General concepts of Approximation theory

We have an unknown object (function, matrix, curve, ...) and we have only a limited information about this object ... few function values, few Fourier coefficients

We would like to recover this object as good as possible

- limited information
- quick algorithm
- small memory used
- small error

Example: • $f \in L_2([0,1])$, periodic

- We are allowed to choose $x_1, \dots, x_n$ and rec. algorithm $\phi(f(x_1), \dots, f(x_n))$
- $\|f - \phi(f(x_1), \dots, f(x_n))\|_2$ small

$$(*) \quad \inf_{x_1, \dots, x_n} \inf_{\phi: \mathbb{R}^n \to L_2} \sup_{\|f\|_2 \le 1} \|f - \phi(f(x_1), \dots, f(x_n))\|_2 \quad \dots = 1$$

... sub-class! $\|f\|_X \le 1$

Many different concepts are possible:

- $M_{(x_1,\dots,x_m)} = \{f \in X : f(x_1) = \dots = f(x_m) = 0\}$ ... codim $M \leq m$

$$(*) \geq \inf_{\substack{x_1,\dots,x_m \\ \phi}} \sup_{\substack{\|f\|_X \leq 1 \\ f \in M}} \|f - \phi(0,\dots,0)\|_2 \geq \inf_{\substack{x_1,\dots,x_m \\ \phi}} \sup_{\substack{\|f\|_X \leq 1 \\ f \in M}} \|f\|_2$$

$$\|f\|_2 \leq \left\| \frac{f}{2} + \frac{f}{2} - \frac{\phi(0)}{2} + \frac{\phi(0)}{2} \right\|_2 \leq \frac{\|f - \phi(0)\|_2}{2} + \frac{\|\phi(0) - (-f)\|_2}{2}$$

$$\leq \sup_{\substack{\|f\|_X \leq 1 \\ f \in M}} \|f - \phi(0)\|_2$$

- ϕ is linear ... linear algorithms
- approximation from a given subspace (Galerkin methods).

2, General theory of s-numbers

There is more ways, how to measure compactness of an operator and how to approximate it.

The general theory goes under the name of s-numbers.

Definition: A rule $s: T \to \{s_n(T)\}_{n=1}^{\infty}$, which associates to every operator between ~~the~~ two (quasi-)Banach spaces a scalar sequence, is said to be an s-scale if the following holds:

i, $\|T|\mathcal{L}(X,Y)\| = s_1(T) \geq s_2(T) \geq \ldots \geq 0$ for every $T \in \mathcal{L}(X,Y)$,

ii, $s_{m+n-1}(S+T) \leq s_m(T) + s_n(S)$ for all $S, T \in \mathcal{L}(X,Y)$,

iii, $s_n(T_1 T T_0) \leq \|T_1\| \cdot s_n(T) \cdot \|T_0\|$ for all $T_0 \in \mathcal{L}(X_0, X)$, $T \in \mathcal{L}(X,Y)$, $T_1 \in \mathcal{L}(Y, Y_1)$,

iv, if rank $T < n$, then $s_n(T) = 0$,

v, $s_n(id: \ell_2^n \to \ell_2^n) = 1$.

Furthermore $s_n(T)$ is called the n-th s-number of T.

Definition: Let X, Y be (quasi-)Banach spaces and let $T \in \mathcal{L}(X,Y)$.

Then $a_n(T) := \inf\{\|T - A\|_{\mathcal{L}(X,Y)} : A \in \mathcal{L}(X,Y), \text{rank } A < n\}$, $n \in \mathbb{N}$

are called the approximation numbers of T.

Theorem: The approximation numbers form an s-function

Proof: i, is immediate ... $n=1$, $A=0$

ii, Let $\varepsilon>0$ and let $A, B \in \mathcal{L}(X,Y)$:

$$\|S-A \mid \mathcal{L}(X,Y)\| \le (1+\varepsilon) a_m(S)$$

$$\|T-B \mid \mathcal{L}(X,Y)\| \le (1+\varepsilon) a_n(T)$$

Then $\|(S+T)-(A+B)\|_{\mathcal{L}(X,Y)} \le \|S-A\|_{\mathcal{L}(X,Y)} + \|T-B\|_{\mathcal{L}(X,Y)} \le (1+\varepsilon)\{a_m(S)+a_n(T)\}$.

& $\operatorname{rank}(A+B) \le \operatorname{rank}(A) + \operatorname{rank}(B) \le m-1+n-1$.

iii, Again, $\varepsilon>0$, $A \in \mathcal{L}(X,Y)$, $B \in \mathcal{L}(Y,Z)$: $T \in \mathcal{L}(X,Y)$, $R \in \mathcal{L}(Y,Z)$

$$\|T-A\|_{\mathcal{L}(X,Y)} \le (1+\varepsilon) a_n(T), \quad \|R-B\|_{\mathcal{L}(Y,Z)} \le (1+\varepsilon) a_m(R)$$

$$\|R\circ T - (R\circ A - B\circ A + B\circ T)\|_{\mathcal{L}(X,Z)} = \|(R-B)\circ(T-A)\|_{\mathcal{L}(X,Z)}$$

$$\le \|R-B\|_{\mathcal{L}(Y,Z)} \cdot \|T-A\|_{\mathcal{L}(X,Y)} \le (1+\varepsilon)^2 a_n(T) a_m(R)$$

& $\operatorname{rank}[(R-B)\circ A + B\circ T] \le \operatorname{rank} A + \operatorname{rank} B \le n-1+m-1$

... We proved more than iii; : $a_{m+n-1}(R\circ T) \le a_m(R) a_n(T)$.

iv, simple; v, follows from the following Lemma. ☒

Lemma: Let X be a (quasi-) Banach space with $\dim(X) \ge n$.

Then $a_n(id: X \to X) = 1$.

Proof: Let $a_n(id: X \to X) < 1$ ... Hence there is $A \in \mathcal{L}(X,X)$ with

$$\|id - A\|_{\mathcal{L}(X,X)} < 1 \ \& \ \operatorname{rank} A < n.$$

Then the Neumann series of $A = id - (id - A)$ converges and shows that A must be invertible. Hence $\dim X = \operatorname{rank} A < n$... ☒

Theorem: The approximation numbers yield the largest s-scale.

Proof: Let $T \in \mathcal{L}(X,Y)$, $n \in \mathbb{N}$. Then, for every $\varepsilon > 0$, there is $A \in \mathcal{L}(X,Y)$:

$$\|T-A\|_{\mathcal{L}(X,Y)} \leq (1+\varepsilon)\, a_n(T), \quad \operatorname{rank}(A) < n.$$

Then for every s-function: $s_n(T) = s_n(T-A+A) \leq \underbrace{s_n(A)}_{=0} + \|T-A\|_{\mathcal{L}(X,Y)}$

$$\leq \|T-A\|_{\mathcal{L}(X,Y)} \leq (1+\varepsilon)\, a_n(T). \quad \boxtimes$$

Theorem: Let $0 < p \leq \infty$, let $\sigma = (\sigma_1, \sigma_2, \dots)$ with $\sigma_1 \geq \sigma_2 \geq \dots \geq 0$.

Then $D_\sigma : \ell_p \to \ell_p$ is again the diagonal operator

$$D_\sigma x = (\sigma_1 x_1, \sigma_2 x_2, \dots).$$

Then $a_n(D_\sigma) = \sigma_n$, $n \in \mathbb{N}$.

Proof: 1, estimate from above: $n \in \mathbb{N}$ fixed, $D_\sigma^{n-1} x = (\sigma_1 x_1, \dots, \sigma_{n-1} x_{n-1}, 0, \dots)$

$$\Rightarrow a_n(D_\sigma) \leq \|D_\sigma - D_\sigma^{n-1}\|_{\mathcal{L}(\ell_p,\ell_p)} = \sigma_n.$$

2, estimate from below:

denote
$$D_\sigma^{(n)} : \ell_p^n \to \ell_p^n : \ D_\sigma^{(n)} x = (\sigma_1 x_1, \dots, \sigma_{n-1} x_{n-1}, \sigma_n x_n)$$

$$I_p^n : \ell_p^n \to \ell_p^n : \ I_p^n(x) = x$$

$$J_n : \ell_p^n \to \ell_p : \ J_n(x) = (x_1, \dots, x_n, 0, 0, \dots)$$

$$P_n : \ell_p \to \ell_p^n : \ P_n(x) = (x_1, \dots, x_n).$$

For $\sigma_n \neq 0$: $1 = a_n(I_p^n) = a_n\left(\left(D_\sigma^{(n)}\right)^{-1} \circ D_\sigma^{(n)}\right) \leq \|D_\sigma^{(n)}\|^{-1}_{\mathcal{L}(\ell_p^n,\ell_p^n)} \cdot a_n(D_\sigma^{(n)})$

$$= \sigma_n^{-1} \cdot a_n(D_\sigma^{(n)}) = \sigma_n^{-1} \cdot a_n(P_n \circ D_\sigma \circ J_n) \leq \sigma_n^{-1} \cdot \|P_n\| \cdot \|J_n\| \cdot a_n(D_\sigma) =$$

$$= \sigma_n^{-1} \cdot a_n(D_\sigma). \quad \boxtimes$$

Gelfand and Kolmogorov numbers

Definition: Let X, Y be two (quasi-)Banach spaces, let $T \in \mathcal{L}(X,Y)$.

i, The n-th Gelfand number of T is defined as

$$c_n(T) = \inf\{\|T \circ J_M^X\|_{\mathcal{L}(M,Y)} : M \subset\subset X, \operatorname{codim} M < n\},$$

where $J_M^X : M \to X$ is the canonical embedding of M into X.

ii, The n-th Kolmogorov number of T is defined as

$$d_n(T) = \inf\{\|Q_N^Y \circ T\|_{\mathcal{L}(X, Y/N)} : N \subset Y, \dim N < n\},$$

where $Q_N^Y : Y \to Y/N$ is the quotient map.

Remarks: i,
$$c_n(T) = \inf_{\substack{M \subset\subset X \\ \operatorname{codim} M < n}} \sup_{\substack{x \in M \\ \|x\|_X \le 1}} \|Tx\|_Y$$

... we are looking for a large ($\operatorname{codim} M < n$) subspace of X, on which T is small.

ii, • Y/N ... quotient space ... space of cosets $\bar{y} = \{y - z : z \in N\}$

$$\|\bar{y}\|_{Y/N} = \inf\{\|y - z\|_Y : z \in N\}. \quad \text{... (quasi-)norm}$$

• $Q_N^Y(y) = \bar{y}$

iii, We can rewrite also $d_n(T)$:

$$d_n(T) = \inf_{\substack{N \subset\subset Y \\ \dim N < n}} \|Q_N^Y \circ T\|_{\mathcal{L}(X, Y/N)} = \inf_{\substack{N \subset\subset Y \\ \dim N < n}} \sup_{\substack{x \in X \\ \|x\|_X \le 1}} \|Q_N^Y(Tx)\|_{Y/N}$$

$$= \inf_{\substack{N \subset\subset Y \\ \dim N < n}} \sup_{\substack{x \in X \\ \|x\|_X \le 1}} \|\overline{(Tx)}\|_{Y/N} = \inf_{\substack{N \subset\subset Y \\ \dim N < n}} \sup_{\substack{x \in X \\ \|x\|_X \le 1}} \inf_{z \in N} \|Tx - z\|_Y .$$

- Approximation numbers: Approximate Tx for every $x \in X$ in a linear way – by Ax, A is linear
- Kolmogorov numbers: Approximate Tx from an n-dim. subspace, possibly in a non-linear way.
- Gelfand numbers: Non-linear approximation based on n pieces of linear information.

Proposition: Let X, Y be two (quasi-)Banach spaces, let $T \in \mathcal{L}(X, Y)$. Then

$$c_n(T) \le a_n(T), \quad d_n(T) \le a_n(T) \quad \text{for all } n \in \mathbb{N}.$$

Proof: 1, $d_n(T) \le a_n(T)$ ... put $N = \cancel{\operatorname{rank}(A)}\ A(X) = \operatorname{range}(A)$
$\dim N = \operatorname{rank} A < n$ and

$$d_n(T) \le \sup_{x \in B_X} \inf_{y \in N} \|Tx - y\|_Y \le \sup_{x \in B_X} \|Tx - Ax\|_Y = \|T - A\|_{\mathcal{L}(X,Y)} \le (1+\varepsilon) a_n(T).$$

2, $c_n(T) \le a_n(T)$ ... $M = \ker A = \{x \in X : Ax = 0\}$. Then [codim $M < n$ ... Exercises]

$$c_n(T) \le \sup_{x \in \ker A, \|x\|_X \le 1} \|Tx\|_Y = \sup_x \|Tx - Ax\|_Y \le \|T - A\|_{\mathcal{L}(X,Y)} \le (1+\varepsilon) a_n(T).$$

Lemma: Let X be a (quasi-)Banach space with $\dim(X) \geq n$.

Then $c_n(id: X \to X) = d_n(id: X \to X) = 1$.

Proof: Let $n \in \mathbb{N}$, let X be a space with $\dim X \geq n$.

1, c_n: Let $M \subset\subset X$ have $\operatorname{codim}(M) < n$ ... M is non-empty. Then

$$\sup_{\substack{x \in M \\ \|x\|_X \leq 1}} \|id(x)\|_X = 1 \text{ ... holds for every } M: \; c_n(id) \geq \inf_M \ldots \geq 1.$$

2, d_n: Let $\varepsilon > 0$, $N \subset\subset X$ with $\dim N < n$. Then $N \neq X$ and

(Riesz's Lemma) there exists $x_{N,\varepsilon} \in X$ with

$\|x_{N,\varepsilon}\|_X = 1$ and $\|x_{N,\varepsilon} - y\|_X \geq \frac{1}{1+\varepsilon}$ for all $y \in N$.

Hence $$d_n(T) \geq \inf_{\substack{N \subset X \\ \dim N}} \inf_{y \in N} \|id(x_{N,\varepsilon}) - y\|_X \geq \frac{1}{1+\varepsilon}.$$

Theorem: Gelfand numbers and Kolmogorov numbers form an s-scale.

Proof: 1, Gelfand numbers:

i, if $n = 1$, $\operatorname{codim} M < 1 \Rightarrow M = X$

and $c_1(T) = \sup_{x \in B_X} \|Tx\|_Y = \|T\|_{\mathcal{L}(X,Y)}$.

monotonicity $(c_j(T) \geq c_{j+1}(T))$ is clear.

ii, $\varepsilon > 0$: $\exists M_1, M_2 \subset\subset X$: $\operatorname{codim} M_1 < n$, $\operatorname{codim} M_2 < n$

$$\|Sx\|_Y \leq (1+\varepsilon) c_n(S) \|x\|_X, \; x \in M_1, \quad \|Tx\|_Y \leq (1+\varepsilon) c_n(T) \|x\|_X, \; x \in M_2$$

$x \in M_1 \cap M_2$: $\operatorname{codim}(M_1 \cap M_2) \leq m-1+n-1$

& $\|(S+T)x\|_Y \leq \|Sx\|_Y + \|Tx\|_Y \leq (1+\varepsilon)\|x\|_X \left(c_m(S)+c_n(T)\right)$.

iii, Similarly : $M_1 \subset\subset X$, $M_2 \subset\subset Y$, $\operatorname{codim} M_1 < n$, $\operatorname{codim} M_2 < m$: $\|Tx\|_Y \leq (1+\varepsilon) c_n(T) \|x\|_X$, $x \in M_1$,
$\|Ry\|_Z \leq (1+\varepsilon) c_m(R) \|y\|_Y$, $y \in M_2$.

Then for all x's with $x \in M_1$ & $Tx \in M_2$: ... $x \in M_1 \cap T^{-1}(M_2)$

$$\|R(Tx)\|_Z \leq (1+\varepsilon) c_m(R) \|Tx\|_Y \leq (1+\varepsilon)^2 c_m(R) c_n(T) \|x\|_X$$

... $\operatorname{codim}\left[M_1 \cap T^{-1}(M_2)\right] < m+n-1$.

iv, if $\operatorname{rank} T < n$, then $\operatorname{codim} \underbrace{\ker T}_{M} < n$ a $T|_M \equiv 0$ ... $c_n(T) = 0$.

v, follows from the Lemma before.

2, Kolmogorov numbers

i, $n = 1$: $\dim N = 0$ ... $N = \{0\} \subset\subset Y$, hence $d_1(T) = \sup_{x \in B_X} \|Tx - 0\|_Y = \|T\|_{\mathcal{L}(X,Y)}$

monotonicity ($d_j(T) \geq d_{j+1}(T)$) is clear

ii, $N_1, N_2 \subset\subset Y$: $\dim N_1 < m$, $\dim N_2 < n$
$\varepsilon > 0$ $\quad \forall x \in X$: $\exists y_1 \in N_1$: $\|Sx - y_1\|_Y \leq (1+\varepsilon) d_m(S) \|x\|_X$
$\exists y_2 \in N_2$: $\|Tx - y_2\|_Y \leq (1+\varepsilon) d_n(T) \|x\|_X$

... $\|(S+T)x - (y_1+y_2)\|_Y \leq \|Sx - y_1\|_Y + \|Tx - y_2\|_Y \leq (1+\varepsilon)\|x\|_X \left(d_m(S)+d_n(T)\right)$

iii, follows similarly:

$\varepsilon > 0$, $N_1 \subset\subset Y$, $\dim N_1 < n$, $\dim N_2 < m$
$N_2 \subset\subset Z$

$$\forall x \in X: \exists \bar{y} \in N_1: \|Tx - \bar{y}\|_Y \leq (1+\varepsilon) d_n(T) \cdot \|x\|_X,$$
$$\forall y \in Y: \exists z \in N_2: \|Ry - z\|_Z \leq (1+\varepsilon) d_m(R) \cdot \|y\|_Y$$

... Take $x \in X$ ... find $\bar{y}$ to Tx and then z to $R(Tx - \bar{y})$

Then
$$\|R(Tx) - R(\bar{y}) - z\|_Z = \|R(Tx - \bar{y}) - z\|_Z \leq (1+\varepsilon) d_m(R) \|Tx - \bar{y}\|_Y$$
$$\leq (1+\varepsilon)^2 d_m(R) d_n(T) \cdot \|x\|_X.$$

Observe that $R(\bar{y}) + z \in R(N_1) + N_2$ .... subspace with dim $\leq (m-1)+(n-1)$.

iv, For $\operatorname{rank}(T) < n$, take $N = T(X)$, $\dim N < n$ and $y = Tx$ ... $d_n(T) = 0$.

v, from Lemma before ☒

Theorem: Let X, Y be Banach spaces and let $T \in \mathcal{L}(X, Y)$. Then T is compact if and only if $c_n(T) \to 0$ and if and only if $d_n(T) \to 0$.

Proof: 1, Gelfand numbers, T compact

- $\forall \varepsilon > 0$: $T(B_X) \subset \bigcup_{j=1}^{J} (y_j + \varepsilon B_Y)$ for some J and $y_1, \dots, y_J \in Y$
- By Hahn-Banach theorem, there are $\beta_j \in Y'$ with
$$|\beta_j(y_j)| = \|y_j\|_Y \text{ and } \|\beta_j\|_{Y'} = 1, \quad j = 1, \dots, J$$
- Define $\alpha_j \in X'$ by $\alpha_j(x) = \beta_j(T(x))$, $j = 1, \dots, J$; $M := \{x \in X : \alpha_j(x) = 0 \text{ for all } j\text{'s}\}$
- Let $x \in M$ with $\|x\|_X \leq 1$ and $Tx \in y_j + \varepsilon B_Y$ for some j:

$$\|Tx\|_Y \leq \|Tx - y_j\|_Y + \|y_j\|_Y \leq \varepsilon + |\beta_j(y_j)| \leq \varepsilon + |\beta_j(Tx - y_j)| + \overbrace{|\beta_j(Tx)|}^{=0}$$
$$\leq \varepsilon + \|\beta_j\|_{Y'} \cdot \|Tx - y_j\|_Y \leq 2\varepsilon.$$

Step 2, $c_n \to 0$

- For every $\varepsilon > 0$, there is a $M \subset\subset X$ with $\operatorname{codim} M = J < +\infty$ such that $x \in M \Rightarrow \|Tx\|_Y \leq \varepsilon \|x\|_X$

- We can write M as $M = \bigcap_{j=1}^{J} \ker \alpha_j = \{x \in X : \alpha_1(x) = \dots = \alpha_J(x) = 0\}$ for some $\alpha_1, \dots, \alpha_J \in X'$.

- We show that T' is compact (and, therefore, T is also compact).

- Let $\beta \in B_{Y'}$ ... then (by Hahn-Banach theorem), there is $\theta \in X'$ with

$\theta(x) = (T'\beta)(x)$ for $x \in M$ and

$$\begin{array}{l} X \xrightarrow{T} Y \\ X' \xleftarrow{T'} Y' \\ (T'\beta)(x) = \beta(Tx) \end{array}$$

$$\|\theta\|_{X'} = \sup_{x \in B_X} |\theta(x)| = \sup_{\substack{x \in M \\ \|x\|_X \leq 1}} |\theta(x)| = \sup_{\substack{x \in M \\ \|x\|_X \leq 1}} |(T'\beta)(x)|$$

$$= \sup_{\substack{x \in M \\ \|x\|_X \leq 1}} |\beta(Tx)| \leq \|\beta\|_{Y'} \cdot \sup_{\substack{x \in M \\ \|x\|_X \leq 1}} \|Tx\|_Y \leq \varepsilon.$$

- Define $\tilde{M} = \operatorname{Lin}(\alpha_1, \dots, \alpha_J) \subset\subset X'$. As $(\theta - T'\beta)(x) = 0$ for all $x \in M$, (i.e. $M \subset \ker(\theta - T'\beta)$)

we have $\theta - T'\beta \in \tilde{M}$ ... hence $T'\beta = (T'\beta - \theta) + \theta \in \tilde{M} + \varepsilon B_{X'}$.

... this holds for every $\beta \in B_{Y'}$ ... therefore $T'(B_{Y'}) \subset \tilde{M} + \varepsilon B_{X'}$.

and (of course) $T'(B_{Y'}) \subset \|T'\| \cdot B_{X'}$

- The set $(\|T'\| \cdot B_{X'}) \cap \tilde{M}$ is a bounded set in a finite-dimensional subspace

... finite ε-covering $\Rightarrow$ finite 2ε-covering of $T'(B_{Y'})$.

3, Kolmogorov numbers, T compact

$\varepsilon>0$, $T(B_X)\subset\bigcup_{j=1}^{J}(y_j+\varepsilon B_Y)$ for some $J\in\mathbb{N}$ and $y_1,\dots,y_J\in Y$

Put $N:=\mathrm{Lin}\{y_1,\dots,y_J\}$.

Then $d_{J+1}(T)\le\sup_{x\in B_X}\inf_{y\in N}\|Tx-y\|_Y\le\sup_{x\in B_X}\inf_{j=1,\dots,J}\|Tx-y_j\|_Y\le\varepsilon$.

4, $d_n(T)\to 0$

$\varepsilon>0$: $\exists N\subset\subset Y$ with finite dimension: $\sup_{x\in B_X}\inf_{y\in N}\|Tx-y\|_Y\le\varepsilon$.

$\Rightarrow T(B_X)\subset N+\varepsilon B_Y$

and ... $T(B_X)\subset\|T\|\cdot B_Y$

... finite ε-covering of $(\|T\|+\varepsilon)\cdot B_Y\cap N$ ... and 2ε-covering of $T(B_X)$. $\square$

Theorem: Let $T\in\mathcal{L}(X,Y)$, where X and Y are (quasi-)Banach spaces.

i, If X is a Hilbert space, then $c_n(T)=a_n(T)$

ii, If Y is a Hilbert space, then $d_n(T)=a_n(T)$

iii, If X and Y are Hilbert spaces, then $c_n(T)=d_n(T)=a_n(T)$.

Proof: iii, follows from i, and ii,

i, Let $n\in\mathbb{N}$, $\varepsilon>0$ and let $M\subset\subset X$ with $\operatorname{codim} M<n$ and

$$\forall x\in M:\ \|Tx\|_Y\le(1+\varepsilon)c_n(T)\|x\|_X.$$

- Put $M^\perp=\{y\in X:\langle x,y\rangle_X=0 \text{ for all } x\in M\}$ the orthogonal complement of M
- $P_{M^\perp}$ ... the orthogonal projection onto $M^\perp$; $A=T\circ P_{M^\perp}$

Then $\dim M^\perp < n$, rank $A < n$ and

$$a_n(T) \le \|T-A\|_{\mathcal{L}(X,Y)} = \sup_{\substack{x\in X\\ \|x\|_X\le 1}} \|Tx - Ax\|_Y = \sup_{x\in B_X} \|Tx - T(P_{M^\perp}x)\|_Y$$

$$= \sup_{x\in B_X} \|T(\underbrace{x - P_{M^\perp}x}_{=P_M x\in M})\|_Y \le \sup_{x\in B_X} \underbrace{\|x - P_{M^\perp}x\|_X}_{=\|P_M x\|_X \le \|x\|_X \le 1} \cdot (1+\varepsilon)c_n(T) \le (1+\varepsilon)c_n(T).$$

ii, Let $n\in\mathbb{N}$, $\varepsilon>0$, $N\subset\subset Y$, $\dim N < n$:

$$\forall x\in X: \inf_{y\in N} \|Tx-y\|_Y \le (1+\varepsilon)d_n(T)\cdot\|x\|_X.$$

- $A = P_N T$: $a_n(T) \le \|T-A\|_{\mathcal{L}(X,Y)} = \sup_{x\in B_X} \|Tx - Ax\|_Y = \sup_{x\in B_X} \|Tx - P_N(Tx)\|_Y$

$$\le \sup_{x\in B_X} \inf_{y\in N} \|Tx-y\|_Y \le (1+\varepsilon)d_n(T).$$

$\boxtimes$

To compute/estimate s-numbers for given operators might be difficult!

- Lemma 1: Let N be a subspace of ℓ_∞^m with codim $N < n$. Then there exists $x = (x_1, \dots, x_m) \in N$ with $\|x\|_\infty = 1$ and $\#\{k : |x_k| = 1\} \geq m - n + 1$.

Proof: • The set $B_{\ell_\infty^m} \cap N$ is convex

- Let $x \in \underbrace{B_{\ell_\infty^m} \cap N}_{K}$ be an extreme point

  (If $x = \lambda x_1 + (1-\lambda) x_2$, $0 < \lambda < 1$, $x_1, x_2 \in K$, then $x_1 = x_2 = x$)

  Krein-Milman: a convex set is a closed convex hull of its extreme points

- $I := \{k : |x_k| = 1\}$, $M := \{y \in \ell_\infty^m : y_k = 0 \text{ for all } k \in I\}$

- $\# I + \dim M = m$. If $\# I \leq m - n$, then $\dim M \geq n$

  $\Rightarrow M \cap N \neq \{0\}$ ... there is $y \in M \cap N$, $\|y\|_\infty = 1$.

  ... y is in N, $y_k = 0$ if $|x_k| = 1$ $x \pm \delta y \in K = B_{\ell_\infty^m} \cap N$

  ... x is not an extreme point

  $\delta = 1 - \max\{|x_k| : k \notin I\} > 0$. ☒

Lemma 2: Let $0 < q < p \leq \infty$ and $|x_{n+1}| \leq \min(|x_1|, \dots, |x_n|)$, then

$$\frac{\left(\sum_{j=1}^{n+1} |x_j|^q\right)^{1/q}}{\left(\sum_{j=1}^{n+1} |x_j|^p\right)^{1/p}} \geq \frac{\left(\sum_{j=1}^{n} |x_j|^q\right)^{1/q}}{\left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p}}.$$

Proof: Let $\alpha = \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p}$; $\beta := \left(\sum_{j=1}^{n} |x_j|^q\right)^{1/q}$.

... $x_{n+1} = 0$ trivial

As $p > q$: $\left|\frac{x_j}{x_{n+1}}\right|^q \leq \left|\frac{x_j}{x_{n+1}}\right|^p$

... sum up over $j = 1, \dots, n$: $\left(\frac{\beta}{|x_{n+1}|}\right)^q \leq \left(\frac{\alpha}{|x_{n+1}|}\right)^p$

and $\left\{1 + \left(\frac{\beta}{|x_{n+1}|}\right)^q\right\}^{p/q} \geq 1 + \left(\frac{\beta}{|x_{n+1}|}\right)^q \geq 1 + \left(\frac{|x_{n+1}|}{\alpha}\right)^p$

Finally $LHS = \frac{(\beta^q + |x_{n+1}|^q)^{1/q}}{(\alpha^p + |x_{n+1}|^p)^{1/p}} = \frac{\beta}{\alpha} \cdot \underbrace{\frac{(1 + (|x_{n+1}|/\beta)^q)^{1/q}}{(1 + (|x_{n+1}|/\alpha)^p)^{1/p}}}_{\geq 1.} \geq \frac{\beta}{\alpha}$ ☒

Theorem: Let $0 < q \leq p \leq \infty$. Then

$$a_n(id: \ell_p^m \to \ell_q^m) = c_n(id: \ell_p^m \to \ell_q^m) = (m - n + 1)^{1/q - 1/p}.$$

If $1 \leq q \leq p \leq +\infty$, then also

$$d_n(id: \ell_p^m \to \ell_q^m) = (m - n + 1)^{1/q - 1/p}.$$

Proof: • Let $P_{n-1}: \ell_p^m \to \ell_q^m$, $P_{n-1}x = (x_1, \dots, x_{n-1}, 0, \dots, 0)$.

Then
$$c_n(id) \le a_n(id) \le \| Id - P_{n-1} \|_{\mathcal{L}(\ell_p^m, \ell_q^m)} = (m-n+1)^{1/q - 1/p}$$

(holds also for d_n)

• Let $M \subset \ell_p^m$ be a subspace with codim $M < n$. By Lemma 1, there is $x \in M$ with $\|x\|_\infty \le 1$, $\# \underbrace{\{k : |x_k| = +1\}}_{I} \ge m-n+1$.

Then
$$\| id : M \to \ell_q^m \| \ge \frac{\|x\|_q}{\|x\|_p} = \frac{\left(\sum_{j=1}^m |x_j|^q \right)^{1/q}}{\left(\sum_{j=1}^m |x_j|^p \right)^{1/p}} \ge \frac{\left(\sum_{j \in I} |x_j|^q \right)^{1/q}}{\left(\sum_{j \in I} |x_j|^p \right)^{1/p}} \ge (m-n+1)^{1/q - 1/p}.$$

Hence $c_n(id) \ge (m-n+1)^{1/q - 1/p}$.

• The lower bound for d_n follows by duality! ... ? Direct proof?

$$(m-n+1)^{1/p' - 1/q'} \le c_n(id : \ell_{q'}^m \to \ell_{p'}^m) = \inf_{\substack{M \subset \mathbb{R}^m \\ \operatorname{codim} M < n}} \sup_{\substack{x \in M \\ \|x\|_{q'} \le 1}} \|x\|_{p'}$$

$$= \inf_M \sup_x \sup_{\|y\|_p \le 1} |\langle y, x \rangle| = \inf_{\substack{N = M^\perp \\ M \subset \mathbb{R}^m \\ \operatorname{codim} M < n}} \sup_{\|y\|_p \le 1} \sup_{\substack{x \in N^\perp \\ \|x\|_{q'} \le 1}} |\langle y, x \rangle|$$

• Let $x \in N^\perp$, $\|x\|_{q'} \le 1$, $z \in N$ ... $|\langle y, x \rangle| = |\langle y - z, x \rangle| \le \|y - z\|_q \cdot \|x\|_{q'}$

... $|\langle y, x \rangle| \le \inf_{z \in N} \|y - z\|_q$

$$\dots \le \inf_{\substack{N \subset \mathbb{R}^m \\ \dim N < n}} \sup_{\|y\|_p \le 1} \inf_{z \in N} \|y - z\|_q = d_n(id : \ell_p^m \to \ell_q^m).$$

Next, we study $l_1 \to l_\infty$ ...

- $a_n(id: l_1 \to c_0) = 1$ for $n = 1, 2, 3, \dots$
  - estimate from above: $a_n(id) \le \|id\| = 1$ ... $\|x\|_{c_0} \le \|x\|_1$.
  - estimate from below: consider any $A: l_1 \to c_0$ with finite rank. Then A is compact:

    $$\forall \varepsilon > 0 \; \exists N \in \mathbb{N}, y_1, \dots, y_N \in c_0 : A(B_{l_1}) \subset \bigcup_{j=1}^{N} (y_j + \varepsilon B_{c_0})$$

    $y_1, \dots, y_N$ are sequences converging to zero

    ... $\exists m \in \mathbb{N}$: $|y_{j,m}| < \varepsilon$ for $j = 1, \dots, N$

    And $\exists J \in \{1, \dots, N\}$: $\|Ae_m - y_J\|_\infty \le \varepsilon$. Therefore

    $$\|id - A\|_{\mathcal{L}(l_1, c_0)} \ge \|(id - A)e_m\|_\infty \ge |1 - (Ae_m)_m| \ge |1 - y_{J,m}| - |y_{J,m} - (Ae_m)_m|$$

    $$\ge 1 - \varepsilon - \varepsilon = 1 - 2\varepsilon.$$

    $\Rightarrow \|id - A\| \ge 1$ & $a_n(id: l_1 \to c_0) = 1$.

- $a_n(id: l_1 \to l_\infty) = 1/2$ for $n = 2, 3, \dots$
  - let $A_0 y = \frac{e}{2} \cdot \sum_{i=1}^{\infty} y_i$, where $e = (1, 1, \dots)$ ... rank $A_0 = 1$

    and $\|id - A_0\|_{\mathcal{L}(l_1, l_\infty)}$: $\ge \|(id - A)e_1\|_\infty = \|(1, 0, 0, \dots) - (\frac{1}{2}, \frac{1}{2}, \dots)\|_\infty = 1/2$

    $$\le \|(id - A_0)x\|_\infty = \|(x_1, x_2, \dots) - (1, 1, \dots) \cdot \sum_{j=1}^{\infty} x_j / 2\|_\infty \le 1/2.$$

    $$\dots \left| x_j - \frac{1}{2} \sum_{l=1}^{\infty} x_l \right| = \left| \frac{x_j}{2} - \sum_{l \ne j} x_l / 2 \right| \le \|x\|_1 / 2 \le \frac{1}{2}$$

• estimate from below:

suppose that there is $A \in \mathcal{L}(\ell_1, \ell_\infty)$ with finite rank, such that $\|id - A | \mathcal{L}(\ell_1, \ell_\infty)\| = \sup_{j,k \in \mathbb{N}} |(Ae_j - e_j)_k| < 1/2$.

Then $\|Ae_j - Ae_k\|_\infty \geq |(Ae_j)_k - (Ae_k)_k| = |1 - (Ae_j)_k - [1 - (Ae_k)_k]|$

$\geq 1 - |(Ae_j)_k| - |1 - (Ae_k)_k| \geq 1 - 2\|id - A\|_{\mathcal{L}(\ell_1, \ell_\infty)} > 0$

... A is not compact ... contradiction.

Theorem: $a_n(id : \ell_1^m \to \ell_\infty^m) \leq 3\left[\frac{\log_2(m+1)}{n}\right]^{1/2}$, $n = 1, \dots, m$.

Proof: • Khintchine inequalities:

Let $1 \leq p < +\infty$, then there are $A_p, B_p > 0$ with

$$A_p \cdot \|a\|_2 \leq \left(\frac{1}{2^m} \sum_{e \in \{-1,+1\}^m} |\langle a, e\rangle|^p\right)^{1/p} \leq B_p \cdot \|a\|_2$$

... without proof, $B_p \leq ([p/2] + 1)^{1/2}$.

• There are $x_1, \dots, x_m \in \ell_2^n$ with $\|x_i\|_2 = 1$ for all $i = 1, \dots, m$

and $|\langle x_i, x_j\rangle| \leq 2\left[\frac{\log_2 m}{n}\right]^{1/2}$ for $i \neq j$ ... nearly orthogonal vectors

... • $m \leq n$ ... orth. family

• $m \geq n$ & m already proven ... induction!

$$\Rightarrow \sum_{i=1}^{m} \sum_{e \in \{-1,+1\}^m} |\langle x_i, e\rangle|^p \le B_p^p \cdot m \cdot 2^m$$

... there exists some $e \in \{-1,+1\}^m$ with $\sum_{i=1}^{m} |\langle x_i, e\rangle|^p \le B_p^p m$.

$$x_{m+1} = \frac{1}{\sqrt{m}} e. \quad \ldots \|x_{m+1}\|_2 = 1$$

$$|\langle x_i, x_{m+1}\rangle| \le B_p \cdot m^{1/p} \cdot m^{-1/2}, \quad i = 1, \ldots, m.$$

Choosing $p := \log_2 m$: $|\langle x_i, x_{m+1}\rangle| \le 2\left[\frac{\log_2 m}{m}\right]^{1/2}$.

- For $x_1, \ldots, x_m \in \ell_2^m$ as above, put $A = (A_{ij})_{i,j=1}^{m}$, $A_{ij} := \langle x_i, x_j\rangle$

$$a_{m+1}(id : \ell_1^m \to \ell_\infty^m) \le \|id - A\|_{\mathcal{L}(\ell_1^m, \ell_\infty^m)} = \sup_{1 \le k \le m} \|(id - A)e_k\|_\infty$$

$$= \max_{1 \le j \ne k \le m} |A_{j,k}| \le 2\left[\frac{\log_2 m}{m}\right]^{1/2} \le 3\left(\frac{\log_2(m+1)}{m+1}\right)^{1/2}.$$