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Abstract of the Doctoral Thesis

Fine properties of Sobolev embeddings

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Autoreferát byl rozeslán dne ....

Contents

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $1 \le p \le \infty$ and let k be a natural number. We denote by $W_p^k(\Omega)$ the Sobolev spaces of functions from $L_p(\Omega)$ with all distributive derivatives of order smaller or equal to k in $L_p(\Omega)$. If

$$1 \le p_1, p_2 \le \infty, \qquad k_1 - k_2 \ge n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ \tag{1.1}$$

and the boundary of Ω is Lipschitz then $W_{p_1}^{k_1}(\Omega)$ is continuously embedded into $W_{p_2}^{k_2}(\Omega)$, i.e.

$$W_{p_1}^{k_1}(\Omega) \hookrightarrow W_{p_2}^{k_2}(\Omega). \tag{1.2}$$

This theorem goes back to Sobolev [22].

If the inequality in (1.1) is strict, the embedding is even compact, cf. [20] and [15]. During the second half of the last century, this fact (and its numerous generalisations) found its applications in many areas of modern analysis, especially in connection with partial differential (and pseudo-differential) equations. For this reason, the study of spaces of smooth functions became an important part of functional analysis with (1.2) playing a central role. There is a vast literature on function spaces of Sobolev type and all of them deal also with many variants of the Sobolev embedding. We refer at least to [1], [19], [16], [23], [17] and [10].

The thesis is composed of 5 papers [27]–[31]. In these papers we studied several aspects of the Sobolev embedding (and some of its generalisations) and presented some new results.

In the following sections, we describe our achievements.

2 *Optimal Sobolev embeddings on* $\mathbb{R}^n$ Publ. Mat. 51 (2007), 17-44.

Let us first recall the concept of the non-increasing rearrangement.

We denote by $\mathfrak{M}(\mathbb{R}^n)$ the set of real-valued Lebesgue-measurable functions on $\mathbb{R}^n$ finite almost everywhere and by $\mathfrak{M}_+(\mathbb{R}^n)$ the class of non-negative functions in $\mathfrak{M}(\mathbb{R}^n)$. Finally, $\mathfrak{M}_+(0,\infty,\downarrow)$ denotes the set of all non-increasing functions from $\mathfrak{M}_+(0,\infty)$. Given $f \in \mathfrak{M}(\mathbb{R}^n)$ we define its non-increasing rearrangement by

$$f^*(t) = \inf\{\lambda > 0 : |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}| \le t\}, \quad 0 < t < \infty. \tag{2.1}$$

For a set $A \subset \mathbb{R}^n$ we denote by $|A|$ its Lebesgue measure. A detailed treatment of rearrangements may be found in [3].

We also recall some basic aspects of the theory of Banach function norms. For details, see again [3].

Definition 2.1. A functional $\varrho : \mathfrak{M}_+(0,\infty) \to [0,\infty]$ is called a *Banach function norm* on $(0,\infty)$ if, for all $f, g, f_n, (n = 1, 2, \dots)$, in $\mathfrak{M}_+(0,\infty)$, for all constants $a \ge 0$ and for all measurable subsets E of $(0,\infty)$, it satisfies the following axioms

$$\begin{aligned}
(A_1) \quad & \varrho(f)=0 \quad \text{if and only if} \quad f=0 \text{ a.e.};\\
& \varrho(af)=a\varrho(f);\\
& \varrho(f+g)\le \varrho(f)+\varrho(g);\\
(A_2) \quad & \text{if} \quad 0\le g\le f \text{ a.e. then} \quad \varrho(g)\le\varrho(f);\\
(A_3) \quad & \text{if} \quad 0\le f_n\uparrow f \text{ a.e. then} \quad \varrho(f_n)\uparrow\varrho(f);\\
(A_4) \quad & \text{if} \quad |E|<\infty \quad \text{then} \quad \varrho(\chi_E)<\infty;\\
(A_5) \quad & \text{if} \quad |E|<\infty \quad \text{then} \quad \int_E f\le C_E\varrho(f)
\end{aligned}$$

with some constant $0<C_E<\infty$, depending on ϱ and E but independent of f.

If, in addition, $\varrho(f)=\varrho(f^*)$, we say that ϱ is a *rearrangement-invariant* (*r.i.*) *Banach function norm.* We often use the notions *norm* and *r.i. norm* to shorten the notation.

Definition 2.2. Let ϱ_R and ϱ_D be two r. i. norms. We set

$$L^{\varrho_R}(\mathbb{R}^n)=\left\{u\in L^1_{\text{loc}}(\mathbb{R}^n): ||u|L^{\varrho_R}(\mathbb{R}^n)||=\varrho_R(u^*)<\infty\right\} \tag{2.2}$$

and

$$W^1_{\varrho_D}(\mathbb{R}^n)=\left\{u\in L^1_{\text{loc}}(\mathbb{R}^n): ||u|W^1_{\varrho_D}(\mathbb{R}^n)||=\varrho_D(u^*)+\varrho_D(|\nabla u|^*)<\infty\right\}. \tag{2.3}$$

The space L^{ϱ_R} is called a *rearrangement-invariant Banach function space.* It follows directly from its definition that if $u^*=v^*$ for two measurable functions u and v, then $||u|L^{\varrho_R}(\mathbb{R}^n)||=||v|L^{\varrho_R}(\mathbb{R}^n)||$. Hence, the norm depends only on the size of the function values, not on a specific distribution of these values. The space $W^1_{\varrho_D}(\mathbb{R}^n)$ is called the *Sobolev space associated to* L^{ϱ_D}. Here, ∇u denotes the gradient of a function u.

Our aim is to study the embedding

$$W^1_{\varrho_D}(\mathbb{R}^n)\hookrightarrow L^{\varrho_R}(\mathbb{R}^n). \tag{2.4}$$

The embedding (2.4) is equivalent to

$$\varrho_R(u^*)\le c[\varrho_D(u^*)+\varrho_D(|\nabla u|^*)], \quad u\in W^1_{\varrho_D}(\mathbb{R}^n). \tag{2.5}$$

The inequality (2.5) is the main subject of our study.

We are interested in two main questions:

1. Suppose that the 'range' norm ϱ_R is given. We want to find the optimal (that is, essentially smallest) norm ϱ_D for which (2.5) holds. The optimality means that if (2.5) holds with ϱ_D replaced by some other rearrangement-invariant norm σ, then there exists a constant $C>0$ such that $\varrho_D(u^*)\le C\sigma(u^*)$ for all functions $u\in L^1_{\text{loc}}(\mathbb{R}^n)$.

2. Suppose that the 'domain' norm ϱ_D is given. We would like to construct the corresponding optimal 'range' norm ϱ_R. This means that the ϱ_R will be the essentially largest rearrangement-invariant norm for which (2.5) holds.

The first step in the study of (2.5) is a reduction of (2.5) to the boundedness of certain Hardy operators.

Theorem 2.3. *Let ϱ_D, ϱ_R be two r.i. Banach function norms on $(0,\infty)$. Then the inequality*

$$\varrho_R(u^*) \le c[\varrho_D(u^*) + \varrho_D(|\nabla u|^*)], \quad u \in W^1_{\varrho_D}(\mathbb{R}^n), \tag{2.6}$$

holds if and only if there is a constant $K > 0$ such that

$$\varrho_R\left(\int_t^\infty f(s)s^{1/n-1}\mathrm{d}s\right) \le K\varrho_D\left(f(t) + \int_t^\infty f(s)s^{1/n-1}\mathrm{d}s\right) \tag{2.7}$$

for all $f \in \mathfrak{M}_+(0,\infty)$.

The main tool in the proof is the following generalisation of the Pólya—Szegö principle from [7, (4.3)]:

$$\int_0^t \left[-s^{1-1/n}\frac{\mathrm{d}u^*}{\mathrm{d}s}\right]^*(s)\mathrm{d}s \le c\int_0^t |\nabla u|^*(s)\mathrm{d}s, \tag{2.8}$$

which holds for every $t > 0$ and every weakly differentiable function u such that $(\nabla u) \in L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ and

$$|\{x \in \mathbb{R}^n : |u(x)| > s\}| < \infty \quad \text{for all} \quad s > 0.$$

Up to this place, our approach follows [10]. But unlike there, (2.7) involves two different integral operators and therefore it is still not suitable for further investigation. Therefore we will derive another equivalent version of (2.6). In (2.7) we substitute

$$g(t) = f(t) + \int_t^\infty f(s)s^{1/n-1}\mathrm{d}s, \quad f \in \mathfrak{M}_+(0,\infty), \quad t > 0. \tag{2.9}$$

We shall need also the inverse substitution. Namely, if g is defined by (2.9), then

$$f(t) = g(t) - e^{nt^{1/n}}\int_t^\infty g(s)s^{1/n-1}e^{-ns^{1/n}}\mathrm{d}s. \tag{2.10}$$

Finally, we sum up (2.9) and (2.10) and obtain

$$\int_t^\infty f(s)s^{1/n-1}\mathrm{d}s = e^{nt^{1/n}}\int_t^\infty g(u)u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u \text{ for a.e. } t > 0. \tag{2.11}$$

This substitution can now be used to reformulate (2.6).

Theorem 2.4. *Let ϱ_D, ϱ_R be two r.i. Banach function norms on $(0,\infty)$. Then, (2.6) is equivalent to*

$$\varrho_R\left(e^{nt^{1/n}}\int_t^\infty g(u)u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u\right) \le c\varrho_D(g) \text{ for all } g \in \mathbf{G}, \tag{2.12}$$

where $\mathbf{G}$ is a new class of functions, defined by

$$\begin{aligned} \mathbf{G} = \Big\{g \in \mathfrak{M}_+(0,\infty) :\ & \text{there is a function } f \in \mathfrak{M}_+(0,\infty) \text{ such that} \\ & g(t) = f(t) + \int_t^\infty f(s)s^{1/n-1}\mathrm{d}s \text{ for all } t > 0\Big\} \\ = \Big\{g \in \mathfrak{M}_+(0,\infty) : g(t) - e^{nt^{1/n}} & \int_t^\infty g(s)s^{1/n-1}e^{-ns^{1/n}}\mathrm{d}s \ge 0 \text{ for all } t > 0\Big\}. \end{aligned} \tag{2.13}$$

Hence the inequality (2.6) is equivalent to the boundedness of the Hardy-type operator

$$(Gg)(u) = e^{nu^{1/n}} \int_u^\infty g(s)s^{1/n-1}e^{-ns^{1/n}}\mathrm{d}s, \quad u > 0, \tag{2.14}$$

on the set $\mathbf{G}$, the image of the positive cone $\mathfrak{M}_+(0,\infty)$ under the operator

$$f \to f(t) + \int_t^\infty f(s)s^{1/n-1}\mathrm{d}s.$$

Before we proceed any further we shall state some basic properties of the class $\mathbf{G}$.

Remark 2.5. (i) $\mathbf{G}$ contains all non-negative non-increasing functions.

(ii) For every g from $\mathbf{G}$, Gg is non-increasing.

(iii) The set $\mathbf{G}$ is a *convex cone*, that is, for every $\alpha, \beta > 0$ and $g_1, g_2 \in \mathbf{G}$, we have $\alpha g_1 + \beta \alpha_2 \in \mathbf{G}$.

Remark 2.6. (i) To show some applications we prove that $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{\frac{np}{n-p},p}(\mathbb{R}^n)$ for $1 \le p < n$. In this case, we have $\varrho_R(f) = ||f^*(t)t^{-1/n}||_p$ and $\varrho_D(f) = ||f||_p$. Using Remark 2.5 (ii) and the boundedness of classical Hardy operators on L^p we get for every function $g \in \mathbf{G}$ that

$$\begin{aligned}\varrho_R(Gg) = ||t^{-1/n}(Gg)^*(t)||_p &= \left|\left|t^{-1/n}e^{nt^{1/n}}\int_t^\infty g(u)u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u\right|\right|_p \\ &\le \left|\left|t^{-1/n}\int_t^\infty g(u)u^{1/n-1}\mathrm{d}u\right|\right|_p \le c||t^{-1/n}g(t)t^{1/n}||_p = c||g||_p = c\varrho_D(g).\end{aligned}$$

(ii) Another application of the obtained results is the embedding $W^1(L^{n,1})(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$. In this case

$$\begin{aligned}\varrho_R(Gg) = \sup_{t>0}(Gg)(t) = (Gg)(0) &= \int_0^\infty g(u)u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u \\ &\le \int_0^\infty g(u)u^{1/n-1}\mathrm{d}u \le \int_0^\infty g^*(u)u^{1/n-1}\mathrm{d}u = \varrho_D(g)\end{aligned}$$

for every function $g \in \mathbf{G}$. Now we used Remark 2.5 (ii).

(iii) Both these applications recover well-known results. They demonstrate some important aspects of this method. First, the second basic property of the class $\mathbf{G}$ (c.f. Remark 2.5, (ii)) lies in the roots of every Sobolev embedding. Second, the boundedness of Hardy operators plays a crucial role in this theory.

Now we can describe the solution of one of the main problems stated before. We shall construct the optimal domain norm ϱ_D to a given range norm ϱ_R.

Theorem 2.7. *Let the norm ϱ_R satisfy*

$$\varrho_R\left(G(g^{**})\right) \le c\varrho_R\left(G(g^*)\right), \quad g \in \mathfrak{M}_+(0,\infty). \tag{2.15}$$

Then the optimal domain norm ϱ_D corresponding to ϱ_R is defined by

$$\varrho_D(g) := \varrho_R\left(G(g^{**})\right), \quad g \in \mathfrak{M}_+(0,\infty). \tag{2.16}$$

Next, we solve the converse problem. Namely, the norm ϱ_D is now considered to be fixed and we are searching for the optimal ϱ_R. First of all we shall introduce some notation. We recall (2.14) and define

$$(Gg)(t) = e^{nt^{1/n}} \int_t^\infty g(s)s^{1/n-1}e^{-ns^{1/n}}\mathrm{d}s, \quad g \in \mathfrak{M}_+(0,\infty), \quad t > 0, \tag{2.17}$$

$$(Hh)(t) = t^{1/n-1}e^{-nt^{1/n}} \int_0^t h(s)e^{ns^{1/n}}\mathrm{d}s, \quad h \in \mathfrak{M}_+(0,\infty), \quad t > 0, \tag{2.18}$$

$$E(s) = e^{-ns^{1/n}} \int_0^s e^{nu^{1/n}}\mathrm{d}u, \quad s > 0. \tag{2.19}$$

The operators G and H are mutually dual in the following sense

$$\int_0^\infty h(t)Gg(t)\mathrm{d}t = \int_0^\infty g(u)Hh(u)\mathrm{d}u \quad \text{for all} \quad g,h \in \mathfrak{M}_+(0,\infty). \tag{2.20}$$

Theorem 2.8. *Assume that the r.i. norm ϱ_D satisfies*

$$\varrho_D\left(\int_s^\infty f(u)\frac{E(u)}{u}u^{1/n-1}\mathrm{d}u\right) \le c\varrho_D(f), \quad f \in \mathfrak{M}_+(0,\infty). \tag{2.21}$$

and that its dual norm ϱ_D' satisfies

$$\varrho_D'(H(h^{**})) \le c\varrho_D'(H(h^*)), \quad h \in \mathfrak{M}_+(0,\infty). \tag{2.22}$$

Then the optimal range norm in (2.12) associated to ϱ_D is given as the dual norm to $\varrho_D'(H(f^{**}))$. Or, equivalently, the dual of the optimal range norm can be described by $\varrho_R'(f) := \varrho_D'(H(f^{**}))$.

We also derive sufficient conditions for (2.21) and (2.22). In general, we follow the idea of [10, Theorem 4.4]. First of all, for every function $f \in \mathfrak{M}_+(0,\infty)$, we define the dilation operator E by

$$(E_sf)(t) = f(st), \quad t > 0, \quad s > 0.$$

It is well known, [3, Chapter 3, Prop. 5.11], that for every r.i. norm ϱ on $\mathfrak{M}_+(0,\infty)$ and every $s > 0$ the operator E_s satisfies

$$\varrho(E_sf) \le c\varrho(f), \quad f \in \mathfrak{M}_+(0,\infty).$$

The smallest possible constant c in this inequality (which depends of course on s) is denoted by $h_\varrho(s)$. Hence

$$h_\varrho(s) = \sup_{f \not\equiv 0} \frac{\varrho(E_sf)}{\varrho(f)}.$$

Using this notation, we may give a characterisation of (2.15) and (2.22).

Theorem 2.9. *If a rearrangement–invariant norm ϱ_R satisfies $\int_0^1 s^{-1/n}h_{\varrho_R}(s)\mathrm{d}s < \infty$, then it also satisfies (2.15).*

Theorem 2.10. *If an r.i. norm σ satisfies $\int_0^1 s^{-1/n} h_\sigma(s)\mathrm{d}s < \infty$ then it satisfies also (2.22) with ϱ_D' replaced by σ.*

We will now present some applications of our results.

Example 2.11. Let

$$\varrho_R(f) = \varrho_\infty(f) = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} |f(x)|.$$

Then $h_{\varrho_R}(s) = 1$ and, according to Theorem 2.9, (2.15) is satisfied and the optimal domain norm is given by

$$\varrho_D(f) \approx \sup_{t>0} (Gf^*)(t) = \int_0^\infty f^*(s) s^{1/n-1} e^{-ns^{1/n}} \mathrm{d}s, \quad f \in \mathfrak{M}(\mathbb{R}^n).$$

This norm is essentially smaller than $\varrho_{n,1}(f) = \int_0^\infty t^{1/n-1} f^*(t)\mathrm{d}t$, hence this result improves the second example from Remark 2.6. Now, an easy calculation shows that

$$\varrho_D(f) \approx f^*(1) + \int_0^1 f^*(t) t^{1/n-1}\mathrm{d}t \approx \varrho_\infty(f^*\chi_{(1,\infty)}) + \varrho_{n,1}(f^*\chi_{(0,1)}), \quad f \in \mathfrak{M}(\mathbb{R}^n).$$

Example 2.12. Let

$$\varrho_D(f) = \varrho_1(f) = \int_{\mathbb{R}^n} |f(x)|\mathrm{d}x.$$

In that case, $\varrho_D' = \varrho_\infty$, whence $h_{\varrho_D'}(s) = 1$. So, by Theorem 2.10, (2.22) is satisfied. It is a simple exercise to verify (2.21). Using Theorem 2.8, the optimal range norm can be described as the dual norm to

$$\sigma(f) = \varrho_\infty(Hf^*) = \varrho_\infty\left(t^{1/n-1} e^{-nt^{1/n}} \int_0^t f^*(s) e^{ns^{1/n}} \mathrm{d}s \right).$$

The optimal range norm

$$\varrho_R(g) = \sigma'(g) = \sup_{f:\varrho_\infty(Hf^*)\le 1} \int_0^\infty f^*(t) g^*(t)\mathrm{d}t,$$

is equivalent to

$$\varrho_R(g) = \sup_{f:\varrho_\infty(Hf^*)\le 1} \int_0^\infty f^*(t) g^*(t)\mathrm{d}t \approx \int_0^1 g^*(t) t^{-1/n}\mathrm{d}t + \int_1^\infty g^*(t)\mathrm{d}t.$$

Finally, we consider the case of limiting Sobolev embedding, where ϱ_D is set to be $\varrho_D(f) = \varrho_n(f) = \left(\int_{\mathbb{R}^n} |f(x)|^n \mathrm{d}x\right)^{1/n}$. In that case, $\varrho_D'(f) = \varrho_{n'}(f)$, where n' is the conjugated exponent to n, namely $\frac{1}{n} + \frac{1}{n'} = 1$. Direct calculation shows that $h_{\varrho_D'}(s) = s^{-1/n'}$ and $\int_0^1 s^{-1/n} h_{\varrho_D'}(s)\mathrm{d}s = \infty$. Moreover, standard examples ($h(s) = \frac{1}{s|\log s|^2}\chi_{(0,1/2)}(s)$) show that (2.22) is not satisfied.

To include this important case into the frame of our work, we develop a finer theory of an optimal range space. This is described in the following assertion.

Theorem 2.13. *Let ϱ_D be a given r.i. norm such that* (2.21) *holds and*

$$\varrho_D'(H\chi_{(0,1)}) < \infty. \tag{2.23}$$

Set

$$\sigma(h) = \varrho_D'(Hh^*), \quad h \in \mathfrak{M}_+(0,\infty).$$

Then,

$$\varrho_R := \sigma' \tag{2.24}$$

is an r.i. norm which satisfies (2.12) *and which is optimal for* (2.12)*.*

Let us apply Theorem 2.13 to the limiting Sobolev embeddings with

$$\varrho_D(f) = \varrho_n(f) = \Big(\int_0^\infty |f^*(t)|^n \mathrm{d}t\Big)^{1/n}.$$

It may be shown, that (2.23) and (2.21) are satisfied in this case. So, Theorem 2.13 is applicable and gives the optimal range norm. The result is presented in the next Theorem.

Theorem 2.14. *Let $\varrho_D = \varrho_n$. Then, the optimal range norm, ϱ_R, satisfies*

$$\varrho_R(f) \approx \varrho_n(f) + \lambda(f^*\chi_{(0,1)}), \tag{2.25}$$

where

$$\lambda(g) := \left(\int_0^1 \left(\frac{g^*(t)}{\log(\frac{e}{t})}\right)^n \frac{\mathrm{d}t}{t}\right)^{\frac{1}{n}}, \qquad g \in \mathfrak{M}(0,1).$$

Remark 2.15. We note that λ from Theorem 2.14 is the well-known norm discovered in various contexts independently by Maz'ya [17], Hanson [13] and Brézis–Wainger [5].

3 *A remark on better-lambda inequality* Math. Ineq. Appl. 10 (2007), 335-341.

The classical Riesz potentials are defined for every real number $0 < \gamma < n$ as a convolution operators $(I_\gamma f)(x) = (\tilde{I}_\gamma * f)(x)$, where $x \in \mathbb{R}^n$ and $\tilde{I}_\gamma(x) = |x|^{\gamma-n}$. This definition coincides with the usual one up to some multiplicative constant c_γ which is not interesting for our purpose. Burkholder and Gundy invented in [6] the technique involving distribution function later known as *good λ-inequality.* This inequality dealt with level sets of singular integral operators and of maximal operator. Later, Bagby and Kurtz discovered in [2] that the reformulation of good λ-inequality in terms of non-increasing rearrangement contains more information.

We generalise their approach in the following way. For every Young's function Φ satisfying the Δ_2-condition we define the Riesz potential

$$(I_\Phi f)(x) = \int_{\mathbb{R}^n} \tilde{\Phi}^{-1}\left(\frac{1}{|x-y|^n}\right) f(y)\mathrm{d}y,$$

where $\tilde{\Phi}$ is the Young's function conjugated to Φ and $\tilde{\Phi}^{-1}$ is its inverse. Instead of the classical Hardy-Littlewood maximal operator we work with a generalised maximal operator

$$(M_\varphi f)(x) = \sup_{Q \ni x} \frac{1}{\varphi(|Q|)} \int_Q |f(y)| \mathrm{d}y,$$

where φ is a given nonnegative function on $(0, \infty)$ and the supremum is taken over all cubes Q containing x with sides parallel to the coordinate axes such that $\varphi(|Q|) > 0$. For every measurable set $\Omega \subset \mathbb{R}^n$ we denote by $|\Omega|$ its Lebesgue measure.

We prove that under some restrictive conditions on function Φ one can obtain an inequality combining the nonincreasing rearrangement of $I_\Phi f$ and $M_{\tilde{\Phi}^{-1}} f$. We also show that this restrictive condition cannot be left out.

Definition 3.1. 1. Let $\phi : [0, \infty) \to [0, \infty)$ be a non-decreasing and right-continuous function with $\phi(0) = 0$ and $\lim_{t \to \infty} \phi(t) = \infty$. Then the function Φ defined by

$$\Phi(t) = \int_0^t \phi(s) \mathrm{d}s, \quad t \geq 0$$

is said to be a *Young's function*.

2. A Young's function is said to satisfy the Δ_2—*condition* if there is $c > 0$ such that

$$\Phi(2t) \leq c \, \Phi(t), \quad t \geq 0.$$

3. A Young's function is said to satisfy the ∇_2—*condition* if there is $l > 1$ such that

$$\Phi(t) \leq \frac{1}{2l} \Phi(lt), \quad t \geq 0.$$

4. Let Φ be a Young's function, represented as the indefinite integral of ϕ. Let

$$\psi(s) = \sup\{u : \phi(u) \leq s\}, \quad s \geq 0.$$

Then the function

$$\tilde{\Phi}(t) = \int_0^t \psi(s) \mathrm{d}s, \quad t \geq 0,$$

is called the *complementary Young's function* of Φ.

Assume now that a Young's function Φ satisfies the Δ_2—condition. Using the classical O'Neil inequality (see [18]) we obtain

$$(I_\Phi f)^*(t) \leq c \left\{ \tilde{\Phi}^{-1} \left(\frac{1}{t} \right) \int_0^t f^*(u) \mathrm{d}u + \int_t^\infty f^*(u) \tilde{\Phi}^{-1} \left(\frac{1}{u} \right) \mathrm{d}u \right\}, \tag{3.1}$$

We shall derive a better λ-inequality connecting the operators I_Φ and $M_{\tilde{\Phi}^{-1}}$.

Theorem 3.2. *Let us suppose that a Young's function Φ satisfies the Δ_2—condition. Let us further suppose that there is a constant $c_1 > 0$ such that*

$$\tilde{\Phi}^{-1}(s)\tilde{\Phi}^{-1}(1/s) < c_1, \quad s > 0. \tag{3.2}$$

Then there is a constant $c_2 > 0$, such that for every function f and every positive number t

$$(I_\Phi f)^*(t) \leq (I_\Phi |f|)^*(t) \leq c_2 \,(M_{\tilde{\Phi}^{-1}} f)^*(t/2) + (I_\Phi |f|)^*(2t) \tag{3.3}$$

In the following example we will show that the assumption (3.2) cannot be omitted.

Theorem 3.3. *There is a Young's function Φ satisfying the Δ_2—condition for which*

$$\sup_{f,t>0} \frac{(I_\Phi f)^*(t) - (I_\Phi f)^*(2t)}{(M_{\tilde{\Phi}^{-1}} f)^*(t/2)} = \infty.$$

4 *A new proof of Jawerth-Franke embedding* to appear in Rev. Mat. Complut.

In this paper, we considered an analogue of a Sobolev embedding generalised to Besov and Triebel-Lizorkin spaces. Let us first give their definition.

Let $S(\mathbb{R}^n)$ be the Schwartz space of all complex-valued rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^n$ and let $S'(\mathbb{R}^n)$ be its dual - the space of all tempered distributions.

endowed with the norm For $\psi \in S(\mathbb{R}^n)$ we denote by

$$\widehat{\psi}(\xi) = (F\psi)(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^n} e^{-i<x,\xi>}\psi(x)\mathrm{d}x, \quad x \in \mathbb{R}^n,$$

its Fourier transform and by $\psi^\vee$ or $F^{-1}\psi$ its inverse Fourier transform.

We give a Fourier-analytic definition of Besov and Triebel-Lizorkin spaces, which relies on the so-called *dyadic resolution of unity*. Let $\varphi \in S(\mathbb{R}^n)$ with

$$\varphi(x) = 1 \quad \text{if} \quad |x| \leq 1 \quad \text{and} \quad \varphi(x) = 0 \quad \text{if} \quad |x| \geq \frac{3}{2}. \tag{4.1}$$

We put $\varphi_0 = \varphi$ and $\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$ for $j \in \mathbb{N}$ and $x \in \mathbb{R}^n$. This leads to the identity

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1, \qquad x \in \mathbb{R}^n.$$

Definition 4.1. (i) Let $s \in \mathbb{R}, 1 \leq p, q \leq \infty$. Then $B^s_{pq}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$||f|B^s_{pq}(\mathbb{R}^n)|| = \left(\sum_{j=0}^{\infty} 2^{jsq}||(\varphi_j \widehat{f})^\vee|L_p(\mathbb{R}^n)||^q\right)^{1/q} < \infty \tag{4.2}$$

(with the usual modification for $q = \infty$).

(ii) Let $s \in \mathbb{R}, 1 \le p < \infty, 1 \le q \le \infty$. Then $F^s_{pq}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$||f|F^s_{pq}(\mathbb{R}^n)|| = \left|\left| \left(\sum_{j=0}^{\infty} 2^{jsq} |(\varphi_j \widehat{f})^\vee(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n) \right|\right| < \infty \tag{4.3}$$

(with the usual modification for $q = \infty$).

Remark 4.2. These spaces have a long history. In this context we recommend [19], [24], [25] and [26] as standard references. We point out that the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ are independent of the choice of ψ in the sense of equivalent norms. Special cases of these two scales include Lebesgue spaces, Sobolev spaces, Hölder-Zygmund spaces and many other important function spaces. We omit any detailed discussion.

The classical Sobolev embedding theorem can be extended to these two scales.

Theorem 4.3. *Let* $-\infty < s_1 < s_0 < \infty$ *and* $0 < p_0 < p_1 \le \infty$ *with*

$$s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}. \tag{4.4}$$

(i) If $0 < q_0 \le q_1 \le \infty$*, then*

$$B^{s_0}_{p_0q_0}(\mathbb{R}^n) \hookrightarrow B^{s_1}_{p_1q_1}(\mathbb{R}^n). \tag{4.5}$$

(ii) If $0 < q_0, q_1 \le \infty$ *and* $p_1 < \infty$*, then*

$$F^{s_0}_{p_0q_0}(\mathbb{R}^n) \hookrightarrow F^{s_1}_{p_1q_1}(\mathbb{R}^n). \tag{4.6}$$

We observe that there is no condition on the fine parameters q_0, q_1 in (4.6). This surprising effect was first observed in full generality by Jawerth, [14]. Using (4.6), we may prove

$$F^{s_0}_{p_0q}(\mathbb{R}^n) \hookrightarrow F^{s_1}_{p_1p_1}(\mathbb{R}^n) = B^{s_1}_{p_1p_1}(\mathbb{R}^n) \quad \text{and} \quad B^{s_0}_{p_0p_0}(\mathbb{R}^n) = F^{s_0}_{p_0p_0}(\mathbb{R}^n) \hookrightarrow F^{s_1}_{p_1q}(\mathbb{R}^n)$$

for every $0 < q \le \infty$. But Jawerth ([14]) and Franke ([12]) showed that these embeddings are not optimal and may be improved.

Theorem 4.4. *Let* $-\infty < s_1 < s_0 < \infty$*,* $0 < p_0 < p_1 \le \infty$ *and* $0 < q \le \infty$ *with* (4.4)*.*

(i) Then

$$F^{s_0}_{p_0q}(\mathbb{R}^n) \hookrightarrow B^{s_1}_{p_1p_0}(\mathbb{R}^n). \tag{4.7}$$

(ii) If $p_1 < \infty$*, then*

$$B^{s_0}_{p_0p_1}(\mathbb{R}^n) \hookrightarrow F^{s_1}_{p_1q}(\mathbb{R}^n). \tag{4.8}$$

The original proofs (see [14] and [12]) use interpolation techniques. We rely on a different method. First, we observe that using (for example) the wavelet decomposition method, (4.7) and (4.8) is equivalent to

$$f^{s_0}_{p_0q} \hookrightarrow b^{s_1}_{p_1p_0} \quad \text{and} \quad b^{s_0}_{p_0p_1} \hookrightarrow f^{s_1}_{p_1q} \tag{4.9}$$

under the same restrictions on parameters s_0, s_1, p_0, p_1, q as in Theorem 4.4. Here, b^s_{pq} and f^s_{pq} stands for the sequence spaces of Besov and Triebel-Lizorkin type. We prove (4.9) directly using the technique of the non-increasing rearrangement on a rather elementary level.

We introduce the sequence spaces associated with the Besov and Triebel-Lizorkin spaces. Let $m \in \mathbb{Z}^n$ and $\nu \in \mathbb{N}_0$. Then $Q_{\nu m}$ denotes the closed cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes, centred at $2^{-\nu}m$, and with side length $2^{-\nu}$. By $\chi_{\nu m} = \chi_{Q_{\nu m}}$ we denote the characteristic function of $Q_{\nu m}$. If

$$\lambda = \{\lambda_{\nu m} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\},$$

$-\infty < s < \infty$ and $0 < p, q \le \infty$, we set

$$||\lambda|b^s_{pq}|| = \Big(\sum_{\nu=0}^{\infty} 2^{\nu(s-\frac{n}{p})q}\Big(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|^p\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}} \tag{4.10}$$

appropriately modified if $p = \infty$ and/or $q = \infty$. If $p < \infty$, we define also

$$||\lambda|f^s_{pq}|| = \Bigg|\Bigg|\Big(\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n} |2^{\nu s}\lambda_{\nu m}\chi_{\nu m}(\cdot)|^q\Big)^{1/q}|L_p(\mathbb{R}^n)\Bigg|\Bigg|. \tag{4.11}$$

The connection between the function spaces $B^s_{pq}(\mathbb{R}^n)$, $F^s_{pq}(\mathbb{R}^n)$ and the sequence spaces b^s_{pq}, f^s_{pq} may be given by various decomposition techniques, we refer to [26, Chapters 2 and 3] for details and further references.

As a result of these characterisations, (4.7) and (4.8) are equivalent to (4.9).

We gave a new proof of Theorem 4.4. Instead of interpolation, we used the technique of the non-increasing rearrangement on a rather elementary level. It means, we gave the direct proof of the following embedding theorems for sequence spaces of Besov and Triebel-Lizorkin type.

Theorem 4.5. *Let* $-\infty < s_1 < s_0 < \infty$, $0 < p_0 < p_1 \le \infty$ *and* $0 < q \le \infty$. *Then*

$$f^{s_0}_{p_0 q} \hookrightarrow b^{s_1}_{p_1 p_0} \quad \text{if} \quad s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}. \tag{4.12}$$

Theorem 4.6. *Let* $-\infty < s_1 < s_0 < \infty, 0 < p_0 < p_1 < \infty$ *and* $0 < q \le \infty$. *Then*

$$b^{s_0}_{p_0 p_1} \hookrightarrow f^{s_1}_{p_1 q} \quad \text{if} \quad s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}. \tag{4.13}$$

Theorems 4.5 and 4.6 are sharp in the following sense.

Theorem 4.7. *Let* $-\infty < s_1 < s_0 < \infty$, $0 < p_0 < p_1 \le \infty$ *and* $0 < q_0, q_1 \le \infty$ *with*

$$s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}.$$

(i) If

$$f^{s_0}_{p_0 q_0} \hookrightarrow b^{s_1}_{p_1 q_1}, \tag{4.14}$$

then $q_1 \ge p_0$.

(ii) If $p_1 < \infty$ *and*

$$b^{s_0}_{p_0 q_0} \hookrightarrow f^{s_1}_{p_1 q_1}, \tag{4.15}$$

then $q_0 \le p_1$.

Remark 4.8. Using (any of) the usual decomposition techniques, the same statements hold true also for the function spaces. These results were first proved in [21].

5 *Sampling numbers and function spaces* J. Compl. 23 (2007), 773-792.

If the inequality in (1.1) is strict, then the embedding (1.2) is compact. The quality of this compactness may be in some sense described by many techniques. We mention at least the approximation numbers, Gelfand numbers or entropy numbers. We shall concentrate on other approximation quantities, namely the so-called *sampling numbers*.

First, we give the definition of Besov and Triebel-Lizorkin spaces on domains.

Let Ω be a bounded domain. Let $D(\Omega) = C_0^\infty(\Omega)$ be the collection of all complex-valued infinitely-differentiable functions with compact support in Ω and let $D'(\Omega)$ be its dual - the space of all complex-valued distributions on Ω.

Let $g \in S'(\mathbb{R}^n)$. Then we denote by $g|\Omega$ its restriction to Ω:

$$(g|\Omega) \in D'(\Omega), \qquad (g|\Omega)(\psi) = g(\psi) \quad \text{for} \quad \psi \in D(\Omega).$$

Definition 5.1. Let Ω be a bounded domain in $\mathbb{R}^n$. Let $s \in \mathbb{R}$, $1 \le p, q \le \infty$ with $p < \infty$ in the F-case. Let A^s_{pq} stand either for B^s_{pq} or F^s_{pq}. Then

$$A^s_{pq}(\Omega) = \{f \in D'(\Omega) : \exists g \in A^s_{pq}(\mathbb{R}^n) : g|\Omega = f\}$$

and

$$||f|A^s_{pq}(\Omega)|| = \inf ||g|A^s_{pq}(\mathbb{R}^n)||,$$

where the infimum is taken over all $g \in A^s_{pq}(\mathbb{R}^n)$ such that $g|\Omega = f$.

We now introduce the concept of sampling numbers.

Definition 5.2. Let Ω be a bounded Lipschitz domain. Let $G_1(\Omega)$ be a space of continuous functions on Ω and $G_2(\Omega) \subset D'(\Omega)$ be a space of distributions on Ω. Suppose that the embedding

$$id : G_1(\Omega) \hookrightarrow G_2(\Omega)$$

is compact.

For $\{x_j\}_{j=1}^k \subset \Omega$ we define the *information map*

$$N_k : G_1(\Omega) \to \mathbb{C}^n, \qquad N_k f = (f(x_1), \dots, f(x_k)), \quad f \in G_1(\Omega).$$

For any (linear or nonlinear) mapping $\varphi_n : \mathbb{C}^k \to G_2(\Omega)$ we consider

$$S_k : G_1(\Omega) \to G_2(\Omega), \qquad S_k = \varphi_k \circ N_k.$$

(i) Then, for all $k \in \mathbb{N}$, the $k-$th *sampling number* $g_k(id)$ is defined by

$$g_k(id) = \inf_{S_k} \sup\{||f - S_k f|G_2(\Omega)|| : ||f|G_1(\Omega)|| \le 1\}, \tag{5.1}$$

where the infimum is taken over all k-tuples $\{x_j\}_{j=1}^k \subset \Omega$ and all (linear or nonlinear) φ_k.

(ii) For all $k \in \mathbb{N}$ the $k-$th *linear sampling number* $g_k^{\text{lin}}(id)$ is defined by (5.1), where now only linear mappings φ_k are admitted.

The study of sampling numbers of the Sobolev embeddings of spaces of Besov and Triebel-Lizorkin type is dived into three steps.

Step 1: **The case** $s_2 > 0$

In this subsection, we discuss the case where $\Omega = I^n = (0,1)^n$ is the unit cube, $G_1(\Omega) = A^{s_1}_{p_1q_1}(\Omega)$ and $G_2(\Omega) = A^{s_2}_{p_2q_2}(\Omega)$ with $s_1 > \frac{n}{p_1}$ and $s_1 - n\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big)_+ > s_2 > 0$. Here, $A^s_{pq}(\Omega)$ stands either for a Besov space $B^s_{pq}(\Omega)$ or a Triebel-Lizorkin space $F^s_{pq}(\Omega)$, see Definition 5.1 for details. We start with the most simple and most important case, namely when $p_1 = p_2 = q_1 = q_2$.

Proposition 5.3. *Let $\Omega = I^n = (0,1)^n$. Let $G_1(\Omega) = B^{s_1}_{pp}(\Omega)$ and $G_2(\Omega) = B^{s_2}_{pp}(\Omega)$ with $1 \le p \le \infty$,*

$$s_1 > \frac{n}{p}, \quad \textit{and} \quad s_1 > s_2 > 0.$$

Then

$$g_k^{\text{lin}}(id) \lesssim k^{-\frac{s_1-s_2}{n}}.$$

The proof of this statement requires unfortunately several techniques from the theory of function spaces like characterisation by differences, local polynomial approximation and multiplier assertions. See [29] for details.

Using the real interpolation method, the results could be easily extended.

Proposition 5.4. *Let $\Omega = I^n = (0,1)^n$. Let $G_1(\Omega) = A^{s_1}_{p_1q_1}(\Omega)$ and $G_2(\Omega) = A^{s_2}_{p_2q_2}(\Omega)$ with $1 \le p_1, p_2, q_1, q_2 \le \infty$ ($p_1, p_2 < \infty$ in the F-case),*

$$s_1 > \frac{n}{p_1}, \quad \textit{and} \quad s_1 - n\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big)_+ > s_2 > 0. \tag{5.2}$$

Then

$$g_k^{\text{lin}}(id) \lesssim k^{-\frac{s_1-s_2}{n} + \left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+}. \tag{5.3}$$

It turns out, that these estimates are sharp. Namely, we have

Theorem 5.5. *Let $\Omega = I^n = (0,1)^n$. Let $G_1(\Omega) = A^{s_1}_{p_1q_1}(\Omega)$ and $G_2(\Omega) = A^{s_2}_{p_2q_2}(\Omega)$ with $1 \le p_1, p_2, q_1, q_2 \le \infty$ ($p_1, p_2 < \infty$ in the F-case) and (5.2) Then*

$$g_k(id) \approx g_k^{\text{lin}}(id) \approx k^{-\frac{s_1-s_2}{n} + \left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+}. \tag{5.4}$$

Step 2: **The case** $s_2 = 0$

In the case $s_2 = 0$, new phenomena come into play. The same method can be applied also in this case. Unfortunately, there appears a gap between the estimates from below and from above. The exact formulation is as follows.

Theorem 5.6. *Let $\Omega = I^n = (0,1)^n$. Let*

$$id : G_1(\Omega) \hookrightarrow G_2(\Omega)$$

with

$$G_1(\Omega) = B^s_{p_1 q_1}, \quad G_2(\Omega) = B^0_{p_2 q_2}$$

and

$$1 \le p_1, q_1, p_2, q_2 \le \infty, \quad s > \frac{n}{p_1}.$$

Then

$$k^{-\frac{s}{n} + (\frac{1}{p_1} - \frac{1}{p_2})_+} \lesssim g_k(id) \lesssim g_k^{\text{lin}}(id) \lesssim k^{-\frac{s}{n} + (\frac{1}{p_1} - \frac{1}{p_2})_+} (1 + \log k)^{1/q_2}, \qquad k \in \mathbb{N}. \tag{5.5}$$

This effect was studied in detail in [30], see below.

Step 3: **The case $s_2 < 0$**

As in the last case, we consider the situation when $s_2 < 0$.

Theorem 5.7. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Let*

$$id : G_1(\Omega) = A^{s_1}_{p_1 q_1}(\Omega) \hookrightarrow G_2(\Omega) = A^{s_2}_{p_2 q_2}(\Omega)$$

with $1 \le p_1, p_2, q_1, q_2 \le \infty$ (with $p_1, p_2 < \infty$ in the F-case) and

$$s_1 > \frac{n}{p_1}, \qquad s_2 < 0.$$

If $p_1 \ge p_2$, then

$$g_k(id) \approx g_k^{\text{lin}}(id) \approx k^{-\frac{s_1}{n}}. \tag{5.6}$$

If $p_1 < p_2$ and $s_2 > \frac{n}{p_2} - \frac{n}{p_1}$, then

$$g_k(id) \approx g_k^{\text{lin}}(id) \approx k^{-\frac{s_1}{n} + \frac{s_2}{n} + \frac{1}{p_1} - \frac{1}{p_2}}. \tag{5.7}$$

If $p_1 < p_2$ and $\frac{n}{p_2} - \frac{n}{p_1} > s_2$, then

$$g_k(id) \approx g_k^{\text{lin}}(id) \approx k^{-\frac{s_1}{n}}. \tag{5.8}$$

These estimates can be applied in connection with elliptic differential operators, which was the actual motivation for this research, c.f. [8] and [9]. Let us briefly introduce this setting. Let

$$\mathcal{A} : H \to G$$

be a bounded linear operator from a Hilbert space H to another Hilbert space G. We assume that $\mathcal{A}$ is boundedly invertible, hence

$$\mathcal{A}(u) = f$$

has a unique solution for every $f \in G$. A typical application is an operator equation, where $\mathcal{A}$ is an elliptic differential operator, and we assume that

$$\mathcal{A} : H_0^s(\Omega) \to H^{-s}(\Omega),$$

where Ω is a bounded Lipschitz domain, $H_0^s(\Omega)$ is a function space of Sobolev type with fractional order of smoothness $s > 0$ of functions vanishing on the boundary and H^{-s} is a function space of Sobolev type with negative smoothness $-s < 0$. The classical example is the Poisson equation

$$-\Delta u = f \quad \text{in} \quad \Omega \qquad \text{and} \qquad u = 0 \quad \text{on} \quad \partial\Omega.$$

Here, $s = 1$ and

$$\mathcal{A} = -\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$$

is bounded and boundedly invertible. We want to approximate the *solution operator* $u = S(f)$ using only function values of f.

We define the k-th linear sampling number of the identity $id : H^{-1+t}(\Omega) \to H^{-1}(\Omega)$ by

$$g_k^{\text{lin}}(id : H^{-1+t}(\Omega) \to H^{-1}(\Omega)) = \inf_{S_k} ||id - S_k|\mathcal{L}(H^{-1+t}(\Omega), H^{-1}(\Omega))||, \tag{5.9}$$

where t is a positive real number with $-1 + t > \frac{n}{2}$, and the k-th linear sampling number of $S : H^{-1+t}(\Omega) \to H^1(\Omega)$ by

$$g_k^{\text{lin}}(S : H^{-1+t}(\Omega) \to H^1(\Omega)) = \inf_{S_k} ||S - S_k|\mathcal{L}(H^{-1+t}(\Omega), H^1(\Omega))||. \tag{5.10}$$

The infimum in (5.9) and (5.10) runs over all linear operators S_k of the form (1.1) and $\mathcal{L}(X, Y)$ stands for the space of bounded linear operators between two Banach spaces X and Y, equipped with the classical operator norm.

It turns out that these quantities are equivalent (up to multiplicative constants which do not depend neither on f nor on k) and are of the asymptotic order

$$g_k^{\text{lin}}(S : H^{-1+t}(\Omega) \to H^1(\Omega)) \approx g_k^{\text{lin}}(id : H^{-1+t}(\Omega) \to H^{-1}(\Omega)) \approx k^{-\frac{-1+t}{n}}.$$

We refer to [8] and [9] for a detailed discussion of this approach. The estimates of sampling numbers of an embedding between two function spaces translates therefor into estimates of sampling numbers of the solution operator S. We observe that the more regular f, the faster is the decay of the linear sampling numbers of the solution operator S. Let us also point out that optimal linear methods (not restricted to use only the function values of f) achieve asymptotically a better rate of convergence, namely $k^{-\frac{t}{n}}$. Hence, the limitation to the sampling operators results in a serious restriction. One has to pay at least $k^{1/n}$ in comparison with optimal linear methods.

Using our estimates of sampling numbers of identities between Besov and Triebel-Lizorkin spaces, this result may be generalised as follows.[1] If $p \geq 2$, $1 \leq q \leq \infty$ and $-1 + t > \frac{d}{p}$ then

$$g_k^{\text{lin}}(S : B_{pq}^{-1+t}(\Omega) \to H^1(\Omega)) \approx g_k^{\text{lin}}(id : B_{pq}^{-1+t}(\Omega) \to H^{-1}(\Omega)) \approx k^{-\frac{-1+t}{n}}.$$

[1] Although the results are stated only for Besov spaces, they are proved also for Triebel-Lizorkin spaces, which include also fractional Sobolev spaces as a special case.

If $p < 2$ with $\dfrac{1}{p} > \dfrac{1}{n} + \dfrac{1}{2}$, $1 \le q \le \infty$ and $-1 + t > \frac{n}{p}$ then

$$g_k^{\text{lin}}(S : B_{pq}^{-1+t}(\Omega) \to H^1(\Omega)) \approx g_k^{\text{lin}}(id : B_{pq}^{-1+t}(\Omega) \to H^{-1}(\Omega)) \approx k^{-\frac{t}{n}+\frac{1}{p}-\frac{1}{2}}.$$

Finally, if $p < 2$ with $\dfrac{1}{p} < \dfrac{1}{n} + \dfrac{1}{2}$, $1 \le q \le \infty$ and $-1 + t > \frac{n}{p}$ then

$$g_k^{\text{lin}}(S : B_{pq}^{-1+t}(\Omega) \to H^1(\Omega)) \approx g_k^{\text{lin}}(id : B_{pq}^{-1+t}(\Omega) \to H^{-1}(\Omega)) \approx k^{-\frac{-1+t}{n}}.$$

We prove the same results also for the nonlinear sampling numbers $g_k(S)$. Altogether, the regularity information of f may now be described by an essentially broader scale of function spaces.

6 *Dilation operators and sampling numbers* to appear in J. of Function Spaces and Appl.

This paper is divided into two parts. In the first part, we consider the dilation operators

$$T_k : f \to f(2^k \cdot), \quad k \in \mathbb{N},$$

in the framework of Besov spaces $B_{pq}^s(\mathbb{R}^n)$. Their behaviour is well known if $1 \le p, q \le \infty$ and $s > 0$, cf. [11, 2.3.1]. As mentioned there, the case $s = 0$ remained open. Some partial results can be found in [4]. For $1 \le p, q \le \infty$ we supply the final answer to this problem showing that

$$||T_k|\mathcal{L}(B_{pq}^0(\mathbb{R}^n))|| \approx 2^{-k\frac{d}{p}} \cdot \begin{cases} k^{\frac{1}{q}-\frac{1}{p}}, & \text{if } 1 < p < \infty \text{ and } p \ge \max(q,2), \\ k^{\frac{1}{q}-\frac{1}{2}}, & \text{if } 1 < p < \infty \text{ and } 2 \ge \max(p,q), \\ 1, & \text{if } 1 < p < \infty \text{ and } q \ge \max(p,2), \\ k^{\frac{1}{q}}, & \text{if } p = 1 \text{ or } p = \infty, \end{cases} \tag{6.1}$$

where $||T_k|\mathcal{L}(B_{pq}^0(\mathbb{R}^n))||$ denotes the norm of the operator T_k from $B_{pq}^0(\mathbb{R}^n)$ into itself. One observes that for $1 < p < \infty$ the number 2 plays an exceptional role. This effect has its origin in the Littlewood-Paley decomposition theorem.

The second part of the paper deals with applications to estimates of sampling numbers. Let us briefly sketch this approach.

Let $\Omega = (0,1)^n$ and let $B_{pq}^s(\Omega)$ denote the Besov spaces on Ω, see Definition 5.1 for details. We try to approximate $f \in B_{p_1q_1}^{s_1}(\Omega)$ in the norm of another Besov space, say $B_{p_2q_2}^{s_2}(\Omega)$, by a linear sampling method

$$S_k f = \sum_{j=1}^{n} f(x_j) h_j, \tag{6.2}$$

where $h_j \in B_{p_2q_2}^{s_2}(\Omega)$ and $x_j \in \Omega$. To give a meaning to the pointwise evaluation in (6.2), we suppose that

$$s_1 > \frac{n}{p_1}.$$

Then the embedding $B^{s_1}_{p_1q_1}(\Omega) \hookrightarrow C(\bar{\Omega})$ holds true and the pointwise evaluation represents a bounded operator. Second, we always assume that the embedding $B^{s_1}_{p_1q_1}(\Omega) \hookrightarrow B^{s_2}_{p_2q_2}(\Omega)$ is compact. This is true if, and only if,

$$s_1 - s_2 > n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ .$$

Concerning the parameters p_1, p_2, q_1, q_2 we always assume that they belong to $[1, \infty]$.

We measure the worst case error of $S_k f$ on the unit ball of $B^{s_1}_{p_1q_1}(\Omega)$, given by

$$\sup\{||f - S_k f|B^{s_2}_{p_2q_2}(\Omega)|| : ||f|B^{s_1}_{p_1q_1}(\Omega)|| \leq 1\}. \tag{6.3}$$

The same worst case error may be considered also for nonlinear sampling methods

$$S_k f = \varphi(f(x_1), \dots, f(x_k)), \tag{6.4}$$

where $\varphi : \mathbb{C}^k \to B^{s_2}_{p_2q_2}(\Omega)$ is an arbitrary mapping. We shall discuss the decay of (6.3) for linear (6.2) and nonlinear (6.4) sampling methods.

The case $s_2 \neq 0$ was considered in [29], but the interesting limiting case $s_2 = 0$ was left open so far. It is the aim of this paper to close this gap. It was already pointed out in [29], see especially (2.6) in [29] for details, that the estimates from above for the dilation operators T_k on the target space $B^{s_2}_{p_2q_2}(\mathbb{R}^n)$ have their direct counterparts in estimates from above for the decay of sampling numbers. Using this method, which will not be repeated here, a direct application of (6.1) supplies the estimates

$$g^{\text{lin}}_k(id) \lesssim k^{-\frac{s}{d}} \cdot \begin{cases} (\log k)^{\frac{1}{q_2} - \frac{1}{p}}, & \text{if } 1 < p < \infty \text{ and } p \geq \max(q_2, 2), \\ (\log k)^{\frac{1}{q_2} - \frac{1}{2}}, & \text{if } 1 < p < \infty \text{ and } 2 \geq \max(p, q_2), \\ 1, & \text{if } 1 < p < \infty \text{ and } q_2 \geq \max(p, 2), \\ (\log k)^{\frac{1}{q_2}}, & \text{if } p = 1 \text{ or } p = \infty, \end{cases} \tag{6.5}$$

where $g^{\text{lin}}_k(id)$ with $2 \leq k \in \mathbb{N}$ are the linear sampling numbers of the embedding

$$id : B^s_{pq_1}(\Omega) \to B^0_{pq_2}(\Omega), \qquad s > \frac{n}{p}.$$

Surprisingly, all estimates in (6.5) are sharp.

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[30] J. Vybíral, *Dilation operators and sampling numbers*, to appear in J. of Function Spaces and Appl.

[31] J. Vybíral, *A new proof of Jawerth-Franke embedding*, to appear in Rev. Mat. Complut.

7 Publications relevant to the thesis

- *Optimal Sobolev embeddings on* $\mathbb{R}^n$
  Publ. Mat. 51 (2007), 17-44.
- *A remark on better-lambda inequality*
  Math. Ineq. Appl. 10 (2007), 335-341.
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