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Modern Approximation Theory

Summer term 2022

Exercises: 4. Sheet

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1. Let $t > 0$ be a real number and let $d \geq 2$ be an integer. Show that there exists $\mathcal{M} \subset \mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : \|x\|_2 = 1\}$ such that

   (i) $\#\mathcal{M} \leq (1 + 2/t)^d$,

   (ii) $\forall z \in \mathbb{S}^{d-1}\ \exists\, x \in \mathcal{M} : \|x - z\|_2 \leq t$.

   Hint: Take $x^1 \in \mathbb{S}^{d-1}$ arbitrarily. If $x^1, \dots, x^j \in \mathbb{S}^{d-1}$ were already chosen, take $x^{j+1} \in \mathbb{S}^{d-1}$ arbitrarily such that $\|x^{j+1} - x^l\|_2 > t$ for all $l = 1, \dots, j$. Repeat this, as long as it goes. Then use that $B(x^i, t/2)$ are all disjoint and included in $B(0, 1 + t/2)$ and compare the volumes.

2. Let $\omega \sim \mathcal{N}(0,1)$ be a standard Gaussian variable. Show that

$$\mathbb{E}e^{\lambda\omega^2} = \frac{1}{\sqrt{1-2\lambda}}, \quad -\infty < \lambda < 1/2.$$

3. Let $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m$ and let $\omega_1, \dots, \omega_m \sim \mathcal{N}(0,1)$ be independent. Show that

$$\langle \lambda, \omega \rangle = \lambda_1\omega_1 + \dots + \lambda_m\omega_m \sim \|\lambda\|_2 \cdot \mathcal{N}(0,1) = \mathcal{N}(0, \|\lambda\|_2^2).$$

4. Prove the Lemma of Johnson and Lindestrauss:
   Let $0 < \varepsilon < 1$ and let m, N and d be positive integers with

$$m \geq 4\Big(\varepsilon^2/2 - \varepsilon^3/3\Big)^{-1} \ln N.$$

   Then for every set $\{x^1, \dots, x^N\} \subset \mathbb{R}^d$ there is a mapping $f : \mathbb{R}^d \to \mathbb{R}^m$ with

$$(1-\varepsilon)\|x^i - x^j\|_2^2 \leq \|f(x^i) - f(x^j)\|_2^2 \leq (1+\varepsilon)\|x^i - x^j\|_2^2, \quad i, j \in \{1, \dots, N\}.$$

   Hint: Choose $f(x) = Ax$, where A is a Gaussian matrix. Then use the concentration inequality for one Az with $z = (x^i - x^j)/\|x^i - x^j\|_2$ and the union bound.

5. Finish the proof of the RIP for Gaussian matrices, i.e., show that if $C' > 0$ is given then there exists $C > 0$ such that if

$$m \geq C\delta^{-2}\Big(s\ln\Big(\frac{eN}{s}\Big) + \ln\Big(\frac{2}{\varepsilon}\Big)\Big),$$

   then

$$\binom{N}{s} \cdot 9^s \cdot 2\exp(-C'm\delta^2) < 1.$$