· Carl's inequality for quasi-Banach space

Theorem: Let X be a p-Banach space, let Y be a q-Banach space ($0 < p, q \le 1$). Then for every $\alpha > 0$ there is a constant $\gamma_\alpha > 0$ such that

$$\sup_{1 \le k \le n} k^\alpha e_k(T) \le \gamma_\alpha \sup_{1 \le k \le n} k^\alpha c_k(T)$$

holds for every $T \in \mathcal{L}(X, Y)$ and all $n \in \mathbb{N}$.

Remarks: · Carl - for Banach spaces
· Donoho - CS - paper ... $0 < p < 1$

· Rauhut & Foucart .... book
· A. Hinrichs, A. Kolleck & J.V.

· $e_n(T)$ ... entropy numbers

· $c_n(T)$ ... Gelfand numbers: $c_n(T) = \inf_{\substack{M \subset X \\ \operatorname{codim} M < n}} \underbrace{\sup_{\substack{x \in M \\ \|x\|_X \le 1}} \|Tx\|_Y}_{\|T|_M\|}$

codim .... algebraic

Preparations: · If X a p-Banach space, $M \subset\subset X$, then

$$X/M = \{[z] = z + M : z \in X\}; \quad \|[z]\|_{X/M} = \|z + M\|_{X/M}$$
$$= \inf\{\|y\|_X : y \in z + M\}$$
$$= \inf\{\|z - y\|_X : y \in M\}$$

Claim: X/M is a p-BS.

$[x], [y] \in X/M$, $\varepsilon > 0$

Then there are $z^x, z^y \in M$: $\|x - z^x\|_X \le (1+\varepsilon)\|[x]\|_{X/M}$
$\|y - z^y\|_X \le (1+\varepsilon)\|[y]\|_{X/M}$

$$\Rightarrow \|[x]+[y]\|_{X/M}^p = \|[x+y]\|_{X/M}^p \le \|x + y - z^x - z^y\|_X^p \le$$

$$\leq \|x-z^x\|_X^p + \|y-z^y\|_X^p \leq (1+\varepsilon)\{\|[x]\|_{X/M}^p + \|[y]\|_{X/M}^p\}.$$
$$\varepsilon \to 0$$

- (Known fact): If X is real m-dim. p-Banach space then
$$e_n(id: X \to X) \leq 4^{1/p} \cdot 2^{-\frac{n-1}{m}}, \quad n \geq 1.$$

- We show for $n \in \mathbb{N}$: $\boxed{n^\alpha e_n(T) \leq \gamma_\alpha \sup_{1 \leq k \leq n} k^\alpha c_k(T)}$

- Multiply T by a const. $c_k(T) \leq k^{-\alpha}$ .... $\|T\| = c_1(T) \leq 1$.

- <u>Lifting property</u>: Let $M \subset\subset X$ with finite codim, let $\varepsilon > 0$ be fixed.

Let $\mathcal{M} \subset X/M$ be an ε-net of the unit ball of X/M:

$\forall [x] \in B_{X/M}$ there is $[x'] \in \mathcal{M}$ and

$$\varepsilon > \|[x]-[x']\|_{X/M} = \inf_{z \in M} \|x-x'-z\|_X \qquad 0<\varepsilon<1$$

$\forall x \in B_X$ ... $\|[x]\|_{X/M} \leq \|x\|_X^{\leq 1}$ ... $[x] \in B_{X/M}$ $\exists [x'] \in \mathcal{M}$
$\exists z \in M : \|x-x'-z\|_X < \varepsilon$

- $\|[x']\|_{X/M}^p \leq \underbrace{\|[x']-[x]\|_{X/M}^p}_{\varepsilon^p < 1} + \underbrace{\|[x]\|_{X/M}^p}_{\leq 1} \leq \varepsilon^p + 1 < 2.$

$\mathcal{M} \subset 2^{1/p} \cdot B_{X/M}$

- For every $[\omega] \in \mathcal{M}$: $\|[\omega]\|_{X/M} < 2^{1/p}$ ... $\tilde{\omega} \in [\omega]$ ... $\tilde{\omega} - \omega \in M$
  - $\|\tilde{\omega}\|_X < 2^{1/p}$

  ... We collect $\tilde{\omega}$'s in $\mathcal{N} \subset X$. $\mathcal{N} \subset 2^{1/p} \cdot B_X$
  - $\# \mathcal{M} = \# \mathcal{N}$

- $\forall x \in B_X$ ... $[x] \in B_{X/M}$ .... $[\omega] \in \mathcal{M}$ .... <u>$\exists \tilde{\omega} \in \mathcal{N}$</u>, <u>$z \in M$</u>

$$\boxed{\|x-\tilde{\omega}-z\|_X < \varepsilon}$$

• $m_1, m_2, \dots, m_N$ $\quad (m_j = 2^{N-j})$; $\quad M_1, \dots, M_N$ …. codim $M_j = m_j$

$$\|T|_{M_j}\| \le (\cancel{1+}\varepsilon)\, c_{m_{j+1}}(T)$$

Step 1: m_1; $\quad X/M_1$ … $\dim X/M_1 = \operatorname{codim} M_1 = m_1$

$$e_\ell(id: X/M_1 \to X/M_1) \le \overbrace{4^{1/p}\cdot 2^{-\frac{\ell-1}{m_1}}}^{\varepsilon_1}, \quad \ell \ge 1$$

$$k_1 \;\dots\dots\; \mathcal{M}_1 \subset 2^{1/p}\cdot B_{X/M_1} \xrightarrow{\text{lifted}} \mathcal{N}_1 \subset 2^{1/p} B_X$$

$$\forall x \in B_X : \exists x_1 \in \mathcal{N}_1, z_1 \in M_1 : \boxed{\|x-(x_1+z_1)\|_X < \varepsilon_1} = 4^{1/p}\cdot 2^{-\frac{k_1-1}{m_1}}$$

$$k_j = \lceil 2^{N-j}(2/p+\beta)+1 \rceil$$

$$\varepsilon_1 = 4^{1/p}\cdot 2^{-2/p-\beta} = 2^{-\beta}; \quad \|z_1\|_X^p \le \overbrace{\underbrace{\|x-(x_1+z_1)\|_X^p}_{<\varepsilon_1^p \le 1} + \underbrace{\|x\|_X^p}_{\le 1} + \underbrace{\|x_1\|_X}_{\le 2}}^{\le 4}$$

$\boxed{\beta > \alpha}$

$$\|z_1\|_X \le 4^{1/p}$$

Step 2: $\left\| \frac{x-(x_1+z_1)}{\varepsilon_1} \right\|_X \le 1$

$$m_2 = 2^{N-2}, \; M_2, \; id: X/M_2 \to X/M_2 \;\dots\; e_{k_2} \le \varepsilon_2 = \underbrace{4^{1/p}\cdot 2^{-\frac{k_2-1}{m_2}}}_{\le 2^{-\beta}}$$

$$\rightsquigarrow \mathcal{M}_2 \xrightarrow{\text{lifting}} \mathcal{N}_2 \subset 2^{1/p}\cdot B_X \qquad \#\mathcal{N}_2 \le 2^{k_2-1}$$

$$\Rightarrow \left\| \frac{x-(x_1+z_1)}{\varepsilon_1} - (\underset{\in \mathcal{N}_2}{x_2} + \underset{\in M_2}{z_2}) \right\|_X \le \varepsilon_2$$

$$\|x-(x_1+z_1)-\varepsilon_1(x_2+z_2)\|_X \le \varepsilon_1\cdot\varepsilon_2$$

$$\|z_2\|_X^p \le \overbrace{\underbrace{\|u\|_X^p}_{\le 1} + \underbrace{\left\|\frac{x-(x_1+z_1)}{\varepsilon_1}\right\|_X^p}_{\le 1} + \underbrace{\|x_2\|_X^p}_{\le 2}}^{\le 4}$$

Step 3: $m_3 = 2^{N-3}$, M_3, $\mathcal{M}_3$, $\mathcal{N}_3$; k_3, $\varepsilon_3 \le 2^{-\beta}$

$$\left\| \frac{x-(x_1+z_1)}{\varepsilon_1\cdot\varepsilon_2} - \frac{x_2+z_2}{\varepsilon_2} - (\overset{\in\mathcal{N}_3}{x_3}+\overset{\in\mathcal{M}_3}{z_3})\right\|_X \le \varepsilon_3 \quad \dots \|\overset{\in\mathcal{M}_j}{z_j}\|_X \le 4^{1/p}$$

$$\|x-\delta_0(x_1+z_1)-\delta_1(x_2+z_2)-\delta_2(x_3+z_3)\|_X \le \delta_3 \qquad \delta_0=1,\ \delta_j = \prod_{\ell=1}^{j}\varepsilon_\ell \le 2^{-j\beta}$$

up to N

$$m_1 = 2^{N-1}, \dots, m_N = 2^{N-N} = 1\ ;\ M_1, M_2, \dots, M_N$$

$$\forall x\in B_X:\ \Big\| x-\sum_{j=1}^{N}\delta_{j-1}(\overset{\in\mathcal{N}_j}{x_j}+\overset{\in\mathcal{M}_j}{z_j})\Big\|_X \le \delta_N\ ;\ \mathcal{N}_j \subset 2^{1/p}B_X,\ \|z_j\|_X \le 4^{1/p}$$

$$\Big\|Tx - T\Big(\sum_{j=1}^{N}\delta_{j-1}x_j\Big)\Big\|_Y^q = \Big\|T\Big(x-\sum_{j=1}^{N}\delta_{j-1}(x_j+z_j)\Big)+\sum_{j=1}^{N}\delta_{j-1}T(z_j)\Big\|_Y^q$$

$$\le \underbrace{\Big\|T\Big(x-\sum_{j=1}^{N}\delta_{j-1}(x_j+z_j)\Big)\Big\|_Y^q}_{\le \|T\|^q\cdot\|x-\sum_{j=1}^{N}\delta_{j-1}(x_j+z_j)\|_X^q} + \underbrace{\sum_{j=1}^{N}\delta_{j-1}^q\|Tz_j\|_Y^q}_{\le \sum_{j=1}^{N}\delta_{j-1}^q\cdot\|T|_{\mathcal{M}_j}\|^q\cdot\|z_j\|_X^q}$$

$z_j \in \mathcal{M}_j$

$$\underbrace{\|T\|^q\cdot\Big\|x-\sum_{j=1}^{N}\delta_{j-1}(x_j+z_j)\Big\|_X^q}_{\le(\|T\|\cdot\delta_N)^q} \qquad \underbrace{\sum_{j=1}^{N}\delta_{j-1}^q\cdot\|T|_{\mathcal{M}_j}\|^q\cdot\|z_j\|_X^q}_{\le\sum_{j=1}^{N}\delta_{j-1}^q\cdot c_{m_j+1}(T)^q\cdot 4^{q/p}}$$

$$\le (\|T\|\cdot\delta_N)^q + \sum_{j=1}^{N}\delta_{j-1}^q c_{m_j+1}(T)^q\cdot 4^{q/p}$$

$c_k(T)\le k^{-\alpha}$

$m_j = 2^{N-j}$

$$\le 2^{-N\beta q} + \sum_{j=1}^{N}2^{-(j-1)\beta q}\cdot\underbrace{(m_j+1)^{-\alpha q}}_{\le 2^{-(N-j)\alpha q}}\cdot 4^{q/p}$$

$$\le 2^{-N\beta q} + \sum_{j=1}^{N}2^{-jq(\beta-\alpha)}\cdot 2^{\beta q}\cdot 4^{q/p}\cdot 2^{-N\alpha q}$$

$$\le \underbrace{2^{-N\beta q}}_{\le 2^{-N\alpha q}} + 2^{\beta q}\cdot 4^{q/p}\cdot 2^{-N\alpha q}\underbrace{\sum_{j=1}^{N}2^{-jq\overbrace{(\beta-\alpha)}^{>0}}}_{C_{\alpha,\beta}}$$

$$\le C'_{\alpha,\beta,p,q}\cdot 2^{-N\alpha q}$$

$$\Rightarrow \mathcal{N} = \left\{ T\left(\sum_{k=1}^{N} \sigma_{k-1} x_k\right) : x_k \in \mathcal{N}_k \right\}$$

$$\#\mathcal{N} = \#\mathcal{N}_1 \cdot \ldots \cdot \#\mathcal{N}_N \leq 2^{k_1 - 1} \cdot \ldots \cdot 2^{k_N - 1}$$

$$\sum_{j=1}^{N} (k_j - 1) \cong \sum_{j=1}^{N} 2^{N-j(2/p+\beta)} \leq C \cdot 2^N$$

$$\boxed{n \simeq C \cdot 2^N}$$

$$e_n(T)^q \lesssim 2^{-N\alpha q} \quad \ldots \quad e_n(T) \lesssim 2^{-N\alpha} \lesssim n^{-\alpha}$$

Rest ... follow by monotonicity .... ∎