1, $\mathcal{L}(X,Y)$ ... $T \in \mathcal{L}(X,Y)$

$$\|T\| = \sup_{x \in B_X} \|Tx\|_Y$$

... $S, T \in \mathcal{L}(X,Y)$ ... $\|S+T\|_{\mathcal{L}(X,Y)} = \sup_{x \in B_X} \|(S+T)x\|_Y$

$$\leq \sup_{x \in B_X} \left(\|Sx\|_Y^p + \|Tx\|_Y^p\right)^{1/p} = \left[\sup_{x \in B_X} \left(\|Sx\|_Y^p + \|Tx\|_Y^p\right)\right]^{1/p}$$

$$\leq \left(\sup_{x \in B_X} \|Sx\|_Y^p + \sup_{x \in B_X} \|Tx\|_Y^p\right)^{1/p} = \left(\|S\|^p + \|T\|^p\right)^{1/p}$$

... $\|S+T\|^p \leq \|S\|^p + \|T\|^p$.

2, $T \in \mathcal{L}(X,X)$, $\|T\|_{\mathcal{L}(X,X)} < 1$ $\Rightarrow$ $(id - T)$ is invertible

$\|T^n\| \leq \|T\|^n$ ... The series $\sum_{n=0}^{\infty} T^n$ converges ... $\left\|\sum_{n=l}^{m} T^n\right\|$

$$\leq \sum_{n=l}^{m} \|T\|^n < \varepsilon \ldots$$

$$(id - T)\sum_{n=0}^{N} T^n = \sum_{n=0}^{N} T^n - \sum_{n=1}^{N+1} T^n = T^0 - T^{N+1} = id - \underset{\downarrow \atop 0}{T^{N+1}}$$

- X ... p-Banach space $\left\|\sum_{n=l}^{m} T^n\right\|^p \leq \sum_{n=l}^{m} \|T^n\|^p \leq \sum_{n=l}^{m} \left[\|T\|^p\right]^n < \varepsilon$

OK.

3, X, Y ... quasi-Banach spaces

$N \subset\subset X$ ... linear subspace ($x, y \in N$, $\alpha, \beta \in \mathbb{R} \Rightarrow \alpha x + \beta y \in N$)

$\dim N$ ... cardinality of a basis ... finite combinations form N

a, $N_1, N_2 \subset\subset X$ ... $\dim N_1 < m_1$ ... $\exists x_1, \ldots, x_{m_1-1} : \{\alpha_1 x_1 + \ldots + \alpha_{m_1-1} x_{m_1-1} ; \alpha \in \mathbb{R}^{m_1-1}\} = N_1$

$\dim N_2 < m_2$ ... $\exists y_1, \ldots, y_{m_2-1}$ ... $= N_2$

$$N_1 + N_2 = \{\alpha_1 x_1 + \ldots + \alpha_{m_1-1} x_{m_1-1} + \beta_1 y_1 + \ldots + \beta_{m_2-1} y_{m_2-1} : \alpha \in \mathbb{R}^{m_1-1}, \beta \in \mathbb{R}^{m_2-1}\}$$

$\Rightarrow x_1, \ldots, x_{m_1-1}, y_1, \ldots, y_{m_2-1}$ is a basis ... $\underbrace{\dim N_1 + N_2}$ $\leq m_1 + m_2 - 2$
$< m_1 + m_2 - 1$

b, $\operatorname{codim} M_1 = \dim X/M_1$ .... $X = M_1 + N_1$, $M_1 \cap N_1 = \{0\}$
$M_1 \subset\subset X$ $\quad \|$
$\dim N_1$ ... how many vectors do I need to add to a basis of M_1 to have a basis of X.

We know that

$$\exists \nu : M_1 \cap M_2 + \mathrm{Lin}\{x_1, \ldots, x_\nu\} = M_1$$

$$M_2 + \mathrm{Lin}\{v_1, \ldots, v_\beta\} = X \quad \ldots \beta = \operatorname{codim} M_2$$

intersect with M_1

$$M_1 \cap \{M_2 + \mathrm{Lin}\{v_1, \ldots, v_\beta\}\} = X \cap M_1 = M_1$$

/ 3b

$M_1 \cap M_2$ ... basis $\{r_\alpha\}_{\alpha \in A}$

- add $\{x_1, \dots, x_\mu\}$ ... basis of M_1
- add $\{y_1, \dots, y_\nu\}$ ... basis of M_2

... $\{r_\alpha\}_{\alpha\in A} \cup \{x_j, 1 \leq j \leq \mu\} \cup \{y_j; 1 \leq j \leq \nu\}$ ... basis of $M_1 \oplus M_2$

indep: $\underbrace{\sum c_\alpha r_\alpha}_{\in M_1} + \underbrace{\sum a_j x_j}_{\in M_1} + \underbrace{\sum b_j y_j}_{\in M_1} = 0 \Rightarrow \in M_1$ ... $b \equiv 0$ ... etc.

$M_1 \oplus M_2$ & $\{z_1, \dots, z_m\}$ ... basis of X

- $\operatorname{codim} M_1 = \nu + m$, $\operatorname{codim} M_2 = \mu + m$, $\operatorname{codim} M_1 \oplus M_2 = \mu + \nu + m$.

3c, $A\in\mathcal{L}(X,Y)$, rank $A<n$; $M=\ker A=\{x\in X: Ax=0\}$

$A(X)\subset\subset Y$; rank $A=\dim A(X)$

• $M+N=X$, $\dim N=\operatorname{codim} M$ … $N\cap M=\{0\}$

$\downarrow$

basis $\{x_1,\dots,x_r\}$ … $\{Ax_1,\dots,Ax_r\}\subset A(X)$

⋮

basis of $A(X)$

• $y\in A(X)$ … $y=Ax$ & $x=x_M+\alpha_1x_1+\dots+\alpha_rx_r$

$A(x_M)=0$ … $A(\alpha_1x_1+\dots+\alpha_rx_r)=$

$=\alpha_1Ax_1+\dots+\alpha_rAx_r=y$

• $\alpha_1Ax_1+\dots+\alpha_rAx_r=0$

$A(\underbrace{\alpha_1x_1+\dots+\alpha_rx_r}_{\in N\cap M})=0$ … $\alpha_1x_1+\dots+\alpha_rx_r=0$ … $\alpha\equiv0$

codim $M=\dim N=\dim A(X)=$ rank A.

d, $T\in\mathcal{L}(X,Y)$, $N\subset\subset X$, $\dim N<n$

$N=\operatorname{Lin}\{x_1,\dots,x_r\}$ … $T(N)=\operatorname{Lin}\{Tx_1,\dots,Tx_r\}$

… $\dim(T(N))\le r=\dim N$.

e, $T\in\mathcal{L}(X,Y)$, $M\subset\subset Y$, codim $M<n$ … ? codim $T^{-1}(M)<n$?

$T^{-1}(M)=\{x\in X: Tx\in M\}$ … lin. subspace

$T^{-1}(M)+\underset{\operatorname{Lin}\{x_1,\dots,x_r\}}{\overset{}{N}}=X$, $T^{-1}(M)\cap N=\{0\}$; $\{Tx_1,\dots,\overset{\notin M}{Tx_r}\}$ lin. indep.

$T(\underbrace{\alpha_1x_1+\dots+\alpha_rx_r}_{\in N\cap T^{-1}(M)})=0$ … $\alpha\equiv0$

$\Rightarrow M \cap \mathrm{Lin}\{Tx_1, \dots, Tx_\nu\} = \{0\}$ ... $\operatorname{codim} M \geq \nu = \operatorname{codim} T^{-1}(M)$

f_1 X ... Banach space, $\alpha, \alpha_1, \dots, \alpha_J \in X'$

$$\alpha \in \mathrm{Lin}(\alpha_1, \dots, \alpha_J) \Leftrightarrow \left(\bigcap_{j=1}^{J} \ker \alpha_j\right) \subset \ker \alpha$$

... $\Rightarrow$ clear: $\alpha = \lambda_1 \alpha_1 + \dots + \lambda_J \alpha_J$, $x \in \bigcap_{j=1}^{J} \ker \alpha_j$ ... $\alpha_1(x) = \dots = \alpha_J(x) = 0$

$$\Rightarrow \alpha(x) = \lambda_1 \alpha_1(x) + \dots + \lambda_J \alpha_J(x) = 0$$

$$\Rightarrow x \in \ker \alpha.$$

$\Leftarrow$: • $n=1$ ... or $J=1$ ... $\ker \alpha_1 \subset \ker \alpha \overset{?}{\Rightarrow} \alpha = \lambda \alpha_1$

• $\alpha = 0$ ... OK

• $\alpha \neq 0$ ... $X \neq \ker \alpha \supset \ker \alpha_1$ ... $\alpha_1 \neq 0$ as well

then $\operatorname{codim} \ker \alpha = \operatorname{codim} \ker \alpha_1 = 1$ ... $\ker \alpha = \ker \alpha_1$

$X = \ker \alpha_1 + \mathrm{Lin}(\bar{x})$ ... $\alpha(\bar{x}) = \lambda \alpha_1(\bar{x})$

$$\lambda = \frac{\alpha(\bar{x})}{\alpha_1(\bar{x})}$$

$x = u + \mu \bar{x}$ $\quad \alpha(x) = \alpha(u) + \mu \alpha(\bar{x}) = 0 + \mu \cdot \lambda \alpha_1(\bar{x})$

$$= \lambda \alpha_1(\mu \bar{x}) = \lambda \alpha_1(x) \quad \forall x \in X.$$

Induction step: holds for J, proof for $J+1$

$\alpha_1, \dots, \alpha_{J+1} \in X'$, $\alpha \in X'$ with $\ker(\alpha) \supset \bigcap_{j=1}^{J+1} \ker(\alpha_j)$

we want $\alpha \in \mathrm{Lin}(\alpha_1, \dots, \alpha_{J+1})$

if $\bigcap_{j=1}^{J} \ker(\alpha_j) = \bigcap_{j=1}^{J+1} \ker(\alpha_j)$ ... $\ker(\alpha_{J+1}) \supset \bigcap_{j=1}^{J} \ker \alpha_j$

α_{J+1} is in the span of $(\alpha_1, \dots, \alpha_J)$

& α is in the span of $(\alpha_1, \dots, \alpha_J)$

• if not: $\bigcap_{j=1}^{J} \ker(\alpha_j)$ is strictly larger than $\bigcap_{j=1}^{J+1} \ker(\alpha_j)$

... $\exists \bar{x} \in$ which is not in

$\alpha_1(\bar{x}) = \dots = \alpha_J(\bar{x}) = 0$ $\qquad$ $\alpha_{J+1}(\bar{x}) \neq 0$

$\ker\left(\alpha + \lambda \alpha_{J+1}\right) \supset \bigcap_{j=1}^{J} \ker(\alpha_j)$ ... then $\alpha + \lambda\alpha_{J+1} \in \mathrm{Lin}(\alpha_1, \dots, \alpha_J)$

• $\alpha_1(x) = \dots = \alpha_J(x) = \alpha_{J+1}(x) = 0 \implies \alpha(x) = 0$

• Take x with $\alpha_1(x) = \dots = \alpha_J(x) = 0$ ... if $\alpha_{J+1}(x) = 0$ ... then $\alpha(x) = 0$

& $(\alpha + \lambda\alpha_{J+1})(x) = 0$

y & $\alpha_{J+1}(y) = 0$

• if $\alpha_{J+1}(x) \neq 0$ ... $x = y + \nu\bar{x}$

$$(\alpha + \lambda\alpha_{J+1})(x) = (\alpha + \lambda\alpha_{J+1})(y) + (\alpha + \lambda\alpha_{J+1})(\bar{x})$$

$$= \underbrace{\alpha(y)}_{=0} + \lambda \underbrace{\alpha_{J+1}(y)}_{=0} + \alpha(\bar{x}) + \lambda\alpha_{J+1}(\bar{x}) \overset{!}{=} 0$$

$$\lambda = -\frac{\alpha(\bar{x})}{\alpha_{J+1}(\bar{x})}$$

... OK.

4, $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\,dt, \; s>0$

- $\Gamma(s+1) = s\Gamma(s)$

$$\overset{\|}{\int_0^\infty \underset{v'}{e^{-t}}\underset{u}{t^{s}}\,dt} = \cancel{\int_0^\infty t \cdot e^{-t} t^{s-1}\,dt}$$

$$\|$$

$$\left[t^s \cdot (-e^{-t})\right]_0^\infty - s\int_0^\infty t^{s-1}(-e^{-t})\,dt = s\Gamma(s)$$

- $B(\alpha,\beta) = \int_0^1 (1-t)^{\alpha-1} t^{\beta-1}\,dt$

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty\int_0^\infty t^{\alpha-1} s^{\beta-1} e^{-t-s}\,dt\,ds$$

$$s = ur, \quad t = u(1-r) \quad \Big\| \quad s+t = u, \quad r = \frac{s}{s+t}$$

$$\begin{pmatrix} \frac{\partial s}{\partial u}, & \frac{\partial s}{\partial r} \\ \frac{\partial t}{\partial u}, & \frac{\partial t}{\partial r} \end{pmatrix} = \begin{pmatrix} r; & u \\ 1-r, & -u \end{pmatrix} \; \ldots \; \det = -ru-(1-r)u$$

$$= -ru-u+ru = -u$$

$$= \int_0^1\int_0^\infty [u(1-r)]^{\alpha-1}(ur)^{\beta-1} e^{-\cancel{ur}-u(1-\cancel{r})} \cdot u\,du\,dr$$

$$= \int_0^\infty u^{\alpha-1}\cdot u^{\beta-1}\cdot u\, e^{-u}\,du \int_0^1 (1-r)^{\alpha-1} r^{\beta-1}\,dr = \Gamma(\alpha+\beta)\,B(\alpha,\beta)$$

$$\Rightarrow B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

5, $A_m = \{(t_1,\dots,t_m)\in\mathbb{R}^m,\ t_j \ge 0,\ t_1+\dots+t_m \le 1\}$

$m=1$ .... $A_1=[0;1]$ ... $\displaystyle\int_{A_1} t_1^{1/p-1}\,dt_1 = \left[\frac{t^{1/p}}{1/p}\right]_{t=0}^{t=1} = p\cdot t^{1/p}\Big|_{t=1} = p$

$$p^1\cdot\frac{\Gamma(1/p+1)^1}{\Gamma(1/p+1)} = p$$

$m-1 \longrightarrow m$:

$$\int_{A_m}\prod_{j=1}^{m} t_j^{1/p-1}\,dt = \int_0^1 t_m^{1/p-1}\int_{\substack{(t_1,\dots,t_{m-1})\\ t_j\ge 0,\ t_1+\dots+t_{m-1}\le 1-t_m}}\prod_{j=1}^{m-1} t_j^{1/p-1}\,d(t_1,\dots,t_{m-1})\,dt_m$$

$$s_j=\frac{t_j}{1-t_m},\quad j=1,\dots,m-1$$

$$=\int_0^1 t_m^{1/p-1}\int_{A_{m-1}}\prod_{j=1}^{m-1}\left[s_j(1-t_m)\right]^{1/p-1}ds\cdot(1-t_m)^{m-1}\,dt_m$$

$$=\underbrace{\int_0^1 t_m^{1/p-1}(1-t_m)^{m-1}\cdot(1-t_m)^{(m-1)(1/p-1)}\,dt_m}_{B\left(1/p;\ \frac{m-1}{p}+1\right)\ =\ \frac{\Gamma(1/p)\,\Gamma\left(\frac{m-1}{p}+1\right)}{\Gamma\left(\frac{m}{p}+1\right)}}\cdot\underbrace{\int_{A_{m-1}}\prod_{j=1}^{m-1}s_j^{1/p-1}\,ds}_{p^{m-1}\frac{\Gamma(1/p+1)^{m-1}}{\Gamma\left(\frac{m-1}{p}+1\right)}}$$

$$=\frac{\frac{1}{p}\Gamma(1/p)\,p^m\,\Gamma(1/p+1)^{m-1}}{\Gamma\left(\frac{m}{p}+1\right)}$$

$$=p^m\cdot\frac{\Gamma(1/p+1)\,\Gamma(1/p+1)^{m-1}}{\Gamma\left(\frac{m}{p}+1\right)}=p^m\frac{\Gamma(1/p+1)^m}{\Gamma\left(\frac{m}{p}+1\right)}.$$

6, $2^n \cdot \{x \in \mathbb{R}^n,\ x \geq 0,\ \sum_{i=1}^{n} |x_i|^p \leq 1\}$

$$t_i = |x_i|^p$$

$$dt_i = p \cdot x_i^{p-1} dx_i \quad \ldots \quad \frac{dt_i}{p} \cdot \left(t_i^{1/p}\right)^{1-p} = \frac{1}{p} t_i^{1/p - 1} dt_i$$

$$\Rightarrow \int_{B_{\ell_p^n}^+} 1 \prod_{i=1}^{n} dx_i = \frac{1}{p^n} \int_{A_n} \prod_{i=1}^{n} t_i^{1/p-1} dt = \frac{\Gamma(1/p+1)^n}{\Gamma(n/p+1)}$$

$$\Rightarrow vol(B_p^n) = \frac{2^n \cdot \Gamma(1/p+1)^n}{\Gamma(n/p+1)}.$$

7, $Y \subset\subset X$ closed proper subspace, $0 < \alpha < 1$

$\Rightarrow \exists x : \|x\|_X = 1$ & $dist(x, Y) \geq \alpha$.

Proof... Take $w \notin Y$. Y is closed $\Rightarrow dist(w, Y) > 0$

choose $\underset{\in Y}{w_0}$ with $\quad 0 < dist(w, Y) \leq \|w - w_0\|_X < \frac{1}{\varepsilon} dist(w, Y)$.

Put $\quad x := \dfrac{w - w_0}{\|w - w_0\|_X}, \ \ldots \|x\|_X = 1$

$y \in Y$: $\quad \|x - y\|_X = \left\| \dfrac{w - w_0}{\|w - w_0\|} - y \right\| = \dfrac{1}{\|w - w_0\|} \|w - \overbrace{w_0 - y\|w - w_0\|}^{\in Y}\|_X \geq \dfrac{dist(w, Y)}{\|w - w_0\|} > \varepsilon.$