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Modern Approximation Theory

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Exercises: 4. Sheet

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1. Let N be a positive integer and consider the set of N-th roots of unity in $\mathbb{C}$:
$$\mathbb{Z}_N = \{1, e^{2\pi i/N}, e^{2\cdot 2\pi i/N}, \dots, e^{(N-1)\cdot 2\pi i/N}\}.$$
Show that
   - $\mathbb{Z}_N$ is an Abelian group (with the usual multiplication).
   - $\mathbb{Z}_N$ is isomorph with the set $\{0, 1, \dots, N-1\}$ with summation modulo N.
   - The vectors $(e_l)_{l=0}^{N-1}$ defined by
   $$e_l(k) = e^{2\pi i lk/N} \quad \text{for } l = 0, 1, \dots, N-1 \text{ and } k = 0, 1, \dots, N-1.$$
   are orthogonal with respect to the usual inner product.
2. *Discrete Fourier transform (DFT)* $\mathcal{F}x$ of a vector $x \in \mathbb{C}^N$ is defined as
$$\mathcal{F}x := \mathbb{F}_N x,$$
where $\mathbb{F}_N$ is the N-dimensional *Fourier matrix*
$$\mathbb{F}_N = \left(e^{-2\pi i k\ell/N}\right)_{k,\ell=0,\dots,N-1}.$$
3. *Cyclic convolution* $x * y \in \mathbb{C}^N$ of signals $x, y \in \mathbb{C}^N$ over $\mathbb{Z}_N$ is defined as
$$(x * y)_k := \sum_{\ell=0}^{N-1} x_{(k-\ell) \bmod N}\, y_\ell, \qquad k \in \mathbb{Z}_N = \{0, \dots, N-1\}.$$
4. Show that it holds
$$\mathcal{F}(x * y) = (\mathcal{F}x) \cdot (\mathcal{F}y),$$
where "$\cdot$" is the entry-wise product of two vectors.
5. Derive the formula for the DFT of shift and modulation of a given vector, i.e.
$$\mathcal{F}(\{x_n e^{2\pi i \frac{nm}{N}}\}) \quad \text{a} \quad \mathcal{F}(\{x_{n-m}\}).$$
6. Show that
$$\left(\frac{1}{\sqrt{N}}\mathbb{F}_N\right)^4 = Id.$$
7. Prove the following Lemma:
If $x \in \mathbb{C}^N$ is an s-sparse vector such that s of its consecutive Fourier coefficients vanish, then $x = 0$.
Hint: Rewrite the exercise as a Vandermond matrix.
8. Implement the Prony method and test its stability and robustness (i.e. test the defects of sparsity and noise).

4. Sheet

1, $\cdot\ \mathbb{Z} = \{ e^{2\pi i j/N} : j = 0, \ldots, N-1 \}$

$$e^{2\pi i k/N} \cdot e^{2\pi i j/N} = e^{2\pi i (j+k)/N} = e^{2\pi i (j+k \bmod N)/N}$$

$\cdot$ $$\langle e_j, e_\ell \rangle = \sum_{k=0}^{N-1} e_j(k)\, \overline{e_\ell(k)} = \sum_{k=0}^{N-1} e^{2\pi i (j-\ell)k/N}$$

$$= \sum_{k=0}^{N-1} \left[e^{2\pi i (j-\ell)/N} \right]^k \begin{cases} j=\ell \ldots\ldots N \\ j\neq\ell \ldots \quad \dfrac{1-\left[e^{2\pi i (j-\ell)/N}\right]^N}{1-e^{2\pi i (j-\ell)/N}} = 0 \end{cases}$$

2, $\mathbb{F}_N = \left(e^{-2\pi i k\ell/N} \right)_{k,\ell = 0, \ldots, N-1}$; $\quad (x*y)_k = \sum_{\ell=0}^{N-1} x_{k-\ell \bmod N}\, y_\ell$

4, $$\left[\mathcal{F}(x*y)\right]_u = \left[\mathbb{F}_N (x*y)\right]_u = \sum_{v=0}^{N-1} e^{-2\pi i u v/N} (x*y)_v$$

$$= \sum_{v=0}^{N-1} e^{-2\pi i u v/N} \cdot \sum_{\ell=0}^{N-1} x_{v-\ell}\, y_\ell = \sum_{\ell=0}^{N-1} y_\ell\, e^{-2\pi i u \ell/N} \underbrace{\sum_{v=0}^{N-1} x_{v-\ell}\, e^{-2\pi i u (v-\ell)/N}}_{\text{indep. on } \ell}$$

$$= (\mathcal{F}y)_u \cdot (\mathcal{F}x)_u .$$

5, $m = 0, \dots, N-1$, $y_n = x_n e^{2\pi i \cdot \frac{mn}{N}}$

a, $(\mathcal{F}y)_k = \sum\limits_{l=0}^{N-1} e^{-2\pi i k l/N} y_l = \sum\limits_{l=0}^{N-1} e^{-2\pi i k l/N} \cdot e^{2\pi i l m/N} x_l$

$$= \sum_{l=0}^{N-1} e^{-2\pi i l(k-m)/N} x_l = (\mathcal{F}x)_{k-m}$$

b, $y_n = x_{n-m}$, $(\mathcal{F}y)_k = \sum\limits_{l=0}^{N-1} e^{-2\pi i k l/N} y_l = \sum\limits_{l=0}^{N-1} e^{-2\pi i k l/N} \cdot x_{l-m}$

$$= e^{-2\pi i k m/N} \sum_{l=0}^{N-1} e^{-2\pi i k(l-m)/N} x_{l-m} = e^{-2\pi i k m/N} \cdot (\mathcal{F}x)_k$$

6, $(\mathbb{F}_N^2)_{uv} = \sum_{\alpha=0}^{N-1} (\mathbb{F}_N)_{u\alpha} (\mathbb{F}_N)_{\alpha v} = \sum_{\alpha=0}^{N-1} e^{-2\pi i u\alpha/N} e^{-2\pi i \alpha v/N}$

$$= \sum_{\alpha=0}^{N-1} e^{-2\pi i \alpha(u+v)/N} = \begin{cases} (u+v) \bmod N = 0 & \ldots\ N \\ (u+v) \bmod N \neq 0 & \ldots\ 0 \end{cases}$$

$$(\mathbb{F}_N^4)_{kl} = \sum_{m=0}^{N-1} (\mathbb{F}_N^2)_{k,m} (\mathbb{F}_N^2)_{m,l} = \sum_{m=0}^{N-1} N \cdot \delta_{N-k,m}\ N \delta_{N-m,l}$$

$m = N-k \ldots l = N-m = k \qquad \ldots\ N^2 \cdot I$

7, $x \in \mathbb{C}^N \ldots$ s-sparse $\ldots\ x_{i_1}, \ldots, x_{i_s} \neq 0$, otherwise 0

$(\mathcal{F}x)_j, \ldots, (\mathcal{F}x)_{j+s-1} = 0 \qquad j$ fixed

$$(\mathcal{F}x)_{j+k} = \sum_{l=0}^{N-1} e^{-2\pi i l(j+k)/N} x_l = \sum_{m=1}^{s} e^{-2\pi i\, i_m (j+k)/N} x_{i_m}$$

$$= \sum_{m=1}^{s} e^{-2\pi i (j+k) i_m/N} x_{i_m} =$$

$$= \sum_{m=1}^{s} \underbrace{\left[e^{-2\pi i\, i_m/N}\right]^k}_{t_m^k} \underbrace{\left[e^{-2\pi i j\, i_m/N} x_{i_m}\right]}_{y_m}$$

$$= \sum_{m=1}^{s} t_m^k y_m \ldots \begin{bmatrix} 1 & \ldots & 1 \\ t_1 & \ldots & t_s \\ \vdots & & \\ t_1^{s-1} & \ldots & t_s^{s-1} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_s \end{bmatrix} \ldots$$

all t's are diff.
the matrix is regular
$\Rightarrow y \equiv 0.$