Odvození jpdf pro vlastní čísla Gaussovských matic

- Máme jpdf podle elementů H (resp. horního trojúhelníku)

$$\rho(H) = \prod_{i=1}^{N} \frac{e^{-H_{ii}^2/2}}{\sqrt{2\pi}} \prod_{i<j} \frac{e^{-H_{ij}^2}}{\sqrt{\pi}} = \prod_{i,j=1}^{N} e^{-H_{ij}^2/2} \cdot (2\pi)^{-N/2} \cdot \pi^{-\frac{N(N-1)}{4}}$$

- Z toho chceme odvodit hustotu na množině spektr H:

$$\tau(x_1, \dots, x_N) = \frac{1}{Z_N} e^{-\frac{1}{2}\|x\|_2^2} \prod_{j<k} |x_j - x_k|,$$

$$Z_N = (2\pi)^{N/2} \prod_{j=1}^{N} \frac{\Gamma(1+j/2)}{\Gamma(1+1/2)}$$

Poznámky: • uspořádané $(x_1 < x_2 < \dots < x_N)$ vs. neuspořádané spektrum … $(x_1, \dots, x_N)$ obecné, neurčené jednoznačně

… $\tau(x_1, \dots, x_N) = \tau(x_{\sigma(1)}, \dots, x_{\sigma(N)})$, σ permutace

→ liší se jen o faktor $N!$

- pro $A \subset \mathbb{R}^N$ inv. vůči permutacím $(x \in A \Leftrightarrow (x_{\sigma(1)}, \dots, x_{\sigma(N)}) \in A)$ bude platit $\mathbb{P}(\sigma(H) \in A) = \int_A \tau(x)\,dx$

- Vztah mezi H a $\sigma(H) = x$ je dán jako $H = O X O^T$

kde O … ortogonální … $OO^T = O^T O = I$

$X = \mathrm{diag}(x_1, \dots, x_N)$ … $x = \mathrm{diag}(X)$.

Nejprve zkusíme v jednodušší formě: polární souřadnice

$(x,y) \ \dots\ (r,\varphi) \qquad \Psi(r,\varphi) = (r\cos\varphi, r\sin\varphi) = (x,y)$

Mějme náhodný vektor V s $\mathbb{P}(V\in B) = \int_B \rho(x,y)\,d(x,y)$

Chceme najít hustotu τ tak, aby: $\int_{\{(r,\varphi):\,\Psi(r,\varphi)\in B\}} \tau(r,\varphi)\,d(r,\varphi)$

$$\int_B \rho(x,y)\,d(x,y) = \int_{\{(r,\varphi):\,\Psi(r,\varphi)\in B\}} \tau(r,\varphi)\,d(r,\varphi) = \int_{\Psi^{-1}(B)} \tau(r,\varphi)\,d(r,\varphi)$$

Substituce v hustotě na $\mathbb{R}^2$

$$\int_B \rho(x,y)\,d(x,y) = \int_{\Psi^{-1}(B)} \rho(r\cos\varphi, r\sin\varphi)\cdot r\cdot dr\,d\varphi$$

$$\Rightarrow \tau(r,\varphi) = \rho(r\cos\varphi, r\sin\varphi)\cdot r$$

- vyjádřit x,y pomocí r,φ
- Jakobián $r = |J((x,y),(r,\varphi))| = \left|\det\begin{pmatrix} \frac{\partial x}{\partial r}, & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r}, & \frac{\partial y}{\partial \varphi}\end{pmatrix}\right|$.
- nechť τ nezávisí na φ : $\rho(x,y) = \bar\rho(x^2+y^2)$

$$\tau(r,\varphi) = r\cdot\rho(r\cos\varphi, r\sin\varphi) = r\cdot\bar\rho(r^2)$$

- Pokud B je rot. invariantní ... $\Psi^{-1}(B) = C\times[0,2\pi]$

$$\int_{\Psi^{-1}(B)} \tau(r,\varphi)\,dr\,d\varphi = \int_C\int_0^{2\pi} r\cdot\bar\rho(r^2)\,d\varphi\,dr = \int_C r\cdot\bar\rho(r^2)\,dr\cdot 2\pi$$

$$\bar\tau(r) = 2\pi\cdot r\,\bar\rho(r^2)$$

Nyní $\Upsilon : (X, \sigma) \longrightarrow H = \sigma X \sigma^T$

$H = H(X, \sigma)$

- Vyjádříme $\rho(H) = \rho(H_{11}(X,\sigma), \ldots, H_{NN}(X,\sigma))$

Lze několika způsoby:

1, Lemma: Pro $A \in \mathbb{R}^{m \times N}$, $(\varphi_j)_{j=1}^N$, $(\psi_j)_{j=1}^N$ ort. báze $\mathbb{R}^N$

je $\sum_{j=1}^N \|A\varphi_j\|_2^2 = \sum_{j=1}^N \|A\psi_j\|_2^2 = \|A\|_F^2 = \sum_{i,j} A_{ij}^2$

Důkaz: $\psi_j = \sum_{k=1}^N \langle \psi_j, \varphi_k \rangle \varphi_k$; $A\psi_j = \sum_{k=1}^N \langle \psi_j, \varphi_k \rangle A\varphi_k$

a tedy $\sum_{j=1}^N \|A\psi_j\|_2^2 = \sum_{j=1}^N \left\langle \sum_{k=1}^N \langle \psi_j, \varphi_k \rangle A\varphi_k, \sum_{l=1}^N \langle \psi_j, \varphi_l \rangle A\varphi_l \right\rangle$

$$= \sum_{k=1}^N \sum_{l=1}^N \langle A\varphi_k, A\varphi_l \rangle \underbrace{\sum_{j=1}^N \langle \psi_j, \varphi_k \rangle \langle \psi_j, \varphi_l \rangle}_{= \langle \varphi_k, \varphi_l \rangle = \delta_{k,l}}$$

$$= \sum_{k=1}^N \langle A\varphi_k, A\varphi_k \rangle = \sum_{k=1}^N \|A\varphi_k\|_2^2 \quad \ldots \quad \varphi_k = e_k \text{ kan. báze}$$

$$\Rightarrow \quad = \|A\|_F^2$$

Důsledek: $A \in \mathbb{R}^{N \times N}$ symetrická $\Rightarrow \|A\|_F^2 = \sum_{j=1}^N \lambda_j(A)^2$

$\ldots (\varphi_k)_{k=1}^N$ je báze z vl. vektorů $\ldots A\varphi_k = \lambda_k \varphi_k$

$$\Rightarrow \|A\|_F^2 = \sum_{k=1}^N \|A\varphi_k\|_2^2 = \sum_{k=1}^N \lambda_k(A)^2.$$

Celkem je tedy $\rho(H) \cong \prod_{i,j=1}^{N} e^{-H_{ij}^2/2} = e^{-\|H\|_F^2/2} = e^{-\|x\|_2^2/2}$.

2, Pozn. Stejně se dokáže: $A \in \mathbb{R}^{N\times N}$, $(\varphi_j)_{j=1}^N$, $(\psi_j)_{j=1}^N$ orb. $\mathbb{R}^N$, pak

$$\sum_{j=1}^{N} \langle A\psi_j, \psi_j \rangle = \sum_{k=1}^{N} \langle A\varphi_k, \varphi_k \rangle = tr(A)$$

Dále platí: $A, B \in \mathbb{R}^{m\times N}$: $tr(A^TB) = \sum_{k=1}^{N} (A^TB)_{kk} =$

$$= \sum_{k=1}^{N}\sum_{j=1}^{m} (A^T)_{kj} B_{jk} = \sum_{j=1}^{m}\sum_{k=1}^{N} A_{jk} B_{jk} = \langle A, B \rangle_F$$

$$= \langle B, A \rangle_F = tr(B^TA)$$ ... stopa je cyklická

- $tr(H^2) = \sum_{k=1}^{N} (H^2)_{kk} = \sum_{k=1}^{N}\sum_{l=1}^{N} H_{kl} \cdot H_{lk} \overset{H=H^T}{=} \sum_{k,l=1}^{N} H_{kl}^2 = \|H\|_F^2$

- Celkem je tedy $\rho(H) \cong \exp(-\|H\|_F^2/2) = \exp(-tr(H^2)/2)$

$$= \exp[-tr((OXO^T)(OXO^T))/2] =$$

$$= \exp[-tr(OX^2O^T)/2] = \exp(-tr(O^TOX^2)/2)$$

↑ $tr(ABC) = tr(CAB) = tr(BCA)$ jsou-li součiny definovány

$$= \exp(-tr(X^2)/2) = \exp(-\|x\|_2^2/2).$$

3, Poslední způsob - hrubou silou

$$H = OXO^T, \quad H_{ij} = (OXO^T)_{ij} = \sum_{l=1}^{N} O_{il} (XO^T)_{lj}$$

$$= \sum_{l=1}^{N} O_{il} \sum_{k=1}^{N} X_{lk} (O^T)_{kj} = \sum_{l,k=1}^{N} X_{lk} O_{il} O_{jk} \overset{l=k}{=} \sum_{k=1}^{N} x_k O_{ik} O_{jk}$$

a
$$\sum_{i,j=1}^{N} H_{ij}^2 = \sum_{i,j=1}^{N} \left(\sum_{k=1}^{N} x_k O_{ik} O_{jk} \right)^2 =$$

$$= \sum_{i,j=1}^{N} \sum_{k,l=1}^{N} x_k O_{ik} O_{jk} \, x_l O_{il} O_{jl} = \sum_{k,l=1}^{N} x_k x_l \sum_{i,j=1}^{N} O_{ik} O_{jk} O_{il} O_{jl}$$

$$= \sum_{k,l=1}^{N} x_k x_l \left(\sum_{i=1}^{N} O_{ik} O_{il} \right) \left(\sum_{j=1}^{N} O_{jk} O_{jl} \right) = \sum_{k=1}^{N} x_k^2 .$$

• Budeme muset vypočítat $J(H,(x,O))$ ... vyjde

$$|J(H,(x,O))| = \left| \prod_{j<k} (x_j - x_k) \right| = \prod_{j<k} |x_j - x_k|$$

... nezávisí na O!

V analogii s polárními souřadnicemi:

• Nechť $\mathcal{B}$ je "rotačně invariantní" podmnožina
$$\mathcal{M}=\{M\in\mathbb{R}^{N\times N}: M=M^T\}$$
tedy $\forall M\in\mathcal{B}\ \forall O: OMO^T\in\mathcal{B}$

Pak
$$\int_{\mathcal{B}}\varrho(H)\,dH=\int_{\mathcal{B}}\varrho(H_{11},\dots,H_{NN})\,dH_{11}\dots dH_{NN}$$
$$=\int_{\{(x,O):\,\Psi(x,O)=OXO^T\in\mathcal{B}\}}\varrho(H_{11}(x,O),\dots,H_{NN}(x,O))\cdot|J(H,\{x,O\})|\cdot dx\,dO$$
$$=\int_{\Psi^{-1}(\mathcal{B})}e^{-\|x\|_2^2/2}\cdot\prod_{j<k}|x_j-x_k|\cdot dx\,dO\cdot(2\pi)^{-N/2}\cdot\pi^{-N(N-1)/4}\quad\dots\ \Psi^{-1}(\mathcal{B})=C\times\Omega_N$$
$$\Omega_N=\{O: OO^T=O^TO=I\}$$
$$=\int_C\int_{\Omega_N}e^{-\|x\|_2^2/2}\cdot\prod_{j<k}|x_j-x_k|\,dO\,dx\cdot c_N$$
$$=\int_C e^{-\|x\|_2^2/2}\prod_{j<k}|x_k-x_j|\,dx\cdot\underbrace{\int_{\Omega_N}1\,dO}_{vol(\Omega_N)}\cdot c_N$$

$$\tau(x_1,\dots,x_N)=e^{-\|x\|_2^2/2}\cdot\prod_{j<k}|x_j-x_k|\cdot\frac{c_N\cdot vol(\Omega_N)}{2^N\cdot N!}$$

- Faktor $2^N \cdot N!$ odpovídá počtu (x, σ), které se zobrazují na stejné H ... permutace $x_1, \dots, x_N$, volba znaménka N vlastních vektorů.

- Pokud bychom u pol. souřadnic měli $\varphi \in [0, 4\pi]$, museli bychom hustotu vydělit dvěma ...

- Vzápětí ukážeme, že $vol(\Omega_N) = \prod_{j=1}^{N} \frac{2\pi^{j/2}}{\Gamma(j/2)}$ ... Pak bude

$$\frac{1}{Z_N} = (2\pi)^{-N/2} \pi^{-N(N-1)/4} \cdot \left(\prod_{j=1}^{N} \frac{2\pi^{j/2}}{\Gamma(j/2)} \right) \cdot \frac{1}{2^N \cdot N!} =$$

$$(2\pi)^{-N/2} \underbrace{\pi^{-N(N-1)/4} \cdot \pi^{N(N+1)/4}}_{\pi^{N/2}} \cdot \frac{1}{\Gamma(1/2)\cdot 1/2 \cdots \Gamma(N/2)\cdot N/2 \cdot 2^N}$$

$$= (2\pi)^{-N/2} \cdot \frac{(\sqrt{\pi}/2)^N}{\Gamma(1/2+1) \cdots \Gamma(N/2+1)} \quad \dots \quad \sqrt{\pi}/2 = \Gamma(1 + \tfrac{1}{2})$$

Zbývá
- $vol(\Omega_N)$
- $|J(H, (x, \sigma))|$

Výpočet vol(Ω_N)

• 1, jednotková koule a sféra v $\mathbb{R}^n$

$$B_n = \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\}; \quad S^{n-1} = \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$$

$$\int_{\mathbb{R}^n} e^{-\|x\|_2^2/2} dx = \int_{\mathbb{R}^n} e^{-x_1^2/2} \dots e^{-x_n^2/2} dx_1 \dots dx_n = \left(\int_{\mathbb{R}} e^{-t^2/2} dt\right)^n = (2\pi)^{n/2}$$

$$\| $$

$$\int_0^\infty \int_{rS^{n-1}} e^{-r^2/2} dA\, dr = \int_0^\infty e^{-r^2/2} r^{n-1} A_{n-1} dr = A_{n-1} \int_0^\infty e^{-t} (2t)^{\frac{n-1}{2} - \frac{1}{2}} dt$$

$$t = r^2/2$$
$$dt = r\,dr$$

$$= A_{n-1} \cdot 2^{\frac{n}{2}-1} \Gamma(n/2)$$

$$\Rightarrow A_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}; \quad V_n = \int_0^1 \frac{2\pi^{n/2}}{\Gamma(n/2)} r^{n-1} dr = \frac{2\pi^{n/2}}{n\Gamma(n/2)} = \frac{\pi^{n/2}}{\Gamma(n/2+1)}.$$

Pozn: • $\mathrm{vol}(R \cdot B) = \int_{R \cdot B} 1\, dx \underset{x = Ry}{=} \int_B R^n dy = R^n \cdot \mathrm{vol}(B).$

• $$A_{n-1} = \lim_{\varepsilon \to 0} \frac{\mathrm{vol}[(1+\varepsilon)B] - \mathrm{vol}[(1-\varepsilon)B]}{2\varepsilon} =$$

$$= \lim_{\varepsilon \to 0} \frac{(1+\varepsilon)^n - (1-\varepsilon)^n}{2\varepsilon} \mathrm{vol}(B) = \lim_{\varepsilon \to 0} \frac{2n\varepsilon}{2\varepsilon} \mathrm{vol}(B)$$
$$= n \cdot \mathrm{vol}(B) = nV_n$$

• obecně $\mathrm{area}(r S^{n-1}) = \frac{d}{dr} \mathrm{vol}(r B_n)$

• Formule $\int_{\mathbb{R}^n} f(x)\,dx = \int_0^\infty \int_{r\mathbb{S}^{n-1}} f(x)\,dA\,dr$

$f = \chi_{\{a \le \|x\| \le b\}}$: $vol(b\cdot B) - vol(a\cdot B) = [b^n - a^n]\cdot vol(B)$

& $\int_a^b area(r\,\mathbb{S}^{n-1})\,dr = area(\mathbb{S}^{n-1}) \int_a^b r^{n-1}\,dr = \frac{b^n - a^n}{n} area(\mathbb{S}^{n-1})$

platí pro jednoduché rad. symetrické funkce,
tedy i pro lin. kombinace ... tedy pro $f \in L_1(\mathbb{R}^n)$ rad. sym.

2) $\Omega_N = \{\, O \in \mathbb{R}^{N\times N} : O O^T = O^T O = I \,\}$
počet stupňů volnosti ... dimenze Ω_N?

první řádek : $N-1$ ... N & $\|x\|_2^2 = 1$
druhý řádek : $N-2$ ... N & $\|a_2\|_2^2 = 1$ & $\langle a_1, a_2 \rangle = 0$
⋮
N-tý řádek : $(N-1)-(N-1) = 0$ ... jen 2 vektory možné

$D_N = dim(\Omega_N) = \frac{N(N-1)}{2}$

první řádek: $A_{N-1} = area(\mathbb{S}^{N-1}) = \frac{2\pi^{N/2}}{\Gamma(N/2)}$

druhý řádek : $A_{N-2} = \frac{2\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N-1}{2})}$

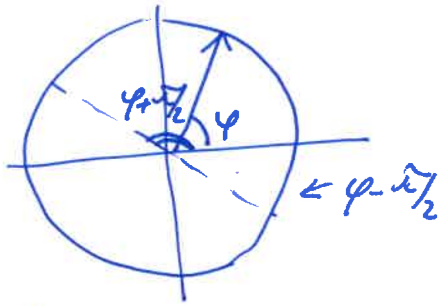
⋮

N-tý řádek : $2 = \frac{2\pi^{1/2}}{\Gamma(1/2)}$

$$vol(\Omega_N) = \prod_{j=1}^{N} \frac{2\pi^{j/2}}{\Gamma(j/2)}$$

Pro $N=2$ .... $vol(\Omega_N) \overset{?}{=} \frac{2\pi^{1/2}}{\Gamma(1/2)} \cdot \frac{2\pi^{2/2}}{\Gamma(2/2)} = 2\cdot 2\pi = 4\pi$

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} : \quad \sigma_{11}^2 + \sigma_{12}^2 = 1, \quad \sigma_{21}^2 + \sigma_{22}^2 = 1, \quad \sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22} = 0$$



$$\begin{pmatrix} \cos\varphi, \sin\varphi \\ \sin\varphi, -\cos\varphi \end{pmatrix}, \text{ nebo } \begin{pmatrix} \cos\varphi, \sin\varphi \\ -\sin\varphi, \cos\varphi \end{pmatrix} \; \dots \; 2\pi \cdot 2 = 4\pi$$

$\varphi \in [0, 2\pi]$ $\qquad$ $\varphi \in [0, 2\pi]$

$$vol(\Omega_2) = \int_{\mathbb{R}^4} \prod_{i,j=1}^{2} d\sigma_{ij} \, \delta\left(\sqrt{\sigma_{11}^2+\sigma_{12}^2}-1\right) \delta\left(\sqrt{\sigma_{21}^2+\sigma_{22}^2}-1\right) \delta\left(\sigma_{11}\sigma_{21}+\sigma_{12}\sigma_{22}\right)$$

$\sigma_{11} = r\cos\varphi \qquad \sigma_{21} = R\cos\psi$

$\sigma_{12} = r\sin\varphi \qquad \sigma_{22} = R\sin\psi$

$$\Rightarrow \int_0^{2\pi} d\varphi \int_0^{2\pi} d\psi \int_0^{\infty} dr \int_0^{\infty} dR \; r \cdot R \cdot \delta(r-1)\delta(R-1) \underbrace{\delta\big(rR(\cos\varphi\cos\psi + \sin\varphi\sin\psi)\big)}_{= \delta(\cos(\varphi-\psi))}$$

$$= \int_0^{2\pi}\int_0^{2\pi} \delta(\cos(\varphi-\psi)) \, d\varphi \, d\psi = \int_0^{2\pi} 2 \, d\psi = 4\pi.$$

$\forall \psi \; \exists 2\varphi$

Ještě potřebujeme vypočítat Jakobián $J(H, \{x, O\})$

$H = O X O^T$

- Nejprve pro $N=2$

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix} = \begin{pmatrix} \cos\varphi, & \sin\varphi \\ -\sin\varphi, & \cos\varphi \end{pmatrix} \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \begin{pmatrix} \cos\varphi; & -\sin\varphi \\ \sin\varphi; & \cos\varphi \end{pmatrix}$$

$$= \begin{pmatrix} \cos\varphi, & \sin\varphi \\ -\sin\varphi, & \cos\varphi \end{pmatrix} \begin{pmatrix} x_1\cos\varphi, & -x_1\sin\varphi \\ x_2\sin\varphi, & x_2\cos\varphi \end{pmatrix}$$

$$= \begin{pmatrix} x_1\cos^2\varphi + x_2\sin^2\varphi, & (-x_1+x_2)\sin\varphi\cos\varphi \\ (x_2-x_1)\sin\varphi\cos\varphi, & x_1\sin^2\varphi + x_2\cos^2\varphi \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial H_{11}}{\partial x_1}, & \frac{\partial H_{11}}{\partial x_2}, & \frac{\partial H_{11}}{\partial \varphi} \\ \frac{\partial H_{12}}{\partial x_1}, & \frac{\partial H_{12}}{\partial x_2}, & \frac{\partial H_{12}}{\partial \varphi} \\ \frac{\partial H_{22}}{\partial x_1}, & \frac{\partial H_{22}}{\partial x_2}, & \frac{\partial H_{22}}{\partial \varphi} \end{pmatrix} = + \begin{pmatrix} \cos^2\varphi & ; \sin^2\varphi; & +2(x_2-x_1)\cos\varphi\sin\varphi \\ -\sin\varphi\cos\varphi; & \sin\varphi\cos\varphi; & (x_2-x_1)(\cos^2\varphi-\sin^2\varphi) \\ \sin^2\varphi; & \cos^2\varphi; & (x_1-x_2)(2\sin\varphi\cos\varphi) \end{pmatrix}$$

$$\det \overset{I+III\to I}{=} (x_2-x_1)\cdot \det\begin{pmatrix} 1 & ; 1; & 0 \\ -\sin\varphi\cos\varphi, & \sin\varphi\cos\varphi, & \cos^2\varphi-\sin^2\varphi \\ \sin^2\varphi, & \cos^2\varphi, & -2\sin\varphi\cos\varphi \end{pmatrix}$$

$$\overset{II-I\to II}{=} (x_2-x_1)\det\begin{pmatrix} 1; & 0; & 0 \\ -\sin\varphi\cos\varphi; & 2\sin\varphi\cos\varphi, & \cos^2\varphi-\sin^2\varphi \\ \sin^2\varphi, & \cos^2\varphi-\sin^2\varphi, & -2\sin\varphi\cos\varphi \end{pmatrix}$$

$= -(x_2-x_1)$

$$= (x_2-x_1)\{-4\sin^2\varphi\cos^2\varphi - (\cos^2\varphi-\sin^2\varphi)\} = (x_2-x_1)\{-\cos^4\varphi-\sin^4\varphi-2\sin^2\varphi\cos^2\varphi\}$$

Obecné $N \geq 2$: neschůdné

odvodíme v řeči diferenciálů

$$f(x+h) = f(x) + \alpha h + o(h) \dots \qquad df(x,y) = \alpha dh_1 + \beta dh_2$$

$$df = \alpha dh \quad \dots \quad \alpha = \frac{\partial f}{\partial x} \qquad \frac{\partial f}{\partial x} = \alpha, \ \frac{\partial f}{\partial y} = \beta$$

- $f(x+dx) - f(x) = \alpha dx$

$$H = O X O^T$$

- O podléhá vazbě $OO^T = O^T O = I$

Tedy $(O+\partial O)(O+\partial O)^T = (O+\partial O)^T(O+\partial O) = I$

$$(O+\partial O)(O^T + (\partial O)^T) = (O^T + (\partial O)^T)(O+\partial O) = I \qquad \cdot (\partial O)^T = \partial(O^T)$$

$$\partial O \cdot O^T + O \cdot (\partial O)^T = \partial O^T \cdot O + O^T \partial O = 0$$

$$\Downarrow$$

$$\partial O^T = -O^T (\partial O) O^T$$

- $H + \partial H = (O+\partial O)(X+\partial X)(O+\partial O)^T$

$$= H + \partial O \cdot X \cdot O^T + O \partial X (O^T) + O X (\partial O)^T$$

$$\Rightarrow \partial H = \partial O \cdot X \cdot O^T + O \cdot \partial X \cdot O^T + \underbrace{O X (\partial O)^T}_{-O X O^T (\partial O) O^T}$$

$$= O \left[O^T \cdot \partial O \cdot X + \partial X - X O^T (\partial O) \right] O^T$$

$\partial \Omega = O^T \partial O$: $\quad \partial H = O \underbrace{\left[\partial \Omega \cdot X + \partial X - X \partial \Omega \right]}_{\partial \hat{H}} O^T$

- ∂H a $\partial \hat{H}$ se liší jen o ortonormální transformaci
- $\partial \Omega$ a ∂O se – " –

Místo ∂H podle $\partial X, \partial O$ budeme uvažovat pouze

$\partial \hat{A}$ podle $\partial X, \partial \Omega$!

$$\frac{d\hat{A}_{ij}}{dx_k} = \begin{cases} 1 & \text{pro } i=j=k \\ 0 & \dots \text{jinak} \end{cases} \quad ; \quad \frac{d\hat{A}_{ij}}{d\Omega_{kl}} = \begin{cases} 0 & \text{pro } i \neq k, \text{ nebo } j \neq l \\ x_j - x_i & \text{pro } i=k, j=l \end{cases}$$

	$/dx_1$ .... $/dx_N$	$/d\Omega_{12}$ .... $/d\Omega_{N-1,N}$
$d\hat{A}_{11}$ ⋮ $d\hat{A}_{NN}$	$\begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix}$	0
$d\hat{A}_{12}$ ⋮ $d\hat{A}_{N-1,N}$	0	$\begin{matrix} x_2 - x_1 & & \\ & \ddots & \\ & & x_N - x_{N-1} \end{matrix}$

$$\Rightarrow \det = \prod_{i<j} x_j - x_i$$

- Pozn. $O^T \partial O$ je antisymetrická:

$$0 = \partial O^T \cdot \partial O + \partial O^T \cdot O = O^T \cdot \partial O + (O^T \partial O)^T.$$

Obecný postup pro N=2

$$H \longleftrightarrow (x_1, x_2, \varphi)$$

$$X = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}; \quad O = \begin{pmatrix} \cos\varphi, & +\sin\varphi \\ -\sin\varphi, & \cos\varphi \end{pmatrix}$$

$$\partial X = \begin{pmatrix} dx_1 & 0 \\ 0 & dx_2 \end{pmatrix}; \quad \partial O = \begin{pmatrix} -\sin\varphi, & \cos\varphi \\ -\cos\varphi, & -\sin\varphi \end{pmatrix} d\varphi$$

$$\partial\Omega = O^T \cdot \partial O = \begin{pmatrix} \cos\varphi; & -\sin\varphi \\ \sin\varphi; & \cos\varphi \end{pmatrix} \cdot \begin{pmatrix} -\sin\varphi; & \cos\varphi \\ -\cos\varphi; & -\sin\varphi \end{pmatrix} d\varphi$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} d\varphi$$

$$\partial\hat{H} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} d\varphi + \begin{pmatrix} dx_1 & 0 \\ 0 & dx_2 \end{pmatrix} - \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} d\varphi$$

$$= \begin{pmatrix} dx_1 & (x_2 - x_1) d\varphi \\ (x_2 - x_1) d\varphi & dx_2 \end{pmatrix} \qquad \begin{pmatrix} \frac{\partial\hat{H}}{\partial x_1} & & \\ & \ddots & \\ & & \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & x_2 - x_1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\det = x_1 - x_2$$