1, $c_{00} = \{\{\lambda_j\}_{j=1}^{\infty} \subset \mathbb{R} : \exists J \in \mathbb{N} : \forall j > J : \lambda_j = 0\}$; $\|\lambda\|_{c_{00}} = \max_{j \in \mathbb{N}} |\lambda_j|$

Say $\lambda_j^J = \frac{1}{j}$, $j \leq J$

$J < K$: $\|\lambda^J - \lambda^K\|_{c_{00}} = \frac{1}{J+1}$ ... is a Cauchy sequence

... does not converge for $J \to \infty$.

2, Banach space ... $\mathcal{C}_0([0;1]) = \{f \in \mathcal{C}([0;1]), f(0) = f(1) = 0\}$

& sup-norm

$T : f \to \int_0^1 f(x)dx$; $|T(f)| \leq \int_0^1 |f(x)|dx \leq 1$ $\forall f$ with $\|f\| \leq 1$

$\|T\| = \sup_f T(f) = 1$, $\int_0^1 f(x)dx = 1$ ... impossible in $\mathcal{C}_0([0;1])$

3, • $|f_\lambda(\gamma)| = \left|\sum_{j=1}^{\infty} \lambda_j \gamma_j\right| \leq \sum_{j=1}^{\infty} |\lambda_j| \cdot |\gamma_j| \leq \|\lambda\|_\infty \cdot \|\gamma\|_1$

... $\|f_\lambda\|_{(l_1)'} \leq \|\lambda\|_\infty$ & $f_\lambda \in (l_1)'$

• Let $f \in l_1'$: $\lambda_j = f(e_j)$ ... $f(\alpha_1, \alpha_2, \dots, \alpha_N, 0, \dots)$

$= f\left(\sum_{j=1}^{N} \alpha_j e_j\right) = \sum_{j=1}^{N} \alpha_j \lambda_j$

$\gamma \in l_1$ given, $(\gamma^N)_j = \gamma_j$, $j \leq N$ ... $\gamma^N \to \gamma$ in l_1

$= 0$, $j > N$

$f(\gamma^N) = \sum_{j=1}^{N} \lambda_j \gamma_j \to \sum_{j=1}^{\infty} \lambda_j \gamma_j$.

4, $M \subset X$

$\text{conv}\, M = \bigcap \{K, K \supset M, K \text{ is convex}\}$

$M^* = \{x \in X : \exists n \in \mathbb{N}\ \exists x_1, \ldots, x_n \in M\ \exists \lambda_1, \ldots, \lambda_n \in [0,1], \lambda_1 + \ldots + \lambda_n = 1$
$\& \ x = \lambda_1 x_1 + \ldots + \lambda_n x_n\}$

• M^* is convex $\quad x, y \in M \ldots \quad x = \sum_{j=1}^{N} \lambda_j x_j, \ y = \sum_{j=1}^{M} \mu_j y_j$

$$\theta x + (1-\theta) y = \sum_{j=1}^{N} \theta \lambda_j x_j + \sum_{j=1}^{M} (1-\theta) \mu_j y_j$$

$$\ldots \sum \theta \lambda_j + \sum (1-\theta) \mu_j = \theta + 1 - \theta = 1.$$

$\Downarrow$

$\text{conv}\, M \subset M^*$

• $M_n^* \ldots M^* = \bigcup_{n=1}^{\infty} M_n^* \quad \ldots \text{conv}\, M \supset M_2^*$ by def.

$\text{conv}\, M \supset M_n^*$ by induction

$\Rightarrow \text{conv}\, M \supset \bigcup M_n^* = M^*.$

5, a, $\sum_{j=a}^{b} \binom{j}{a} = \underbrace{\binom{a}{a} + \binom{a+1}{a} + \ldots +}_{\binom{b}{a+1}} \binom{b}{a} = \binom{b}{a+1} + \binom{b}{a} = \binom{b+1}{a+1}$

... induction

b, $1 \le m \le k \quad \ldots \# = \#\{(s_1, \ldots, s_m) \in \mathbb{N}_0^m : s_1 + \ldots + s_m \le k - m\}$

$s = t - 1$

... put m black dots onto k pos. ... s_1 ... empty spots before 1^{st} black

$s_2 \ldots$

$\vdots$

$\Rightarrow \binom{m}{k}.$

... or by induction

c, $t=(t_1,\dots,t_m)\in\mathbb{Z}^m$, $|t_1|+\dots+|t_m|\le k$

j coordinates zero, m-j nonzero

$$\sum_{j=0}^{m}\binom{m}{j}\binom{k}{m-j}2^{m-j}$$

signs of non-zero coordinates

d, $e_n(id: l_1^m\to l_\infty^m)$

$$A_k^m=\left\{\left(\frac{t_1}{k},\dots,\frac{t_m}{k}\right):(t_1,\dots,t_m)\in\mathbb{Z}^m,\ |t_1|+\dots+|t_m|=k\right\}$$

$$\subseteq B_1^m=B_{l_1^m}$$

- $x,y\in A_k^m$, $x\neq y$: $\|x-y\|_\infty\ge\frac{1}{k}$

$2^{n-1}<\#A_k^m$ ... one l_∞-ball must cover 2 or more points from A_k^m

$\Downarrow$

$$e_n\ge\frac{1}{2k}$$

- on the other hand, $2^{n-1}\ge\#A_k^m\Rightarrow e_n\le 1/k$

... centers in A_k^m.

6, a, Let $F \in L_p([0,1])'$ and let f on $[0,1]$ be with $\underset{\alpha =}{F(f)} \neq 0$, $f \neq 0$

We write $f = f_{11} + f_{12}$, where $f_{11} = f \cdot \chi_{[0,a]}$, $f_{12} = f \chi_{[a,1]}$

$$\int_0^1 |f(x)|^p dx = 2\int_0^a |f(x)|^p dx = 2\int_a^1 |f(x)|^p dx$$

Then: $\|f_{11}\|_p^p = \frac{1}{2}\|f\|_p^p = \|f_{12}\|_p^p$

and $\underset{\alpha}{\underset{\|}{F(f)}} = F(f_{11}) + F(f_{12})$ ... $|F(f_{11})| \geq \frac{|\alpha|}{2}$

or $|F(f_{12})| \geq \frac{|\alpha|}{2}$.

Assume $|F(f_{11})| \geq \frac{|\alpha|}{2}$

Then $f_{11} = f_{21} + f_{22}$, $\|f_{21}\|_p^p = \frac{1}{2}\|f_{11}\|_p^p = \frac{1}{4}\|f\|_p^p = \|f_{22}\|_p^p$

$\frac{|\alpha|}{2} \leq |F(f_{11})| \leq |F(f_{21})| + |F(f_{22})|$ ... Assume $|F(f_{21})| \geq \frac{|\alpha|}{4}$

... we get a sequence $\|f_n\|_p^p = \frac{1}{2^n}\|f\|_p^p$

$$|F(f_n)| \geq \frac{|\alpha|}{2^n}$$

$$\Rightarrow \|F\| \geq \sup_n \frac{|F(f_n)|}{\|f_n\|_p} \geq \frac{|\alpha|/2^n}{\left[\frac{1}{2^n}\|f\|_p^p\right]^{1/p}} = \frac{1}{\|f\|_p} \cdot \alpha \cdot 2^{n(-1+1/p)}$$

$= +\infty$.

6b, $(l_p)' = l_\infty$

- $\lambda \in l_\infty$, $f_\lambda(\gamma) = \sum \lambda_i \gamma_i$: $\|f_\lambda\| \leq \sup_{\|\gamma\|_p \leq 1} |\sum \lambda_i \gamma_i|$

$$\leq \sup_{\|\gamma\|_p \leq 1} \|\lambda\|_\infty \sum_{i=1}^{\infty} |\gamma_i| \leq \|\lambda\|_\infty \cdot \sup_{\|\gamma\|_p \leq 1} \sum |\gamma_i|^p = \|\lambda\|_\infty$$

- Let $f \in (l_p)'$, $\lambda_j = f(e^j)$ — show that $f(\gamma) = \sum \lambda_j \gamma_j$.

c, • $B_{l_p^m} \subset B_{l_1^m}$ & $B_{l_1^m}$ is convex => conv $B_{l_p^m} \subset B_{l_1^m}$

- $B_{l_1^m} = \mathrm{conv}\{\pm e_1, \dots, \pm e_m\} \subset \mathrm{conv}\, B_{l_p^m}$

- conv $B_{l_p} \subset B_{l_1}$ ... as before: $B_{l_p} \subset B_{l_1}$ & B_{l_1} is convex.

$\vdots$

$\sum_{i=1}^{m} \lambda_i x_i$, $\sum \lambda_i = 1$, $x_i \in B_{l_p}$ ... $x \in B_{l_1} \setminus l_p$ is not in conv B_{l_p}.

d, $\|\chi_{[0,2]}\|_p = 2^{1/p}$; $\|\chi_{[0,1]}\|_p = 1 = \|\chi_{[1,2]}\|_p$.

7, Banach limit $\quad p(x) := \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} x_k \quad , x\in \ell_\infty$

a, $\lim x_n \overset{=\alpha}{\text{exists}} \Rightarrow p(x)=\alpha$

b, p is sublinear $\quad p(x+y) = \limsup \frac{1}{n}\sum_{k=1}^{n}(x_k+y_k)$

$\leq p(x)+p(y)$

c, $(\mathrm{Lim}) \in (\ell_\infty)'$ ... $\mathrm{Lim}(x)\leq p(x)$ ... Hahn-Banach

d, • $p(x) \leq \limsup_{n\to\infty} x_n$ .. $\mathrm{Lim}(x)\leq p(x)$

• $\underset{\text{linear}}{\mathrm{Lim}(-x)} \leq p(-x) = \limsup_{n\to\infty}\left(-\frac{1}{n}\sum_{k=1}^{n} x_k\right) = -\liminf\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)$

$-\mathrm{Lim}(x)$

e, $\mathrm{Lim}(Sx)-\mathrm{Lim}(x) = \mathrm{Lim}(Sx-x) \leq \limsup \frac{1}{n}\sum_{k=1}^{n}(Sx)_k - x_k$

$= \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}(x_{k+1}-x_k) = \limsup_{n\to\infty} \frac{1}{n}(x_{n+1}-x_1)$

$\to 0$ (x bounded)

f, $x=(1,0,1,0,\dots)$ ... $x+Sx = \mathbb{1}$

$\mathrm{Lim}(x)+\mathrm{Lim}(Sx) = 1 \Rightarrow \mathrm{Lim}(x) = \frac{1}{2}$

$= \mathrm{Lim}(Sx)$

$\mathrm{Lim}(x\cdot Sx) = \mathrm{Lim}(0) = 0$

g, $\mathrm{Lim}(0,\dots,0,1,\dots,1,\dots) = 1$ $\qquad$ h, easy

but $\lim_{n\to\infty} \sum_{j=n+1}^{\infty} \lambda_j = 0$ for every $\lambda\in\ell_1$.