Carl's inequality for quasi-Banach spaces

Theorem: Let X be a p-Banach space and Y be a q-Banach space ($0 < p, q \le 1$). Then for any $\alpha > 0$ there exists a constant $\gamma_\alpha > 0$ such that

$$\sup_{1 \le k \le n} k^\alpha e_k(T) \le \gamma_\alpha \sup_{1 \le k \le n} k^\alpha c_k(T)$$

holds for all $T \in \mathcal{L}(X, Y)$.

$e_n(T)$ ... entropy numbers

$c_n(T)$ ... Gelfand numbers

We start with preparations:

- $$c_n(T) = \inf_{\substack{M \subset X \\ \operatorname{codim} M < n}} \sup_{\substack{x \in M \\ \|x\|_X \le 1}} \|Tx\|_Y$$

  is defined as before ... codim is algebraic notion.

- If X is a p-space and $M \subset X$ is a subspace, then

  $$X/M = \{z + M : z \in X\}, \quad \|z + M\|_{X/M} = \inf\{\|y\|_X : y \in z + M\}$$

  is well-defined and a p-space again. ... $[x] = x + M$

Indeed, let $[x], [y] \in X/M$ and $\varepsilon > 0$. Then there are $z^x, z^y \in M$ with $\|x - z^x\|_X \le (1+\varepsilon)\|[x]\|_{X/M}$, $\|y - z^y\|_X \le (1+\varepsilon)\|[y]\|_{X/M}$

and $\|\underbrace{[x+y]}_{[x]+[y]}\|^p_{X/M} \le \|x + y - z^x - z^y\|^p_X \le \|x - z^x\|^p_X + \|y - z^y\|^p_X \le (1+\varepsilon)^p \{\|[x]\|^p_{X/M} + \|[y]\|^p_{X/M}\}$.

• (Known fact): If X is real m-dimensional p-Banach space, then
$$e_n(id: X \to X) \le 4^{1/p} \cdot 2^{-\frac{n-1}{m}} \quad \text{for all } n \in \mathbb{N}.$$

• We show for every $n \in \mathbb{N}$: $n^{\alpha} e_n(T) \le \gamma_\alpha \sup\limits_{1 \le k \le n} k^{\alpha} c_k(T)$

• Multiplying T with a constant, we may assume that $c_k(T) \le k^{-\alpha}$ (and especially $\|T\| = c_1(T) \le 1$).

• Lifting property of nets:

• Let $M \subset X$ be a space with finite codimension. Let $\varepsilon > 0$ be fixed. Let $\mathcal{M} \subset X/M$ be an ε-net of the unit ball of X/M.

This means that for any $[x] \in B_{X/M}$ there is $[x'] \in \mathcal{M}$ with $\|[x]-[x']\|_{X/M} = \inf\limits_{z \in M} \|x - x' - z\|_X < \varepsilon$.

Then also $\|[x']\|^p_{X/M} \le \|[x']-[x]\|^p_{X/M} + \|[x]\|^p_{X/M} \le \varepsilon^p + 1 \le 2$

$$\Rightarrow \mathcal{M} \subset 2^{1/p} \cdot B_{X/M}.$$

For every $\underbrace{[w]}_{= [\tilde{w}]} \in \mathcal{M}$, there is a representer $\tilde{w} \in [w]$ with $\|\tilde{w}\|_X \le 2^{1/p}$ ... we collect them into $\mathcal{N} \subset 2^{1/p} \cdot B_X$.

We summarise: For any $x \in B_X$, $\|[x]\|_{X/M} \le \|x\|_X$ ... therefore $[x] \in B_{X/M}$ there exists $\tilde{w} \in \mathcal{N}$ such that
$$\|x - \tilde{w} - z\|_X < \varepsilon \text{ for some } z \in M.$$

• We now iterate this construction. Let $m_1 = 2^{N-1}$ $\dots$ $m_j = 2^{N-j}$

1, Let M_1 be the optimal subspace for $c_{m_1}(T)$: • $\operatorname{codim} M_1 < m_1$

• $\|T|_{M_1}\| \le (1+\varepsilon)\, c_{m_1}(T)$

• X/M_1 is $m_1 - 1$-dim. space

$\dots\ e_n(id : X/M_1 \to X/M_1) \le 4^{1/p}\, 2^{-\frac{n-1}{m_1}} \quad \forall n \ge 1.$

We fix k_1 later on and get $\mathcal{M}_1 \subset 2^{1/p} B_{X/M_1}$ and $\mathcal{N}_1 \subset 2^{1/p} B_X$

such that $\varepsilon_1 = (1+\varepsilon)\, 4^{1/p} \cdot 2^{-\frac{k_1 - 1}{m_1}}$, $\# \mathcal{M}_1 = \# \mathcal{N}_1 \le 2^{k_1 - 1}$.

$\dots\ \forall x \in B_X : \exists x_1 \in \mathcal{N}_1, z_1 \in M_1 :$ • $\|x - (x_1 + z_1)\|_X < \varepsilon_1$.

• $\|z_1\|_X^p \le \underbrace{\|x - (x_1 + z_1)\|_X^p}_{< \varepsilon_1^p \le 1} + \underbrace{\|x\|_X^p}_{\le 1} + \underbrace{\|x_1\|_X^p}_{\le 2} \le 4.$

2, M_2: $m_2 = 2^{N-2}$, $\operatorname{codim} M_2 < m_2$, $\varepsilon_2 = (1+\varepsilon)\, 4^{1/p}\, 2^{-\frac{k_2 - 1}{m_2}}$, $\# \mathcal{M}_2 = \# \mathcal{N}_2 \le 2^{k_2 - 1}$

$\left\| \frac{x - (x_1 + z_1)}{\varepsilon_1} \right\|_X < 1$: $\exists x_2 \in \mathcal{N}_2, z_2 \in M_2$:

$\left\| \frac{x - (x_1 + z_1)}{\varepsilon_1} - (x_2 + z_2) \right\|_X < \varepsilon_2 \ \dots\ \| x - (x_1 + z_1) - \varepsilon_1 (x_2 + z_2) \|_X \le \varepsilon_1 \cdot \varepsilon_2$

$\vdots$

$\forall x \in B_X\ \exists x_j \in \mathcal{N}_j, z_j \in M_j$:

$\| x - (x_1 + z_1) - \varepsilon_1 (x_2 + z_2) - \dots - \varepsilon_1 \cdots \varepsilon_{n-1} (x_n + z_n) \|_X < \prod_{j=1}^{n-1} \varepsilon_j \cdot \varepsilon_n = \prod_{j=1}^{n} \varepsilon_j =: \delta_n$

$\| x - \sum_{j=1}^{n} \delta_{j-1} (x_j + z_j) \|_X \le \delta_n$

Moreover: $$\|z_n\|_X^p \le \Big\| \underbrace{\frac{x-(x_1+z_1)-\dots-\varepsilon_1\cdots\varepsilon_{n-2}(x_{n-1}+z_{n-1})}{\varepsilon_{n-1}} - (x_n+z_n)}_{\le \varepsilon_n \le 1} \Big\|_X^p + \|x_n\|_X^p \le 4.$$

and $$\Big\|Tx-\sum_{k=1}^n \delta_{k-1}\, Tx_k\Big\|_Y^q \le \Big\|Tx-\sum_{k=1}^n \delta_{k-1}\, T(x_k+z_k)+\sum_{k=1}^n \delta_{k-1}\, Tz_k\Big\|_Y^q$$

$$\le \Big\|Tx-\sum_{k=1}^n \delta_{k-1}\, T(x_k+z_k)\Big\|_Y^q + \sum_{k=1}^n \delta_{k-1}^q \|Tz_k\|_Y^q$$

$$\le \|T\|^q \cdot \Big\|x-\sum_{k=1}^n \delta_{k-1}(x_k+z_k)\Big\|_X^q + \sum_{k=1}^n \delta_{k-1}^q \cdot \|Tz_k\|_Y^q$$

$$\le (\|T\|\cdot\delta_n)^q + 4^{q/p} \sum_{k=1}^n \delta_{k-1}^q \cdot c_{m_k}(T)^q \cdot (1+\varepsilon)^q.$$

Hence, every Tx, $x\in B_X$ can be well-approximated by the elements of the net

$$\mathcal{N} = \Big\{ T\Big(\sum_{k=1}^n \delta_{k-1} x_k\Big) : x_k \in \mathcal{N}_k \Big\}$$

with card. $$\#\mathcal{N} \le \prod_{k=1}^n (\#\mathcal{N}_k) \le 2^{k_1+\dots+k_n-n}$$

and $$\delta_k = \prod_{j=1}^k \varepsilon_k = 2^{k/p - \sum_{j=1}^k \frac{k_j-1}{m_j}}$$

Finally, we choose $k_j = \lceil 2^{N-j}(2/p+\beta)+1 \rceil$, $0<\alpha<\beta$,

$j=1,\dots,N$

and get $k_1+\dots+k_N+1-N \le C\cdot 2^N = n$

and $e_n(T)^q \le (\overbrace{\|T\|\delta_N}^{\le 1})^q + 4^{q/p}\cdot \sum_{k=1}^{N} \delta_{k-1}^q \cdot c_{m_k}(T)^q$

$$\le 2^{-\beta N q} + 4^{q/p}\sum_{k=1}^{N} 2^{-\beta(k-1)q}\cdot 2^{-\alpha \overset{(N-k)}{k} q}$$

$2^{-kq(\beta-\alpha)}$

$\varepsilon_j = 4^{1/p}\cdot 2^{-\frac{k_j-1}{m_j}} \le 2^{-\beta}$

$\delta_j \le 2^{-\beta j}$

$$\lesssim 2^{-N\alpha q}.$$

Hence, for $n \simeq C\cdot 2^N$, we have $e_n(T) \lesssim n^{-\alpha}$.