Jan Vybiral

Modern Approximation Theory

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Exercises: 2. Sheet

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1. Let X, Y be two (quasi-)Banach spaces. Show that if Y is a p-Banach space for some $0 < p \le 1$, then also $\mathcal{L}(X,Y)$ is a p-Banach space. (Prove only the p-triangle inequality, not the completeness).

2. Let X be a Banach space and let $T \in \mathcal{L}(X,X)$ be a linear operator with $\|T\|_{\mathcal{L}(X,X)} < 1$. Show that $(id - T) \in \mathcal{L}(X,X)$ is invertible and that $(id-T)^{-1}$ is given by the Neumann series

$$\sum_{n=0}^{\infty} T^n = id + T + T^2 + \dots.$$

   Does the same hold if X is a quasi-Banach space? Or a p-Banach space?

3. Let X, Y be two quasi-Banach spaces. Show that

   a) If $N_1, N_2 \subset\subset X$ with $\dim N_1 < n_1$ and $\dim N_2 < n_2$, then $\dim N_1 + N_2 < m + n - 1$.

   b) If $M_1, M_2 \subset\subset X$ with $\operatorname{codim} M_1 < n_1$ and $\operatorname{codim} M_2 < n_2$, then $\operatorname{codim} M_1 \cap M_2 < m + n - 1$.

   c) Let $A \in \mathcal{L}(X,Y)$ with $\operatorname{rank} A < n$. We define $M = \operatorname{kern} A = \{x \in X : Ax = 0\}$. Show that $\operatorname{codim} M < n$.

   d) Let $T \in \mathcal{L}(X,Y)$ and $N \subset\subset X$ be a subspace with $\dim N < n$. Show that, $\dim T(N) < n$.

   e) Let $T \in \mathcal{L}(X,Y)$ and $M \subset\subset Y$ be a subspace with $\operatorname{codim} M < n$. Show that $\operatorname{codim} T^{-1}(M) < n$.

   f) Let X be a Banach space and $\alpha, \alpha_1, \dots, \alpha_J \in X'$. Show that

$$\alpha \in Lin(\alpha_1, \dots, \alpha_J) \Leftrightarrow \bigcap_{j=1}^{J} \operatorname{kern} \alpha_j \subset \operatorname{kern} \alpha.$$

4. Let

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt, \quad s > 0.$$

   Show that $\Gamma(s+1) = s\Gamma(s)$ and

$$B(\alpha, \beta) = \int_0^1 (1-t)^{\alpha-1} t^{\beta-1} dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \qquad \alpha > 0, \quad \beta > 0.$$

   Hint:

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty \int_0^\infty t^{\alpha-1} s^{\beta-1} e^{-t-s} \, dt \, ds$$

   and put $s = ur, t = u(1-r)$.

5. Let $0 < p < \infty$ and

$$A_m = \{(t_1, \dots, t_m) \in \mathbb{R}^m : t_j \geq 0 \text{ for } j = 1, \dots, m \quad \text{and} \quad t_1 + \dots + t_m \leq 1\}.$$

Show that

$$\int_{A_m} \prod_{j=1}^{m} t_j^{1/p-1} dt = p^m \frac{[\Gamma(1/p+1)]^m}{\Gamma(m/p+1)}.$$

Hint: Induction. Use the formula

$$\int_{A_m} \prod_{j=1}^{m} t_j^{1/p-1} dt = \int_0^1 t_m^{1/p-1} \int_{A_{m-1}} \prod_{j=1}^{m-1} [(1-t_m)s_j]^{1/p-1} (1-t_m)^{m-1} ds dt_m,$$

where $s_j = \frac{t_j}{1-t_m}$, $j = 1, \dots, m-1$.

6. (Dirichlet, 1839) Use the previous exercise to calculate $\mathrm{vol}(B_{\ell_p^m(\mathbb{R})})$.

7. Prove the Riesz's Lemma for Banach spaces: Let X be a normed space and let Y be a closed proper subspace of X. Let $0 < \alpha < 1$. Then there exists $x \in X$ with $\|x\|_X = 1$ and $\mathrm{dist}(x, Y) = \inf\{\|x - y\|_X : y \in Y\} \geq \alpha$.
Is the lemma true also for quasi-Banach spaces?