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Modern Approximation Theory

Summer term 2022

Exercises: ~~4.~~5 Sheet

1. Let $t > 0$ be a real number and let $d \geq 2$ be an integer. Show that there exists $\mathcal{M} \subset \mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : \|x\|_2 = 1\}$ such that

   (i) $\#\mathcal{M} \leq (1 + 2/t)^d$,

   (ii) $\forall z \in \mathbb{S}^{d-1}\ \exists x \in \mathcal{M} : \|x - z\|_2 \leq t$.

   Hint: Take $x^1 \in \mathbb{S}^{d-1}$ arbitrarily. If $x^1, \dots, x^j \in \mathbb{S}^{d-1}$ were already chosen, take $x^{j+1} \in \mathbb{S}^{d-1}$ arbitrarily such that $\|x^{j+1} - x^l\|_2 > t$ for all $l = 1, \dots, j$. Repeat this, as long as it goes. Then use that $B(x^i, t/2)$ are all disjoint and included in $B(0, 1 + t/2)$ and compare the volumes.

2. Let $\omega \sim \mathcal{N}(0, 1)$ be a standard Gaussian variable. Show that

$$\mathbb{E}e^{\lambda\omega^2} = \frac{1}{\sqrt{1 - 2\lambda}}, \quad -\infty < \lambda < 1/2.$$

3. Let $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m$ and let $\omega_1, \dots, \omega_m \sim \mathcal{N}(0, 1)$ be independent. Show that

$$\langle \lambda, \omega \rangle = \lambda_1\omega_1 + \cdots + \lambda_m\omega_m \sim \|\lambda\|_2 \cdot \mathcal{N}(0, 1) = \mathcal{N}(0, \|\lambda\|_2^2).$$

4. Prove the Lemma of Johnson and Lindestrauss:
   Let $0 < \varepsilon < 1$ and let m, N and d be positive integers with

$$m \geq 4\Big(\varepsilon^2/2 - \varepsilon^3/3\Big)^{-1} \ln N.$$

   Then for every set $\{x^1, \dots, x^N\} \subset \mathbb{R}^d$ there is a mapping $f : \mathbb{R}^d \to \mathbb{R}^m$ with

$$(1 - \varepsilon)\|x^i - x^j\|_2^2 \leq \|f(x^i) - f(x^j)\|_2^2 \leq (1 + \varepsilon)\|x^i - x^j\|_2^2, \quad i, j \in \{1, \dots, N\}.$$

   Hint: Choose $f(x) = Ax$, where A is a Gaussian matrix. Then use the concentration inequality for one Az with $z = (x^i - x^j)/\|x^i - x^j\|_2$ and the union bound.

5. Finish the proof of the RIP for Gaussian matrices, i.e., show that if $C' > 0$ is given then there exists $C > 0$ such that if

$$m \geq C\delta^{-2}\Big(s\ln\Big(\frac{eN}{s}\Big) + \ln\Big(\frac{2}{\varepsilon}\Big)\Big),$$

   then

$$\binom{N}{s} \cdot 9^s \cdot 2\exp(-C'm\delta^2) < \cancel{1}\ \varepsilon$$

5. Sheet

1, $x^1 \in \mathbb{S}^{d-1}$ arbitrary

$x^1, \dots, x^j \in \mathbb{S}^{d-1} \dots x^{j+1}$ arbitrary with • $x^{j+1} \in \mathbb{S}^{d-1}$

• $\|x^{j+1} - x^l\|_2 > t, \; l = 1, \dots, j$

• $B(x^i, t/2)$ are disjoint

• and are included in $B(0, (1 + t/2))$

$$N \cdot vol(B(x^i, t/2)) \leq vol(B(0, 1 + t/2))$$

$$N \cdot (t/2)^d \, vol(B(0,1)) \leq (1 + t/2)^d \, vol(B(0,1))$$

$$\Rightarrow N \leq (1 + 2/t)^d$$

• $B(x_j, t)$ cover $\mathbb{S}^{d-1}$!

2, $\mathbb{E}(e^{\lambda \omega^2}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\lambda t^2} e^{-t^2/2} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(\lambda - 1/2) t^2} dt$

$s = \cancel{\sqrt{2\lambda - 1/2}\, t} \sqrt{1 - 2\lambda}\, t$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-s^2/2} \frac{ds}{\sqrt{1-2\lambda}} = \frac{1}{\sqrt{1-2\lambda}}$$

3, standard...

4, $\|z\|_2 = 1 \Rightarrow \mathbb{P}(|\|Az\|_2^2 - 1| > t) \leq 2e^{-\frac{m}{2}[t^2/2 - t^3/3]}$; $A = \frac{1}{\sqrt{m}} (\omega_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,N}}$

$$\mathbb{P}(|\|f(x^i) - f(x^j)\|_2^2 - \|x^i - x^j\|_2^2| > \varepsilon \|x^i - x^j\|_2^2) = \mathbb{P}(|\|Az\|_2^2 - 1| > \varepsilon)$$

$$z = \frac{x^i - x^j}{\|x^i - x^j\|_2}$$

$\bullet\ \mathbb{P}(\text{JL lemma inequality fails}) \le \binom{N}{2} \cdot 2 \cdot e^{-\frac{m}{2}\left[\varepsilon^2/2 - \varepsilon^3/3\right]}$

$$< N^2 e^{-\frac{m}{2}\left[\varepsilon^2/2 - \varepsilon^3/3\right]} < 1$$

for $2\ln N - \frac{m}{2}\left[\varepsilon^2/2 - \varepsilon^3/3\right] < 0$.

5) $\exists C>0: m \ge C\delta^{-2}\left(s\ln\left(\frac{eN}{s}\right) + \ln\left(\frac{2}{\varepsilon}\right)\right) \Rightarrow \binom{N}{s} 9^s \cdot 2\exp(-C' m\delta^2) < \varepsilon$

$\Updownarrow$

$(*)\quad C^{-1}\delta^2 m \ge s\ln\left(\frac{eN}{s}\right) + \ln\left(\frac{2}{\varepsilon}\right)$

$\Uparrow \binom{N}{s} \le \left(\frac{eN}{s}\right)^s$

$\left(\frac{eN}{s}\right)^s \cdot 9^s \cdot 2\exp(-C' m\delta^2) < \varepsilon$

$\Updownarrow$

$s\ln\left(\frac{eN}{s}\right) + s\ln 9 + (-C' m\delta^2) + \ln(2/\varepsilon) < 0 \quad (**)$

We want to show that $\forall C'>0\ \exists C>0: (*) \Rightarrow (**)$

- $s\ln 9 \le \ln 9 \cdot s \cdot \ln\left(\frac{eN}{s}\right) \le C^{-1}\delta^2 m \cdot \ln 9$
- $s\ln\left(\frac{eN}{s}\right) \le C^{-1}\delta^2 m$
- $\ln(2/\varepsilon) \le C^{-1}\delta^2 m$

$$\Rightarrow s\ln\left(\frac{eN}{s}\right) + s\ln 9 + \ln(2/\varepsilon) \le (\ln 9 + 2)\delta^2 m C^{-1} \le C' m\delta^2$$

$$\ldots C \ge \frac{2+\ln 9}{C'} .$$