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Modern Approximation Theory

Summer term 2022

Exercises: 1. Sheet

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1. Let Y be a p-Banach space. Show that
$$aB_Y + bB_Y \subset (a^p + b^p)^{1/p} B_Y.$$
2. Show that $e_n(T) \to 0$ if, and only if, T is compact.
3. Let $0 < p < \infty$. Show that
$$\Gamma(1 + x/p)^{1/x} \sim x^{1/p}, \quad x > 1,$$
where the constant(s) may depend on p.
4. Let $0 < p, q \leq \infty$ and $n \in \mathbb{N}$. Show that
$$\|id : \ell_p^n(\mathbb{R}) \to \ell_p^n(\mathbb{R})\| = \max(1, n^{1/q-1/p}).$$
5. Let $0 < p < q < \infty$. Then $\|x\|_q \leq \|x\|_p^{p/q} \cdot \|x\|_\infty^{1-p/q}$ for all $x \in \mathbb{R}^n$ (or $x \in \ell_p$).
6. Let $1 \leq k \leq n$. Show that
$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k.$$
7. Let $0 < p_0 \leq p \leq p_1 \leq \infty$ and $0 \leq \theta \leq 1$ with
$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$
Show that
$$\|x\|_p \leq \|x\|_{p_0}^{1-\theta} \cdot \|x\|_{p_1}^{\theta}.$$
8. Find two quasi-Banach spaces X and Y and an operator $T \in \mathcal{L}(X, Y)$, such that $\|T\| > e_1(T)$. Can one take one of the spaces to be a Banach space?