Jan Vybiral

Modern Approximation Theory

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Exercises: 3. Sheet

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1. Show that the space

$$c_{00} = \{\{\lambda_j\}_{j=1}^\infty \subset \mathbb{R} \ : \ \exists J \in \mathbb{N}\ \forall j > J : \lambda_j = 0\}, \qquad ||\{\lambda_j\}_{j=1}^\infty||_{c_{00}} = \max_{j\in\mathbb{N}} |\lambda_j|,$$

is not complete.

2. Find a Banach space X and $f \in X'$, such that

$$\|f\|_{X'} > |f(x)|$$

for all $x \in X$ with $\|x\|_X = 1$.

3. Show that $(\ell_1)' = \ell_\infty$ in the sense that
a) for all $\lambda = \{\lambda_j\}_{j=1}^\infty \in \ell_\infty$ is f_λ defined as

$$f_\lambda(\{\gamma_j\}_{j=1}^\infty) = \sum_{j=1}^\infty \lambda_j \gamma_j$$

an element of $(\ell_1)'$;
b) every element $f \in (\ell_1)'$ can be represented as f_λ for some $\lambda \in \ell_\infty$.

4. Let X be a vector space and $M \subset X$. Show that

$\mathrm{conv} M = \{x \in X : \exists n \in \mathbb{N}\ \exists x_1, \dots, x_n \in M\ \exists \lambda_1, \dots, \lambda_n \in [0, 1] \ : \ \lambda_1 + \dots + \lambda_n = 1 \text{ and } x = \lambda_1 x_1 + \dots + \lambda_n x_n\}.$

5. a) Let $a, b \in \mathbb{N}$ with $a \le b$. Show that

$$\sum_{j=a}^b \binom{j}{a} = \binom{b+1}{a+1}.$$

b) Show that $(1 \le m \le k)$:

$$\#\{(t_1, \dots, t_m) \in \mathbb{N}^m : t_1 + \dots + t_m \le k\} = \binom{k}{m}.$$

c) Prove that

$$\#\{(t_1, \dots, t_m) \in \mathbb{Z}^m : |t_1| + \dots + |t_m| \le k\} = \sum_{j=0}^m 2^{m-j} \binom{m}{j} \binom{k}{m-j}.$$

d)* Try to use the last formula directly in the proof of the estimate of $e_n(id : \ell_1^m \to \ell_\infty^m)$.

6. Show that ($0 < p < 1$):
a)* $(L_p([0,1]))' = \{0\}$. That means that the only continuous linear mapping $F : L_p([0,1]) \to \mathbb{R}$ is the trivial $F(f) = 0$ for all $f \in L_p([0,1])$.
b) $(\ell_p)' = \ell_\infty$, $(\ell_p^m)' = \ell_\infty^m$.
c)* $\mathrm{conv} B_{\ell_p^m} = B_{\ell_1^m}$; is it true that also $\mathrm{conv} B_{\ell_p} = B_{\ell_1}$?
d) Is the constant $2^{1/p-1}$ optimal in the triangle inequality for $L_p([0,1])$?

7. **Banach limit**

Consider the function

$$p(x) = \limsup \frac{1}{n} \sum_{k=1}^{n} x_k$$

for $x = \{x_k\} \in \ell_\infty$. Show that:

(a) $p(x) = \lim\limits_{k\to\infty} x_k$ for $x \in c = \{\{x_n\}_{n=1}^\infty : \lim_{n\to\infty} x_n \text{ exists}\}$, $\|x\|_c = \|x\|_\infty$;

(b) p is a sublinear functional;

(c) There exists an extension $\mathrm{Lim} \in (\ell_\infty)'$ of the linear and continuous functional $\lim \in c'$ with $\mathrm{Lim}(x) \le p(x)$ (Hahn-Banach!)

(d) $\liminf x_k \le \mathrm{Lim}(x) \le \limsup x_k$ for $x = \{x_k\} \in \ell_\infty$. Especially, it holds $\mathrm{Lim}(x) \ge 0$ if $x_k \ge 0$ for all $k \in \mathbb{N}$;

(e) It holds that $\mathrm{Lim}(Sx) = \mathrm{Lim}(x)$ for $x = \{x_k\} \in \ell_\infty$, where S is the shift operator $Sx = (x_2, x_3, \dots)$;

(f) Lim is not multiplicative, i.e. there are $x, y \in \ell_\infty$ with $\mathrm{Lim}(x \cdot y) \ne \mathrm{Lim}(x)\mathrm{Lim}(y)$;

(g) Show that $g \notin \ell_1$, therefore $(\ell_\infty)' \ne \ell_1$.

The functional Lim is called the Banach limit on ℓ_∞;

(h) Show that $(\ell_\infty^m)' = \ell_1^m$.