + \cos(2\pi(x-y)) \end{aligned}$$

- H_d - a d-fold tensor product of H_1: $K_x(y) = \prod_{i=1}^{d} [1 + \cos(2\pi(x_i - y_i))]$

- The aim was to show the *curse of dimension* for the integration of these analytic functions:

$$\mathrm{err}_n(\mathrm{INT})^2 \geq 1 - n\,2^{-d}.$$

E. Novak, Intractability results for positive quadrature formulas and extremal problems for trigonometric polynomials, J. Complexity 1999

RKHS - Reproducing Kernel Hilbert Spaces

$L_x(f) = f(x)$ is continuous on H: $f(x) = \langle f, K_x \rangle_H$, $K_x \in H$

Then $K_x(y) = \langle K_x, K_y \rangle_H =: K(x,y)$ - reproducing kernel of H

Proposition

Let H be a RKHS on Ω with kernel K and let $S(f) = \langle f, h \rangle_H$ for some $h \in H$. Then (for all $\alpha > 0$ and $n \in \mathbb{N}$)

- $\big(K(x_j, x_k) - \alpha h(x_j) h(x_k)\big)_{j,k \le n} \succeq 0$ *for all* $x_1, \ldots, x_n \in \Omega$

if and only if

- $\mathrm{err}_n(S)^2 \ge \|h\|_H^2 - \alpha^{-1}$.

Remark: $h = 1 \ldots$

A. Hinrichs, D. Krieg, E. Novak, J. V., J. Compl. 2020

Erich's problem:

- choose $n, d \geq 2$
- choose $x_1, \ldots, x_n \in \mathbb{R}^d$: n arbitrary points in $\mathbb{R}^d$, i.e.

$$\begin{aligned} x_1 &= (x_{1,1}, x_{1,2}, \ldots, x_{1,d}) \\ x_2 &= (x_{2,1}, x_{2,2}, \ldots, x_{2,d}) \\ &\vdots \\ x_n &= (x_{n,1}, x_{n,2}, \ldots, x_{n,d}) \end{aligned}$$

- then the matrix

$$\left\{ \prod_{i=1}^{d} \frac{1 + \cos(x_{j,i} - x_{k,i})}{2} - \frac{1}{n} \right\}_{j,k=1}^{n}$$

is positive semidefinite. **a.k.a. "Conjecture of E. Novak (1999)"**

Some thoughts around:

- Reformulation:

$$\left\{f(x_j - x_k)\right\}_{j,k=1}^{n} \succeq \frac{1}{n}E_n \quad \text{for} \quad f(y) = \prod_{i=1}^{d} \frac{1+\cos(y_i)}{2}, \quad y \in \mathbb{R}^d$$

- Bochner's theorem: $\hat{f} \geq 0 \implies \left\{f(x_j - x_k)\right\}_{j,k=1}^{n} \succeq 0.$
- Easy for $n = 2$ and $n \geq 2^d$
- **Do it yourself:** $n = 3, d = 2$???
- Tested numerically by E. Novak, published in NA (=numerical analysis) Digest - 1997
- **More general conjecture:** Let $f : \mathbb{R}^d \to \mathbb{R}$ have $f(0) = 1$ and $f, \hat{f} \geq 0$. Then $\left\{f(x_j - x_k) - \frac{1}{n}\right\}_{j,k=1}^{n}$ is positive semidefinite for all $n \geq 2$ and all choices of $x_1, \dots, x_n \in \mathbb{R}^d$.

A. Hinrichs, J.V., On positive positive-definite functions and Bochner's Theorem, J. Compl. 2011

Schur's product theorem

The answer is given by the Schur product theorem and its variants!

Issai Schur
Jan. 10, 1875 – Jan. 10, 1941

Theorem (Issai Schur 1911)

If $M, N \in \mathbb{R}^{n \times n}$ are positive semidefinite, then $M \circ N$ is positive semidefinite.

Here, $(M \circ N)_{j,k} = M_{j,k} N_{j,k}$ is the entry-wise (=Schur or Hadamard) product of M and N.

There is a number of different proofs: using trace formula, using Gaussian integration, using eigenvalue decomposition, ...

J. Schur, "Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen",
Journal für die reine und angewandte Mathematik, 1911

§ 4.

In enger Beziehung zu dem Satze V steht folgender Satz, der trotz seiner Einfachheit nicht bekannt zu sein scheint:

VII. *Sind*

$$A = \sum_{p,q=1}^{n} a_{pq} x_p \bar{x}_q, \quad B = \sum_{p,q=1}^{n} b_{pq} x_p \bar{x}_q \quad (a_{pq} = \bar{a}_{qp},\ b_{pq} = \bar{b}_{qp})$$

zwei (nichtnegativ) definite Hermitesche Formen, so besitzt auch die Form

$$C = \sum_{p,q=1}^{n} a_{pq} b_{pq} x_p \bar{x}_q$$

dieselbe Eigenschaft. Bezeichnet man mit α *und* α' *die größte und die kleinste charakteristische Wurzel der Form* A, *ferner mit* b *und* b' *die größte und die kleinste unter den Zahlen* $b_{11}, b_{22}, \ldots, b_{nn}$, *so liegt jede charakteristische Wurzel der Form* C *zwischen* $\alpha' b'$ *und* αb.

Theorem: Let $M \in \mathbb{R}^{n\times n}$ be a positive semidefinite matrix with $M_{j,j} = 1$ for all $j = 1, \ldots, n$. Then

$$M \circ M \succeq \frac{1}{n} \cdot E_n = \frac{1}{n} ee^T,$$

where $e = (1, \ldots, 1)^T$.

J. V., A variant of Schur's product theorem and its applications, Adv. Math. 2020

I.J. Schoenberg, Positive definite functions on spheres, Duke Math. J. 1942

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where $e = (1, \dots, 1)^T$.
Formerly unknown (?) improvement of Schur's theorem.

Pointed out be Dmitryi Bilyk (University of Minnesota):

The generalized version of Schur's theorem follows also from the theory of Gegenbauer polynomials.

$\lambda > -1/2$: $(C_k^\lambda(t))_{k=0}^\infty$ - polynomials of order k on $[-1, 1]$, orthogonal w.r.t. $(1-t^2)^{\lambda-1/2}$.

J. V., A variant of Schur's product theorem and its applications, Adv. Math. 2020

I.J. Schoenberg, Positive definite functions on spheres, Duke Math. J. 1942

Other variants

Variants of Schur's theorem - with nearly the same proof!

1. Entries on the diagonal are not identically equal to one: $\operatorname{diag} M := (M_{1,1}, \ldots, M_{n,n})^T$
   Let $M \in \mathbb{R}^{n\times n}$ be a positive semidefinite matrix. Then

$$M \circ M \succeq \frac{1}{n}(\operatorname{diag} M)(\operatorname{diag} M)^T.$$

2. $\operatorname{rank}(M) = k$:

$$M \circ M \succeq \frac{1}{k}(\operatorname{diag} M)(\operatorname{diag} M)^T.$$

3. Variant for $M \neq N$

Numerical integration - Erich's problem

- Let $n, d \geq 2$ and $x_1, \ldots, x_n \in \mathbb{R}^d$
- The matrices

$$M^i = \left\{ \cos\left(\frac{x_{j,i} - x_{k,i}}{2} \right) \right\}_{j,k=1}^{n}, \quad i = 1, \ldots, d$$

  are positive semidefinite (classical Bochner's theorem)
- Therefore, $M := M^1 \circ \cdots \circ M^d$ is positive semidefinite (classical Schur product theorem)
- Finally,

$$M \circ M = \left\{ \prod_{i=1}^{d} \cos^2\left(\frac{x_{j,i} - x_{k,i}}{2} \right) \right\}_{j,k=1}^{n} \succeq \frac{1}{n} E_n.$$

Comparison of sampling and approximation numbers

H - a RKHS on $[0,1]^d \subset \mathbb{R}^d$; F - its unit ball
We want to compare

- $\mathrm{err}_n(H)$ - error of numerical integration
- $g_n(H)$ - error of approximation using function values
- $a_n(H)$ - error of approximation using linear information

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- $$\mathrm{err}_n(H) = \inf_{\substack{x_1,\dots,x_n \in [0,1]^d \\ c_1,\dots,c_n \in \mathbb{R}}} \sup_{\|f\|_H \le 1} \left| \int_{[0,1]^d} f(x)dx - \sum_{i=1}^n c_i f(x_i) \right|$$
- $$g_n(H) = \inf_{\substack{x_1,\dots,x_n \in [0,1]^d \\ g_1,\dots,g_n \in L_2}} \sup_{\|f\|_H \le 1} \left\| f - \sum_{i=1}^n f(x_i) g_i \right\|_2$$
- $$a_n(H) = \inf_{\substack{L_1,\dots,L_n \in H' \\ g_1,\dots,g_n \in L_2}} \sup_{\|f\|_H \le 1} \left\| f - \sum_{i=1}^n L_i(f) g_i \right\|_2,$$

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By definition:

$$\mathrm{err}_n(H) \leq g_n(H) \quad \text{and} \quad a_n(H) \leq g_n(H).$$

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Warning: The path was not short.

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Starter: Lower bounds in tensor product problems

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We aim to prove the curse of dimension for numerical integration in tensor product problems

F_1 - RKHS on Ω_1; $F_d = F_1 \otimes \cdots \otimes F_1$
S_1 - linear functional on F_1; $S_d = S_1 \otimes \cdots \otimes S_1$

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Assume that there are functions e_1 and e_2 on Ω_1 such that e_1^2, e_2^2 and $\sqrt{2}e_1 e_2$ are orthonormal in F_1 and let $S_1(e_i^2) = \sqrt{2}/2$ and $S_1(e_1 e_2) = 0$. Then

$$\mathrm{err}_n(S_d)^2 \geq 1 - n\,2^{-d}.$$

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Erich's problem:
$e_1(x) = 2^{1/4}\cos(\pi x)$ and $e_2(x) = 2^{1/4}\sin(\pi x)$ on $[0,1]$
$\implies b_1 = h_1 = 1,\ b_2(x) = \cos(2\pi x),\ b_3(x) = \sin(2\pi x).$

- We assume that e_1^2, e_2^2 and $\sqrt{2}e_1e_2$ is an orthonormal basis of F_1
- So is $b_1 = \frac{1}{2}\sqrt{2}(e_1^2 + e_2^2)$, $b_2 = \frac{1}{2}\sqrt{2}(e_1^2 - e_2^2)$, and $b_3 = \sqrt{2}e_1e_2$

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- On F_1, S_1 is represented by $h_1 = b_1$.
- $M_1(x,y) = e_1(x)e_1(y) + e_2(x)e_2(y)$ is a reproducing kernel on Ω_1.

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- The reproducing kernel K_1 of F_1 satisfies for all $x, y \in \Omega_1$

$$K_1(x, y) = \sum_{i=1}^{3} b_i(x)b_i(y) = \left(\sum_{i=1}^{2} e_i(x)e_i(y)\right)^2 = M_1(x, y)^2$$

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- K_d, M_d, h_d - tensor products of K_1, M_1, h_1; $h_d(x) = 2^{-d/2}M_d(x,x)$
- By the Schur product theorem

$$\left(K_d(x_j,x_k) - n^{-1}\,2^d\,h_d(x_j)h_d(x_k)\right)_{j,k\le n}$$

is positive semi-definite for all $x_1,\ldots,x_n \in \Omega_d$.

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- The Proposition gives that

$$\mathrm{err}_n(S_d)^2 \ge 1 - n\,2^{-d}.$$

Lower bounds in tensor product problems

$F_1, \ldots, F_d$ - RKHS on $\Omega_1, \ldots, \Omega_d$; $F^d = F_1 \otimes \cdots \otimes F_d$
$S_1, \ldots, S_d$ - linear functionals on $F_1, \ldots, F_d$; $S^d = S_1 \otimes \cdots \otimes S_d$
unit norms, representers $h_1, \ldots, h_d$

Theorem: Non-homogeneous tensor products
Assume that there are functions f_i and g_i in F_i and a number $\alpha_i \in (0, 1]$ such that (h_i, f_i, g_i) is orthonormal in F_i and $\alpha_i h_i = \sqrt{f_i^2 + g_i^2}$. Then

$$\mathrm{err}_n(S^d)^2 \geq 1 - n \prod_{i=1}^{d} (1 + \alpha_i^2)^{-1}.$$

Numerical integration revisited

H_γ: Sobolev space on $[0,1]$:

$$H_\gamma := \left\{ f(x) = \sum_{j\in\mathbb{Z}} \alpha_j e^{2\pi i j x}, \quad \|f\|_\gamma^2 = \sum_{j\in\mathbb{Z}} \frac{\alpha_j^2}{\gamma_j^2} < \infty \right\}$$

Kernel: $K(x,y) = \sum_{j\in\mathbb{Z}} \gamma_j^2 e^{2\pi i j(x-y)}$

A. Hinrichs, D. Krieg, E. Novak, J.V., Lower bounds for the error of quadrature formulas for Hilbert spaces, J. Compl. 2021

A. Hinrichs, D. Krieg, E. Novak, J.V., Lower bounds for integration and recovery in L_2, J. Compl. 2022

D. Krieg, J.V., New lower bounds for the integration of periodic functions, to appear in JFAA

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Kernel: $K(x,y) = \sum_{j\in\mathbb{Z}} \gamma_j^2 e^{2\pi ij(x-y)}$

$\gamma \in \ell_2$: f is bounded, the series converges uniformly

$$\left| \sum_{|j|\geq m} \alpha_j e^{2\pi ijx} \right| = \left| \sum_{|j|\geq m} \frac{\alpha_j}{\gamma_j} \cdot \gamma_j e^{2\pi ijx} \right| \leq \left(\sum_{j\in\mathbb{Z}} \frac{\alpha_j^2}{\gamma_j^2} \right)^{1/2} \cdot \left(\sum_{|j|\geq m} \gamma_j^2 \right)^{1/2} \to 0$$

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Numerical integration revisited

Theorem: Let $\mu \in \ell_1(\mathbb{Z})$ be a non-negative and non-zero sequence and let

$$\gamma_\ell^2 := (\mu * \mu)_\ell := \sum_{j\in\mathbb{Z}} \mu_j \mu_{j+\ell}, \quad \ell \in \mathbb{Z}.$$

Then for all $n \in \mathbb{N}_0$: $\mathrm{err}_n(H_\gamma, \mathrm{INT})^2 \geq \gamma_0^2 \left(1 - \frac{n\gamma_0^2}{\|\gamma\|_2^2}\right)$.

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Idea of the proof:

- $e_k(x) := e^{2\pi i k x}$ for $k \in \mathbb{Z}$ and $x \in \mathbb{R}$
- $M(x,y) := \sum_{k\in\mathbb{Z}} \mu_k e_k(x-y)$
- $K(x,y) = \left|M(x,y)\right|^2 = \sum_{j,k\in\mathbb{Z}} \mu_j \mu_k e_{k-j}(x-y) = \sum_{\ell\in\mathbb{Z}} \gamma_\ell^2 e_\ell(x-y)$.

Numerical integration revisited

If $\gamma \in \ell_2(\mathbb{Z})$ is given, how do we choose μ?

$$\mu_k := \begin{cases} 0 & \text{if } k < r, \\ \gamma_k^2 \left(\sum_{\ell=r}^{\infty} \gamma_\ell^2 \right)^{-1/2} & \text{if } k \geq r. \end{cases}$$

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Theorem: Let $a \in \ell_2(\mathbb{N}_0)$ be non-negative and non-increasing. If we put $\gamma = (\ldots, a_3, a_1, a_0, a_2, a_4, \ldots)$, we have $a_n(H_\gamma) = a_n$ for all $n \in \mathbb{N}_0$ and

$$g_n(H_\gamma)^2 \geq \frac{1}{8n} \sum_{k \geq n} a_k(H_\gamma)^2$$

for infinitely many values of n.

Numerical integration revisited

Theorem: For $\beta > 1/2$ and $\gamma_n = n^{-1/2} \log^{-\beta}(n+1)$:

$$a_n(H_\gamma) \approx n^{-1/2} \log^{-\beta}(n+1)$$
$$\mathrm{err}_n(H_\gamma) \approx g_n(H_\gamma) \approx n^{-1/2} \log^{-\beta+1/2}(n+1).$$

$\ldots \log^{1/2}(n+1)$ gap between linear information and function values!

Numerical integration revisited: Infinite trace

What if $(a_n(H))_n \notin \ell_2$?

The convergence of the sampling numbers can be extremely slow.

Theorem: Let $a, \tau \in c_0(\mathbb{N}_0)$ be non-increasing with $a \notin \ell_2(\mathbb{N}_0)$. Then there is an example (H, L_2) such that $a_n(H) = a_n$ for all $n \in \mathbb{N}_0$ and

$$g_n(H) \geq \tau_n$$

for all but finitely many values of $n \in \mathbb{N}_0$.

You can choose, e.g., $a_n \asymp n^{-1/2}$ and $\tau_n \asymp \log^{-1/2} n$.

A. Hinrichs, E. Novak, J. Vybiral, Linear information versus function evaluations for L_2-appr., 2008

Sums of squares

Replace $\gamma_\ell^2 := (\mu * \mu)_\ell$ by something more general, which would still allow the Schur technique to work:

$\gamma_\ell^2 := \sum_{i \leq m} (\mu_i * \mu_i)_\ell$ - sum of convolution squares

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Theorem: If $\gamma \in \ell_2(\mathbb{Z}^d)$ is a sum of convolution squares, then

$$\mathrm{err}_n(H_\gamma)^2 \ \geq \ \gamma_0^2 \left(1 - \frac{n\gamma_0^2}{\|\gamma\|_2^2}\right).$$

Theorem: Let $\gamma \in \ell_2(\mathbb{Z})$ be non-negative, symmetric and monotonically decreasing on $\mathbb{N}_0$. Then

$$\mathrm{err}_n(H_\gamma)^2 \ \geq \ \min\left\{\frac{\gamma_0^2}{2}, \ \frac{1}{8n} \sum_{k \geq 4n} \gamma_k^2\right\} \qquad \text{for all} \ \ n \in \mathbb{N}.$$

Extension to $d > 1$

$d > 1$: $\Omega = [0,1]^d$, $e_k(x) = e^{2\pi i k\cdot x}$ $(k \in \mathbb{Z}^d,\ x \in \Omega)$, $\beta > 1/2$

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Isotropic smoothness:

- H_γ as usual, $\gamma_k^\beta = (1+|k|)^{-d/2}\log^{-\beta}(3+|k|), k \in \mathbb{Z}^d$
- If $\beta > 1/2$

$$a_n(H_\gamma) \asymp n^{-1/2}(\log n)^{-\beta}$$

and

$$\mathrm{err}_n(H_\gamma) \approx g_n(H_\gamma) \approx n^{-1/2}(\log n)^{-\beta+1/2}.$$

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Mixed smoothness:

- H_γ as usual, $\gamma_k = \prod_{j=1}^d (1+|k_j|)^{-1/2}\log^{-\beta}(e+|k_j|), \quad k \in \mathbb{Z}^d$
- If $\beta > 1/2$

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and

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Limits of the bump-function technique:

Back to the roots:

Could not we get the same just by bump functions?

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No, we can't!

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Could not we get the same just by bump functions?

No, we can't!

There is no bump function, which could give the lower bound!

- $\gamma_n = n^{-1/2} \log^{-\beta}(n+1)$
- Take any $\varphi \in C([0,1])$ with supp $\varphi \subset [0, 1/(2n)]$ and consider functions of the type $\sum_{j=1}^n \varphi(x - z_j)$
- Theorem: There is a choice of $\{z_1, \ldots, z_n\} \subset [0,1]$ such that

$$\frac{\int_0^1 \sum_{j=1}^n \varphi(x - z_j) dx}{\left\| \sum_{j=1}^n \varphi(x - z_j) \right\|_{H_\gamma}} \leq C\, n^{-1/2} (\log n)^{-\beta}.$$

- Characterization by differences! Symmetric bumps subtract!
- Removing some bumps from $\sum_{j=0}^{2n-1} \varphi(x - j/(2n))$ increases the norm.

Thank you for your attention!