Jan Vybiral

Modern Approximation Theory

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Exercises: 4. Sheet

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1. Let N be a positive integer and consider the set of N-th roots of unity in $\mathbb{C}$:
$$\mathbb{Z}_N = \{1, e^{2\pi i/N}, e^{2\cdot 2\pi i/N} \dots, e^{(N-1)\cdot 2\pi i/N}\}.$$
Show that
   - $\mathbb{Z}_N$ is an Abelian group (with the usual multiplication).
   - $\mathbb{Z}_N$ is isomorph with the set $\{0, 1, \dots, N-1\}$ with summation modulo N.
   - The vectors $(e_l)_{l=0}^{N-1}$ defined by
   $$e_l(k) = e^{2\pi i lk/N} \quad \text{for } l = 0, 1, \dots, N-1 \text{ and } k = 0, 1, \dots, N-1.$$
   are orthogonal with respect to the usual inner product.

2. *Discrete Fourier transform (DFT)* $\mathcal{F}x$ of a vector $x \in \mathbb{C}^N$ is defined as
$$\mathcal{F}x := \mathbb{F}_N x,$$
where $\mathbb{F}_N$ is the N-dimensional *Fourier matrix*
$$\mathbb{F}_N = \left(e^{-2\pi i k\ell/N}\right)_{k,\ell=0,\dots,N-1}.$$

3. *Cyclic convulotion* $x * y \in \mathbb{C}^N$ of signals $x, y \in \mathbb{C}^N$ over $\mathbb{Z}_N$ is defined as
$$(x * y)_k := \sum_{\ell=0}^{N-1} x_{(k-\ell) \bmod N} y_\ell, \qquad k \in \mathbb{Z}_N = \{0, \dots, N-1\}.$$

4. Show that it holds
$$\mathcal{F}(x * y) = (\mathcal{F}x) \cdot (\mathcal{F}y),$$
where "$\cdot$" is the entry-wise product of two vectors.

5. Derive the formula for the DFT of shift and modulation of a given vector, i.e.
$$\mathcal{F}(\{x_n e^{2\pi i \frac{nm}{N}}\}) \quad \text{a} \quad \mathcal{F}(\{x_{n-m}\}).$$

6. Show that
$$\left(\frac{1}{\sqrt{N}}\mathbb{F}_N\right)^4 = Id,$$

7. Prove the following Lemma:
If $x \in \mathbb{C}^N$ is an s-sparse vector such that s of its consecutive Fourier coefficients vanish, then $x = 0$.
Hint: Rewrite the exercise as a Vandermond matrix.

8. Implement the Prony method and test its stability and robustness (i.e. test the defects of sparsity and noise).