Decomposition methods and their applications in the theory of function spaces

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Zusammenfassung der Habilitationsschrift

Decomposition methods and their applications in the theory of function spaces

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Die vorgelegte kumulative Habilitationsschrift *Decomposition methods and their applications in the theory of function spaces* (*Zerlegungstechniken und ihre Anwendungen in der Theorie der Funktionenräume*) präsentiert die Ergebnisse der Arbeiten [1–7].

Studiert werden isotrope Funktionenräume vom Besov und Triebel-Lizorkin Typ sowie einige ihrer Varianten, wie Räume mit dominierend gemischten Glattheitseigenschaften und Räume mit variabler Glattheit und Integrabilität). Die Fourier-analytische Definition der isotropen Funktionenräume $B^s_{p,q}(\mathbb{R}^d)$ und $F^s_{p,q}(\mathbb{R}^d)$ benutzt eine glatte Zerlegung der Eins $(\varphi_j)_{j=0}^\infty$. Daraus ergibt sich eine Zerlegung einer Distribution $f \in S'(\mathbb{R}^d)$

$$f = \sum_{j=0}^\infty \mathcal{F}^{-1}(\varphi_j \mathcal{F} f), \quad \text{Konvergenz in} \quad S'(\mathbb{R}^d).$$

Hierbei steht $\mathcal{F}$ für die Fouriertransformation auf $S'(\mathbb{R}^d)$ und $\mathcal{F}^{-1}$ für ihre Inverse. Die Benutzung der Fouriertransformation direkt in der Definition der betrachteten Funktionenräume hat viele Vorteile für zahlreiche Anwendungen, insbesondere in der Theorie der partiellen Differentialgleichungen.

Erst in den siebziger und achtziger Jahren des 20. Jahrhunderts wurde auch klar, dass die guten Eigenschaften der Besov und Triebel-Lizorkin Skalen eine atomare Charakterisierung dieser Räume erlauben. Die erste Charakterisierung von diesem Typ scheint die von Coifman [8] zu sein. Er betrachtet die Hardy Räume $H^p(\mathbb{R})$ mit $0 < p \le 1$ und beweist, dass eine Distribution $f \in S'(\mathbb{R})$ dann und nur dann in $H^p(\mathbb{R})$ liegt, wenn eine Zerlegung

$$f = \sum_{i=0}^\infty \alpha_i b_i$$

existiert, wobei α_i reelle Zahlen sind, b_i so genannte p-Atome und

$$A\|f|H^p(\mathbb{R})\|^p \le \sum_{i=0}^\infty |\alpha_i|^p \le B\|f|H^p(\mathbb{R})\|^p$$

gilt. Die Konstanten $A, B > 0$ hängen hier nur von p ab.

Durch die Arbeiten von Frazier und Jawerth [9] und [10] wurde diese Technik auch auf viele andere Räume angewendet - im Prinzip auf alle die Räume, welche Triebel in seinem Buch [11] als *gute* Räume bezeichnet. Das sind alle Funktionenräume, die genug Fourier-Multiplikatoren besitzen. Leider zeigte sich, dass viele klassische Räume ($C^m(\mathbb{R}^d)$, $L_1(\mathbb{R}^d)$, $L_\infty(\mathbb{R}^d)$ oder $BV(\mathbb{R}^d)$) diese Definition nicht erfüllen und diese werden deshalb von Triebel als *schlechte* Räume bezeichnet.

Ein weiterer Durchbruch wurde dann in den Arbeiten von Daubechies [12, 13] erzielt. In diesen wird die Konstruktion von Wavelets mit kompaktem Träger beschrieben, welche in der angewandten Mathematik sehr populär geworden sind. Später wurden auch Wavelet-Charakterisierungen von Besov und Triebel-Lizorkin Räumen mit Hilfe von Daubechies-Wavelets gezeigt, vgl. [14] und [15].

Die vorgelegten Arbeiten [1–7] präsentieren die Vor- und Nachteile von einigen dieser Zerlegungen und beschreiben ihre Anwendungen in der Theorie der Funktionenräume.

In der ersten Arbeit [1] werden die Zerlegungstechniken ausgenutzt, um Eigenschaften kompakter Einbettungsoperatoren zwischen isotropen Besov und Triebel-Lizorkin Räumen zu studieren. Falls Ω ein Lipschitz Gebiet ist und die Ungleichung

$$s_1 - s_2 > d\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big)_+$$

erfüllt ist, dann ist die Einbettung

$$\mathcal{I}d : B^{s_1}_{p_1q_1}(\Omega) \to B^{s_2}_{p_2q_2}(\Omega)$$

kompakt. Die Theorie der s-Zahlen (cf. [16]) ordnet dann dem Operator $\mathcal{I}d$ eine monoton-fallende Folge reeller Zahlen $s_n(\mathcal{I}d)$ zu. Das Abfallverhalten dieser Folge beschreibt dann gewisse geometrische Eigenschaften des Operators $\mathcal{I}d$. Das Konzept von s-Zahlen enthält insbesondere die Approximationszahlen, Kolmogorov-Zahlen und Gelfand-Zahlen. Das Verhalten von $s_n(\mathcal{I}d)$ wurde in den letzten dreißig Jahren ausführlich studiert. Die Arbeit [1] sammelt die bekannten Ergebnisse auf dem Gebiet und schließt einige gebliebene Lücken dieser Theorie. Hierbei ist der Ansatz bei allen drei oben genannten Typen von s-Zahlen gleich - die Abschätzung von $s_n(\mathcal{I}d : B^{s_1}_{p_1q_1}(\Omega) \to B^{s_2}_{p_2q_2}(\Omega))$ wird mit Hilfe der Zerlegungstechniken auf Abschätzung von $s_n(id : b^{s_1,\Omega}_{p_1q_1} \to b^{s_2,\Omega}_{p_2q_2})$ reduziert. Hierbei sind die Folgenräume $b^{s,\Omega}_{pq}$ viel anschaulicher und einfacher zu behandeln.

Der zweite Artikel [2] beschäftigt sich mit anderen Eigenschaften isotroper Besov und Triebel-Lizorkin Räume. Falls X ein (quasi-)Banachraum von lokal integrierbaren Funktionen ist, dann definiert man die sogenannte *Wachstums-Envelope-Funktion* (*growth envelope function*) von X als

$$\mathcal{E}^X_G(t) := \sup_{\|f|X\|\le 1} f^*(t), \quad 0 < t < 1.$$

Hier steht f^* für die nicht-wachsende Umordnung von f.

Falls für $0 < t < 1$ und ein $\alpha > 0$ die Relation $\mathcal{E}^X_G(t) \approx t^{-\alpha}$ erfüllt ist, dann definiert man den *Wachstums-Envelope-Index* (*growth envelope index*) u_X als Infimum aller Zahlen $v \in (0,\infty]$, so dass

$$(1) \qquad \left(\int_0^\epsilon \left[\frac{f^*(t)}{\mathcal{E}^X_G(t)}\right]^v \frac{dt}{t}\right)^{1/v} \le c\,\|f|X\|$$

(mit der üblichen Modifikation für $v = \infty$) für ein $\epsilon > 0, c > 0$ und alle $f \in X$ gilt. Das Paar $\mathfrak{E}_G(X) = (\mathcal{E}^X_G, u_X)$ nennt sich dann *Wachstums-Envelope* (*growth envelope*) des Funktionenraums X.

Diese Begriffe wurden von D. D. Haroske eingeführt und ausführlich studiert, vgl. [17] und [18]. Wir benutzen in [2] die atomare und die Wavelet-Zerlegung, um das Verhalten in Grenzfällen zu untersuchen. Die Ergebnisse kann man dann auch als Aussagen über optimale Einbettungen von Besov und Triebel-Lizorkin Räumen in die sogenannten *Lorentzräume* interpretieren.

Die dritte Arbeit [3] beschäftigt sich mit den Einbettungen von Franke und Jawerth [19, 20] (vgl. die Formel (2.10) der Habilitationsschrift) für Räume mit dominierenden gemischten Glattheitseigenschaften. Die Beweismethode von Franke und Jawerth basiert auf einem klugen Interpolationstrick und ist im Fall von Räumen mit dominierend gemischten Glattheitseigenschaften nicht direkt anwendbar. Partielle Ergebnisse wurden in [21, 22] erzielt, aber einige Fälle sind offen geblieben. Unsere Methode modifiziert die Technik von [23] und ermöglicht einen einheitlichen Beweis ohne Einschränkung an die Parameter der Räume.

Die Arbeit [4] studiert Räume von Triebel-Lizorkin Typ, die erst vor kurzem in [24] eingeführt wurden. Diese Räume werden durch Funktionsparameter $s(x), p(x)$ und $q(x)$ beschrieben. Das heißt, dass die Glattheit, Integrabilität und Summabilität in diesen Räumen von dem Ort x abhängt. Die Theorie dieser Räume basiert auf den Lebesgueräumen mit variabler Integrabilität $L_{p(\cdot)}(\mathbb{R}^d)$, die insbesondere durch die Arbeiten von Kováčik, Rákosník und Růžička [25, 26] populär geworden sind. In [4] wurde gezeigt, dass die Einbettung

$$F^{s_0(\cdot)}_{p_0(\cdot),q(\cdot)}(\mathbb{R}^d) \hookrightarrow F^{s_1(\cdot)}_{p_1(\cdot),q(\cdot)}(\mathbb{R}^d)$$

gilt, wenn die Funktionsparameter $s(\cdot), p(\cdot)$ und $q(\cdot)$ gewisse Regularitätsbedingungen erfüllen und die Identität

$$s_0(x) - \frac{d}{p_0(x)} = s_1(x) - \frac{d}{p_1(x)}, \quad x \in \mathbb{R}^d$$

punktweise erfüllt ist.

In den Arbeiten [5, 6] wurden Eigenschaften von diagonalen Spuroperatoren im Rahmen der Funktionenräume mit dominierenden gemischten Glattheitseigenschaften in $\mathbb{R}^2$ und $\mathbb{R}^3$ studiert. Das Verhalten des nicht-diagonalen Spuroperators

$$(\mathrm{tr} f)(t) := f(t, 0), \quad t \in \mathbb{R}$$

in diesen Räumen ist bekannt [27]. Für den diagonalen Spuroperator

$$(\mathrm{tr}_\Gamma f)(t) := f(t, t), \quad t \in \mathbb{R}$$

gab es partielle Ergebnisse im zweidimensionalen Fall, cf. [28] und [29]. Die Arbeit [5] liefert eine (fast) komplette Antwort auf diese Frage im $\mathbb{R}^2$. In [6] hat sich gezeigt, dass im $\mathbb{R}^3$ neue Effekte eintreten und der Spurraum als Summe von drei Räumen gegeben ist. Die Beweise in diesen beiden Arbeiten basieren auf atomaren Zerlegungen.

Die letzte Arbeit [7] widmet sich dem Studium radialer Funktionen. Falls $X(\mathbb{R}^d) \subset S'(\mathbb{R}^d)$ ein (quasi-)Banachraum von Distributionen ist, dann bezeichnen wir mit $RX(\mathbb{R}^d)$ die Menge aller radialen Distributionen von $X(\mathbb{R}^d)$ ausgestattet mit der Norm in $X(\mathbb{R}^d)$. Wir untersuchen die Eigenschaften des Spuroperators

$$(\mathrm{tr} f)(t) := f(t, 0, \dots, 0), \quad t > 0$$

im Rahmen von radialen Besov, Triebel-Lizorkin und Sobolevräumen. Im Fall von Besov und Triebel-Lizorkin Räumen benutzen wir eine angepasste atomare Charakterisierung um $\mathrm{tr}\, XB^s_{p,q}(\mathbb{R}^d)$ und $\mathrm{tr}\, XF^s_{p,q}(\mathbb{R}^d)$ zu beschreiben. Diese wird weiter benutzt, um Regularitätseigenschaften und den Abfall der radialen Funktionen aus diesen Räumen zu untersuchen.

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Preface

This cumulative habilitation thesis presents the work done in seven research articles [28, 68, 72, 93, 95, 97, 98]. The summary introduces the mathematical background of the subject and contains a historical survey of decomposition techniques in the frame of function spaces. After that, the results of the above mentioned papers are discussed. Although I tried to comment also on the proofs of the results and put them into the historical perspective given before, I would like to point the reader to the original papers for full proofs and further references.

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Contents

1 Introduction

The main subject of this habilitation thesis are function spaces and their various decomposition techniques with a special emphasis on the applications of these techniques to more or less classical problems of the theory of function spaces. We start with a brief historical overview, which we use also to introduce some basic notation. As we are not able to cover all the topics of the theory of function spaces in this short survey, we refer to [1, 2, 37, 44, 45, 77, 53, 83] for much more details and further references. Our selection of the topics is mainly governed by our interest in decomposition techniques.

The very first traces of the study of function spaces may be found already in the second half of eighteen century. This period was devoted to the study of classical spaces of continuous and continuously differentiable functions. A new era of function spaces started with the pioneering work of Sobolev [73, 74, 75] (with some forerunners [24, 58]). The theory of distributions became an essential tool, which allowed to achieve new results (e.g. embedding theorems) applicable in the study of partial differential equations.

In later years, the area became an object of a vastly growing interest. More and more function spaces were defined with the help of explicit norms. In the parallel, the advantages of the techniques of Fourier analysis (like Littlewood-Paley theory) became evident. In this connection, the Hardy spaces $H_p(\Delta)$ (cf. Section 3) played a crucial role.

During the 60's and 70's of the last century, the well structured scales of Besov and Triebel-Lizorkin spaces, cf. Definition 2.1, emerged from the variety of function spaces available so far. They exhibit several advantages. Many classical spaces may be identified as Besov or Triebel-Lizorkin spaces for a special choice of parameters. Furthermore, their definition is given in terms of distributions and Fourier analysis and these spaces have "good" properties from the Fourier-analytic point of view, cf. [84, Section 2.2.3]. Also the spaces with fractional (or even negative) smoothness could be incorporated easily into these two scales. On the other hand, the definition of Besov and Triebel-Lizorkin spaces involves a certain smooth dyadic decomposition of unity, which makes it look much more complicated than that of Sobolev spaces.

Further essential breakthrough was achieved in the work of Frazier and Jawerth [22] and [23] (with an important forerunner being [13]). It was discovered that spaces of functions and distributions may be characterized in terms of their decomposition properties. They considered the decomposition formula $f = \sum_Q \langle f, \varphi_Q \rangle \psi_Q$ for all $f \in S'(\mathbb{R}^d)$, where Q runs over all dyadic cubes of $\mathbb{R}^d$ and φ_Q and ψ_Q are shifts of dilations of special functions φ and ψ.

A similar approach was then followed in all other decomposition techniques, which appeared afterwards. They all say, roughly speaking, that a function (or a distribution) f belongs to a certain function space (say $B^s_{p,q}(\mathbb{R}^d)$) if, and only if, it may be written in a form

$$f = \sum_{j,m} \lambda_{j,m} a_{j,m}, \tag{1.1}$$

where $\lambda_{j,m}$ are (real or complex) scalars and $a_{j,m}$ are certain special building blocks. Furthermore, the (quasi-)norm of f in the given function space is in some sense equivalent to the (quasi-)norm of the sequence $\lambda = (\lambda_{j,m})_{j,m}$ in an appropriate sequence space (i.e. $b^s_{p,q}$ in the case of Besov spaces).

Of course, the formula (1.1) gives arise to many questions, like the uniqueness of the decomposition or the linearity of the dependence of λ on f. For example, in the decomposition of Frazier and Jawerth the mapping $f \to \{\langle f, \varphi_Q \rangle\}_Q$ is linear, but it is not an isomorphism between the given function space and the corresponding sequence space.

But three properties of the building blocks $a_{j,m}$ appearing already in [22] and [23] are common to most of all the known decomposition techniques. Those are *smoothness*, *vanishing moment conditions* and *localization*.

• Quite naturally, the basic building blocks are supposed to exhibit at least the same degree of smoothness as the functions (or distributions) in the function space under consideration. Due to the very weak convergence of (1.1) (which is usually assumed to converge in $S'(\mathbb{R}^d)$), the smoothness of the building blocks is not limited from above. As the classical Haar wavelets are not even continuous, the question of minimal smoothness required in (1.1) has also been studied, cf. [88].

• The necessity of the moment conditions becomes clear when dealing with singular distributions. Therefore, the number of moment conditions needed grows with s (the smoothness of the space) decreasing, cf. Theorem 3.6. Let us point out, that one possible way how to achieve (even an infinite number of) vanishing moments is to work with a function, whose Fourier transform has its support bounded away from zero.

• Finally, the localization of the building blocks is also necessary. One may observe that for $p > 1$ overlapping building blocks would allow to consider decompositions of f with arbitrarily small norm of the sequence of coefficients $\lambda = (\lambda_{j,m})_{j,m}$. This corresponds to no localization conditions needed in the decomposition theorem of $H_p(\mathbb{R}^d)$, $0 < p \leq 1$ of Coifman [13], cf. Theorem 3.2.

During last two decades, various different decomposition techniques appeared. They are usually named after the building blocks used, so that we speak about *atomic*, *molecular*, *quarkonial* or *wavelet decomposition*. Furthermore, these decompositions were adapted to a number of different function spaces (anisotropic spaces, spaces with dominating mixed smoothness, spaces of Morrey and Campanato type, ...). Last, but not least, the methods were adapted to spaces on domains. The huge interest in these techniques was driven by the large number of applications based on or making a use of them, i.e. signal processing in many disciplines (like medicine or geology), algorithm design, data compression or numerical analysis to name at least a few of them.

Our approach in this thesis is different. We want to point out, how the theory of decomposition techniques is helping to deal with problems in the theory of function spaces. It turns out (and it has been like that since the work of Frazier and Jawerth), that many classical problems may be much more easily formulated and handled in the language of sequence spaces. We shall deal here mainly with Sobolev and trace embeddings of function spaces and their properties.

The plan of this survey is as follows. In Section 2, we present necessary notation and definitions, Section 3 describes (some of) the decomposition techniques with an emphasis on those which shall be used later on and Section 4 presents the results of the papers, which are part of this cumulative thesis.

2 Definitions and basic notation

In this section we give the necessary notation and the definitions of the function spaces considered in this work.

We denote by $\mathbb{R}$ the set of all real numbers and by $\mathbb{R}^d$ the d-dimensional Euclidean space. Furthermore, $\mathbb{N}$ stands for the set of all natural numbers, $\mathbb{Z}$ for the set of all integers and $\mathbb{C}$ for the set of all complex numbers.

We denote by $S(\mathbb{R}^d)$ the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions equipped with the usual topology and its dual by $S'(\mathbb{R}^d)$.

The Fourier transform of $\varphi \in S(\mathbb{R}^d)$ is given by

$$\mathcal{F}\varphi(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \varphi(x) e^{-i\xi\cdot x} dx, \quad \xi \in \mathbb{R}^d$$

with ins inverse denoted by

$$\mathcal{F}^{-1}\varphi(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \varphi(x) e^{i\xi\cdot x} dx, \quad \xi \in \mathbb{R}^d.$$

Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended to $S'(\mathbb{R}^d)$ by duality. We often write $\hat{\varphi}$ as a shortcut for $\mathcal{F}\varphi$ and $\varphi^\vee$ for $\mathcal{F}^{-1}\varphi$.

Although we are mainly interested in function spaces of Besov and Triebel-Lizorkin type (as defined in Section 2.2), we first collect the definitions of (some of) the classical function spaces.

2.1 Classical spaces

(i) The space of all complex-valued bounded and uniformly continuous functions is denoted by $C(\mathbb{R}^d)$ and is equipped with the norm $\|f|C(\mathbb{R}^d)\| = \sup_{x\in\mathbb{R}^d} |f(x)|$.

Let $m \in \mathbb{N}$. Then we denote by $C^m(\mathbb{R}^d)$ the space of all functions on $\mathbb{R}^d$, such that $D^\alpha f \in C(\mathbb{R}^d)$ for all multiindices α with $|\alpha| \le m$. The norm is then given by $\|f|C^m(\mathbb{R}^d)\| = \max_{|\alpha|\le m} \|D^\alpha f|C(\mathbb{R}^d)\|$.

(ii) The Lebesgue spaces $L_p(\mathbb{R}^d)$, $0 < p \le \infty$ are spaces of measurable functions, for which

$$\|f|L_p(\mathbb{R}^d)\| := \begin{cases} \left(\displaystyle\int_{\mathbb{R}^d} |f(x)|^p dx\right)^{1/p}, & \text{if } 0 < p < \infty \\ \operatorname{ess\,sup}_{x\in\mathbb{R}^d} |f(x)|, & \text{if } p = \infty \end{cases}$$

is finite. Sometimes, we write only $\|f\|_p$ instead of $\|f|L_p(\mathbb{R}^d)\|$ for short.

(iii) Let $1 \le p \le \infty$ and $k \in \mathbb{N}_0$. Then the *Sobolev space* $W_p^k(\mathbb{R}^d)$ is defined by

$$W_p^k(\mathbb{R}^d) = \{f \in S'(\mathbb{R}^d) : D^\alpha f \in L_p(\mathbb{R}^d) \text{ if } |\alpha| \le k\}.$$

Here, the derivatives are interpreted in the distributional sense. One of the cornerstones of the theory of Sobolev spaces is the embedding property (usually called *Sobolev embedding*)

$$W_{p_0}^{k_0}(\mathbb{R}^d) \hookrightarrow W_{p_1}^{k_1}(\mathbb{R}^d) \tag{2.1}$$

if $0 \le k_1 \le k_0$ are natural numbers, $1 \le p_0 \le p_1 < \infty$ and

$$k_0 - \frac{d}{p_0} = k_1 - \frac{d}{p_1}. \tag{2.2}$$

When considering the spaces on domains, then (under conditions which we shall discuss in detail later) (2.1) becomes even compact.

(iv) An essential effort was devoted to the extension of the theory of function spaces also to spaces with fractional (or even negative) smoothness. One of the reasons for that is hidden already in (2.2) - for given p_0, p_1 and k_0, the optimal k_1 may be a fractional real number.

The classical way is represented by *Hölder spaces* $C^s(\mathbb{R}^d)$. Let $s>0$ be not an integer. Then we define

$$C^s(\mathbb{R}^d)=\Big\{f\in C^{[s]}(\mathbb{R}^d): \tag{2.3}$$
$$\|f|C^s(\mathbb{R}^d)\|:=\|f|C^{[s]}(\mathbb{R}^d)\|+\sum_{|\alpha|=[s]}\sup_{x\neq y}\frac{|D^\alpha f(x)-D^\alpha f(y)|}{|x-y|^{\{s\}}}<\infty\Big\}.$$

Here, $s=[s]+\{s\}$ with $0\le\{s\}<1$ is a decomposition of s into its integer and fractional part.

The closely related *Zygmund spaces* $\mathcal{C}^s(\mathbb{R}^d)$ are obtained by replacing the first order by second order differences in (2.3). The definition of the (classical) *Besov spaces* reflects a similar idea. It works with the decomposition of the smoothness parameter $s=[s]^-+\{s\}^+$, where $0<\{s\}^+\le 1$. Let $s>0$ and $1\le p,q<\infty$. Then

$$\Lambda^s_{p,q}(\mathbb{R}^d)=\Big\{f\in W^{[s]^-}(\mathbb{R}^d):\|f|\Lambda^s_{p,q}(\mathbb{R}^d)\|:=\|f|W^{[s]^-}(\mathbb{R}^d)\| \tag{2.4}$$
$$+\sum_{|\alpha|=[s]^-}\left(\int_{\mathbb{R}^d}|h|^{-\{s\}^+q}\|\Delta_h^2D^\alpha f\|_p^q\frac{dh}{|h|^d}\right)^{1/q}<\infty\Big\}, \tag{2.5}$$

where $\Delta_h^2 g$ are the usual second order differences of g. If $q=\infty$, only notational changes are necessary. Let us refer to [84, Section 2.2] for other spaces (i.e. *Slobodeckij spaces* and *Bessel potential spaces*) with fractional smoothness.

2.2 Besov and Triebel-Lizorkin spaces

We give a Fourier-analytic definition of Besov and Triebel-Lizorkin spaces, which relies on the so-called *smooth dyadic resolution of unity*. Let $\varphi\in S(\mathbb{R}^d)$ with

$$\varphi(x)=1\quad\text{if}\quad |x|\le 1\quad\text{and}\quad\varphi(x)=0\quad\text{if}\quad |x|\ge\frac{3}{2}. \tag{2.6}$$

We put $\varphi_0=\varphi$ and $\varphi_j(x)=\varphi(2^{-j}x)-\varphi(2^{-j+1}x)$ for $j\in\mathbb{N}$ and $x\in\mathbb{R}^d$. This leads to the identity

$$\sum_{j=0}^\infty\varphi_j(x)=1,\qquad x\in\mathbb{R}^d.$$

Definition 2.1. (i) Let $s\in\mathbb{R}$ and $0<p,q\le\infty$. Then $B^s_{pq}(\mathbb{R}^d)$ is the collection of all $f\in S'(\mathbb{R}^d)$ such that

$$\|f|B^s_{pq}(\mathbb{R}^d)\|=\left(\sum_{j=0}^\infty 2^{jsq}\|(\varphi_j\widehat{f})^\vee|L_p(\mathbb{R}^d)\|^q\right)^{1/q} \tag{2.7}$$

is finite (with the usual modification for $q=\infty$).

(ii) Let $s\in\mathbb{R}, 0<p<\infty$ and $0<q\le\infty$. Then $F^s_{pq}(\mathbb{R}^d)$ is the collection of all $f\in S'(\mathbb{R}^d)$ such that

$$\|f|F^s_{pq}(\mathbb{R}^d)\|=\left\|\left(\sum_{j=0}^\infty 2^{jsq}|(\varphi_j\widehat{f})^\vee(\cdot)|^q\right)^{1/q}|L_p(\mathbb{R}^d)\right\| \tag{2.8}$$

is finite (with the usual modification for $q=\infty$).

Remark 2.2. (i) The spaces $B^s_{pq}(\mathbb{R}^d)$ and $F^s_{pq}(\mathbb{R}^d)$ are independent on the choice of the function φ as soon as it satisfies (2.6). Unfortunately, if $p=\infty$ in the F-case (which was excluded in Definition 2.1), then this is no longer true and a different approach is necessary. We shall not go into details and refer to the recent monograph [100].

(ii) Let $s\in\mathbb{R}$, $0<p<\infty$ and $0<q\le\infty$. Then the embedding

$$B^s_{p,\min(p,q)}(\mathbb{R}^d)\hookrightarrow F^s_{p,q}(\mathbb{R}^d)\hookrightarrow B^s_{p,\max(p,q)}(\mathbb{R}^d).$$

is an easy consequence of the Definition 2.1.

(iii) Let $-\infty<s_1<s_0<\infty,\quad 0<p_0<p_1<\infty,\quad 0<q_0\le q_1\le\infty$ with

$$s_0-\frac{d}{p_0}=s_1-\frac{d}{p_1}.$$

Then the classical Sobolev embedding (2.1) has its counterpart also for Besov and Triebel-Lizorkin spaces

$$B^{s_0}_{p_0,q_0}(\mathbb{R}^d)\hookrightarrow B^{s_1}_{p_1,q_1}(\mathbb{R}^d)\quad\text{and}\quad F^{s_0}_{p_0,\infty}(\mathbb{R}^d)\hookrightarrow F^{s_1}_{p_1,q_0}(\mathbb{R}^d). \tag{2.9}$$

Furthermore, the Jawerth-Franke embedding [21, 33] states that

$$F^{s_0}_{p_0,\infty}(\mathbb{R}^d)\hookrightarrow B^{s_1}_{p_1,p_0}(\mathbb{R}^d)\quad\text{and}\quad B^{s_0}_{p_0,p_1}(\mathbb{R}^d)\hookrightarrow F^{s_1}_{p_1,q_0}(\mathbb{R}^d). \tag{2.10}$$

(iv) The books [84, 53, 7] describe the stage of the theory of function spaces of Besov and Triebel-Lizorkin type as it stood in the late 1970's. For the more modern aspects of this theory we refer to the books of Triebel [86, 91, 92] and to [100].

(v) We use this place to introduce the symbols

$$\sigma_p=\max(1/p-1,0),\quad \sigma_{pq}=\max(1/p-1,1/q-1,0)$$

and

$$\sigma^d_p=d\max(1/p-1,0),\quad \sigma^d_{pq}=d\max(1/p-1,1/q-1,0).$$

These quantities play an important role in the theory of this spaces and shall be used frequently later on.

(vi) Definition 2.1 covers many of the classical spaces defined by derivatives and/or differences (cf. Section 2.1 for some examples). Especially,

$$\begin{aligned}
&B^s_{\infty,\infty}(\mathbb{R}^d)=\mathcal{C}^s(\mathbb{R}^d)\quad\text{if}\quad s>0,\\
&B^s_{\infty,\infty}(\mathbb{R}^d)=C^s(\mathbb{R}^d)\quad\text{if}\quad s>0,\quad s\notin\mathbb{N},\\
&B^s_{p,q}(\mathbb{R}^d)=\Lambda^s_{p,q}(\mathbb{R}^d)\quad\text{if}\quad s>0,\quad 1\le p<\infty,\quad 1\le q\le\infty,\\
&F^s_{p,2}(\mathbb{R}^d)=W^s_{p,2}(\mathbb{R}^d)\quad\text{if}\quad s>0,\quad s\in\mathbb{N},\quad 1<p<\infty.
\end{aligned}$$

(vii) Definition 2.1 of isotropic Besov and Triebel-Lizorkin spaces has numerous modifications and extensions, which lead to specific function spaces, for example anisotropic spaces, spaces of generalized smoothness or spaces of variable smoothness and/or integrability.

Later on, we shall give the definition of spaces of dominating mixed smoothness, which play an important role in analysis of high-dimensional problems and which is based on a special decomposition of unity involving a certain tensor product structure.

2.3 Spaces on domains

Let Ω be a bounded domain. Then one may easily modify the definitions given in Section 2.1 to obtain function spaces on Ω. Unfortunately, Definition 2.1 relies essentially on the use of Fourier transform and does not allow such an easy modification. Therefore, the Besov and Triebel-Lizorkin spaces on Ω are usually defined by restriction. Let $D(\Omega) = C_0^\infty(\Omega)$ be the collection of all complex-valued infinitely-differentiable functions with compact support in Ω and let $D'(\Omega)$ be its dual - the space of all complex-valued distributions on Ω.

Let $g \in S'(\mathbb{R}^d)$. Then we denote by $g|\Omega$ its restriction to Ω:

$$(g|\Omega) \in D'(\Omega), \qquad (g|\Omega)(\psi) = g(\psi) \quad \text{for} \quad \psi \in D(\Omega).$$

Definition 2.3. Let Ω be a bounded domain in $\mathbb{R}^d$. Let $s \in \mathbb{R}$, $0 < p, q \le \infty$ with $p < \infty$ in the F-case. Let A^s_{pq} stand either for B^s_{pq} or F^s_{pq}. Then

$$A^s_{pq}(\Omega) = \{f \in D'(\Omega) : \exists g \in A^s_{pq}(\mathbb{R}^d) : g|\Omega = f\}$$

and

$$\|f|A^s_{pq}(\Omega)\| = \inf \|g|A^s_{pq}(\mathbb{R}^d)\|,$$

where the infimum is taken over all $g \in A^s_{pq}(\mathbb{R}^d)$ such that $g|\Omega = f$.

Although Definition 2.3 is an easy and convenient way how to define function spaces on domains, an intrinsic characterization of these spaces is necessary on many occasions. It turns out, that under only minor regularity assumptions on Ω (i.e. Lipschitz boundary), the spaces may be characterized by differences (in a fashion similar to Section 2.1). As this will not be needed in the sequel, we only refer to [91, Section 1.11] for details and further references.

We shall later need the existence of a universal extension operator as it was given by Rychkov [60]. This result (with many forerunners for which we refer to references given in [60]) states, that if Ω has Lipschitz boundary then there is a common bounded linear extension operator Ext : $A^s_{p,q}(\Omega) \to A^s_{p,q}(\mathbb{R}^d)$ for all admissible s, p and q. Another important fact will be the existence of atomic and wavelet decomposition techniques adapted to function spaces on domains. We shall return to this point in Section 3.

2.4 Function spaces of dominating mixed smoothness

In many real-life applications it is necessary to consider functions depending on a large number of variables d. Furthermore, many of the usual numerical techniques suffer from the so-called *curse of dimensionality*, i.e. the fact, that its complexity grows very fast (i.e. exponentially) in d. We refer to the recent monographs of Novak and Woźniakowski [51, 52]. It is known since late 1950's that boundedness of certain mixed derivatives is a suitable assumption, which allows to essentially restrict the critical influence of the dimension d.

Sobolev spaces with dominating mixed smoothness $S^{\overline{r}}_p W(\mathbb{R}^d)$ have been introduced in 1962 by S. M. Nikol'skij, see [48, 49], originally in connection with some partial differential equations. If $\overline{r} = (r_1, \dots, r_d) \in \mathbb{N}_0^d$ is a vector of non-negative integers, then the space $S^{\overline{r}}_p W(\mathbb{R}^d)$ is defined as

$$S^{\overline{r}}_p W(\mathbb{R}^d) = \left\{ f \in S'(\mathbb{R}^d) : \|f|S^{\overline{r}}_p W(\mathbb{R}^d)\| := \Big(\sum_{0 \le \alpha \le \overline{r}} \|D^\alpha f\|_p^p \Big)^{1/p} \right\}.$$

Here, the summation runs over all multiindices $\alpha \in \mathbb{N}_0^d$, such that $0 \le \alpha_i \le r_i$ for all $i = 1, \dots, d$. The terms with mixed derivatives (i.e. $\|D^{\overline{r}} f\|_p$) play a crucial role and gave the name to this scale of function spaces. Also spaces of Besov type with $q = \infty$, i.e. the spaces denote below by $S^{\overline{r}}_{p,\infty}B(\mathbb{R}^d)$, were considered by Nikol'skij. A systematic treatment of spaces with dominating mixed smoothness was then given in [3].

Later on there has been some interest in these type of spaces as special cases of vector-valued Sobolev spaces ($S^{r,\dots,r}_p W(\mathbb{R}^d)$ can be interpreted as an iterated version of the Sobolev spaces $W^r_p(\mathbb{R})$), see Grisvard [26], Sparr [76], Schmeißer [62] and Sickel and Ullrich [70, 71]. We refer, e.g., to the monographs of Tikhomirov [82], Temlyakov [80] and Nikol'skij [50] and to the surveys [8] and [11] for more details and further references.

We deal with function spaces of dominating mixed smoothness of Besov and Triebel-Lizorkin type very much in the spirit of [65]. Their definition is based on Definition 2.1 with the decomposition of unity replaced by another one with a tensor product structure.

For $\overline{k} = (k_1, \dots, k_d) \in \mathbb{N}_0^d$ and $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ we define $\varphi_{\overline{k}}(x) = \varphi_{k_1}(x_1) \cdots \varphi_{k_d}(x_d)$. Then, since

$$\sum_{\overline{k} \in \mathbb{N}_0^d} \varphi_{\overline{k}}(x) = 1 \quad \text{for every} \quad x \in \mathbb{R}^d, \tag{2.11}$$

the system $(\varphi_{\overline{k}})_{\overline{k} \in \mathbb{N}_0^d}$ forms a dyadic resolution of unity with the inner tensor product structure.

Definition 2.4. Let $\overline{r} = (r_1, \dots, r_d) \in \mathbb{R}^d$ and $0 < q \le \infty$.
(i) Let $0 < p \le \infty$. Then $S^{\overline{r}}_{p,q}B(\mathbb{R}^d)$ is the collection of all $f \in S'(\mathbb{R}^d)$ such that

$$\|f|S^{\overline{r}}_{p,q}B(\mathbb{R}^d)\|_\varphi = \Big(\sum_{\overline{k} \in \mathbb{N}_0^d} 2^{q\overline{k}\cdot\overline{r}} \|(\varphi_{\overline{k}}\hat{f})^\vee|L_p(\mathbb{R}^d)\|^q\Big)^{1/q} = \|2^{\overline{k}\cdot\overline{r}}(\varphi_{\overline{k}}\hat{f})^\vee|\ell_q(L_p)\| \tag{2.12}$$

is finite.
(ii) Let $0 < p < \infty$. Then $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$ is the collection of all $f \in S'(\mathbb{R}^d)$ such that

$$\|f|S^{\overline{r}}_{p,q}F(\mathbb{R}^d)\|_\varphi = \Big\|\Big(\sum_{\overline{k} \in \mathbb{N}_0^d} |2^{\overline{k}\cdot\overline{r}}(\varphi_{\overline{k}}\hat{f})^\vee(\cdot)|^q\Big)^{1/q}|L_p(\mathbb{R}^d)\Big\| = \|2^{\overline{k}\cdot\overline{r}}(\varphi_{\overline{k}}\hat{f})^\vee|L_p(\ell_q)\| \tag{2.13}$$

is finite.

We shall need later on also other variants of Definition 2.1, cf. Definition 4.18 and Definition 4.25.

3 Decomposition techniques

3.1 Hardy spaces

The history of atomic decompositions is closely related to Hardy spaces H_p. In its original form, the Hardy space $H_p(\Delta)$ is a space of holomorphic functions on the unit disc $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ satisfying

$$\|f|H_p(\Delta)\| := \sup_{0<r<1} \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p dt\right)^{1/p} < \infty.$$

This definition (which goes back to F. Riesz) was extended to functions of real variables by C. Fefferman and E. M. Stein in [20]. The space $H_p(\mathbb{R}^d), 0 < p \le \infty$ is a space of $f \in S'(\mathbb{R}^d)$, such that

$$(M_\Phi f)(x) := \sup_{t>0} |(f * \Phi_t)(x)|, \quad x \in \mathbb{R}^d$$

is in $L_p(\mathbb{R}^d)$. Here $\Phi \in S(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \Phi(x)dx = 1$ is arbitrary and $\Phi_t(x) = t^{-d}\Phi(x/t)$. Furthermore,

$$\|f|H_p(\mathbb{R}^d)\| := \|M_\Phi f|L_p(\mathbb{R}^d)\|$$

is a quasinorm on $H_p(\mathbb{R}^d)$. Different choices of Φ lead to equivalent quasinorms. If $1 < p < \infty$, then $H_p(\mathbb{R}^d)$ coincides with $L_p(\mathbb{R}^d)$. But for $0 < p \leq 1$, one obtains new function spaces of distributions on $\mathbb{R}^d$.

The first atomic decomposition of $H_p(\mathbb{R}^d)$ with $d = 1$ and $0 < p \leq 1$ was given in [13] and generalized to $d > 1$ in [38]. It uses the notion of p-atoms on the real line.

Definition 3.1. Let $0 < p \leq 1$. A p-atom is a real-valued function b on $\mathbb{R}$ such that $\int_{-\infty}^{\infty} b(x)x^k dx = 0$, $0 \leq k \leq [1/p] - 1$, $k \in \mathbb{N}_0$, and the support of which is contained in an interval I for which $\sup_{x\in\mathbb{R}} |b(x)| \leq |I|^{-1/p}$.

The quantity $[1/p]$ is the integer part of $1/p$. The corresponding decomposition theorem then takes the following form.

Theorem 3.2. ([13]) *A distribution f lies in $H^p(\mathbb{R})$, $0 < p \leq 1$ if, and only if, it can be written in the form*

$$f = \sum_{i=0}^{\infty} \alpha_i b_i,$$

where α_i are in $\mathbb{R}$, b_i are p-atoms for $i \in \mathbb{N}$ and

$$A\|f|H^p(\mathbb{R})\|^p \leq \sum_{i=0}^{\infty} |\alpha_i|^p \leq B\|f|H^p(\mathbb{R})\|^p.$$

Here the constants $A, B > 0$ depend only on p.

3.2 Besov and Triebel-Lizorkin spaces

M. Frazier and B. Jawerth extended in [22, 23] the method of Coifman to a huge variety of other function spaces. They studied the decomposition formula $f = \sum_Q \langle f, \varphi_Q\rangle \psi_Q$ for $f \in S'(\mathbb{R}^d)$. Here, Q runs over all dyadic cubes of $\mathbb{R}^d$ and φ_Q and ψ_Q arise through shifting and dilating of special functions φ and ψ. These functions are smooth, rapidly decreasing and possess compactly supported Fourier transform. The mapping

$$S_\varphi : f \to (\langle f, \varphi_Q\rangle)_Q$$

is called φ-transform. Theorem 2.2 of [23] then states, that S_φ maps the homogenous Triebel-Lizorkin space $\dot{F}^s_{p,q}(\mathbb{R}^d)$ into a special sequence space $\dot{f}^s_{p,q}$, which is defined through the (quasi)norm

$$\|\lambda|\dot{f}^s_{p,q}\| := \left\| \left(\sum_Q (|Q|^{-s/n-1/2}|\lambda_Q|)^q \chi_Q(\cdot) \right)^{1/q} \right\|_p,$$

where the sum runs again over all dyadic cubes of $\mathbb{R}^d$, $|Q|$ stands for the Lebesgue measure of Q and χ_Q is the characteristic function of Q.

Furthermore, the inverse φ-transform defined as

$$T_\psi : \lambda = (\lambda_Q)_Q \to \sum_Q \lambda_Q \psi_Q$$

maps $\dot{f}^s_{p,q}$ onto $\dot{F}^s_{p,q}(\mathbb{R}^d)$ and $T_\psi \circ S_\varphi$ is the identity on $\dot{F}^s_{p,q}(\mathbb{R}^d)$.

Remark 3.3. • Frazier and Jawerth worked mainly with the homogenous function spaces and stated only in Section 12 of [23] the necessary modifications needed to deal with inhomogeneous spaces.

- Unfortunately, the φ-transform S_φ is no isomorphism between $\dot{F}^s_{p,q}(\mathbb{R}^d)$ and $\dot{f}^s_{p,q}$, i.e. S_φ does not map $\dot{F}^s_{p,q}(\mathbb{R}^d)$ *onto* $\dot{f}^s_{p,q}$. This was essentially improved using the theory of wavelets.

- The theory of [23] applies exactly to those function spaces which admit some sort of Littlewood-Paley characterization. This is in a very good agreement with the the observation of Triebel (see [84, Section 2.2.3]), who divided the function spaces into *good* and *bad* spaces according to their Fourier-analytic properties. Let us mention on this place that some prominent function spaces (like $L_1(\mathbb{R}^d)$, $L_\infty(\mathbb{R}^d)$ or $C(\mathbb{R}^d)$) are considered as *bad* function spaces from this point of view.

- The condition on vanishing moments of Coifman is incorporated in [23] through the assumption, that the support of the Fourier transform of φ and ψ stays away from zero. The new condition of [23] is that the building blocks ψ_Q are essentially localized on the dyadic cube Q (i.e. rapidly decreasing outside Q). This is reflected in all other decomposition techniques which involve both the vanishing moments condition and some kind of localization of the building blocks.

The central role in the theory of decomposition of function spaces is played by the atomic decomposition. We give the version as presented by Triebel in Section 1.5 of [91]. First, we define the corresponding building blocks. Let us observe, that in contrast with Definition 3.1, the localization of the atoms is required.

Definition 3.4. (i) Let $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^d$. Then we denote by $Q_{\nu m}$ the closed cube in $\mathbb{R}^d$ with sides parallel to the coordinate axes, centered at $2^{-\nu}m$, and with side-length $2^{-\nu+1}$. Furthermore, $c\,Q_{\nu m}$ stands for the cube in $\mathbb{R}^d$ concentric with $Q_{\nu m}$ and with side length $c\,2^{-\nu+1}$.
(ii) Let $K \in \mathbb{N}_0$ and $c \geq 1$. A continuous function $a : \mathbb{R}^d \to \mathbb{C}$ for which there exist all derivatives $D^\alpha a$ if $|\alpha| \leq K$ is called a 1_K-atom if

$$\operatorname{supp} a \subset c\,Q_{0,m} \text{ for some } m \in \mathbb{Z}^d$$

and

$$|D^\alpha a(x)| \leq 1 \text{ for } |\alpha| \leq K.$$

(iii) Let $K \in \mathbb{N}_0, L \geq 0$, and $c \geq 1$. A continuous function $a : \mathbb{R}^d \to \mathbb{C}$ for which there exist all derivatives $D^\alpha a$ if $|\alpha| \leq K$ is called an (K, L)-atom if

$$\operatorname{supp} a \subset c\,Q_{\nu m} \text{ for some } \nu \in \mathbb{N}, m \in \mathbb{Z}^d,$$

$$|D^\alpha(x)a| \leq 2^{|\alpha|\nu} \text{ for } |\alpha| \leq K,$$

and

$$\int_{\mathbb{R}^d} x^\beta a(x)dx = 0 \text{ for } |\beta| < L.$$

Also the sequence spaces used in the frame of Besov and Triebel-Lizorkin spaces are somewhat more complicated compared to Theorem 3.2. We present a version, which reflects all the three parameters of the corresponding function spaces.

Definition 3.5. If $0 < p, q \leq \infty$, $s \in \mathbb{R}$ and

$$\lambda = \{\lambda_{\nu m} \in \mathbb{C} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^d\} \tag{3.1}$$

then we define

$$b^s_{pq} = \left\{\lambda : \|\lambda|b^s_{pq}\| = \left(\sum_{\nu=0}^{\infty} 2^{\nu(s-\frac{d}{p})q}\Big(\sum_{m\in\mathbb{Z}^d} |\lambda_{\nu m}|^p\Big)^{q/p}\right)^{1/q} < \infty\right\} \tag{3.2}$$

and

$$f^s_{pq} = \left\{\lambda : \|\lambda|f^s_{pq}\| = \left\|\left(\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^d} |2^{\nu s}\lambda_{\nu m}\chi_{\nu m}(\cdot)|^q\right)^{1/q} |L_p(\mathbb{R}^d)\right\| < \infty\right\} \tag{3.3}$$

with the usual modification for p and/or q equal to ∞. Here $\chi_{\nu m}$ stands for the characteristic function of $Q_{\nu m}$.

The atomic decomposition of Besov and Triebel-Lizorkin spaces is then given very much in the spirit of Theorem 3.2 and it goes back in a similar form to [22] and [23].

Theorem 3.6. ([91], Theorem 1.19) *(i) Let $0 < p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$. Let $K \in \mathbb{N}_0, L \geq 0$ with*

$$K > s \text{ and } L > \sigma^d_p - s$$

be fixed. Then $f \in S'(\mathbb{R}^d)$ belongs to $B^s_{p,q}(\mathbb{R}^d)$ if, and only if, it can be represented as

$$f = \sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^d} \lambda_{\nu m} a_{\nu m}, \text{ unconditional convergence being in } S'(\mathbb{R}^d), \tag{3.4}$$

where for fixed $c \geq 1$, $a_{\nu m}$ are 1_K-atoms ($\nu = 0$) or (K, L)-atoms ($\nu \in \mathbb{N}$) and $\lambda \in b^s_{pq}$. Furthermore,

$$\|f|B^s_{p,q}(\mathbb{R}^d)\| \approx \inf \|\lambda|b^s_{pq}\|$$

are equivalent quasi-norms where the infimum is taken over all admissible representations (3.4)*.*
(ii) Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$. Let $K \in \mathbb{N}_0, L \geq 0$ with

$$K > s \text{ and } L > \sigma^d_{pq} - s$$

be fixed. Then $f \in S'(\mathbb{R}^d)$ belongs to $F^s_{p,q}(\mathbb{R}^d)$ if, and only if, it can be represented by (3.4)*, where for fixed $c \geq 1$, $a_{\nu m}$ are 1_K-atoms ($\nu = 0$) or (K, L)-atoms ($\nu \in \mathbb{N}$) and $\lambda \in f^s_{p,q}$. Furthermore,*

$$\|f|F^s_{p,q}(\mathbb{R}^d)\| \approx \inf \|\lambda|f^s_{pq}\|$$

are equivalent quasi-norms where the infimum is taken over all admissible representations (3.4)*.*

Nowadays, a large variety of decomposition techniques is available in the literature. We shall present (a variant of) one of the most important one - the wavelet decomposition theorem. It removes some of the obstacles of Theorem 3.6. The first is the implicit definition of atoms - atoms are building blocks satisfying certain properties but may vary from one function to the other. The other sometimes inconvenient feature of Theorem 3.6 is the dependence of the coefficients λ in the optimal decomposition (3.4) on the distribution f. Due to some applications it would be desirable that this dependence is linear. Unfortunately, this does not follow from the theory of atomic decompositions.

We do not aim to give an overview of the vast area of wavelets. We recall only the minimum needed later on and point to [15, 46, 99] as standard references. The following theorem of Daubechies ensures the existence of compactly supported wavelets.

Theorem 3.7. **([14, 15])** *For any $k \in \mathbb{N}$ there are real-valued compactly supported functions*

$$\psi_0, \psi_1 \in C^k(\mathbb{R})$$

satisfying

$$\int_{\mathbb{R}} t^\alpha \psi_1(t)dt = 0, \quad \alpha = 0, 1, \ldots, k-1,$$

such that

$$\{2^{\nu/2}\psi_{\nu m} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}\}$$

with

$$\psi_{\nu m}(t) = \begin{cases} \psi_0(t-m) & \text{if } \nu = 0, m \in \mathbb{Z}, \\ 2^{-\frac{1}{2}}\psi_1(2^{\nu-1}t - m) & \text{if } \nu \in \mathbb{N}, m \in \mathbb{Z} \end{cases}$$

is an orthonormal basis in $L_2(\mathbb{R})$.

Wavelets on $\mathbb{R}^d$ may be obtained as tensor products of one-dimensional wavelets. With their help we obtain the following characterization of Besov and Triebel-Lizorkin spaces.

Theorem 3.8. **([90], Theorem 19)** *Let $0 < p, q \le \infty$, $s \in \mathbb{R}$ and $k \in \mathbb{N}$ with $k > \max(s, \sigma_p^d - s)$. Let ψ_0, ψ_1 be the Daubechies wavelets of smoothness k. Let $E = \{0,1\}^d \setminus (0, \ldots, 0)$. For $e = (e_1, \ldots, e_d) \in E$ let*

$$\Psi_e(x) = \prod_{j=1}^d \psi_{e_j}(x_j), \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

(i) Then

$$\begin{cases} \Psi(x-m) = \prod_{j=1}^d \psi_0(x_j - m_j) & m = (m_1, \ldots, m_d) \in \mathbb{Z}^d, \\ 2^{\frac{\nu-1}{2}d}\Psi_e(2^{\nu-1}x - m) & e \in E, \nu \in \mathbb{N}, m \in \mathbb{Z}^d \end{cases}$$

is an orthonormal basis in $L_2(\mathbb{R}^d)$.

(ii) Let $f \in S'(\mathbb{R}^d)$. Then $f \in B^s_{pq}(\mathbb{R}^d)$ if, and only if, it can be represented as

$$f = \sum_{m \in \mathbb{Z}^d} \lambda_m \Psi(x-m) + \sum_{\nu \in \mathbb{N}} \sum_{e \in E} \sum_{m \in \mathbb{Z}^d} \lambda^e_{\nu m} 2^{-\nu d/2} \Psi_e(2^{\nu-1}x - m), \text{ convergence in } S'(\mathbb{R}^d) \quad (3.5)$$

with

$$\|\lambda|\mathfrak{b}^s_{pq}\| = \Big(\sum_{m \in \mathbb{Z}^d} |\lambda_m|^p\Big)^{\frac{1}{p}} + \left(\sum_{\nu=1}^\infty 2^{\nu(s-\frac{d}{p})q} \sum_{e \in E} \Big(\sum_{m \in \mathbb{Z}^d} |\lambda^e_{\nu m}|^p\Big)^{\frac{q}{p}}\right)^{\frac{1}{q}} < \infty$$

appropriately modified if $p = \infty$ and/or $q = \infty$. The representation in (3.5) is unique, the complex coefficients $(\lambda_m)_{m \in \mathbb{Z}^d}$ and $(\lambda^e_{\nu m})_{e \in E, \nu \in \mathbb{N}_0, m \in \mathbb{Z}^d}$ depend linearly on f and the mapping, which associates to $f \in B^s_{pq}(\mathbb{R}^d)$ the sequence of coefficients, is an isomorphic map of $B^s_{pq}(\mathbb{R}^d)$ onto $\mathfrak{b}^s_{pq}$.

(iii) Let $f \in S'(\mathbb{R}^d)$. Then $f \in F^s_{pq}(\mathbb{R}^d)$ if, and only if, it can be represented as

$$f = \sum_{m \in \mathbb{Z}^d} \lambda_m \Psi(x-m) + \sum_{\nu \in \mathbb{N}} \sum_{e \in E} \sum_{m \in \mathbb{Z}^d} \lambda^e_{\nu m} 2^{-\nu d/2} \Psi_e(2^{\nu-1}x - m), \text{ convergence in } S'(\mathbb{R}^d) \quad (3.6)$$

with

$$\|\lambda|\mathsf{f}^s_{pq}\| = \Big(\sum_{m\in\mathbb{Z}^d}|\lambda_m|^p\Big)^{\frac{1}{p}} + \left\|\left(\sum_{\nu=1}^{\infty}2^{\nu(s-\frac{d}{p})q}\sum_{e\in E}\sum_{m\in\mathbb{Z}^d}|\lambda^e_{\nu m}|^q\chi_{\nu m}(x)\right)^{1/q}\right\|_p < \infty$$

appropriately modified if $p=\infty$ and/or $q=\infty$. The representation in (3.6) *is unique, the complex coefficients $(\lambda_m)_{m\in\mathbb{Z}^d}$ and $(\lambda^e_{\nu m})_{e\in E,\nu\in\mathbb{N}_0,m\in\mathbb{Z}^d}$ depend linearly on f and the mapping, which associates to $f\in F^s_{pq}(\mathbb{R}^d)$ the sequence of coefficients, is an isomorphic map of $F^s_{pq}(\mathbb{R}^d)$ onto f^s_{pq}.*

Remark 3.9. The wavelet decomposition has several very convenient advantages. The decomposition (3.5) is unique and its coefficients depend in a linear way on f. Furthermore, it provides an isomorphism between the corresponding function and sequence spaces. On the other hand, the structure of the compactly supported wavelets from Theorem 3.7 is rather complicated. For example, it is known that the their support must grow linearly with k. In particular, there are no compactly supported infinitely differentiable wavelets.

3.3 Function spaces of dominating mixed smoothness

We shall describe some of the decomposition techniques also in the context of function spaces with dominating mixed smoothness, cf. Definition 2.4. We shall again give only those decompositions necessary later.

For $\overline{\nu}\in\mathbb{N}_0^d$ and $\overline{m}\in\mathbb{Z}^d$ we denote by $Q_{\overline{\nu m}}$ the cube with the center at the point $2^{-\overline{\nu}}\overline{m} = (2^{-\nu_1}m_1,\dots,2^{-\nu_d}m_d)$ with sides parallel to the coordinate axes and of lengths $2^{-\nu_1},\dots,2^{-\nu_d}$. We denote by $\chi_{\overline{\nu m}} = \chi_{Q_{\overline{\nu m}}}$ the characteristic function of $Q_{\overline{\nu m}}$ and by $c\,Q_{\overline{\nu m}}$ we mean a cube concentric with $Q_{\overline{\nu m}}$ with sides c times longer.

Definition 3.10. If $0<p,q\le\infty$, $\overline{r}\in\mathbb{R}^d$ and

$$\lambda = \{\lambda_{\overline{\nu m}}\in\mathbb{C} : \overline{\nu}\in\mathbb{N}_0^d, \overline{m}\in\mathbb{Z}^d\} \tag{3.7}$$

then we define

$$s^{\overline{r}}_{pq}b = \left\{\lambda : \|\lambda|s^{\overline{r}}_{pq}b\| = \Big(\sum_{\overline{\nu}\in\mathbb{N}_0^d}2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})q}\Big(\sum_{\overline{m}\in\mathbb{Z}^d}|\lambda_{\overline{\nu m}}|^p\Big)^{q/p}\Big)^{1/q} < \infty\right\} \tag{3.8}$$

and

$$s^{\overline{r}}_{pq}f = \left\{\lambda : \|\lambda|s^{\overline{r}}_{pq}f\| = \left\|\Big(\sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{m}\in\mathbb{Z}^d}|2^{\overline{\nu}\cdot\overline{r}}\lambda_{\overline{\nu m}}\chi_{\overline{\nu m}}(\cdot)|^q\Big)^{1/q}|L_p(\mathbb{R}^d)\right\| < \infty\right\} \tag{3.9}$$

with the usual modification for p and/or q equal to ∞.

Remark 3.11. We point out that with λ given by (3.7) and $g_{\overline{\nu}}(x) = \sum_{\overline{m}\in\mathbb{Z}^d}\lambda_{\overline{\nu m}}\chi_{\overline{\nu m}}(x)$, we obtain

$$\|\lambda|s^{\overline{r}}_{pq}b\| = \|2^{\overline{\nu}\cdot\overline{r}}g_{\overline{\nu}}|\ell_q(L_p)\|, \qquad \|\lambda|s^{\overline{r}}_{pq}f\| = \|2^{\overline{\nu}\cdot\overline{r}}g_{\overline{\nu}}|L_p(\ell_q)\|.$$

Next we briefly describe the atomic and subatomic decomposition. We refer to [94] for details. Compared to the situation there, we now concentrate on the ”regular” case,

$$\overline{r} > \begin{cases}\sigma_p & \text{in the B-case,}\\ \sigma_{pq} & \text{in the F-case.}\end{cases} \tag{3.10}$$

In this case, no vanishing moment conditions are needed.

Definition 3.12. Let $\overline{K} \in \mathbb{N}_0^d$ and $\gamma > 1$. A $\overline{K}$-times differentiable complex-valued function $a(x)$ is called $\overline{K}$-atom related to $Q_{\overline{\nu}\,\overline{m}}$ if

$$\operatorname{supp} a \subset \gamma Q_{\overline{\nu}\,\overline{m}}, \tag{3.11}$$

and

$$|D^\alpha a(x)| \le 2^{\alpha\cdot\overline{\nu}} \qquad \text{for} \quad 0 \le \alpha \le \overline{K} \tag{3.12}$$

Theorem 3.13. ([94], Theorem 2.4) *Let* $0 < p, q \le \infty$, ($p < \infty$ *in the* $F-$*case*) *and* $\overline{r} \in \mathbb{R}^d$ *with* (3.10). *Fix* $\overline{K} \in \mathbb{N}_0^d$ *with*

$$K_i \ge (1 + [r_i])_+ \qquad i = 1, \dots, d. \tag{3.13}$$

Then $f \in S'(\mathbb{R}^d)$ *belongs to* $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ *if, and only if, it can be represented as*

$$f = \sum_{\overline{\nu}\in\mathbb{N}_0^d} \sum_{\overline{m}\in\mathbb{Z}^d} \lambda_{\overline{\nu m}} a_{\overline{\nu m}}(x), \quad \textit{convergence being in} \quad S'(\mathbb{R}^d), \tag{3.14}$$

where $(a_{\overline{\nu m}}(x))_{\overline{\nu}\in\mathbb{N}_0^d, \overline{m}\in\mathbb{Z}^d}$ *are* $\overline{K}$*-atoms related to* $Q_{\overline{\nu}\,\overline{m}}$ *and* $\lambda \in s^{\overline{r}}_{pq}a$. *Furthermore,*

$$\inf \|\lambda | s^{\overline{r}}_{pq} a\|,$$

where the infimum runs over all admissible representations (3.14), *is an equivalent quasi-norm in* $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$.

One observes, that Theorem 3.13 resembles very much Theorem 3.6 with necessary modifications forced by the tensor product structure of function spaces with dominating mixed smoothness. Another decomposition needed later is the so-called *quarkonial decomposition.* We refer to [87] and [89] for the isotropic version.

Definition 3.14. Let $\psi \in S(\mathbb{R})$ be a non-negative function with

$$\operatorname{supp} \psi \subset \{t \in \mathbb{R} : |t| < 2^\phi\} \tag{3.15}$$

for some $\phi \ge 0$ and

$$\sum_{n\in\mathbb{Z}} \psi(t - n) = 1, \quad t \in \mathbb{R}. \tag{3.16}$$

We define $\Psi(x) = \psi(x_1) \cdots \psi(x_d)$ and $\Psi^\beta(x) = x^\beta \Psi(x)$ for $x = (x_1, \dots, x_d)$ and $\beta \in \mathbb{N}_0^d$. Further let $\overline{r} \in \mathbb{R}^d$ and $0 < p \le \infty$. Then

$$(\beta - qu)_{\overline{\nu m}}(x) = \Psi^\beta(2^{\overline{\nu}} x - \overline{m}), \quad \overline{\nu} \in \mathbb{N}_0^d,\ \overline{m} \in \mathbb{Z}^d \tag{3.17}$$

is called β-quark related to $Q_{\overline{\nu}\,\overline{m}}$.

Theorem 3.15. ([94], Theorem 2.6) *Let* $0 < p, q \le \infty$ (*with* $p < \infty$ *in the* F*-case*) *and* $\overline{r} \in \mathbb{R}^d$ *with* (3.10).

(i) *Let*

$$\lambda = \{\lambda^\beta : \beta \in \mathbb{N}_0^d\} \quad \textit{with} \quad \lambda^\beta = \{\lambda^\beta_{\overline{\nu m}} \in \mathbb{C} : \overline{\nu} \in \mathbb{N}_0^d, \overline{m} \in \mathbb{Z}^d\}$$

and let $\varrho > \phi$, *where* ϕ *is the number from* (3.15). *Then* $f \in S'(\mathbb{R}^d)$ *belongs to* $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ *if, and only if, it can be represented as*

$$f = \sum_{\beta\in\mathbb{N}_0^d} \sum_{\overline{\nu}\in\mathbb{N}_0^d} \sum_{\overline{m}\in\mathbb{Z}^d} \lambda^\beta_{\overline{\nu m}} (\beta - qu)_{\overline{\nu m}}(x), \quad \textit{convergence being in} \quad S'(\mathbb{R}^d), \tag{3.18}$$

where $(\beta - qu)_{\overline{\nu m}}(x)$ *are* β*-quarks related to* $Q_{\overline{\nu}\,\overline{m}}$ *and*

$$\sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|}\|\lambda^\beta|s^{\overline{r}}_{pq}a\| < \infty.$$

Furthermore,

$$\inf \sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|}\|\lambda^\beta|s^{\overline{r}}_{pq}a\| < \infty,$$

where the infimum runs over all admissible representations (3.18)*, is an equivalent quasi-norm in* $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$*.*

The advantage of the quarkonial decomposition is the very simple form of the starting function ψ, cf. (3.15) and (3.16). Its dilations form actually an infinitely differentiable decomposition of unity. The price to pay is the more complicated decomposition (3.18) involving triple sums as well as the more complicated sequence spaces featuring the additional factor $\sup_\beta$.

4 Results of the thesis

This section describes the actual results of the presented thesis, i.e. of the publications [28, 68, 72, 93, 95, 97, 98]. They all have one in common, namely they use the decomposition techniques as described in the previous section to obtain some information about the structure of various function spaces. We shall use the notation as presented above supplemented by new definitions if necessary. We shall not give the technical details of the proofs (which may be found in the references) but we shall comment on the use of the decomposition techniques.

4.1 Widths

To describe the properties of infinite-dimensional objects (like function spaces, or operators between them), one may use several different tools. The prominent role among them is played by the theory of s-numbers as developed by Pietsch, cf. [56]. Roughly speaking, one associates to every linear operator T from one (quasi-)Banach space X into another (quasi-)Banach space Y a (non-increasing) sequence of non-negative real numbers $s_n(T)$. The properties of T are then reflected in the speed of the decay of $s_n(T)$. This approach takes it motivation from approximation theory, where it was intuitively used already in nineteenth century. We refer to [56, 12] for further details.

Let us now give the formal definition of s-numbers. First, we recall the definition of p-Banach spaces.

Definition 4.1. Given $p \in (0,1]$, we say, that the quasi-Banach space Y is a p-Banach space if the inequality

$$\|x + y|Y\|^p \leq \|x|Y\|^p + \|y|Y\|^p$$

is satisfied for all $x, y \in Y$.

Let $T : X \to Y$ (where X and Y are quasi-Banach spaces) be a linear operator and let

$$s_1(T) \geq s_2(T) \geq \cdots \geq 0$$

be a sequence of scalars.

Let W, X, Y, Z be (quasi-)Banach spaces and let Y be a p-Banach space, $0 < p \le 1$. If the rule $s : T \to (s_n(T))_{n\in\mathbb{N}}$ satisfies

(S1) $\|T\| = s_1(T) \ge s_2(T) \ge \cdots \ge 0$,

(S2) $s^p_{m+n-1}(S+T) \le s^p_m(T) + s^p_n(S)$ for all $S, T \in \mathcal{L}(X,Y)$ and $m, n \in \mathbb{N}$,

(S3) $s_n(STU) \le \|S\| s_n(T) \|U\|$ for all $U \in \mathcal{L}(W,X), T \in \mathcal{L}(X,Y), S \in \mathcal{L}(Y,Z)$ and $n \in \mathbb{N}$,

(S4) If rank $T < n$, then $s_n(T) = 0$,

(S5) $s_n(I : \ell_2(n) \to \ell_2(n)) = 1$.

then the $s_n(T)$ are called s-numbers of the operator T.

The property (**S1**) is usually referred to as monotonicity. Furthermore, **(S2)** and **(S3)** reflect subadditivity and submultiplicativity of s-numbers, respectively.

We shall not use **(S4)** and **(S5)** later on. Hence, our approach applies also to rules $s : T \to (s_n(T))_{n\in\mathbb{N}}$ which satisfy only **(S1)-(S3)**. Such rules are called *pseudo-s-numbers* in [55, Chapter 12] and cover also the concept of entropy numbers.

We shall apply the notion of s-numbers to study the properties of the embedding operator between two function spaces.

Let Ω be a bounded Lipschitz domain and let

$$s_1 - s_2 > d\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big)_+. \tag{4.1}$$

Then the embedding

$$\mathcal{I}d : B^{s_1}_{p_1q_1}(\Omega) \to B^{s_2}_{p_2q_2}(\Omega) \tag{4.2}$$

is compact.

Using Theorem 3.8 and the existence of a universal extension operator due to Rychkov [60], the question may be reduced to the corresponding problem on the sequence space level. We obtain

$$s_n(\mathcal{I}d : B^{s_1}_{p_1q_1}(\Omega) \to B^{s_2}_{p_2q_2}(\Omega)) \approx s_n(id : \mathsf{b}^{s,\Omega}_{pq} \to \mathsf{b}^{s,\Omega}_{pq}), \tag{4.3}$$

where $\mathsf{b}^{s,\Omega}_{pq}$ is a certain variant of the spaces b^s_{pq} as described in Theorem 3.8 adapted to function spaces on domains.

The discretization technique was used in connection with s-numbers and embeddings of function spaces already in [43] and [42]. We refer also to [39] and [57] for the survey of the state of the art as it was in the second half of 1980's and to [41] for a more modern presentation. The main aim of the presented paper [95] was to collect the known facts, to extend the results to the case of quasi-Banach spaces and to fill some minor gaps left up to that time. Finally, we remark that the behavior of s-numbers in connection with function spaces with dominating mixed smoothness was studied in the classical book of Temlyakov [80] and in the more recent papers [4, 5, 17, 18].

The rest of this section is devoted to the study of the decay of $s_n(\mathcal{I}d)$ for three different s-numbers, namely *approximation, Kolmogorov and Gelfand numbers.*

4.1.1 Approximation numbers

The approximation numbers of the operator T describe, how well may this operator be approximated (in the operator norm) be finite rank operators.

Definition 4.2. Let X, Y be two quasi-Banach spaces and let $T \in \mathcal{L}(X,Y)$. For $n \in \mathbb{N}$, we define the nth approximation number by

$$a_n(T) = \inf\{\|T - L\| : L \in \mathcal{L}(X,Y), \mathrm{rank}(L) < n\}. \tag{4.4}$$

This definition goes back to Pietsch [54] and Tikhomirov [81].

The estimate of $a_n(\mathcal{I}d)$ is based on (4.3) and the estimates of $a_n(id : \ell_p^n \to \ell_q^n)$. If $p, q \geq 1$, the behavior of these quantities is known, cf. [25].

For $0 < p \leq \infty$, we set

$$p' = \begin{cases} \frac{p}{p-1} & \text{if} \quad 1 < p < \infty, \\ 1 & \text{if} \quad p = \infty, \\ \infty & \text{if} \quad 0 < p \leq 1. \end{cases}$$

Lemma 4.3. *For $1 \leq n \leq m < \infty$ and $1 \leq p_1 < p_2 \leq \infty$, we define*

$$\Phi(m,n,p_1,p_2) := \begin{cases} \left(\min\{1, m^{\frac{1}{p_2}} n^{-\frac{1}{2}}\}\right)^{\frac{\frac{1}{p_1} - \frac{1}{p_2}}{\frac{1}{2} - \frac{1}{p_2}}} & \textit{if} \quad 2 \leq p_1 < p_2 \leq \infty, \\ \max\{m^{\frac{1}{p_2} - \frac{1}{p_1}}, \min\{1, m^{\frac{1}{p_2}} n^{-\frac{1}{2}}\} \cdot \sqrt{1 - \frac{n}{m}}\} & \textit{if} \quad 1 \leq p_1 < 2 \leq p_2 \leq \infty, \\ \max\{m^{\frac{1}{p_2} - \frac{1}{p_1}}, \sqrt{1 - \frac{n}{m}}^{\frac{\frac{1}{p_1} - \frac{1}{p_2}}{\frac{1}{p_1} - \frac{1}{2}}}\} & \textit{if} \quad 1 \leq p_1 < p_2 \leq 2 \end{cases}$$

and

$$\Psi(m,n,p_1,p_2) := \begin{cases} \Phi(m,n,p_1,p_2) & \textit{if} \quad 1 \leq p_1 < p_2 \leq p_1', \\ \Phi(m,n,p_2',p_1') & \textit{if} \quad \max(p_1,p_1') < p_2 \leq \infty. \end{cases}$$

Then if $1 \leq p_1 < p_2 \leq \infty$ and $(p_1,p_2) \neq (1,\infty)$

$$a_n(id : \ell_{p_1}^m \to \ell_{p_2}^m) \approx \Phi(m,n,p_1,p_2), \qquad 1 \leq n \leq m < \infty.$$

The constants of equivalence may depend on p_1 and p_2 but are independent of m and n.

This was complemented in [95] by the following two lemmas with a rather straightforward proof involving only standard techniques.

Lemma 4.4. ([95], Lemma 3.3) *If $1 \leq n \leq m < \infty$ and $0 < p_2 \leq p_1 \leq \infty$, then*

$$a_n(id : \ell_{p_1}^m \to \ell_{p_2}^m) = (m - n + 1)^{\frac{1}{p_2} - \frac{1}{p_1}}.$$

Lemma 4.5. ([95], Lemma 3.4) *Let $0 < p \leq 1$.*
(i) Let $0 < \lambda < 1$. Then there is a number $c_\lambda > 0$ such that

$$a_n(id : \ell_p^m \to \ell_\infty^m) \leq \frac{c_\lambda}{\sqrt{n}} \tag{4.5}$$

holds for all natural numbers n and m with $m^\lambda < n \leq m$.

(ii) There is a number $c > 0$ such that

$$a_n(id : \ell_p^{2n} \to \ell_\infty^{2n}) \geq \frac{c}{\sqrt{n}}, \quad n \geq 1. \tag{4.6}$$

Although the estimates in Lemma 4.4 and Lemma 4.5 are not optimal, it was already sufficient to prove the following statement.

Theorem 4.6. ([95], Theorem 3.5) *Let $-\infty < s_2 < s_1 < \infty$ and $0 < p_1, p_2, q_1, q_2 \le \infty$ with (4.1). Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then (4.2) is compact and for $n \in \mathbb{N}$*

$$a_n(\mathcal{I}d) \approx n^{-\frac{s_1-s_2}{d}+\left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+} \qquad \text{if} \quad \begin{cases} \text{either} & 0 < p_1 \le p_2 \le 2, \\ \text{or} & 2 \le p_1 \le p_2 \le \infty, \\ \text{or} & 0 < p_2 \le p_1 \le \infty, \end{cases} \tag{4.7}$$

$$a_n(\mathcal{I}d) \approx n^{-\frac{s_1-s_2}{d}+\frac{1}{p}-\frac{1}{2}} \qquad \text{if} \quad 0 < p_1 < 2 < p_2 < \infty \tag{4.8}$$
$$\text{and} \quad \frac{s_1-s_2}{d} > \frac{1}{p} = \max\Big(1-\frac{1}{p_2}, \frac{1}{p_1}\Big),$$

$$a_n(\mathcal{I}d) \approx n^{\left(-\frac{s_1-s_2}{d}+\frac{1}{p_1}-\frac{1}{p_2}\right)\cdot\frac{\min(p_1',p_2)}{2}} \qquad \text{if} \quad \frac{s_1-s_2}{d} < \frac{1}{p} = \max\Big(1-\frac{1}{p_2}, \frac{1}{p_1}\Big), \tag{4.9}$$
$$\text{and either} \quad 1 < p_1 < 2 < p_2 = \infty$$
$$\text{or} \quad 0 < p_1 < 2 < p_2 < \infty$$

$$a_n(\mathcal{I}d) \approx n^{-\frac{s_1-s_2}{d}+\frac{1}{p_1}-\frac{1}{2}} \qquad \text{if} \quad 0 < p_1 \le 1 < p_2 = \infty. \tag{4.10}$$

The estimates of Theorem 4.6 may be formulated also for other function spaces. We shall give a result for the Bessel potential spaces, which are for $1 \le p \le \infty$ and $s \in \mathbb{R}$ defined as

$$H_p^s(\mathbb{R}^d) = \{f \in S'(\mathbb{R}^d) : \|f|H_p^s(\mathbb{R}^d)\| = \|\mathcal{F}^{-1}(1+|\xi|^2)^{s/2}\mathcal{F}f|L_p(\mathbb{R}^d)\| < \infty\}.$$

The space $H_p^s(\Omega)$ is then defined again by restriction similarly to Definition 2.3.
The embeddings

$$B^0_{\infty,1}(\Omega) \hookrightarrow C(\Omega) \hookrightarrow L_\infty(\Omega) \hookrightarrow B^0_{\infty,\infty}(\Omega)$$

and

$$B^s_{p,1}(\Omega) \hookrightarrow H^s_p(\Omega) \hookrightarrow B^s_{p,\infty}(\Omega)$$

imply the following version of Theorem 4.6.

Theorem 4.7. ([95], Theorem 5.1) *Let $1 \le p \le \infty$, $s > \frac{d}{p}$ and let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then the embeddings*

$$\mathcal{I}d_1 : H_p^s(\Omega) \to C(\Omega) \tag{4.11}$$

$$\mathcal{I}d_2 : H_p^s(\Omega) \to L_\infty(\Omega) \tag{4.12}$$

are compact and

$$a_n(\mathcal{I}d_1) \approx a_n(\mathcal{I}d_2) \approx n^{-\frac{s}{d}+\frac{1}{p}} \qquad \text{if} \quad 2 \le p \le \infty,$$
$$a_n(\mathcal{I}d_1) \approx a_n(\mathcal{I}d_2) \approx n^{-\frac{s}{d}+\frac{1}{2}} \qquad \text{if} \quad 1 \le p < 2 \quad \text{and} \quad \frac{s}{d} > 1,$$
$$a_n(\mathcal{I}d_1) \approx a_n(\mathcal{I}d_2) \approx n^{\left(-\frac{s}{d}+\frac{1}{p}\right)\cdot\frac{p'}{2}} \qquad \text{if} \quad 1 < p < 2 \quad \text{and} \quad \frac{1}{p} < \frac{s}{d} < 1.$$

4.1.2 Kolmogorov and Gelfand numbers

Other very well known and often used s-widths are Kolmogorov and Gelfand widths. They are defined in the following manner.

Definition 4.8. Let X, Y be two quasi-Banach spaces and let $T \in \mathcal{L}(X, Y)$.
(i) For $n \in \mathbb{N}$, we define the nth Kolmogorov number by

$$d_n(T) = \inf\{\|Q_N^Y T\| : N \subset\subset Y, \dim(N) < n\}. \tag{4.13}$$

Here, Q_N^Y stands for the natural surjection of Y onto the quotient space Y/N.
(ii) For $n \in \mathbb{N}$, we define the nth Gelfand number by

$$c_n(T) = \inf\{\|T J_M^X\| : M \subset\subset X, \mathrm{codim}(M) < n\}. \tag{4.14}$$

Here, J_M^X stands for the natural injection of M into X.

Remark 4.9. The definitions (4.13) and (4.14) may be written in an equivalent way as

$$\begin{aligned} d_n(T) &= \inf_{\substack{N \subset\subset Y \\ \dim N < n}} \sup_{\substack{x \in X \\ \|x\|_X \leq 1}} \inf_{z \in N} \|Tx - z\|_Y, \\ c_n(T) &= \inf_{\substack{M \subset\subset X \\ \mathrm{codim} M < n}} \sup_{\substack{x \in M \\ \|x\|_X \leq 1}} \|Tx\|_Y. \end{aligned}$$

We observe, that the linear approximation used in (4.4) was replaced by a non-linear approximation in the definition of d_n, which is implicitly hidden in the definition of the quotient map Q_N^Y. A similar effect is achieved in the definition of c_n by restricting to suprema over $x \in M$ with $\|x\|_X \leq 1$. This explains, why a_n are sometimes referred to as *linear widths* and d_n and c_n as *non-linear widths*. Furthermore, the inequality $\max\{d_n(T), c_n(T)\} \leq a_n(T)$ follows easily.

The corresponding counterparts of Theorem 4.6 for Kolmogorov and Gelfand widths read as follows.

Theorem 4.10. ([95], Theorem 4.6) *Let $-\infty < s_2 < s_1 < \infty$ and $0 < p_1, p_2, q_1, q_2 \leq \infty$ with (4.1). Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then (4.2) is compact and for $n \in \mathbb{N}$*

$$d_n(\mathcal{I}d) \approx n^{-\frac{s_1-s_2}{d}+\left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+} \quad \text{if} \quad \begin{cases} \text{either} & 0 < p_1 \leq p_2 \leq 2, \\ \text{or} & 0 < p_2 \leq p_1 \leq \infty, \end{cases} \tag{4.15}$$

$$d_n(\mathcal{I}d) \approx n^{-\frac{s_1-s_2}{d}} \quad \text{if} \quad 2 < p_1 \leq p_2 \leq \infty \quad \text{and} \quad \frac{s_1-s_2}{d} > \frac{1}{2}\frac{\frac{1}{p_1}-\frac{1}{p_2}}{\frac{1}{2}-\frac{1}{p_2}}, \tag{4.16}$$

$$d_n(\mathcal{I}d) \approx n^{\frac{p_2}{2}\left(-\frac{s_1-s_2}{d}+\frac{1}{p_1}-\frac{1}{p_2}\right)} \quad \text{if} \quad 2 < p_1 \leq p_2 \leq \infty \quad \text{and} \quad \frac{s_1-s_2}{d} < \frac{1}{2}\frac{\frac{1}{p_1}-\frac{1}{p_2}}{\frac{1}{2}-\frac{1}{p_2}}, \tag{4.17}$$

$$d_n(\mathcal{I}d) \approx n^{\left(-\frac{s_1-s_2}{d}+\frac{1}{p_1}-\frac{1}{2}\right)} \quad \text{if} \quad 0 < p_1 < 2 < p_2 \leq \infty \quad \text{and} \quad \frac{s_1-s_2}{d} > \frac{1}{p_1}, \tag{4.18}$$

$$d_n(\mathcal{I}d) \approx n^{\frac{p_2}{2}\left(-\frac{s_1-s_2}{d}+\frac{1}{p_1}-\frac{1}{p_2}\right)} \qquad \textit{if} \quad 0 < p_1 < 2 < p_2 < \infty \quad \textit{and} \quad \frac{1}{p_1}-\frac{1}{p_2} < \frac{s_1-s_2}{d} < \frac{1}{p_1}. \tag{4.19}$$

Theorem 4.11. ([95], Theorem 4.12) *Let $-\infty < s_2 < s_1 < \infty$ and $0 < p_1, p_2, q_1, q_2 \le \infty$ with (4.1). Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then (4.2) is compact and for $n \in \mathbb{N}$*

$$c_n(\mathcal{I}d) \approx n^{-\frac{s_1-s_2}{d}+\left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+} \qquad \textit{if} \quad \begin{cases} \textit{either} & 2 \le p_1 < p_2 \le \infty, \\ \textit{or} & 0 < p_2 \le p_1 \le \infty, \end{cases} \tag{4.20}$$

$$c_n(\mathcal{I}d) \approx n^{-\frac{s_1-s_2}{d}} \qquad \textit{if} \quad 0 < p_1 < p_2 \le 2 \quad \textit{and} \quad \frac{s_1-s_2}{d} > \frac{1}{2}\frac{\frac{1}{p_1}-\frac{1}{p_2}}{\frac{1}{p_1}-\frac{1}{2}}, \tag{4.21}$$

$$c_n(\mathcal{I}d) \approx n^{\frac{p_1'}{2}\left(-\frac{s_1-s_2}{d}+\frac{1}{p_1}-\frac{1}{p_2}\right)} \qquad \textit{if} \quad 1 < p_1 < p_2 \le 2 \quad \textit{and} \quad \frac{s_1-s_2}{d} < \frac{1}{2}\frac{\frac{1}{p_1}-\frac{1}{p_2}}{\frac{1}{p_1}-\frac{1}{2}}, \tag{4.22}$$

$$c_n(\mathcal{I}d) \approx n^{\left(-\frac{s_1-s_2}{d}+\frac{1}{2}-\frac{1}{p_2}\right)} \qquad \textit{if} \quad 0 < p_1 < 2 < p_2 \le \infty \quad \textit{and} \quad \frac{s_1-s_2}{d} > 1-\frac{1}{p_2}, \tag{4.23}$$

$$c_n(\mathcal{I}d) \approx n^{\frac{p_1'}{2}\left(-\frac{s_1-s_2}{d}+\frac{1}{p_1}-\frac{1}{p_2}\right)} \qquad \textit{if} \quad 1 < p_1 < 2 < p_2 \le \infty \quad \textit{and} \quad \frac{1}{p_1}-\frac{1}{p_2} < \frac{s_1-s_2}{d} < 1-\frac{1}{p_2}. \tag{4.24}$$

The proof of Theorem 4.10 and Theorem 4.11 resembles very much the proof of Theorem 4.6. Of course, the estimates of $a_n(id : \ell_p^n \to \ell_q^n)$ must be replaced by similar estimates for $d_n(id : \ell_p^n \to \ell_q^n)$ and $c_n(id : \ell_p^n \to \ell_q^n)$. Again, if $p, q \ge 1$, then they may be found in the literature, but for $\min(p, q) < 1$ some additional work was necessary.

4.2 Embeddings

The papers [98, 28, 97] study the existence of embeddings between various function spaces. In [98], we studied the so-called *growth envelopes* of Besov and Triebel-Lizorkin spaces as introduced by D. Haroske, cf. [30]. Using the wavelet decomposition Theorem 3.8, it was possible to close some gaps in the limiting situation of parameters and answer an open problem posed by Triebel [87] and Haroske [30].

The second paper presented in this section [28] studies the Jawerth-Franke embedding (2.10) in the frame of function spaces with dominating mixed smoothness. The classical method of Jawerth [33] and Franke [21] uses interpolation theory and faces several serious obstacles when applied to this scale of function spaces. We adapt the alternative proof of [96] (which is based on discretization techniques). This allows to give a rather final answer to this problem and complete the previous work of Krbec, Schmeisser and Sickel [36, 64].

Finally, [97] studies the function spaces of variable smoothness and integrability as introduced recently by L. Diening, P. Hästö and S. Roudenko in [16]. It turns out, that the Sobolev embedding may be proven also in this frame of function spaces with (2.2) replaced by a pointwise condition

$$s_0(x) - \frac{d}{p_0(x)} = s_1(x) - \frac{d}{p_1(x)}, \quad x \in \mathbb{R}^d.$$

4.2.1 Envelopes

Under certain conditions on the parameters involved, the spaces $B^s_{p,q}(\mathbb{R}^d)$ and $F^s_{p,q}(\mathbb{R}^d)$ consist of locally integrable functions. The situation was completely characterized in [69, Theorem 3.3.2] by showing, that

$$B^s_{p,q}(\mathbb{R}^d) \hookrightarrow L_1^{\mathrm{loc}}(\mathbb{R}^d) \Leftrightarrow \begin{cases} \text{either} & s > \sigma^d_p, \\ \text{or} & s = \sigma^d_p, 1 < p \le \infty, 0 < q \le \min(p,2), \\ \text{or} & s = \sigma^d_p, 0 < p \le 1, 0 < q \le 1 \end{cases} \tag{4.25}$$

and

$$F^s_{p,q}(\mathbb{R}^d) \hookrightarrow L_1^{\mathrm{loc}}(\mathbb{R}^d) \Leftrightarrow \begin{cases} \text{either} & s > \sigma^d_p, \\ \text{or} & s = \sigma^d_p, 1 \le p < \infty, 0 < q \le 2, \\ \text{or} & s = \sigma^d_p, 0 < p < 1, 0 < q \le \infty. \end{cases} \tag{4.26}$$

In that case, one may apply to the elements of $A^s_{p,q}(\mathbb{R}^d)$ the technique of nonincreasing rearrangement known from the theory of *Banach function spaces*, cf. [6]. Let us recall its definition

Definition 4.12. Let μ be the Lebesgue measure in $\mathbb{R}^d$. If h is a measurable function on $\mathbb{R}^d$, we define the non-increasing rearrangement of h through

$$h^*(t) = \sup\{\lambda > 0 : \mu\{x \in \mathbb{R}^d : |h(x)| > \lambda\} > t\}, \qquad t \in (0,\infty). \tag{4.27}$$

The use of non-increasing rearrangement is closely connected also to the theory of Lorentz spaces, which are defined as follows.

Definition 4.13. Let $0 < p < \infty$ and $0 < q \le \infty$. Then the Lorentz space $L_{p,q}(\mathbb{R}^d)$ consists of all $f \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$ such that the quantity

$$\|f|L_{p,q}(\mathbb{R}^d)\| = \begin{cases} \left(\displaystyle\int_0^\infty [t^{\frac{1}{p}} f^*(t)]^q \frac{dt}{t}\right)^{1/q}, & 0 < q < \infty, \\ \displaystyle\sup_{0<t<\infty} t^{\frac{1}{p}} f^*(t), & q = \infty \end{cases}$$

is finite

Remark 4.14. We shall recall two (almost obvious) properties of Lorentz spaces.

- If $0 < p = q < \infty$, then $L_{p,p}(\mathbb{R}^d) = L_p(\mathbb{R}^d)$.
- If $0 < p < \infty$ and $0 < q_0 \le q_1 \le \infty$, then $L_{p,q_0}(\mathbb{R}^d) \hookrightarrow L_{p,q_1}(\mathbb{R}^d)$.

Non-increasing rearrangement was used by D. Haroske and H. Triebel (see [29], [30], [87] and references given there) to introduce a new way to classify the function spaces of Besov and Triebel-Lizorkin type. Their *growth envelope function* of X is defined by

$$\mathcal{E}^X_G(t) := \sup_{\|f|X\| \le 1} f^*(t), \quad 0 < t < 1,$$

where f^* denotes the non-increasing rearrangement of f.

In the case where $\mathcal{E}_G^X(t) \approx t^{-\alpha}$ for $0 < t < 1$ and some $\alpha > 0$ the *growth envelope index* u_X is given as the infimum of all numbers v, $0 < v \le \infty$, such that

$$\left(\int_0^{\epsilon} \left[\frac{f^*(t)}{\mathcal{E}_G^X(t)} \right]^v \frac{dt}{t} \right)^{1/v} \le c \, \|f|X\| \tag{4.28}$$

(with the usual modification for $v = \infty$) holds for some $\epsilon > 0, c > 0$ and all $f \in X$. The pair $\mathfrak{E}_G(X) = (\mathcal{E}_G^X, u_X)$ is called *growth envelope* of the function space X.

In the case $\sigma_p^d < s$, the growth envelopes of $A^s_{p,q}(\mathbb{R}^d)$ are known, cf. [87, Theorem 15.2] and [30, Theorem 8.1]. If $s = \sigma_p^d$ and (4.25) or (4.26) is fulfilled in the B or F case, respectively, then the growth function is given by $t^{-\frac{1}{\max(p,1)}}$, but the known information about the growth index u is not complete, cf. [87, Remarks 12.5, 15.1] and [30, Prop. 8.12, 8.14 and Remark 8.15].

The growth index of $B^{\sigma_p^d}_{p,q}(\mathbb{R}^d)$ satisfies

$$\begin{cases} q \le u \le p & \text{if } 1 \le p < \infty \text{ and } 0 < q \le \min(p,2), \\ q \le u \le 1 & \text{if } 0 < p < 1 \text{ and } 0 < q \le 1. \end{cases} \tag{4.29}$$

The growth index of $F^{\sigma_p^d}_{p,q}(\mathbb{R}^d)$ satisfies $p \le u \le 1$ if $0 < p < 1$ and $0 < q \le \infty$ and is equal to p, if $1 \le p < \infty$ and $0 < q \le 2$.

The growth envelopes of $B^0_{\infty,q}$ defined on torus $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ with $1 \le q \le 2$ were identified recently by Seeger and Trebels in [66] and are equivalent to $|\log t|^{1/q'}$ for $0 < t \le 1/2$. We fill the remaining gaps for $p < \infty$ by proving the following

Theorem 4.15. ([95], Theorem 1.1) *(i) Let $1 \le p < \infty$ and $0 < q \le \min(p,2)$. Then*

$$\mathfrak{E}_G(B^0_{p,q}) = (t^{-\frac{1}{p}}, p).$$

(ii) Let $0 < p < 1$ and $0 < q \le 1$. Then

$$\mathfrak{E}_G(B^{\sigma_p^d}_{p,q}) = (t^{-1}, q).$$

(iii) Let $0 < p < 1$ and $0 < q \le \infty$. Then

$$\mathfrak{E}_G(F^{\sigma_p^d}_{p,q}) = (t^{-1}, p).$$

These results are closely related to optimal embeddings into the scale of Lorentz spaces. Although the Lorentz spaces do not allow any easy way of discretization (up to the case $1 < p = q < \infty$, when $L_{p,q}(\mathbb{R}^d) = L_p(\mathbb{R}^d)$), the use of this method still allowed to prove the following theorem.

Theorem 4.16. ([95], Theorem 1.2) *(i) Let $1 \le p < \infty$ and $0 < q \le \min(p,2)$. Then*

$$B^0_{p,q}(\mathbb{R}^d) \hookrightarrow L_p(\mathbb{R}^d).$$

(ii) Let $0 < p < 1$ and $0 < q \le 1$. Then

$$B^{\sigma_p^d}_{p,q}(\mathbb{R}^d) \hookrightarrow L_{1,q}(\mathbb{R}^d). \tag{4.30}$$

(iii) Let $0 < p < 1$ and $0 < q \le \infty$. Then

$$F^{\sigma_p^d}_{p,q}(\mathbb{R}^d) \hookrightarrow L_{1,p}(\mathbb{R}^d)$$

and all these embeddings are optimal with respect to the second fine parameter of the scale of the Lorentz spaces.

Remark 4.17. (i) The embedding (4.30) improves [69, Theorem 3.2.1] and [67, Theorem 2.2.3], where the embedding $B^{n(\frac{1}{p}-1)}_{p,q}(\mathbb{R}^d) \hookrightarrow L_1(\mathbb{R}^d)$ is proved for all $0 < p < 1$ and $0 < q \le 1$.

(ii) The proof of Theorem 4.15 (i) and Theorem 4.16 (i) is based on a construction of a sequence of special functions defined in the terms of their wavelet coefficients. The arguments justifying Theorem 4.15 (ii) and Theorem 4.16 (ii) rely on atomic decompositions. Finally, Theorem 4.15 (iii) and Theorem 4.16 (iii) follows easily by Franke-Jawerth embedding (2.10).

4.2.2 Spaces of dominating mixed smoothness

The main aim of the paper [28] is to prove the analogue of the Jawerth-Franke embedding (2.10) in the frame of function spaces with dominating mixed smoothness. Let us mention, that partial results in this direction were already known, cf. [36, 64]. Unfortunately, the original technique of Jawerth [33] and Franke [21] is based on a clever interpolation trick and can not be easily applied to function spaces of dominating mixed smoothness. Although the authors of [36, 64] succeeded to overcome numerous technical obstacles, the results still contain several gaps. We follow a different strategy. We adapt the alternative proof of (2.10) given in [96].

We work with a slightly more general function spaces compared to those presented in Definition 2.4. Namely, let $N \ge 2$ be a natural number and let $d_1, \dots, d_N$ be natural numbers. We set $\overline{d} = (d_1, \dots, d_N)$ and $d = d_1 + \dots + d_N$. For $i = 1, \dots, N$ we define $\big(\varphi^i_j\big)_{j=0}^{\infty} \subset S(\mathbb{R}^{d_i})$ as described in Section 2.2 and put for $\overline{k} = (k_1, \dots, k_N) \in \mathbb{N}_0^N$ and $x = (x^1, \dots, x^N) \in \mathbb{R}^d$

$$\varphi_{\overline{k}}(x) := \varphi^1_{k_1}(x^1) \cdots \varphi^N_{k_N}(x^N). \tag{4.31}$$

As

$$\sum_{\overline{k}\in\mathbb{N}_0^N} \varphi_{\overline{k}}(x) = \left(\sum_{k_1=0}^{\infty} \varphi_{k_1}(x^1)\right) \cdots \left(\sum_{k_N=0}^{\infty} \varphi_{k_N}(x^N)\right) = 1$$

for all $x = (x^1, \dots, x^N) \in \mathbb{R}^d$, we see that $\big(\varphi_{\overline{k}}\big)_{\overline{k}\in\mathbb{N}_0^N}$ forms also a decomposition of unity on $\mathbb{R}^d$ with the tensor product structure.

Definition 4.18. Let $\overline{r} \in \mathbb{R}^N$, $0 < q \le \infty$ and $\varphi = \big(\varphi_{\overline{k}}\big)_{\overline{k}\in\mathbb{N}_0^N}$ be as above.

1. Let $0 < p \le \infty$. Then $S^{\overline{r}}_{p,q}B(\mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_N})$ is the set of all $f \in S'(\mathbb{R}^d)$, such that

$$\big\|f\big|S^{\overline{r}}_{p,q}B(\mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_N})\big\|_{\varphi} := \left(\sum_{\overline{k}\in\mathbb{N}_0^N} 2^{\overline{k}\cdot\overline{r}q} \big\|\big(\varphi_{\overline{k}}\widehat{f}\big)^{\vee}\big|L_p(\mathbb{R}^d)\big\|^q\right)^{1/q} \tag{4.32}$$

   is finite.

2. Let $0 < p < \infty$. Then $S^{\overline{r}}_{p,q}F(\mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_N})$ is the set of all $f \in S'(\mathbb{R}^d)$, such that

$$\big\|f\big|S^{\overline{r}}_{p,q}F(\mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_N})\big\|_{\varphi} := \left\|\left(\sum_{\overline{k}\in\mathbb{N}_0^N} 2^{\overline{k}\cdot\overline{r}q} \big|\big(\varphi_{\overline{k}}\widehat{f}\big)^{\vee}(\cdot)\big|^q\right)^{1/q}\middle|L_p(\mathbb{R}^d)\right\| \tag{4.33}$$

   is finite.

Of course, if $d_1 = \cdots = d_N = 1$ we obtain the spaces considered in Definition 2.4. The analogue of (2.10) in the frame of these function spaces is then contained in the following theorem.

Theorem 4.19. ([28], Theorem 1.2) *Let $\overline{r}^0, \overline{r}^1 \in \mathbb{R}^N$, $0 < p_0 < p_1 \le \infty$ and $0 < q_0, q_1 \le \infty$ with*

$$r_j^0 - \frac{d_j}{p_0} = r_j^1 - \frac{d_j}{p_1}, \quad j = 1, \ldots, N. \tag{4.34}$$

1. *Then*

$$S^{\overline{r}^0}_{p_0,q_0}F(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) \hookrightarrow S^{\overline{r}^1}_{p_1,q_1}B(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) \tag{4.35}$$

   if, and only if, $p_0 \le q_1$.

2. *If $p_1 < \infty$, then*

$$S^{\overline{r}^0}_{p_0,q_0}B(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) \hookrightarrow S^{\overline{r}^1}_{p_1,q_1}F(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) \tag{4.36}$$

   if, and only if, $q_0 \le p_1$.

Remark 4.20. Let us comment briefly on the technique of the proof of Theorem 4.19.

- The wavelet characterization of the spaces introduced in Definition 4.18 was given in [94] if $d_1 = d_2 = \cdots = d_N = 1$ (i.e. for the spaces considered in the Definition 2.4) and in the general case in [27]. This allows to reduce the question to the sequence space level.
- Another technique used in [96] was the concept of non-increasing rearrangement, as described already in Definition 4.12. In the setting of function spaces with dominating mixed smoothness, the so-called multivariate rearrangements may serve as a suitable replacement. This was already observed by Krbec and Schmeisser in [36]. It is defined in the following way.

  Let $f : (0,\infty)^{k-1} \times \mathbb{R}^{d_k} \times \cdots \times \mathbb{R}^{d_N} \to \mathbb{C}$, $k \le N$, be a measurable function. We set

$$(R_k f)(t_1, \ldots, t_{k-1}, s, y^{k+1}, \ldots, y^N) = [f(t_1, \ldots, t_{k-1}, \cdot, y^{k+1}, \ldots, y^N)]^*(s),$$
$$s > 0, \quad t_1, \ldots, t_{k-1} \in (0,\infty), \quad y^i \in \mathbb{R}^{d_i}, i = k+1, \ldots, N.$$

  We define the *multivariate non-increasing rearrangement* of $f : \mathbb{R}^d \to \mathbb{C}$ by

$$(Rf)(\overline{s}) = (R_N \circ \cdots \circ R_1 f)(\overline{s}), \quad \overline{s} = (s_1, \ldots, s_N) \in (0,\infty)^N.$$

The following reformulation of Theorem 4.19 is sometimes more convenient.

Theorem 4.21. ([28], Theorem 1.4) *Let $\overline{r}^0, \overline{r}, \overline{r}^1 \in \mathbb{R}^N$, $0 < p_0 < p < p_1 \le \infty$ with*

$$r_j^0 - \frac{d_j}{p_0} = r_j - \frac{d_j}{p} = r_j^1 - \frac{d_j}{p_1}, \quad j = 1, \ldots, N.$$

Let $0 < q, u, v \le \infty$. Then

$$S^{\overline{r}^0}_{p_0,u}B(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) \hookrightarrow S^{\overline{r}}_{p,q}F(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) \hookrightarrow S^{\overline{r}^1}_{p_1,v}B(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N})$$

if, and only if, $0 < u \le p \le v \le \infty$.

Furthermore, (4.35) was used in [28] to characterize those spaces $S^{\overline{r}}_{p,q}B(\mathbb{R}^{\overline{d}})$ and $S^{\overline{r}}_{p,q}F(\mathbb{R}^{\overline{d}})$ which are embedded in $C(\mathbb{R}^d)$ and $L_u(\mathbb{R}^d)$, $1 < u \le \infty$. This approach was applied already in [36], cf. also [63].

Theorem 4.22. ([28], Theorem 1.5) *(i) Let* $\overline{r} \in \mathbb{R}^N$, $0 < p \leq \infty$ *and* $0 < q \leq \infty$. *Then the following three assertions are equivalent.*

(a) $\quad S^{\overline{r}}_{p,q}B(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) \hookrightarrow C(\mathbb{R}^d)$,

(b) $\quad S^{\overline{r}}_{p,q}B(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) \hookrightarrow L_\infty(\mathbb{R}^d)$,

(c) $$\begin{cases} r_i - \dfrac{d_i}{p} > 0 \quad \textit{for all} \quad i = 1, \ldots, N \quad \textit{or} \\ r_i - \dfrac{d_i}{p} \geq 0 \quad \textit{for all} \quad i = 1, \ldots, N \quad \textit{and} \quad 0 < q \leq 1. \end{cases}$$

(ii) Let $\overline{r} \in \mathbb{R}^N$, $0 < p < \infty$ *and* $0 < q \leq \infty$. *Then the following three assertions are equivalent.*

(a′) $\quad S^{\overline{r}}_{p,q}F(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) \hookrightarrow C(\mathbb{R}^d)$,

(b′) $\quad S^{\overline{r}}_{p,q}F(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) \hookrightarrow L_\infty(\mathbb{R}^d)$,

(c′) $$\begin{cases} r_i - \dfrac{d_i}{p} > 0 \quad \textit{for all} \quad i = 1, \ldots, N \quad \textit{or} \\ r_i - \dfrac{d_i}{p} \geq 0 \quad \textit{for all} \quad i = 1, \ldots, N \quad \textit{and} \quad 0 < p \leq 1. \end{cases}$$

We consider a similar problem also for L_u, $1 < u < \infty$. Due to the Littlewood-Paley theory the number 2 plays an exceptional role if $1 < u < \infty$.

Theorem 4.23. *(i) Let* $\overline{r} \in \mathbb{R}^N$, $1 < u < \infty$, $0 < p \leq \infty$ *and* $0 < q \leq \infty$. *Then* $S^{\overline{r}}_{p,q}B(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) \hookrightarrow L_u(\mathbb{R}^d)$ *if, and only if,* $p \leq u$ *and*

$$\begin{cases} r_i > \dfrac{d_i}{p} - \dfrac{d_i}{u} \quad \textit{for all} \quad i = 1, \ldots, N \quad \textit{or} \\ r_i \geq \dfrac{d_i}{p} - \dfrac{d_i}{u} \quad \textit{for all} \quad i = 1, \ldots, N, \quad 0 < p < u \quad \textit{and} \quad 0 < q \leq u \quad \textit{or} \\ r_i \geq 0 \quad \textit{for all} \quad i = 1, \ldots, N, \quad p = u \quad \textit{and} \quad 0 < q \leq \min(u, 2). \end{cases}$$

(ii) Let $\overline{r} \in \mathbb{R}^N$, $1 < u < \infty$, $0 < p < \infty$ *and* $0 < q \leq \infty$. *Then* $S^{\overline{r}}_{p,q}F(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) \hookrightarrow L_u(\mathbb{R}^d)$ *if, and only if,* $p \leq u$ *and*

$$\begin{cases} r_i > \dfrac{d_i}{p} - \dfrac{d_i}{u} \quad \textit{for all} \quad i = 1, \ldots, N \quad \textit{or} \\ r_i \geq \dfrac{d_i}{p} - \dfrac{d_i}{u} \quad \textit{for all} \quad i = 1, \ldots, N \quad \textit{and} \quad 0 < p < u \quad \textit{or} \\ r_i \geq 0 \quad \textit{for all} \quad i = 1, \ldots, N, \quad p = u \quad \textit{and} \quad 0 < q \leq 2. \end{cases}$$

4.2.3 Spaces of variable smoothness and integrability

The last work presented in this section, namely the paper [97], studies the spaces of variable smoothness and integrability as introduced recently by L. Diening, P. Hästö, and S. Roudenko in [16].

The definition of these spaces is based on the Lebesgue spaces of variable integrability. The modern era of interest in these spaces dates back essentially to the paper by Kováčik and Rákosník [35].

Definition 4.24. Let $p : \mathbb{R}^d \to (0, \infty)$ be a measurable function. Then the space $L_{p(\cdot)}(\mathbb{R}^d)$ consists of all measurable functions $f : \mathbb{R}^d \to [-\infty, \infty]$ such that $\|f|L_{p(\cdot)}(\mathbb{R}^d)\| < \infty$, where

$$\|f|L_{p(\cdot)}(\mathbb{R}^d)\| = \inf\{\lambda > 0 : \int_{\mathbb{R}^d} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \leq 1\}$$

is the Minkowski functional of the set $\{f : \int_{\mathbb{R}^d} |f(x)|^{p(x)} dx \le 1\}$.

To ensure, that $L_{p(\cdot)}(\mathbb{R}^d)$ are quasi-Banach spaces, we assume that

$$p^- := \inf_{x\in\mathbb{R}^d} p(x) > 0.$$

Furthermore, to avoid the known difficulties of the Triebel-Lizorkin scale for $p = \infty$, we require also that

$$p^+ = \sup_{x\in\mathbb{R}^d} p(x) < \infty,$$

hence we assume, that

$$0 < p^- := \inf_{z\in\mathbb{R}^d} p(z) \le p(x) \le \sup_{z\in\mathbb{R}^d} p(z) =: p^+ < \infty, \quad x \in \mathbb{R}^d. \tag{4.37}$$

This allows to define Triebel-Lizorkin spaces of variable smoothness and integrability by assuming, that s, p and q in Definition 2.1 are (locally integrable) functions of x.

Definition 4.25. Let $s : \mathbb{R}^d \to \mathbb{R}$, $p : \mathbb{R}^d \to (0, \infty)$ and $q : \mathbb{R}^d \to (0, \infty]$ be measurable functions. Then $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^d)$ is the collection of all $f \in S'(\mathbb{R}^d)$ such that

$$\|f|F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^d)\| = \left\| \left(\sum_{j=0}^{\infty} 2^{js(\cdot)q(\cdot)} |(\varphi_j \widehat{f})^\vee(\cdot)|^{q(\cdot)} \right)^{1/q(\cdot)} |L_{p(\cdot)}(\mathbb{R}^d) \right\| < \infty \tag{4.38}$$

(with the usual modification for $q(x) = \infty$). Here, the sequence $(\varphi_j)_{j\in\mathbb{N}_0}$ is the decomposition of unity used in Definition 2.1.

This definition places (almost) no conditions on the functional parameters s, p and q. Unfortunately, in that case the spaces may depend on the choice of the decomposition of unity - an effect very well from the theory of $F^s_{\infty,q}$-spaces, cf. [100]. Therefore we pose some regularity restrictions (identical to those made in [16]).

Definition 4.26. Let g be a continuous function on $\mathbb{R}^d$.
(i) We say, that g is *1-locally* log-*Hölder continuous*, abbreviated $g \in C^{\log}_{1-\mathrm{loc}}(\mathbb{R}^d)$, if there exists $c > 0$ such that

$$|g(x) - g(y)| \le \frac{c}{\log(e + 1/\|x - y\|_\infty)} \quad \text{for all} \quad x, y \in \mathbb{R}^d \quad \text{with} \quad \|x - y\|_\infty \le 1.$$

Here, $\|z\|_\infty = \max\{|z_1|, \dots, |z_n|\}$ denotes the maximum norm of $z \in \mathbb{R}^d$.
(ii) We say, that g is *locally* log-*Hölder continuous*, abbreviated $g \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^d)$, if there exists $c > 0$ such that

$$|g(x) - g(y)| \le \frac{c}{\log(e + 1/|x - y|)}, \quad x, y \in \mathbb{R}^d.$$

(iii) We say, that g is *globally* log-*Hölder continuous*, abbreviated $g \in C^{\log}(\mathbb{R}^d)$, if it is locally log-Hölder continuous and there exists $c > 0$ and $g_\infty \in \mathbb{R}$ such that

$$|g(x) - g_\infty| \le \frac{c}{\log(e + |x|)}, \quad x \in \mathbb{R}^d.$$

Definition 4.27. (Standing assumptions of [16]). Let p and q be positive functions on $\mathbb{R}^d$ such that $\frac{1}{p}, \frac{1}{q} \in C^{\log}(\mathbb{R}^d)$ and let $s \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^d)$ with $s(x) \ge 0$ and let $s(x)$ have a limit at infinity.

Remark 4.28. Our approach in [97] was based on the results of [16]. Especially, to ensure that the norm (4.38) does not depend on the choice of the decomposition of unity, it was necessary to pose the standing assumptions throughout. Later on, Kempka [34] proved, that (4.38) gives equivalent quasi-norms for different decompositions of unity also for a wider range of parameters.

We introduce the sequence spaces associated with the Triebel-Lizorkin spaces of variable smoothness and integrability. We shall use again the notation of the dyadic cubes as given in Definition 3.4. If

$$\gamma = \{\gamma_{jm} \in \mathbb{C} : j \in \mathbb{N}_0, m \in \mathbb{Z}^d\},$$

$-\infty < s(x) < \infty$, $0 < p(x) < \infty$ and $0 < q(x) \le \infty$ for all $x \in \mathbb{R}^d$, we define

$$\begin{aligned} \|\gamma|f^{s(\cdot)}_{p(\cdot),q(\cdot)}\| &= \left\|\left(\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^d} 2^{js(\cdot)q(\cdot)}|\gamma_{jm}|^{q(\cdot)}\chi_{jm}(\cdot)\right)^{1/q(\cdot)}|L_{p(\cdot)}(\mathbb{R}^d)\right\| \qquad (4.39)\\ &= \left\|\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^d} 2^{js(\cdot)}|\gamma_{jm}|\chi_{jm}(\cdot)|L_{p(\cdot)}(\ell_{q(\cdot)})\right\|. \end{aligned}$$

Establishing the connection between the function spaces $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^d)$ and the sequence spaces $f^{s(\cdot)}_{p(\cdot),q(\cdot)}$ was the main aim of [16]. Following [22] and [23], these authors investigated the properties of the φ-transform (as discussed briefly in Section 3.2 and denoted by S_φ) and obtained the following result.

Theorem 4.29. ([16], Corollary 3.9) *Under the Standing assumptions of [16]*

$$\|f|F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^d)\| \approx \|S_\varphi f|f^{s(\cdot)}_{p(\cdot),q(\cdot)}\|$$

with constants independent of $f \in F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^d)$.

Although the technique of non-increasing rearrangement fails in many aspects in the frame of variable-exponent Lebesgue spaces, it was possible to use some ideas from [96] and to prove the following embedding theorem for the sequence spaces.

Theorem 4.30. ([97], Theorem 3.1) *Let* $-\infty < s_1(x) \le s_0(x) < \infty$, $0 < p_0(x) \le p_1(x) < \infty$ *for all* $x \in \mathbb{R}^d$ *with* $0 < p_0^- \le p_1^+ < \infty$ *and*

$$s_0(x) - \frac{d}{p_0(x)} = s_1(x) - \frac{d}{p_1(x)}, \quad x \in \mathbb{R}^d.$$

Let $q(x) = \infty$ *for all* $x \in \mathbb{R}^d$ *or* $0 < q^- \le q(x) < \infty$ *for all* $x \in \mathbb{R}^d$ *and* $s_0, \frac{1}{p_0} \in C^{\log}_{1-\mathrm{loc}}(\mathbb{R}^d)$. *Then*

$$f^{s_0(\cdot)}_{p_0(\cdot),q(\cdot)} \hookrightarrow f^{s_1(\cdot)}_{p_1(\cdot),q(\cdot)}.$$

If the first summability index $q(\cdot)$ should be replaced by ∞ (as one would guess from (2.9)), we have to assume, that $s_0(x)$ is strictly larger then $s_1(x)$, i.e. $\inf_{x\in\mathbb{R}^d}(s_0(x) - s_1(x)) > 0$. Then it is possible to prove the following variant of Theorem 4.30.

Theorem 4.31. ([97], Theorem 3.2) *Let* $-\infty < s_1(x) < s_0(x) < \infty$ *and* $0 < p_0(x) < p_1(x) < \infty$ *for all* $x \in \mathbb{R}^d$ *with* $0 < p_0^- < p_1^+ < \infty$,

$$s_0(x) - \frac{d}{p_0(x)} = s_1(x) - \frac{d}{p_1(x)}, \quad x \in \mathbb{R}^d$$

and

$$\varepsilon := \inf_{x\in\mathbb{R}^d}(s_0(x)-s_1(x)) = d\inf_{x\in\mathbb{R}^d}\left(\frac{1}{p_0(x)}-\frac{1}{p_1(x)}\right) > 0. \tag{4.40}$$

Let $s_0, \frac{1}{p_0} \in C^{\log}_{1-\mathrm{loc}}(\mathbb{R}^d)$*. Then, for every* $0<q\le\infty$*,*

$$f^{s_0(\cdot)}_{p_0(\cdot),\infty} \hookrightarrow f^{s_1(\cdot)}_{p_1(\cdot),q}.$$

Using the theory of [16], our results can be translated immediately into embeddings of function spaces.

Theorem 4.32. ([97], Theorem 3.4) *Let* s_0, s_1, p_0, p_1 *and* q *be continuous functions satisfying the Standing assumptions of [16] with* $s_0(x)\ge s_1(x)$ *and* $p_0(x)\le p_1(x)$ *for all* $x\in\mathbb{R}^d$ *and*

$$s_0(x)-\frac{d}{p_0(x)} = s_1(x)-\frac{d}{p_1(x)}, \quad x\in\mathbb{R}^d.$$

(i) Then

$$F^{s_0(\cdot)}_{p_0(\cdot),q(\cdot)}(\mathbb{R}^d) \hookrightarrow F^{s_1(\cdot)}_{p_1(\cdot),q(\cdot)}(\mathbb{R}^d).$$

(ii) If moreover

$$\inf_{x\in\mathbb{R}^d}(s_0(x)-s_1(x)) = d\inf_{x\in\mathbb{R}^d}\left(\frac{1}{p_0(x)}-\frac{1}{p_1(x)}\right) > 0,$$

then

$$F^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)}(\mathbb{R}^d) \hookrightarrow F^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)}(\mathbb{R}^d).$$

The proof of Theorem 4.32 follows directly from the corresponding estimates on the sequence space level (cf. Theorem 4.30 and Theorem 4.31) and the properties of the φ-transform (cf. Theorem 4.29). One may observe, that the conditions posed on the sequence space level are much milder then those of Theorem 4.29.

Using the recent results of Kempka [34], one could probably obtain a connection between $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^d)$ and $f^{s(\cdot)}_{p(\cdot),q(\cdot)}$ for a larger set of parameters, which would then lead to an improvement of Theorem 4.32. Nevertheless, this direction is open for further investigations.

4.3 Traces of spaces with dominating mixed smoothness

This section presents the results of [93] and [72]. Both the papers deal with the properties of the (diagonal) trace operator in the frame of function spaces with dominating mixed smoothness as introduced in Definition 2.4 - the first one in the plane, the second one in $\mathbb{R}^3$. Let us briefly introduce the problem.

The (off-diagonal) trace operator is defined as

$$(\mathrm{tr}\, f)(t) := f(t,0), \quad t\in\mathbb{R}. \tag{4.41}$$

We denote by $\Gamma = \{(t,t)\in\mathbb{R}^2 : t\in\mathbb{R}\}$ the diagonal of $\mathbb{R}^2$ and define also the diagonal trace operator

$$(\mathrm{tr}_\Gamma\, f)(t) := f(t,t), \quad t\in\mathbb{R}. \tag{4.42}$$

In the frame of the isotropic function spaces $B^s_{p,q}(\mathbb{R}^d)$ and $F^s_{p,q}(\mathbb{R}^d)$, the mapping properties of tr and tr_Γ coincide. In the frame of function spaces with dominating smoothness the situation is more complicated. But let us first clarify some details concerning (4.41) and (4.42).

Both (4.41) and (4.42) are well defined as soon as f is continuous. To avoid this restriction, we assume, that

$$\| \operatorname{tr} f|X(\mathbb{R})\| \le c\,\|f|S^{\overline{r}}_{p,q}A(\mathbb{R}^2)\|, \quad f \in S(\mathbb{R}^2)$$

holds for some quasi-Banach space $X(\mathbb{R})$ of distributions with $X(\mathbb{R}) \hookrightarrow S'(\mathbb{R})$. If $S(\mathbb{R}^2)$ is dense in $S^{\overline{r}}_{p,q}A(\mathbb{R}^2)$, then tr extends uniquely to an operator $\operatorname{tr} : S^{\overline{r}}_{p,q}A(\mathbb{R}^2) \to X(\mathbb{R})$. Furthermore, the condition that $S(\mathbb{R}^2)$ is dense in $S^{\overline{r}}_{p,q}A(\mathbb{R}^2)$ may be sometimes circumvented by the use of trivial embeddings. The symbol $\operatorname{tr} S^{\overline{r}}_{p,q}A(\mathbb{R}^2) = X(\mathbb{R})$ is used to denote that $\operatorname{tr} : S^{\overline{r}}_{p,q}A(\mathbb{R}^2) \to X(\mathbb{R})$ and, moreover, there is an (linear, bounded) extension operator $\operatorname{ext} : X(\mathbb{R}) \to S^{\overline{r}}_{p,q}A(\mathbb{R}^2)$ such that $\operatorname{tr} \circ \operatorname{ext} = id$. The same holds true for tr_Γ and similar arguments apply of course also to function spaces in higher dimensions $d > 2$.

The properties of the off-diagonal trace operator tr in the frame of function spaces with dominating mixed smoothness are well known, cf. [65, Section 2.4.2].

Proposition 4.33. *Let* $0 < q \le \infty$.
(i) *Let* $0 < p \le \infty$ *and* $r_3 > 1/p$ *and let* T *be the following trace operator*

$$T : f(x_1, x_2, x_3) \to f(x_1, x_2, 0).$$

Then $T\, S^{r_1,r_2,r_3}_{p,q}B(\mathbb{R}^3) = S^{r_1,r_2}_{p,q}B(\mathbb{R}^2)$.
(ii) *Let* $0 < p < \infty$ *and* $r_3 > \frac{1}{p}$. *Then* $T S^{r_1,r_2,r_3}_{p,q}F(\mathbb{R}^3) = S^{r_1,r_2}_{p,q}F(\mathbb{R}^2)$.

4.3.1 Two-dimensional case

We study the behavior of the diagonal trace operator tr_Γ as defined above. In [85] Triebel proved that, for $1 \le p \le \infty$, $\operatorname{tr}_\Gamma S^{(r_1,r_2)}_{p,1}B(\mathbb{R}^2) = B^{\varrho}_{p,1}(\mathbb{R})$, where $\varrho = \min(r_1, r_2, r_1 + r_2 - \frac{1}{p}) > 0$. The q-dependence was studied in [59], where Rodriguez proved that $\operatorname{tr}_\Gamma S^{(r_1,r_2)}_{p,q}B(\mathbb{R}^2) = B^{\varrho}_{p,q}(\mathbb{R})$, where

$$0 < p \le \infty, 0 < q < \infty, \varrho > \sigma_p \text{ and } \min(r_1, r_2) \ne \frac{1}{p}.$$

In the "limiting case" $\min(r_1, r_2) = \frac{1}{p}$ the same result is proven for $q \le \min(1, p)$.

It was one of the main aims of [93] to complete the information provided by [85] and [59] and fill the remaining gaps. Surprisingly enough, it turned out that in some cases the trace space of $S^{(r_1,r_2)}_{p,q}B(\mathbb{R}^2)$ is the so-called space of generalized smoothness, cf. [47, 19]. These spaces are defined very much in the spirit of Definition 2.1 with 2^{js} replaced by more general sequences, i.e. $(j+1)^\alpha 2^{js}$, where $\alpha \in \mathbb{R}$ is an additional parameter. The spaces are then denoted by $A^{(s,\alpha)}_{p,q}(\mathbb{R})$ with $A \in \{B, F\}$. If we want to emphasize, that we consider these space on the diagonal Γ, we write rather $A^{(s,\alpha)}_{p,q}(\Gamma)$ instead.

The sequence spaces corresponding to these function spaces are then given by

Definition 4.34. If $0 < p, q \le \infty$, $r, \alpha \in \mathbb{R}$ and

$$\lambda = \{\lambda_{\mu n} \in \mathbb{C} : \mu \in \mathbb{N}_0, n \in \mathbb{Z}\}$$

then we define

$$b^{(r,\alpha)}_{pq} = \left\{ \lambda : \|\lambda|b^{(r,\alpha)}_{pq}\| = \left(\sum_{\mu \in \mathbb{N}_0} (\mu+1)^{\alpha q} 2^{\mu(r-\frac{1}{p})q} \Big(\sum_{n \in \mathbb{Z}} |\lambda_{\mu n}|^p \Big)^{q/p} \right)^{1/q} < \infty \right\}$$

and

$$f_{pq}^{(r,\alpha)} = \left\{ \lambda : \|\lambda | f_{pq}^{(r,\alpha)}\| = \left\| \left(\sum_{\mu\in\mathbb{N}_0} \sum_{n\in\mathbb{Z}} |(\mu+1)^\alpha 2^{\mu r} \lambda_{\mu n} \chi_{\mu n}(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R})\right\| < \infty \right\}$$

with the usual modification for p and/or q equal to ∞.

The following theorem was proved in [93] using quarkonial decomposition, cf. Definition 3.14 and Theorem 3.15, adapted to function spaces of dominating mixed smoothness. We refer to [94] for the corresponding version.

Theorem 4.35. ([93], Theorem 3.1) *Let $0 < p, q \le \infty$, and $\overline{r} = (r_1, r_2) \in \mathbb{R}^2$ with*

$$0 < r_1 \le r_2, \varrho = \min\Big(r_1, r_1 + r_2 - \frac{1}{p}\Big) > \sigma_p.$$

If $r_2 \neq \frac{1}{p}$ or $r_2 = \frac{1}{p}$ and $q \le \min(1,p)$ then

$$\mathrm{tr}_\Gamma\, S^{\overline{r}}_{p,q}B(\mathbb{R}^2) = B^{\varrho}_{p,q}(\Gamma).$$

If $r_2 = \frac{1}{p}$, $1 \le \min(p,q)$ then

$$\mathrm{tr}_\Gamma\, S^{\overline{r}}_{p,q}B(\mathbb{R}^2) = B^{(r_1,\frac{1}{q}-1)}_{p,q}(\Gamma).$$

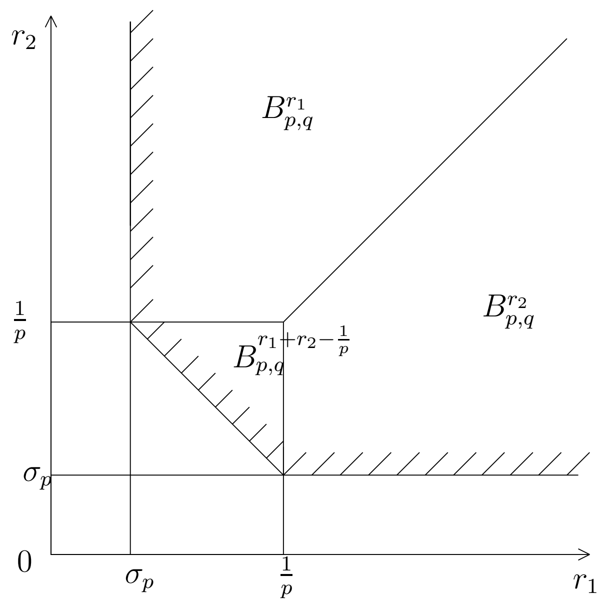
Finally, if $r_2 = \frac{1}{p}$, $p \le \min(1,q)$ then

$$\mathrm{tr}_\Gamma : S^{\overline{r}}_{p,q}B(\mathbb{R}^2) \to B^{(r_1,\frac{1}{q}-\frac{1}{p})}_{p,q}(\Gamma)$$

and

$$\mathrm{ext} : B^{(r_1,\min(\frac{1}{q}-1,0))}_{p,q}(\Gamma) \to S^{\overline{r}}_{p,q}B(\mathbb{R}^2).$$

The following diagram illustrates the behavior of tr_Γ in the non-limiting case.



The other main aim of [93] was to study the behavior in the frame of the Triebel-Lizorkin spaces of dominating mixed smoothness, where virtually nothing was known before. We observe, that in some (but not all) cases the trace space does not depend on q (an effect very well known in the isotropic case, cf. [23]).

Theorem 4.36. ([93], Theorem 4.1) *Let*

$$0 < p < \infty, \quad 0 < q \le \infty, \quad 0 < r_1 \le r_2$$

with

$$\varrho = \min\Big(r_1, r_1 + r_2 - \frac{1}{p}\Big) > \sigma_{p,q}.$$

If $r_2 > \frac{1}{p}$ *then*

$$\mathrm{tr}_\Gamma\, S^{\overline{r}}_{p,q}F(\mathbb{R}^2) = F^{\varrho}_{p,q}(\Gamma). \tag{4.43}$$

If $r_2 < \frac{1}{p}$ *then*

$$\mathrm{tr}_\Gamma\, S^{\overline{r}}_{p,q}F(\mathbb{R}^2) = F^{\varrho}_{p,p}(\Gamma) = B^{\varrho}_{p,p}(\Gamma). \tag{4.44}$$

If $r_2 = \frac{1}{p}$ *and* $p \le \min(1, q)$ *then*

$$\mathrm{tr}_\Gamma\, S^{\overline{r}}_{p,q}F(\mathbb{R}^2) = F^{r_1}_{p,q}(\Gamma). \tag{4.45}$$

If $r_2 = \frac{1}{p}$ *and* $q < p \le 1$ *then*

$$\mathrm{tr}_\Gamma\, S^{\overline{r}}_{p,q}F(\mathbb{R}^2) = F^{r_1}_{p,p}(\Gamma). \tag{4.46}$$

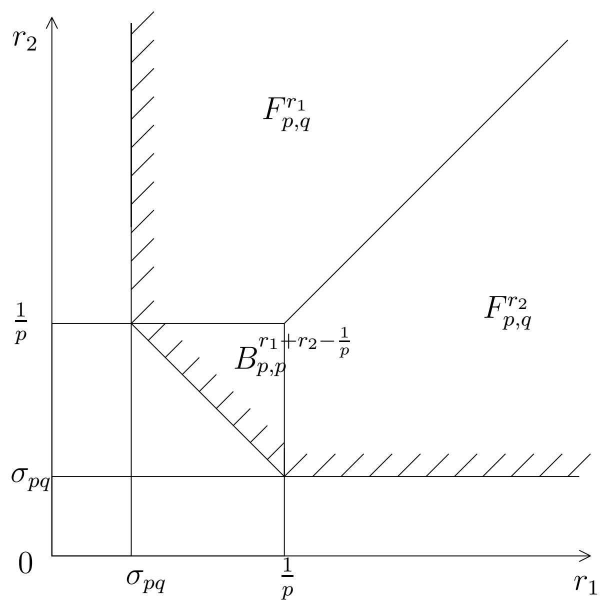
If $r_2 = \frac{1}{p}$ *and* $1 \le p \le q$ *then*

$$\mathrm{tr}_\Gamma : S^{\overline{r}}_{p,q}F(\mathbb{R}^2) \to F^{(r_1, \frac{1}{q}-1)}_{p,q}(\Gamma). \tag{4.47}$$

Finally, if $r_2 = \frac{1}{p}$ *and* $p \ge \max(1, q)$ *then*

$$\mathrm{tr}_\Gamma : S^{\overline{r}}_{p,q}F(\mathbb{R}^2) \to F^{(r_1, \frac{1}{p}-1)}_{p,p}(\Gamma). \tag{4.48}$$

We again illustrate the behavior of tr_Γ in the non-limiting cases by a simple diagram.



4.3.2 Three-dimensional case

The paper [72] studies a similar question in $\mathbb{R}^3$. Surprisingly enough, new fundamental effects appear. To be able to demonstrate these effects in a most simple way, we first consider the case of Sobolev spaces with $p = 2$. In this case, we use the techniques of Fourier analysis and proceed directly. Later on, we use the machinery of atomic decompositions to generalize the results to the full scale of Besov and Triebel-Lizorkin spaces of dominating mixed smoothness.

We consider the trace with respect to the hyperplane

$$\Gamma := \Big\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \quad x_1 + x_2 + x_3 = 0\Big\}$$

with Γ as a model case for a hyperplane in an oblique position. However, taking the trace with respect to the hyperplane

$$\Gamma_{\overline{\gamma}} := \{(x_1, x_2, x_3) : \quad \gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_3 = 0\}, \qquad \overline{\gamma} = (\gamma_1, \gamma_2, \gamma_3)\,,$$

where $\gamma_1 \cdot \gamma_2 \cdot \gamma_3 \neq 0$, would give us the same result (up to the norms of considered operators). This statement relies on the fact, that the mapping

$$f(x_1, x_2, x_3) \to f(\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3), \quad \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \neq 0,$$

is a bounded bijective mapping of $S_2^{\overline{r}}W(\mathbb{R}^3)$ onto itself (cf. Definition 4.37 for the definition of $S_2^{\overline{r}}W(\mathbb{R}^3)$).

The trace operator shall depend on the choice of the orthogonal basis in Γ. Let us give the details. Let

$$\mathcal{O} = \{\vec{\sigma}_1, \vec{\sigma}_2\}, \quad \vec{\sigma}_1 = (\sigma_{1,1}, \sigma_{1,2}, \sigma_{1,3}) \in \Gamma, \quad \vec{\sigma}_2 = (\sigma_{2,1}, \sigma_{2,2}, \sigma_{2,3}) \in \Gamma, \quad \vec{\sigma}_1 \perp \vec{\sigma}_2 \tag{4.49}$$

be an orthogonal basis of Γ. Then we associate to this basis the corresponding "orthogonal" trace operator

$$(\mathrm{tr}_{\mathcal{O}}\, f)(z_1, z_2) = f(z_1\vec{\sigma}_1 + z_2\vec{\sigma}_2), \quad z_1, z_2 \in \mathbb{R}. \tag{4.50}$$

The "natural" trace operators

$$\begin{aligned}
(\mathrm{tr}_1\, f)(x_2, x_3) &= f(-x_2 - x_3, x_2, x_3), && (4.51)\\
(\mathrm{tr}_2\, f)(x_1, x_3) &= f(x_1, -x_1 - x_3, x_3), && (4.52)\\
(\mathrm{tr}_3\, f)(x_1, x_2) &= f(x_1, x_2, -x_1 - x_2) && (4.53)
\end{aligned}$$

and the trace operator $\mathrm{tr}_{\mathcal{O}}\, f$, see (4.49) and (4.50), are connected through

$$\begin{aligned}
(\mathrm{tr}_{\mathcal{O}}\, f)(z_1, z_2) &= f(z_1\vec{\sigma}_1 + z_2\vec{\sigma}_2) = f\left(\sigma_{1,1}z_1 + \sigma_{2,1}z_2, \sigma_{1,2}z_1 + \sigma_{2,2}z_2, \sigma_{1,3}z_1 + \sigma_{2,3}z_2\right)\\
&= (\mathrm{tr}_1\, f)\left(\sigma_{1,2}z_1 + \sigma_{2,2}z_2, \sigma_{1,3}z_1 + \sigma_{2,3}z_2\right)\\
&= (\mathrm{tr}_1\, f)(\mathcal{R}_1\vec{z}), \qquad (4.54)
\end{aligned}$$

where

$$\mathcal{R}_1 = \begin{pmatrix} \sigma_{1,2} & \sigma_{2,2} \\ \sigma_{1,3} & \sigma_{2,3} \end{pmatrix} \quad \text{and} \quad \vec{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \tag{4.55}$$

Analogously one obtains

$$(\mathrm{tr}_{\mathcal{O}}\, f)(z_1, z_2) = (\mathrm{tr}_2\, f)(\mathcal{R}_2\vec{z}) = (\mathrm{tr}_3\, f)(\mathcal{R}_3\vec{z})\,, \tag{4.56}$$

with

$$\mathcal{R}_2 = \begin{pmatrix} \sigma_{1,1} & \sigma_{2,1} \\ \sigma_{1,3} & \sigma_{2,3} \end{pmatrix}, \quad \mathcal{R}_3 = \begin{pmatrix} \sigma_{1,1} & \sigma_{2,1} \\ \sigma_{1,2} & \sigma_{2,2} \end{pmatrix}. \tag{4.57}$$

The linear independence of the vectors $\vec{\sigma}_1, \vec{\sigma}_2$, combined with $\vec{\sigma}_1, \vec{\sigma}_2 \in \Gamma$, ensure that the matrices $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ are regular. In what follows we shall determine the regularity of $\mathrm{tr}_{\mathcal{O}} f$ as well as of $\mathrm{tr}_i f$, $i = 1, 2, 3$. For that, we shall need certain new spaces.

Let $\mathcal{M}$ be a 2×2-matrix,

$$\mathcal{M} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \quad \text{with} \quad \det \mathcal{M} \neq 0. \tag{4.58}$$

Definition 4.37. (i) Let $1 < p < \infty$ and $\overline{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$. Then $S_p^{\overline{r}} W(\mathbb{R}^d)$ is defined as

$$S_p^{\overline{r}} W(\mathbb{R}^d) = \Big\{ f \in S'(\mathbb{R}^d) : \mathcal{F}^{-1}\Big((1 + |\xi_1|^2)^{r_1/2} \, \ldots \, (1 + |\xi_d|^2)^{r_d/2} \, \mathcal{F} f(\xi)\Big)(\,\cdot\,) \in L_p(\mathbb{R}^d)\Big\}.$$

Furthermore, $S_p^{\overline{r}} W(\mathbb{R}^d)$ is equipped with the norm

$$\| \, f \, | S_p^{\overline{r}} W(\mathbb{R}^d) \| := \Big\| \, \mathcal{F}^{-1}\Big(\prod_{i=1}^{d} (1 + |\xi_i|^2)^{r_i/2} \, \mathcal{F} f(\xi) \Big)(\,\cdot\,) \, \Big| L_p(\mathbb{R}^d) \Big\|. \tag{4.59}$$

(ii) Let $\mathcal{M}$ be as in (4.58). Let $r_1, r_2 \in \mathbb{R}$. Then $S_2^{r_1,r_2} W(\mathcal{M}, \mathbb{R}^2)$ denotes the collection of all tempered distributions $f \in S'(\mathbb{R}^2)$ such that $f \circ \mathcal{M} \in S_2^{r_1,r_2} W(\mathbb{R}^2)$. We endow this class with the norm

$$\| \, f \, | S_2^{r_1,r_2} W(\mathcal{M}, \mathbb{R}^2) \| := \| \, f \circ \mathcal{M} \, | S_2^{r_1,r_2} W(\mathbb{R}^2) \| \, .$$

Using the well-known properties of the Fourier transform, it was possible to obtain the description of the properties of $\mathrm{tr}_{\mathcal{O}}$ as described by the following theorem.

Theorem 4.38. ([72], Theorem 2.9) *Let $\mathcal{O}$ be an orthogonal basis of Γ and let $\mathcal{R}_i, i = 1, 2, 3$ be matrices associated with $\mathcal{O}$ by (4.49), (4.55) and (4.57).*
Let $\overline{r} = (r_1, r_2, r_3) \in \mathbb{R}^3$ with $r_i \neq 1/2, i = 1, 2, 3$ and

$$\min\left(r_1, r_2, r_3, r_1 + r_2 - \frac{1}{2}, r_1 + r_3 - \frac{1}{2}, r_2 + r_3 - \frac{1}{2} \right) > 0\,. \tag{4.60}$$

Then

$$\mathrm{tr}_{\mathcal{O}} \in \mathcal{L}\Big(S_2^{\overline{r}} W(\mathbb{R}^3), S^1(\mathbb{R}^2) + S^2(\mathbb{R}^2) + S^3(\mathbb{R}^2) \Big), \tag{4.61}$$

where

$$S^1(\mathbb{R}^2) := \begin{cases} S_2^{r_2,r_3} W(\mathcal{R}_1^{-1}, \mathbb{R}^2), & \text{if } r_1 > \frac{1}{2}\,, \\ S_2^{r_2,r_3+r_1-\frac{1}{2}} W(\mathcal{R}_1^{-1}, \mathbb{R}^2) \cap S_2^{r_2+r_1-\frac{1}{2},r_3} W(\mathcal{R}_1^{-1}, \mathbb{R}^2), & \text{if } r_1 < \frac{1}{2}\,, \end{cases}$$

and similarly for S^2 and S^3.
Conversely, to each function $g \in S^1(\mathbb{R}^2) + S^2(\mathbb{R}^2) + S^3(\mathbb{R}^2)$ there exists a function $f \in S_2^{\overline{r}} W(\mathbb{R}^3)$ such that $\mathrm{tr}_{\mathcal{O}} f = g$.

Let us give more details on the sum of the three spaces appearing in (4.61). It may be illustrated from a different point of view on the Fourier side. For simplicity we concentrate on the situation $\min(r_1, r_2, r_3) > 1/2$.

Let again $\mathcal{O}$ be an orthogonal basis of $\Gamma = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$ and let $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ be the matrices associated with $\mathcal{O}$. First, we notice that $g_3 \in S_2^{r_1,r_2}W(\mathcal{R}_3^{-1}, \mathbb{R}^2)$ if, and only if,

$$\underbrace{\Big[1 + (\sigma_{2,2}\xi_1 - \sigma_{1,2}\xi_2)^2\Big]^{r_1/2}\Big[1 + (\sigma_{2,1}\xi_1 - \sigma_{1,1}\xi_2)^2\Big]^{r_2/2}}_{m_3(\xi_1,\xi_2)} \mathcal{F}g_3(\xi_1, \xi_2) \in L_2(\mathbb{R}^2)\,,$$

Similarly, $g_1 \in S_2^{r_2,r_3}W(\mathcal{R}_1^{-1}, \mathbb{R}^2)$ if, and only if,

$$\underbrace{\Big[1 + \Big(\sigma_{2,3}\xi_1 - \sigma_{1,3}\xi_2\Big)^2\Big]^{r_2/2}\Big[1 + \Big(\sigma_{2,2}\xi_1 - \sigma_{1,2}\xi_2\Big)^2\Big]^{r_3/2}}_{m_1(\xi_1,\xi_2)} \mathcal{F}g_1(\xi_1, \xi_2) \in L_2(\mathbb{R}^2)$$

and $g_2 \in S_2^{r_1,r_3}W(\mathcal{R}_2^{-1}, \mathbb{R}^2)$ if, and only if,

$$\underbrace{\Big[1 + \Big(\sigma_{2,3}\xi_1 - \sigma_{1,3}\xi_2\Big)^2\Big]^{r_1/2}\Big[1 + (\sigma_{2,1}\xi_1 - \sigma_{1,1}\xi_2)^2\Big]^{r_3/2}}_{m_2(\xi_1,\xi_2)} \mathcal{F}g_2(\xi_1, \xi_2) \in L_2(\mathbb{R}^2)\,.$$

In view of these characterizations we define

$$m(\xi_1, \xi_2) := \min\Big(m_1(\xi_1, \xi_2), m_2(\xi_1, \xi_2), m_3(\xi_1, \xi_2)\Big)\,. \tag{4.62}$$

and

$$L_2(\mathbb{R}^2, m) := \Big\{g \in L_2(\mathbb{R}^2) : \quad m\,\mathcal{F}g \in L_2(\mathbb{R}^2)\Big\}$$

equipped with the natural norm

$$\|\, g\,|L_2(\mathbb{R}^2, m)\| := \|\, m\,\mathcal{F}g\,|L_2(\mathbb{R}^2)\|\,.$$

Equipped with this notation, we may restate the result of Theorem 4.38 in the following way.

Theorem 4.39. ([72], Theorem 2.11) *Let $\mathcal{O}$ be an orthogonal basis of Γ and let $\mathcal{R}_i, i = 1, 2, 3$ be matrices associated with $\mathcal{O}$ by* (4.49), (4.55) *and* (4.57). *Suppose that $r_i > 1/2$, $i = 1, 2, 3$. Then there exists a continuous function m such that $\mathrm{tr}_{\mathcal{O}}$ becomes a retraction of $S_2^{r_1,r_2,r_3}W(\mathbb{R}^3)$ onto $L_2(\mathbb{R}^2, m)$. There is a bounded linear extension operator $\mathrm{ext} \in \mathcal{L}(L_2(\mathbb{R}^2, m), S_2^{r_1,r_2,r_3}W(\mathbb{R}^3))$ such that $\mathrm{tr}_{\mathcal{O}} \circ \mathrm{ext} = I$ (identity on $L_2(\mathbb{R}^2, m)$).*

Let us mention that the proofs of Theorems 4.38 and 4.39 involve properties of the Fourier transform (i.e. Plancherel identity), which are not available for $p \neq 2$. Therefore, a direct generalization of this method to such p's seems to be impossible. Nevertheless, the powerful technique of atomic decompositions allowed this step. The description of $\mathrm{tr}_{\mathcal{O}}$ in the frame of Besov and Triebel-Lizorkin spaces is based on the modification of Definition 4.37.

Definition 4.40. Let $0 < p, q \leq \infty$ with $0 < p < \infty$ in the F-case. Let $\mathcal{R}$ be a (2,2)-matrix with $\det \mathcal{R} \neq 0$. Then we put

$$S^{\overline{r}}_{p,q}A(\mathcal{R}, \mathbb{R}^2) := \Big\{f \in S'(\mathbb{R}^2) : f \circ \mathcal{R} \in S^{\overline{r}}_{p,q}A(\mathbb{R}^2)\Big\}\,,$$
$$\|f|S^{\overline{r}}_{p,q}A(\mathcal{R}, \mathbb{R}^2)\| := \|\, f \circ \mathcal{R}\,|S^{\overline{r}}_{p,q}A(\mathbb{R}^2)\|\,.$$

Recall that for $p = q = 2$ we have coincidence of $S^{\overline{r}}_{2,2}B(\mathcal{R}, \mathbb{R}^2)$ with $S^{\overline{r}}_2 W(\mathcal{R}, \mathbb{R}^2)$ in the sense of equivalent norms, cf. [65, Thm. 2.3.1]. By means of these classes we are able to describe the trace classes for Besov as well as for Lizorkin-Triebel classes.

The counterpart of Theorem 4.38 for Besov spaces is as follows.

Theorem 4.41. ([72], Theorem 3.10) *Let $\mathcal{O}$ be an orthogonal basis of Γ and let $\mathcal{R}_i, i = 1,2,3$ be matrices associated with $\mathcal{O}$ by (4.49), (4.55) and (4.57).*
Let $0 < p, q \le \infty$ and $\overline{r} = (r_1, r_2, r_3) \in \mathbb{R}^3$ with $r_i \neq \frac{1}{p}, i = 1,2,3$ and

$$\min\left(r_1, r_2, r_3, r_1 + r_2 - \frac{1}{p}, r_1 + r_3 - \frac{1}{p}, r_2 + r_3 - \frac{1}{p}\right) > \sigma_p. \tag{4.63}$$

Then

$$\mathrm{tr}_{\mathcal{O}} \in \mathcal{L}\Big(S^{\overline{r}}_{p,q}B(\mathbb{R}^3), S^1(\mathbb{R}^2) + S^2(\mathbb{R}^2) + S^3(\mathbb{R}^2)\Big), \tag{4.64}$$

where

$$S^1(\mathbb{R}^2) := \begin{cases} S^{r_2,r_3}_{p,q}B(\mathcal{R}_1^{-1}, \mathbb{R}^2), & \text{if } r_1 > \frac{1}{p}, \\ S^{r_2,r_3+r_1-\frac{1}{p}}_{p,q}B(\mathcal{R}_1^{-1}, \mathbb{R}^2) \cap S^{r_2+r_1-\frac{1}{p},r_3}_{p,q}B(\mathcal{R}_1^{-1}, \mathbb{R}^2), & \text{if } r_1 < \frac{1}{p}, \end{cases}$$

and similarly for S^2 and S^3.
Conversely, to each function $g \in S^1(\mathbb{R}^2) + S^2(\mathbb{R}^2) + S^3(\mathbb{R}^2)$ there exists a function $f \in S^{\overline{r}}_{p,q}B(\mathbb{R}^3)$ such that $\mathrm{tr}_{\mathcal{O}} f = g$.

The proof is based on the atomic decomposition of spaces with dominating mixed smoothness as described in Theorem 3.13 combined with a certain discrete variant of (4.62).

Now we turn to the Triebel-Lizorkin classes. To prove an analog of Theorem 4.41 for these spaces we can proceed in the same way as in case of the Besov spaces.

Theorem 4.42. ([72], Theorem 3.14) *Let $\mathcal{O}$ be an orthogonal basis of Γ and let $\mathcal{R}_i, i = 1,2,3$ be matrices associated with $\mathcal{O}$ by (4.49), (4.55) and (4.57). Let $0 < p < \infty$ and $0 < q \le \infty$. Let $\overline{r} = (r_1, r_2, r_3) \in \mathbb{R}^3$ with*

$$\min(r_1, r_2, r_3) > \max\left(\frac{1}{p}, \sigma_{pq}\right). \tag{4.65}$$

Then

$$\mathrm{tr}_{\mathcal{O}} \in \mathcal{L}\Big(S^{\overline{r}}_{p,q}F(\mathbb{R}^3),\ S^{r_2,r_3}_{p,q}F(\mathcal{R}_1^{-1}, \mathbb{R}^2) + S^{r_1,r_3}_{p,q}F(\mathcal{R}_2^{-1}, \mathbb{R}^2) + S^{r_1,r_2}_{p,q}F(\mathcal{R}_3^{-1}, \mathbb{R}^2)\Big). \tag{4.66}$$

Conversely, to each function $g \in S^{r_2,r_3}_{p,q}F(\mathcal{R}_1^{-1}, \mathbb{R}^2) + S^{r_1,r_3}_{p,q}F(\mathcal{R}_2^{-1}, \mathbb{R}^2) + S^{r_1,r_2}_{p,q}F(\mathcal{R}_3^{-1}, \mathbb{R}^2)$ there exists a function $f \in S^{\overline{r}}_{p,q}F(\mathbb{R}^3)$ such that $\mathrm{tr}_{\mathcal{O}} f = g$.

4.4 Radial subspaces

The last section presents the results of [68]. The study of radial functions was initiated at the end of seventies through the works of Strauss [79] and Lions [40]. The radial lemma of Strauss reads as follows.

Lemma 4.43. ([79]) *Let $d \ge 2$. Every radial function $f \in W^1_2(\mathbb{R}^d)$ is almost everywhere equal to a function $\tilde{f}$, continuous for $x \neq 0$, such that*

$$|\tilde{f}(x)| \le c\,|x|^{\frac{1-d}{2}}\, \| f \,|W^1_2(\mathbb{R}^d)\|, \quad x \in \mathbb{R}^d,$$

where c depends only on d.

The *Radial Lemma* contains three different assertions:

(a) the existence of a representative of f, which is continuous outside the origin;

(b) the decay of f near infinity;

(c) the limited unboundedness near the origin.

The aim of [68] was to study the interplay between regularity and decay properties of radial functions in the broad frame of the scales of Besov and Triebel-Lizorkin spaces. We follow a simple philosophy, namely that the properties of a radial function $f : \mathbb{R}^d \to \mathbb{C}$ are completely determined by its trace

$$(\mathrm{tr}\, f)(t) := f(t,0,\dots,0)\,, \qquad t \in \mathbb{R}.$$

The corresponding extension operator is then defined by

$$(\mathrm{ext}\, g)(x) := g(|x|)\,, \qquad x \in \mathbb{R}^d.$$

Both tr and ext are defined pointwise only under natural regularity restrictions on f and g, respectively. We shall later discuss the existence of the trace in a distributional sense. We shall use the following notation. If E is a space of functions or distributions on $\mathbb{R}^d$, then we denote by RE the subspace of radial functions or distributions of E equipped with the same (quasi-)norm.

4.4.1 Trace spaces of radial subspaces

Although the following result may be very well known, we were not able to find the reference. It presents in a simple way our strategy.

Theorem 4.44. ([68], Theorem 1) *Let $d \geq 2$. For $m \in \mathbb{N}_0$ the mapping* tr *is a linear isomorphism of $RC^m(\mathbb{R}^d)$ onto $RC^m(\mathbb{R})$ with inverse* ext*.*

The proof of this theorem proceeds by mathematical induction and makes use of direct calculations only. This comes as no surprise since the spaces $C^m(\mathbb{R}^d)$ fall into the category of "bad" spaces with no convenient decomposition technique available.

Using real interpolation it is not difficult to derive the following result for the spaces of Hölder-Zygmund type.

Theorem 4.45. ([68], Theorem 2) *Let $s > 0$ and let $0 < q \leq \infty$. Then the mapping* tr *is a linear isomorphism of $RB^s_{\infty,q}(\mathbb{R}^d)$ onto $RB^s_{\infty,q}(\mathbb{R})$ with inverse* ext*.*

The description of the properties of the trace operator in the frame of Besov and Triebel-Lizorkin spaces with $p < \infty$ is predetermined by the simple case of Lebesgue spaces.

Lemma 4.46. *Let $d \geq 2$.*
(i) *Let $0 < p < \infty$. Then* tr $: RL_p(\mathbb{R}^d) \to RL_p(\mathbb{R}, |t|^{d-1})$ *is an linear isomorphism with inverse* ext*.*
(ii) *Let $p = \infty$. Then* tr $: RL_\infty(\mathbb{R}^d) \to RL_\infty(\mathbb{R})$ *is an linear isomorphism with inverse* ext*.*

In particular this means, that whenever the Besov-Triebel-Lizorkin space $A^s_{p,q}(\mathbb{R}^d)$ is contained in $L_1(\mathbb{R}^d) + L_\infty(\mathbb{R}^d)$, then tr is well-defined on its radial subspace. Furthermore, it is known, that

$$B^s_{p,q}(\mathbb{R}^d)\,,\ F^s_{p,q}(\mathbb{R}^d) \hookrightarrow L_1(\mathbb{R}^d) + L_\infty(\mathbb{R}^d)$$

if $s > d \max(0, \frac{1}{p}-1)$, see, e.g., [69]. The proof of Lemma 4.46 involves only a simple substitution.

The trace spaces of the radial subspaces of $B^s_{p,q}(\mathbb{R}^d)$ and $F^s_{p,q}(\mathbb{R}^d)$ are spaces *defined* in terms of atoms. To explain this we need to introduce an appropriate notion of atom as well as appropriately adapted sequence spaces.

Definition 4.47. Let $L \geq 0$ be an integer. Let I be a set either of the form $I = [-a, a]$ or of the form $I = [-b, -a] \cup [a, b]$ for some $0 < a < b < \infty$. An even function $g \in C^L(\mathbb{R})$ is called an even L-atom centered at I if

$$\max_{t \in \mathbb{R}} |b^{(n)}(t)| \leq |I|^{-n}\, , \qquad 0 \leq n \leq L$$

and if either

$$\operatorname{supp} g \subset [-\frac{3\,a}{2}, \frac{3a}{2}] \qquad \text{in case} \quad I = [-a, a]\, ,$$

or

$$\operatorname{supp} g \subset [-\frac{3b-a}{2}, -\frac{3a-b}{2}] \cup [\frac{3a-b}{2}, \frac{3b-a}{2}] \qquad \text{in case} \quad I = [-b, -a] \cup [a, b]\, .$$

Definition 4.48. Let $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. Let

$$\chi^{\#}_{j,k}(t) := \begin{cases} 1 & \text{if} \quad 2^{-j}k \leq |t| \leq 2^{-j}(k+1)\, , \\ 0 & \text{otherwise,} \end{cases} \qquad t \in \mathbb{R}\, .$$

Then we define

$$b^s_{p,q,d} := \left\{ \lambda = (\lambda_{j,k})_{j,k} \; : \quad \|\, \lambda \,|b^s_{p,q,d}\| = \left(\sum_{j=0}^{\infty} 2^{j(s-\frac{d}{p})q} \left(\sum_{k=0}^{\infty} (1+k)^{d-1}\, |\lambda_{j,k}|^p \right)^{q/p} \right)^{1/q} < \infty \right\}$$

and

$$\begin{aligned} f^s_{p,q,d} \quad &:= \quad \Bigg\{ \lambda = (\lambda_{j,k})_{j,k} \; : \\ \|\, \lambda \,|f^s_{p,q,d}\| \quad &= \quad \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} \sum_{k=0}^{\infty} |\lambda_{j,k}|^q\, \chi^{\#}_{j,k}(\cdot) \right)^{1/q} |L_p(\mathbb{R}, |t|^{d-1}) \right\| < \infty \Bigg\}\, , \end{aligned}$$

respectively.

Observe $b^s_{p,q,d} = f^s_{p,q,d}$ in the sense of equivalent quasi-norms. Also the weight $|t|^{d-1}$ from Lemma 4.46 has its direct counterpart in the definition of $b^s_{p,q,d}$ and $f^s_{p,q,d}$. Equipped with these sequence spaces we define now function spaces on $\mathbb{R}$.

Definition 4.49. Let $0 < p < \infty$, $0 < q \leq \infty$, $s > 0$ and $L \in \mathbb{N}_0$.
(i) Then $TB^s_{p,q}(\mathbb{R}, L, d)$ is the collection of all functions $g : \mathbb{R} \to \mathbb{C}$ such that there exists a decomposition

$$g(t) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \lambda_{j,k}\, g_{j,k}(t) \tag{4.67}$$

(convergence in $L_{\max(1,p)}(\mathbb{R}, |t|^{d-1})$), where the sequence $(\lambda_{j,k})_{j,k}$ belongs to $b^s_{p,q,d}$ and the functions $g_{j,k}$ are even L-atoms centered at either $[-2^{-j}, 2^{-j}]$ if $k = 0$ or at

$$[-2^{-j}(k+1), -2^{-j}k] \;\cup\; [2^{-j}k, 2^{-j}(k+1)]$$

if $k > 0$. We put

$$\| g \,| TB^s_{p,q}(\mathbb{R}, L, d)\| := \inf \Big\{ \| (\lambda_{j,k}) \,| b^s_{p,q,d}\| : \quad (4.67) \quad \text{holds} \Big\}.$$

(ii) Then $TF^s_{p,q}(\mathbb{R}, L, d)$ is the collection of all functions $g : \mathbb{R} \to \mathbb{C}$ such that there exists a decomposition (4.67), where the sequence $(\lambda_{j,k})_{j,k}$ belongs to $f^s_{p,q,d}$ and the functions $g_{j,k}$ are as in (i). We put

$$\| g \,| TF^s_{p,q}(\mathbb{R}, L, d)\| := \inf \Big\{ \| (\lambda_{j,k}) \,| f^s_{p,q,d}\| : \quad (4.67) \quad \text{holds} \Big\}.$$

The following theorem shows, that the spaces $TA^s_{p,q}(\mathbb{R}, L, d)$ are the trace spaces of $RA^s_{p,q}(\mathbb{R}^d)$.

Theorem 4.50. ([68], Theorem 3) *Let $d \geq 2$, $0 < p < \infty$ and $0 < q \leq \infty$.*
(i) *Suppose $s > \sigma^d_p$ and $L \geq [s] + 1$. Then the mapping* tr *is a linear isomorphism of $RB^s_{p,q}(\mathbb{R}^d)$ onto $TB^s_{p,q}(\mathbb{R}, L, d)$ with inverse* ext.
(ii) *Suppose $s > \sigma^d_{p,q}$ and $L \geq [s] + 1$. Then the mapping* tr *is a linear isomorphism of $RF^s_{p,q}(\mathbb{R}^d)$ onto $TF^s_{p,q}(\mathbb{R}, L, d)$ with inverse* ext.

Of course, the description of the trace spaces given in Theorem 4.50 suffers from the fact that the spaces $TA^s_{p,q}(\mathbb{R}, L, d)$ were *defined* through their atomic decomposition properties. It would be highly desirable to have some intrinsic characterization of these spaces - for example if one could identify them with some weighted Besov and Triebel-Lizorkin spaces very much in the same manner as Lemma 4.46 describes the trace spaces of radial Lebesgue spaces as weighted Lebesgue spaces.

Unfortunately, this is not always possible. The following theorem shows that even the weighted Lebesgue space obtained in Lemma 4.46 is not always a space of distributions. To allow a comparison between trace spaces of radial Besov and Triebel-Lizorkin spaces and weighted spaces of the same type, we therefore need to know first, when the trace operator maps $RA^s_{p,q}(\mathbb{R}^d)$ into $S'(\mathbb{R})$. The answer is described by the following

Theorem 4.51. ([68], Theorem 8) *Let $d \geq 2$, $0 < p < \infty$, and $0 < q \leq \infty$.*
(a) *Let $d \geq 2$ and let $0 < p < \infty$. Then $RL_p(\mathbb{R}, |t|^{d-1}) \subset S'(\mathbb{R})$ if and only if $d < p$.*
(b) *Let $s > \sigma^d_p$ and $L \geq [s] + 1$. Then the following assertions are equivalent:*
- (i) *The mapping* tr *maps $RB^s_{p,q}(\mathbb{R}^d)$ into $S'(\mathbb{R})$.*
- (ii) *The mapping* tr : *$RB^s_{p,q}(\mathbb{R}^d) \to S'(\mathbb{R})$ is continuous.*
- (iii) *We have $TB^s_{p,q}(\mathbb{R}, L, d) \hookrightarrow S'(\mathbb{R})$.*
- (iv) *We have either $s > d(\frac{1}{p} - \frac{1}{d})$ or $s = d(\frac{1}{p} - \frac{1}{d})$ and $q \leq 1$.*

(c) *Let $s > \sigma^d_{p,q}$ and $L \geq [s] + 1$. Then following assertions are equivalent:*
- (i) *The mapping* tr *maps $RF^s_{p,q}(\mathbb{R}^d)$ into $S'(\mathbb{R})$.*
- (ii) *The mapping* tr : *$RF^s_{p,q}(\mathbb{R}^d) \to S'(\mathbb{R})$ is continuous.*
- (iii) *We have $TF^s_{p,q}(\mathbb{R}, L, d) \hookrightarrow S'(\mathbb{R})$.*
- (iv) *We have either $s > d(\frac{1}{p} - \frac{1}{d})$ or $s = d(\frac{1}{p} - \frac{1}{d})$ and $0 < p \leq 1$.*

Weighted function spaces of Besov and Triebel-Lizorkin type, denoted by $B^s_{p,q}(\mathbb{R}, w)$ and $F^s_{p,q}(\mathbb{R}, w)$, respectively, are a well-developed subject in the literature, we refer to [9, 10, 61]. Fourier analytic definitions as well as characterizations by atoms are given under various restrictions on the weights, see e.g. [9, 10, 31, 32, 65]. In this subsection we are interested in these spaces with respect to the weights $w_{d-1}(t) := |t|^{d-1}$, $t \in \mathbb{R}$, $d \geq 2$. Of course, these weights belong to the Muckenhoupt class $\mathcal{A}_\infty$, more exactly $w_{d-1} \in \mathcal{A}_r$ for any $r > d$, see [78]. It turns out, that if the trace operator maps into $S'(\mathbb{R})$, then it is really possible to identify the trace spaces with corresponding weighted spaces.

Theorem 4.52. ([68], Theorem 9) *Let $d \geq 2$, $0 < p < \infty$, and $0 < q \leq \infty$.*
(i) *Suppose $s > \sigma_p^d$ and let $L \geq [s]+1$. If $TB^s_{p,q}(\mathbb{R}, L, d) \hookrightarrow S'(\mathbb{R})$ (see Theorem 4.51), then $TB^s_{p,q}(\mathbb{R}, L, d) = RB^s_{p,q}(\mathbb{R}, w_{d-1})$ in the sense of equivalent quasi-norms.*
(ii) *Suppose $s > \sigma_{p,q}^d$ and let $L \geq [s]+1$. If $TF^s_{p,q}(\mathbb{R}, L, d) \hookrightarrow S'(\mathbb{R})$ (see Theorem 4.51), then $TF^s_{p,q}(\mathbb{R}, L, d) = RF^s_{p,q}(\mathbb{R}, w_{d-1})$ in the sense of equivalent quasi-norms.*

The following diagram summarizes the situation.

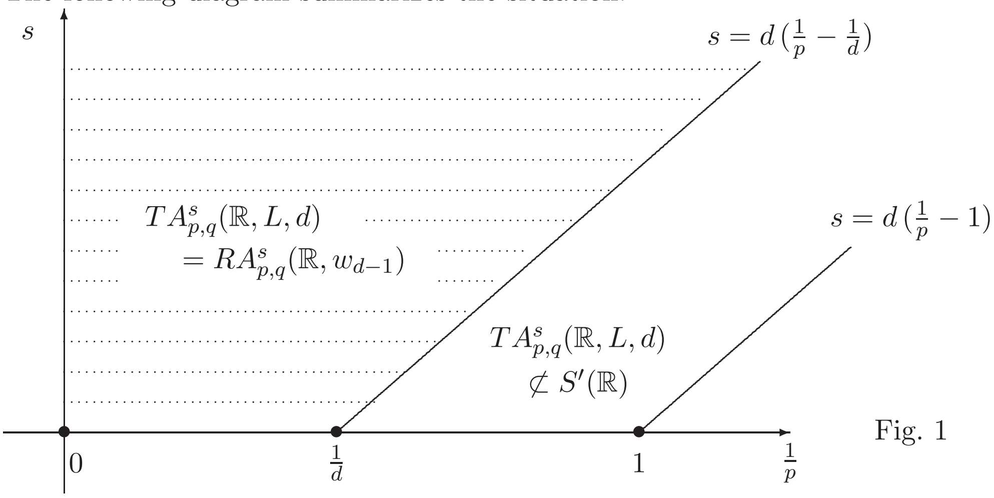


Fig. 1

The trace spaces of Sobolev spaces may be described more directly. As suggested by Lemma 4.46, we can expect, that a crucial role should be played by the weight $w_{d-1}(t) = |t|^{d-1}$. We obtain the following characterization.

Theorem 4.53. ([68], Theorem 6) *Let $d \geq 2$ and $1 \leq p < \infty$.*
(i) *The mapping* tr *is a linear isomorphism (with inverse* ext*) of $RW^1_p(\mathbb{R}^d)$ onto the closure of $RC_0^\infty(\mathbb{R})$ with respect to the norm*

$$\| g \,| L_p(\mathbb{R}, |t|^{d-1})\| + \| g' \,| L_p(\mathbb{R}, |t|^{d-1})\| \,.$$

(ii) *The mapping* tr *is a linear isomorphism (with inverse* ext*) of $RW^2_p(\mathbb{R}^d)$ onto the closure of $RC_0^\infty(\mathbb{R})$ with respect to the norm*

$$\| g \,| L_p(\mathbb{R}, |t|^{d-1})\| + \| g' \,| L_p(\mathbb{R}, |t|^{d-1})\| + \| g'/r \,| L_p(\mathbb{R}, |t|^{d-1})\| + \| g'' \,| L_p(\mathbb{R}, |t|^{d-1})\| \,.$$

Surprisingly enough, the generalization to higher order Sobolev spaces remains open.

4.4.2 Regularity and decay properties

The characterizations obtained so far allow several direct corollaries. For example, if the distribution of f is supported outside of origin, the regularity of its trace may be described in a surprisingly simple way.

Corollary 4.54. ([68], Corollary 1) *Let $\tau > 0$. Let $d \geq 2$, $0 < p < \infty$, and $0 < q \leq \infty$.*
(i) *We suppose $s > \sigma_p^d$. If $f \in RB^s_{p,q}(\mathbb{R}^d)$ such that*

$$\operatorname{supp} f \subset \{x \in \mathbb{R}^d \; : \quad |x| \geq \tau\} \tag{4.68}$$

then its trace f_0 belongs to $B^s_{p,q}(\mathbb{R})$. Furthermore, there exists a constant c (not depending on f and τ) such that

$$\| f_0 \,|B^s_{p,q}(\mathbb{R})\| \le c\,\tau^{-(d-1)/p}\,\| f\,|B^s_{p,q}(\mathbb{R}^d)\| \tag{4.69}$$

holds for all such functions f and all $\tau > 0$.
(ii) *We suppose $s > \sigma^d_{p,q}$. If $f \in RF^s_{p,q}(\mathbb{R}^d)$ such that (4.68) holds, then its trace f_0 belongs to $F^s_{p,q}(\mathbb{R})$. Furthermore, there exists a constant c (not depending on f and τ) such that*

$$\| f_0 \,|F^s_{p,q}(\mathbb{R})\| \le c\,\tau^{-(d-1)/p}\,\| f\,|F^s_{p,q}(\mathbb{R}^d)\| \tag{4.70}$$

holds for all such functions f all $\tau > 0$.

If the support of f stays away from zero and from infinity, the situation becomes very simple. Roughly speaking, in that case the trace of $RA^s_{p,q}(\mathbb{R}^d)$ is $RA^s_{p,q}(\mathbb{R})$.

Corollary 4.55. ([68], Corollary 2) *Let $d \ge 2$, $0 < p < \infty$, $0 < q \le \infty$ and $0 < a < b < \infty$.*
(i) *We suppose $s > \sigma^d_p$. If $g \in RB^s_{p,q}(\mathbb{R})$ such that*

$$\operatorname{supp} g \subset \{x \in \mathbb{R} \;:\quad a \le |x| \le b\} \tag{4.71}$$

then the radial function $f := \operatorname{ext} g$ belongs to $RB^s_{p,q}(\mathbb{R}^d)$ and there exist positive constants A, B such that

$$A\,\| g\,|B^s_{p,q}(\mathbb{R})\| \le \| f\,|B^s_{p,q}(\mathbb{R}^d)\| \le B\,\| g\,|B^s_{p,q}(\mathbb{R})\|\,.$$

(ii) *We suppose $s > \sigma^d_{p,q}$. If $g \in RF^s_{p,q}(\mathbb{R})$ such that (4.71) holds, then the radial function $f := \operatorname{ext} g$ belongs to $RF^s_{p,q}(\mathbb{R}^d)$ and there exist positive constants A, B such that*

$$A\,\| g\,|F^s_{p,q}(\mathbb{R})\| \le \| f\,|F^s_{p,q}(\mathbb{R}^d)\| \le B\,\| g\,|F^s_{p,q}(\mathbb{R})\|\,.$$

Furthermore, following corollary describes the uniform continuity of radial functions.

Corollary 4.56. ([68], Corollary 4) *Let $\tau > 0$. Let $d \ge 2$, $0 < p < \infty$, and $0 < q \le \infty$.*
(i) *If either $s > 1/p$ or $s = 1/p$ and $q \le 1$ then $f \in RB^s_{p,q}(\mathbb{R}^d)$ is uniformly continuous on the set $|x| \ge \tau$.*
(ii) *If either $s > 1/p$ or $s = 1/p$ and $p \le 1$ then $f \in RF^s_{p,q}(\mathbb{R}^d)$ is uniformly continuous on the set $|x| \ge \tau$.*

By looking at the restrictions in Corollary 4.56 we introduce the following set of parameters.

Definition 4.57. (i) We say that the triple (s, p, q) belongs to the set $U(B)$ if (s, p, q) satisfies the restrictions in part (i) of Corollary 4.56.
(ii) The triple (s, p, q) belongs to the set $U(F)$ if (s, p, q) satisfies the restrictions in part (ii) of Corollary 4.56.

These sets of parameters determine the decay properties of f at infinity.

Theorem 4.58. ([68], Theorem 10) *Let $d \ge 2$, $0 < p < \infty$, and $0 < q \le \infty$.*
(i) *Suppose $(s, p, q) \in U(A)$. Then there exists a constant c such that*

$$|x|^{(d-1)/p}\,|f(x)| \le c\,\| f\,|A^s_{p,q}(\mathbb{R}^d)\| \tag{4.72}$$

holds for all $|x| \ge 1$ and all $f \in RA^s_{p,q}(\mathbb{R}^d)$.
(ii) *Suppose $(s, p, q) \in U(A)$. Then*

$$\lim_{|x|\to\infty} |x|^{\frac{d-1}{p}}\,|f(x)| = 0 \tag{4.73}$$

holds for all $f \in RA^s_{p,q}(\mathbb{R}^d)$.
(iii) *Suppose* $(s,p,q) \in U(A)$. *Then there exists a constant* $c > 0$ *such that for all* x, $|x| > 1$, *there exists a smooth radial function* $f \in RA^s_{p,q}(\mathbb{R}^d)$, $\| f \,|RA^s_{p,q}(\mathbb{R}^d)\| = 1$, *such that*

$$|x|^{\frac{d-1}{p}} \, |f(x)| \geq c \,. \tag{4.74}$$

(iv) *Suppose* $(s,p,q) \notin U(A)$ *and* $\frac{1}{p} > \sigma^d_p$. *We assume also that* $\frac{1}{p} > \sigma^d_q$ *in the* F*-case. Then, for all sequences* $(x^j)_{j=1}^\infty \subset \mathbb{R}^d \setminus \{0\}$ *such that* $\lim_{j\to\infty} |x^j| = \infty$, *there exists a radial function* $f \in RA^s_{p,q}(\mathbb{R}^d)$, $\| f \,|RA^s_{p,q}(\mathbb{R}^d)\| = 1$, *such that* f *is unbounded in any neighborhood of* x^j, $j \in \mathbb{N}$.

The regularity properties by zero may be characterized in a similar way. It turns out, that $s > d/p$ implies, that a radial $f \in A^s_{p,q}(\mathbb{R}^d)$ is bounded. In the limiting case $s = d/p$, the situation is again different for B and F spaces.

Lemma 4.59. ([68], Lemma 3) (i) *The embedding* $RB^s_{p,q}(\mathbb{R}^d) \hookrightarrow L_\infty(\mathbb{R}^d)$ *holds if and only if either* $s > d/p$ *or* $s = d/p$ *and* $q \leq 1$.
(ii) *The embedding* $RF^s_{p,q}(\mathbb{R}^d) \hookrightarrow L_\infty(\mathbb{R}^d)$ *holds if and only if either* $s > d/p$ *or* $s = d/p$ *and* $p \leq 1$.

Hence, unboundedness can only happen in case $s \leq d/p$. In that case, we obtain the following.

Theorem 4.60. ([68], Theorem 13) *Let* $d \geq 2$, $0 < p < \infty$ *and* $0 < q \leq \infty$.
(i) *Suppose* $(s,p,q) \in U(A)$ *and* $s < \frac{d}{p}$. *Then there exists a constant c such that*

$$|x|^{\frac{d}{p}-s} \, |f(x)| \leq c \, \| f \,|RA^s_{p,q}(\mathbb{R}^d)\| \tag{4.75}$$

holds for all $0 < |x| \leq 1$ *and all* $f \in RA^s_{p,q}(\mathbb{R}^d)$.
(ii) *Let* $\sigma^d_p < s < d/p$. *There exists a constant* $c > 0$ *such that for all* x, $0 < |x| < 1$, *there exists a smooth radial function* $f \in RA^s_{p,q}(\mathbb{R}^d)$, $\| f \,|RA^s_{p,q}(\mathbb{R}^d)\| = 1$, *such that*

$$|x|^{\frac{d}{p}-s} \, |f(x)| \geq c \tag{4.76}$$

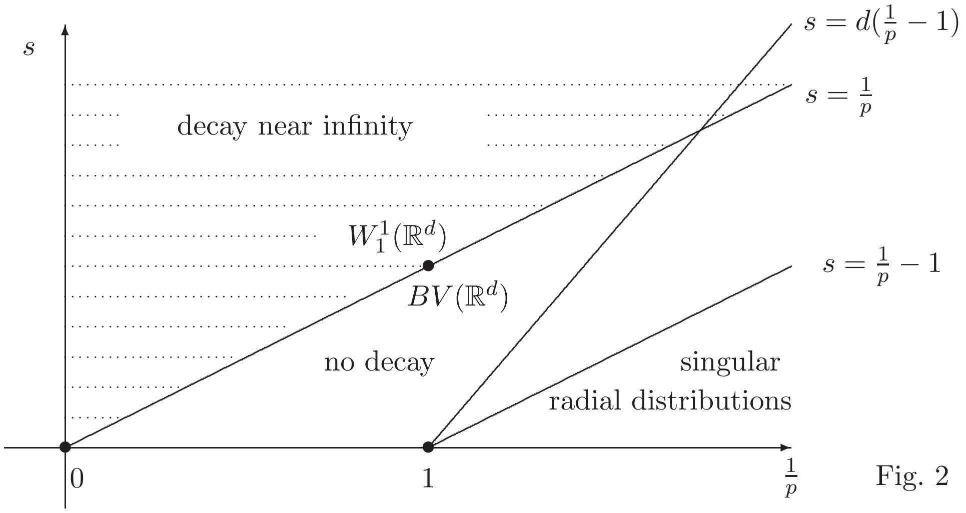


Fig. 2

4.4.3 Functions with bounded variation

The space of functions with bounded variation plays an important role in the calculus of variation, signal processing and other branches of pure and applied analysis. Unfortunately, this space behaves badly in connection with Fourier analysis and, in particular, no decomposition technique of this space is known. Therefore, the trace space of $BV(\mathbb{R}^d)$ is identified by a completely different method. The trace space is then a certain weighted space of functions with bounded variation. Let us first define this new space.

Definition 4.61. (i) A function $\varphi \in C([0,\infty))$ belongs to $C_c^1([0,\infty))$ if it is continuously differentiable on $\mathbb{R}^+$, has compact support, satisfies $\varphi(0) = 0$ and $\lim_{t\to 0^+} \varphi'(t) = \varphi'(0) = \lim_{t\to 0^+} \dfrac{\varphi(t)}{t}$ exists and is finite.
(ii) A function $g \in L_1(\mathbb{R}^+, t^{d-1})$ is said to belong to $BV(\mathbb{R}^+, t^{d-1})$ if there is a signed Radon measure ν on $\mathbb{R}^+$ such that

$$\int_0^\infty g(t)\,[\varphi(s)s^{d-1}]'(t)\,dt = -\int_0^\infty \varphi(t)\,t^{d-1}\,d\nu(t)\,, \qquad \forall \varphi \in C_c^1([0,\infty)) \tag{4.77}$$

and

$$\|\,g\,|BV(\mathbb{R}^+, t^{d-1})\| := \|\,g\,|L_1(\mathbb{R}^+, t^{d-1})\| + \int_0^\infty r^{d-1}\,d|\nu|(r) \tag{4.78}$$

is finite.

By using these new spaces we can prove the following trace theorem.

Theorem 4.62. ([68], Theorem 12) *Let g be a measurable function on $\mathbb{R}^+$. Then $\operatorname{ext} g \in BV(\mathbb{R}^d)$ if, and only if, $g \in BV(\mathbb{R}^+, t^{d-1})$ and*

$$\|\operatorname{ext} g\,|BV(\mathbb{R}^d)\| \asymp \|\,g\,|BV(\mathbb{R}^+, t^{d-1})\,\|\,.$$

The following statement is known to hold for functions $f \in W_1^1(\mathbb{R}^d)$ - in that case, the proof is rather straightforward and goes back to Lions [40]. With the help of the previous characterization, it may be extended to a larger class, namely to $BV(\mathbb{R}^d) \supset W_1^1(\mathbb{R}^d)$. In view of Theorem 4.58, the limiting situation $s = 1/p$ is of particular interest.

Theorem 4.63. ([68], Theorem 11) *Let $d \ge 2$. Then there exists constant c such that*

$$|x|^{d-1}\,|f(x)| \le c\,\|\,f\,|BV(\mathbb{R}^d)\| \tag{4.79}$$

holds for all $|x| > 0$ and all $f \in RBV(\mathbb{R}^d)$. Also

$$\lim_{|x|\to\infty} |x|^{d-1}\,|f(x)| = 0 \tag{4.80}$$

is true for all $f \in RBV(\mathbb{R}^d)$.

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6 Publications relevant to the thesis

This cumulative habilitation consitst of seven papers which are listed below in the order of their appearance in this summary.

[1] J. Vybíral, *Widths of embeddings in function spaces*, J. Compl. 24 (2008), 545–570.

[2] J. Vybíral, *On sharp embeddings of Besov and Triebel-Lizorkin spaces in the subcritical case*, Proc. Amer. Math. Soc. 138 (2010), 141–146.

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Ehrenwörtliche Erklärung

Ich erkläre hiermit, daß mir die Habilitationsordnung der Friedrich-Schiller-Universität Jena vom 30. April 1997 bekannt ist.

Ferner erkläre ich, daß ich die vorliegende Arbeit ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe. Die aus anderen Quellen direkt oder indirekt übernommenen Daten und Konzepte sind unter Angabe der Quelle gekennzeichnet.

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Widths of embeddings in function spaces

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Abstract

We study the approximation, Gelfand and Kolmogorov numbers of embeddings in function spaces of Besov and Triebel-Lizorkin type. Our aim here is to provide sharp estimates in several cases left open in the literature and give a complete overview of the known results. We also add some historical remarks.

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1 Introduction

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $1 \leq p \leq \infty$ and let k be a natural number. We denote by $W_p^k(\Omega)$ the Sobolev spaces of functions from $L_p(\Omega)$ with all distributive derivatives of order smaller or equal to k in $L_p(\Omega)$. If $p_2 < \infty$,

$$k_1 - k_2 \geq d\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+, \tag{1.1}$$

and the boundary of Ω is Lipschitz then $W_{p_1}^{k_1}(\Omega)$ is continuously embedded into $W_{p_2}^{k_2}(\Omega)$. This theorem goes back to Sobolev [55].

If the inequality in (1.1) is strict, the embedding is even compact, cf. [48] and [31]. During the second half of the last century, this fact (and its numerous generalisations) found its applications in many areas of modern analysis, especially in connection with partial differential (and pseudo-differential) equations.

Later on, mathematicians started to be interested in measuring the *quality of compactness* of the embedding

$$I : W_{p_1}^{k_1}(\Omega) \hookrightarrow W_{p_2}^{k_2}(\Omega).$$

The very first question is, of course, how to measure compactness. During the years, several methods were developed. The most popular one assigns to I a non-increasing sequence of non-negative real numbers, say $\{s_n(I)\}_{n\in\mathbb{N}}$, often based on specific approximation quantities, and measures the decay of s_n as n tends to infinity.

Let us present this approach on the following example. Let X and Y be Banach spaces and let $T : X \to Y$ be a bounded linear operator between them. Then the nth approximation number of T is defined by

$$a_n(T) = \inf\{||T - L|| : L \in \mathcal{L}(X,Y),\ \text{rank } (L) < n\}, \quad n \in \mathbb{N}, \tag{1.2}$$

where $\mathcal{L}(X,Y)$ is the space of all bounded linear operators mapping X into Y endowed with the classical operator norm and rank L denotes the dimension of $L(X)$. Hence, we measure how well the operator T may be approximated by finite rank operators. If $\lim\limits_{n\to\infty} a_n(T) = 0$, then T is compact. And in some sense, the faster the sequence $\{a_n(T)\}_{n\in\mathbb{N}}$ tends to zero, the more compact T is.

There are many other ways, how to define a sequence $\{s_n(T)\}_{n\in\mathbb{N}}$ for an operator $T \in \mathcal{L}(X,Y)$ such that the decay of $\{s_n\}$ describes in some sense the compactness of T; we refer to [43, 44, 6], where the axiomatic theory of the so-called s-numbers can be found.

It was observed by many authors, that even in the most simple case

$$id : \ell_{p_1}^m \to \ell_{p_2}^m, \quad m \in \mathbb{N}$$

it is surprisingly difficult to calculate (or at least estimate) the approximation numbers, as well as the other s-numbers, corresponding to id. The complexity of the problem may be demonstrated by the fact, that in several cases the proofs are based on probabilistic arguments and no optimal constructive approximation procedure is known up to now.

As a part of the good news is that these results may be combined with the discretization technique of Maĭorov [37] to get direct counterparts for embeddings between function spaces. Nowadays, there are many discretization techniques well known and studied in the literature.

Let us mention at least spline and wavelet decompositions and the φ-transform, cf. [8, 7, 49, 64, 23, 11, 16, 17].

The research in this area was complicated also by another regretful phenomena, namely communication problems between several groups working on the field. This effect was already pointed out by Caetano [4] and Pietsch [45, Section 6.2.6]. Also the separation of the Russian mathematical school causes some obstacles. Many breakthroughs achieved by Kashin, Gluskin and others were published in Russian. The nicely written dissertation of Lubitz [36] was written in German, never translated into English and never published.

The aim of this paper is rather extensive. We wish to

- give an overview of known results in this area,
- collect some historical references,
- close several minor gaps left open until now,
- present the power of the discretization method, but also its limits,
- provide an easy reference to the results about function spaces.

Several overviews may already be found in the literature, cf. [46, 34, 35, 45]. Unfortunately, they sometimes restrict themselves to $d = 1$, state the results only implicitly, or deal only with integer smoothness parameters $s_1, s_2 \in \mathbb{N}$. Here, leaded by the needs of possible applications, we shall study three types of s-numbers, namely approximation, Kolmogorov and Gelfand numbers, with respect to embeddings of function spaces defined on Lipschitz domains. This generalisation is not particularly interesting from the standpoint of functional analysis, but is of course crucial as far as the applications are concerned.

I would like to thank to my colleagues from Jena, Aicke Hinrichs, Erich Novak, Winfried Sickel and Hans Triebel, for many valuable discussions on the topic.

2 Function and sequence spaces

2.1 Notation

We use standard notation: $\mathbb{N}$ denotes the collection of all natural numbers, $\mathbb{Z}$ the collection of all integers, $\mathbb{R}^d$ is the Euclidean d-dimensional space, where $d \in \mathbb{N}$, and $\mathbb{C}$ stands for the complex plane. Let $S(\mathbb{R}^d)$ be the Schwartz space of all complex-valued rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^d$ and let $S'(\mathbb{R}^d)$ be its dual, the space of all tempered distributions.

Furthermore, $L_p(\mathbb{R}^d)$ with $0 < p \leq \infty$, are the classical Lebesgue spaces endowed with the (quasi-)norm

$$||f|L_p(\mathbb{R}^d)|| = \begin{cases} \left(\int_{\mathbb{R}^d} |f(x)|^p dx\right)^{1/p}, & 0 < p < \infty, \\ \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} |f(x)|, & p = \infty. \end{cases}$$

For $\psi \in S(\mathbb{R}^d)$ we denote by

$$\widehat{\psi}(\xi) = (F\psi)(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i<x,\xi>}\psi(x)dx, \quad x \in \mathbb{R}^d,$$

its Fourier transform and by $\psi^\vee$ or $F^{-1}\psi$ its inverse Fourier transform. Through duality, F and F^{-1} are extended to $S'(\mathbb{R}^d)$.
If $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ are two sequences of non-negative real numbers, we write $a_n \lesssim b_n$ if there is a constant $c > 0$, such that $a_n \le c\, b_n$ for all natural numbers n. The symbols $a_n \gtrsim b_n$ and $a_n \approx b_n$ are defined similarly.

2.2 Function spaces

We give a Fourier-analytic definition of Besov and Triebel-Lizorkin spaces, which relies on the so-called *smooth dyadic resolution of unity.* Let $\varphi \in S(\mathbb{R}^d)$ with

$$\varphi(x) = 1 \quad \text{if} \quad |x| \le 1 \quad \text{and} \quad \varphi(x) = 0 \quad \text{if} \quad |x| \ge \frac{3}{2}. \tag{2.1}$$

We put $\varphi_0 = \varphi$ and $\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$ for $j \in \mathbb{N}$ and $x \in \mathbb{R}^d$. This leads to the identity

$$\sum_{j=0}^\infty \varphi_j(x) = 1, \qquad x \in \mathbb{R}^d.$$

Definition 2.1. (i) Let $s \in \mathbb{R}, 0 < p, q \le \infty$. Then $B^s_{pq}(\mathbb{R}^d)$ is the collection of all $f \in S'(\mathbb{R}^d)$ such that

$$||f|B^s_{pq}(\mathbb{R}^d)|| = \left(\sum_{j=0}^\infty 2^{jsq}||(\varphi_j\widehat{f})^\vee|L_p(\mathbb{R}^d)||^q\right)^{1/q} < \infty \tag{2.2}$$

(with the usual modification for $q = \infty$).
(ii) Let $s \in \mathbb{R}, 0 < p < \infty, 0 < q \le \infty$. Then $F^s_{pq}(\mathbb{R}^d)$ is the collection of all $f \in S'(\mathbb{R}^d)$ such that

$$||f|F^s_{pq}(\mathbb{R}^d)|| = \left|\left|\left(\sum_{j=0}^\infty 2^{jsq}|(\varphi_j\widehat{f})^\vee(\cdot)|^q\right)^{1/q}|L_p(\mathbb{R}^d)\right|\right| < \infty \tag{2.3}$$

(with the usual modification for $q = \infty$).

Remark 2.2. We recommend [40, 59, 60, 51, 61] as standard references with respect to these classes of distributions. Extensive historical overviews, remarks and comments may be found in [60, Chapter 1], [61, Chapter 1] and [45, Chapter 6.7]. Let us mention that the spaces $B^s_{pq}(\mathbb{R}^d)$ and $F^s_{pq}(\mathbb{R}^d)$ do not depend on the choice of φ in the sense of equivalent (quasi-)norms. Many classical function spaces are included in these two scales.

1. If $1 < p < \infty$, then the Littlewood-Paley theorem states that

$$F^0_{p2}(\mathbb{R}^d) = L_p(\mathbb{R}^d).$$

2. Let $1 < p < \infty$ and $s \in \mathbb{N}$. Then

$$F^s_{p2}(\mathbb{R}^d) = W^s_p(\mathbb{R}^d)$$

are the classical Sobolev spaces.

3. Let $s > 0, s \notin \mathbb{N}$. Then
$$B^s_{\infty\infty}(\mathbb{R}^d) = \mathcal{C}^s(\mathbb{R}^d)$$
are the Hölder-Zygmund spaces.

On the other hand, many important function spaces (especially $L_1(\mathbb{R}^d), L_\infty(\mathbb{R}^d)$, $BV(\mathbb{R})$ - the space of functions with bounded variation and $C^k(\mathbb{R}^d)$ - the space of functions with all partial derivatives of order smaller or equal to k uniformly continuous and bounded) are *not* included.

If X and Y are two topological vector spaces, we write $X \hookrightarrow Y$ if X is continuously embedded in Y. The following embeddings describe the interplay between these function spaces and the Besov scale.

$$\begin{gathered} B^0_{11}(\mathbb{R}^d) \hookrightarrow L_1(\mathbb{R}^d) \hookrightarrow B^0_{1\infty}(\mathbb{R}^d), \\ B^0_{\infty 1}(\mathbb{R}^d) \hookrightarrow C(\mathbb{R}^d) \hookrightarrow L_\infty(\mathbb{R}^d) \hookrightarrow B^0_{\infty\infty}(\mathbb{R}^d), \\ B^k_{\infty 1}(\mathbb{R}^d) \hookrightarrow C^k(\mathbb{R}^d) \hookrightarrow B^k_{\infty\infty}(\mathbb{R}^d). \end{gathered} \tag{2.4}$$

In many cases it will be possible to use the Fourier-analytical methods in the framework of Besov spaces and afterwards, simply by applying these simple continuous embeddings, to derive the same results also for the "bad" spaces $L_1(\mathbb{R}^d), L_\infty(\mathbb{R}^d)$ and $C^k(\mathbb{R}^d)$. The same procedure may be used also for the Triebel-Lizorkin scale because of

$$B^s_{p,\min(p,q)}(\mathbb{R}^d) \hookrightarrow F^s_{pq}(\mathbb{R}^d) \hookrightarrow B^s_{p,\max(p,q)}(\mathbb{R}^d). \tag{2.5}$$

Remark 2.3. If $0 < p_1 \leq p_2 \leq \infty$, $0 < q_1, q_2 \leq \infty$ and $s_2 \leq s_1$, then the following version of the Sobolev embedding is true, see [2], [40, Chapters 3 and 11] and [58, Section 2.8.1].

$$B^{s_1}_{p_1,q_1}(\mathbb{R}^d) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^d), \quad \text{if} \quad s_1 - \frac{d}{p_1} > s_2 - \frac{d}{p_2}.$$

There are several modifications of this embedding, which result in compact mappings. The first possibility is to restrict to function spaces on smooth bounded domains, the second involves *weighted spaces* and another one considers the so-called *radial spaces*, i.e. spaces of radial symmetric functions. We concentrate on the first possibility and refer to [61, Chapter 6] and [54] for the second and third approach.

Let Ω be a bounded domain. Let $D(\Omega) = C_0^\infty(\Omega)$ be the collection of all complex-valued infinitely-differentiable functions with compact support in Ω and let $D'(\Omega)$ be its dual - the space of all complex-valued distributions on Ω.

Let $g \in S'(\mathbb{R}^d)$. Then we denote by $g|\Omega$ its restriction to Ω:

$$(g|\Omega) \in D'(\Omega), \qquad (g|\Omega)(\psi) = g(\psi) \quad \text{for} \quad \psi \in D(\Omega).$$

Definition 2.4. Let Ω be a bounded domain in $\mathbb{R}^d$. Let $s \in \mathbb{R}$, $0 < p, q \leq \infty$ with $p < \infty$ in the F-case. Let A^s_{pq} stand either for B^s_{pq} or F^s_{pq}. Then

$$A^s_{pq}(\Omega) = \{f \in D'(\Omega) : \exists g \in A^s_{pq}(\mathbb{R}^d) : g|\Omega = f\}$$

and

$$||f|A^s_{pq}(\Omega)|| = \inf ||g|A^s_{pq}(\mathbb{R}^d)||,$$

where the infimum is taken over all $g \in A^s_{pq}(\mathbb{R}^d)$ such that $g|\Omega = f$.

Intrinsic characterization of $B^s_{p,q}(\Omega), s > \sigma_p = d\left(\frac{1}{p} - 1\right)_+ = d \max\left(\frac{1}{p} - 1, 0\right)$ are known to exist in case of Lipschitz domains, see [12, 13, 14] and [61, Section 1.11.9].

2.3 Sequence spaces

In this section we comment on the discretization techniques mentioned in the Introduction. First, we describe the situation on $\mathbb{R}^d$. Therefore, we introduce the sequence spaces b^s_{pq} and give a wavelet decomposition theorem for Besov spaces on $\mathbb{R}^d$. Good references in our context are [8, 11, 23, 38, 39, 63, 64].

Second, we deal with bounded domains $\Omega \subset \mathbb{R}^d$. The wavelet decomposition techniques may be adapted also to these function spaces, cf. [9, 61], but unfortunately, there are still open problems in this setting. To avoid these gaps, we use the theory on $\mathbb{R}^d$ and combine it with suitable extension and restriction operators.

Theorem 2.5. *For any $k \in \mathbb{N}$ there are real-valued compactly supported functions*

$$\psi_0, \psi_1 \in C^k(\mathbb{R})$$

satisfying

$$\int_{\mathbb{R}} t^\alpha \psi_1(t)dt = 0, \quad \alpha = 0, 1, \dots, k-1,$$

such that

$$\{2^{\nu/2}\psi_{\nu m} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}\}$$

with

$$\psi_{\nu m}(t) = \begin{cases} \psi_0(t-m) & \text{if } \nu = 0, m \in \mathbb{Z}, \\ 2^{-\frac{1}{2}}\psi_1(2^{\nu-1}t - m) & \text{if } \nu \in \mathbb{N}, m \in \mathbb{Z} \end{cases}$$

is an orthonormal basis in $L_2(\mathbb{R})$.

Remark 2.6. This theorem was first proven by Daubechies in [10]. The functions ψ_0 and ψ_1 are therefore usually called Daubechies wavelets. We refer to [63, Theorem 19] for the proof of the next theorem.

Theorem 2.7. *Let $0 < p, q \le \infty$, $s \in \mathbb{R}$ and $k \in \mathbb{N}$ with $k > \max(s, \sigma_p - s)$. Let ψ_0, ψ_1 be the Daubechies wavelets of smoothness k. Let $E = \{0,1\}^d \setminus (0, \dots, 0)$. For $e = (e_1, \dots, e_d) \in E$ let*

$$\Psi_e(x) = \prod_{j=1}^d \psi_{e_j}(x_j), \quad x = (x_1, \dots, x_d) \in \mathbb{R}^d.$$

(i) Then

$$\begin{cases} \Psi(x-m) = \prod_{j=1}^d \psi_0(x_j - m_j) & m = (m_1, \dots, m_d) \in \mathbb{Z}^d, \\ 2^{\frac{\nu-1}{2}d}\Psi_e(2^{\nu-1}x - m) & e \in E, \nu \in \mathbb{N}, m \in \mathbb{Z}^d \end{cases}$$

is an orthonormal basis in $L_2(\mathbb{R}^d)$.

(ii) Let $f \in S'(\mathbb{R}^d)$. Then $f \in B^s_{pq}(\mathbb{R}^d)$ if, and only if, it can be represented as

$$f = \sum_{m \in \mathbb{Z}^d} \lambda_m \Psi(x-m) + \sum_{\nu \in \mathbb{N}} \sum_{e \in E} \sum_{m \in \mathbb{Z}^d} \lambda^e_{\nu m} 2^{-\nu d/2} \Psi_e(2^{\nu-1}x - m) \tag{2.6}$$

with

$$||\lambda|\mathrm{b}^s_{pq}|| = \Big(\sum_{m \in \mathbb{Z}^d} |\lambda_m|^p\Big)^{\frac{1}{p}} + \Big(\sum_{\nu=1}^\infty 2^{\nu(s-\frac{d}{p})q} \sum_{e \in E}\Big(\sum_{m \in \mathbb{Z}^d} |\lambda^e_{\nu m}|^p\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}} < \infty$$

appropriately modified if $p = \infty$ and/or $q = \infty$. The representation in (2.6) *is unique, the complex coefficients* $\{\lambda_m\}_{m\in\mathbb{Z}^d}$ *and* $\{\lambda^e_{\nu m}\}_{e\in E,\nu\in\mathbb{N}_0,m\in\mathbb{Z}^d}$ *depend linearly on f and the mapping, which associates to $f \in B^s_{pq}(\mathbb{R}^d)$ the sequence of coefficients, is an isomorphic map of $B^s_{pq}(\mathbb{R}^d)$ onto b^s_{pq}.*

2.4 s-numbers

Given $p \in (0, 1]$, we say, that the quasi-Banach space Y is a p-Banach space if the inequality

$$||x + y|Y||^p \le ||x|Y||^p + ||y|Y||^p, \qquad x, y \in Y.$$

is satisfied.

We recall a few basic facts of the theory of s-numbers. We refer to [44, 6] for further details. In this theory, one associates to every linear operator $T : X \to Y$ (X and Y quasi-Banach spaces) a sequence of scalars

$$s_1(T) \ge s_2(T) \ge \cdots \ge 0.$$

Let W, X, Y, Z be (quasi-)Banach spaces and let Y be a p-Banach space, $0 < p \le 1$. If the rule $s : T \to \{s_n(T)\}_{n\in\mathbb{N}}$ satisfies

(S1) $||T|| = s_1(T) \ge s_2(T) \ge \cdots \ge 0$,

(S2) $s^p_{m+n-1}(S + T) \le s^p_m(T) + s^p_n(S)$ for all $S, T \in \mathcal{L}(X, Y)$ and $m, n \in \mathbb{N}$,

(S3) $s_n(STU) \le ||S||s_n(T)||U||$ for all $U \in \mathcal{L}(W, X), T \in \mathcal{L}(X, Y), S \in \mathcal{L}(Y, Z)$ and $n \in \mathbb{N}$,

(S4) If rank $T < n$, then $s_n(T) = 0$,

(S5) $s_n(I : \ell_2(n) \to \ell_2(n)) = 1$.

then the $s_n(T)$ are called s-numbers of the operator T.

Let us point out, that we shall not use **(S4)** and **(S5)** in what follows. Hence, our approach applies also to rules $s : T \to \{s_n(T)\}_{n\in\mathbb{N}}$ which satisfy only **(S1)**-**(S3)**. Such rules are called *pseudo-s-numbers* in [43, Chapter 12] and cover also the concept of entropy numbers with $||T|| \ge s_1(T)$ in **(S1)**.

Let

$$\mathcal{I}d : B^{s_1}_{p_1q_1}(\Omega) \to B^{s_2}_{p_2q_2}(\Omega) \tag{2.7}$$

be compact, i.e.

$$s_1 - s_2 > d\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big)_+. \tag{2.8}$$

We denote by

$$\text{ext} : B^{s_1}_{p_1q_1}(\Omega) \to B^{s_1}_{p_1q_1}(\mathbb{R}^d) \tag{2.9}$$

a bounded linear extension operator. A convenient reference for this is Rychkov, cf. [52], but see also the references given there. Here we use the Lipschitz smoothness of $\partial\Omega$. The natural restriction will be denoted by

$$\text{re} : B^{s_2}_{p_2q_2}(\mathbb{R}^d) \to B^{s_2}_{p_2q_2}(\Omega).$$

Clearly, it also represents a bounded linear operator.

Let $k > \max(s_1, \sigma_{p_1} - s_1, s_2, \sigma_{p_2} - s_2)$ be a natural number and let $\mathcal{W}$ be the mapping which associates to each $f \in B^{s_1}_{p_1q_1}(\mathbb{R}^d)$ its wavelet coefficients with respect to the Daubechies wavelets of smoothness k, as described in Theorem 2.7. Our choice of k ensures, that Theorem 2.7 may be applied to both, $B^{s_1}_{p_1q_1}(\mathbb{R}^d)$ and $B^{s_2}_{p_2q_2}(\mathbb{R}^d)$, simultaneously and that $\mathcal{W}^{-1}$ is a bounded linear operator, which maps $\mathsf{b}^{s_2}_{p_2q_2}$ isomorphically onto $B^{s_2}_{p_2q_2}(\mathbb{R}^d)$.

Finally, we adapt the sequence spaces b^s_{pq} to the function spaces on domains.

Definition 2.8. (i) Let $M = \{M_\nu\}_{\nu=0}^\infty$ be a sequence of non-negative integers. We say, that M is admissible, if there is some $\nu_0 \in \mathbb{N}_0$ and two positive real constants c_1, c_2 such that

$$M_\nu = 0 \qquad \text{for all} \quad \nu < \nu_0$$

and

$$c_1 2^{\nu d} \le M_\nu \le c_2 2^{\nu d}, \qquad \nu \ge \nu_0.$$

(ii) If $0 < p, q \le \infty$, $s \in \mathbb{R}$, $E = \{0,1\}^d \setminus (0,\dots,0)$, $M = \{M_\nu\}_{\nu=0}^\infty$ is an admissible sequence and

$$\lambda = \{\lambda_k : k = 1,\dots,M_0\} \cup \{\lambda^e_{\nu k} : e \in E, \nu \in \mathbb{N}, k \in M_\nu\},$$

we set

$$||\lambda|\mathsf{b}^{s,M}_{pq}|| = \Big(\sum_{k=1}^{M_0} |\lambda_k|^p\Big)^{\frac{1}{p}} + \Big(\sum_{\nu=1}^{\infty} 2^{\nu(s-\frac{d}{p})q} \sum_{e\in E}\Big(\sum_{k=1}^{M_\nu} |\lambda^e_{\nu k}|^p\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}}, \tag{2.10}$$

again appropriately modified if $p = \infty$ and/or $q = \infty$.

Let now Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ and let the number $k \in \mathbb{N}$ describing the smoothness of the wavelets be fixed. Then we collect those wavelets, whose support intersects $\bar{\Omega}$:

$$\mathcal{M}_\nu = \begin{cases} \{m \in \mathbb{Z}^d : \operatorname{supp} \Psi(\cdot - m) \cap \bar{\Omega} \neq \emptyset\} & \text{if } \nu = 0, \\ \{m \in \mathbb{Z}^d : \exists e \in E : \operatorname{supp} \Psi_e(2^{\nu-1} \cdot - m) \cap \bar{\Omega} \neq \emptyset\} & \text{if } \nu \ge 1. \end{cases}$$

We observe that the sequence $M = \{M_\nu\}_{\nu=0}^\infty$ with

$$M_\nu = \#(\mathcal{M}_\nu) = \text{number of elements of } \mathcal{M}_\nu, \quad \nu \in \mathbb{N}_0,$$

is an admissible sequence in the sense of Definition 2.8.

With a slight abuse of notation, there is a natural projection operator $P : \mathsf{b}^s_{pq} \to \mathsf{b}^{s,M}_{pq}$ and a natural embedding operator $Q : \mathsf{b}^{s,M}_{pq} \to \mathsf{b}^s_{pq}$.

Using the weak multiplicativity property **(S3)** of s-numbers and the commutative diagram

$$\begin{array}{ccccccc} B^{s_1}_{p_1q_1}(\Omega) & \xrightarrow{\text{ext}} & B^{s_1}_{p_1q_1}(\mathbb{R}^d) & \xrightarrow{\mathcal{W}} & \mathsf{b}^{s_1}_{p_1q_1} & \xrightarrow{P} & \mathsf{b}^{s_1,M}_{p_1q_1} \\ {\scriptstyle \mathcal{I}d}\big\downarrow & & & & & & \big\downarrow{\scriptstyle id} \\ B^{s_2}_{p_2q_2}(\Omega) & \xleftarrow{\text{re}} & B^{s_2}_{p_2q_2}(\mathbb{R}^d) & \xleftarrow{\mathcal{W}^{-1}} & \mathsf{b}^{s_2}_{p_2q_2} & \xleftarrow{Q} & \mathsf{b}^{s_2,M}_{p_2q_2} \end{array}$$

we conclude that

$$s_n(\mathcal{I}d) \lesssim s_n(id), \qquad n \in \mathbb{N}.$$

To obtain the reverse inequality, we first set

$$\mathcal{M}'_\nu = \begin{cases} \{m \in \mathbb{Z}^d : \operatorname{supp} \Psi(\cdot - m) \subset \Omega\} & \text{if } \nu = 0, \\ \{m \in \mathbb{Z}^d : \forall e \in E : \operatorname{supp} \Psi_e(2^{\nu-1} \cdot - m) \subset \Omega\} & \text{if } \nu \geq 1. \end{cases} \tag{2.11}$$

Again, we observe, that the sequence $M' = \{M'_\nu\}_{\nu=0}^\infty$ with

$$M'_\nu = \#(\mathcal{M}'_\nu) = \text{number of elements of } \mathcal{M}'_\nu, \quad \nu \in \mathbb{N}_0,$$

is an admissible sequence in the sense of Definition 2.8.
If we use **(S3)** and

$$\begin{array}{ccccccc} \mathsf{b}^{s_1,M'}_{p_1q_1} & \xrightarrow{Q'} & \mathsf{b}^{s_1}_{p_1q_1} & \xrightarrow{\mathcal{W}^{-1}} & B^{s_1}_{p_1q_1}(\mathbb{R}^d) & \xrightarrow{\text{re}} & B^{s_1}_{p_1q_1}(\Omega) \\ {\scriptstyle id'}\big\downarrow & & & & & & \big\downarrow{\scriptstyle \mathcal{I}d} \\ \mathsf{b}^{s_2,M'}_{p_2q_2} & \xleftarrow{P'} & \mathsf{b}^{s_2}_{p_2q_2} & \xleftarrow{\mathcal{W}} & B^{s_2}_{p_2q_2}(\mathbb{R}^d) & \xleftarrow{\text{ext}} & B^{s_2}_{p_2q_2}(\Omega), \end{array}$$

we get the inequality.

$$s_n(id') \lesssim s_n(\mathcal{I}d), \quad n \in \mathbb{N}.$$

Hence

$$s_n(id') \lesssim s_n(\mathcal{I}d) \lesssim s_n(id), \quad n \in \mathbb{N}. \tag{2.12}$$

It tells us, roughly speaking, that we may restrict ourselves to sequence spaces and all the results translate also into the language of function spaces. Before we start with the study of $s_n(id)$ and $s_n(id')$, we make another simplification. The (finite) sum over $e \in E$ in (2.10) comes from the theory of multivariate wavelet decompositions, but has no influence on the s-numbers.
If $M = \{M_\nu\}_{\nu=0}^\infty$ is an admissible sequence, we set

$$||\lambda|b^{s,M}_{pq}|| = \left(\sum_{\nu=0}^\infty 2^{\nu(s-\frac{d}{p})q} \left(\sum_{k=1}^{M_\nu} |\lambda_{\nu k}|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}.$$

It follows that

$$s_n(\mathcal{I}d : B^{s_1}_{p_1q_1}(\Omega) \to B^{s_2}_{p_2q_2}(\Omega)) \approx s_n(id : \mathsf{b}^{s,M}_{pq} \to \mathsf{b}^{s,M}_{pq}) \approx s_n(id : b^{s,M}_{pq} \to b^{s,M}_{pq}). \tag{2.13}$$

Remark 2.9. The formulas (2.12) and (2.13) represent the main result of this section and is of a crucial importance for our study of s-numbers of (2.7). We have proved (2.13) under the assumption that Ω is a bounded domain in $\mathbb{R}^d$ with Lipschitz boundary. Using more sophisticated tools from the theory of function spaces, it may be proven that (2.13) holds also for more general classes of domains, at least under some restrictions on the parameters $s_1, s_2, p_1, p_2, q_1, q_2$. A detailed inspection of our proof shows, that (2.13) is true anytime there is a bounded linear extension operator (2.9) and its counterpart for $B^{s_2}_{p_2q_2}(\Omega)$. We refer to [62, Section 4.3.4] for a detail treatment of these questions.

3 Approximation numbers

Definition 3.1. Let X, Y be two quasi-Banach spaces and let $T \in \mathcal{L}(X,Y)$. For $n \in \mathbb{N}$, we define the nth approximation number by

$$a_n(T) = \inf\{||T - L|| : L \in \mathcal{L}(X,Y), \operatorname{rank}(L) < n\}.$$

In the setting of Banach spaces, this definition goes back to Pietsch [41] and Tikhomirov [57]. The generalisation to quasi-Banach spaces may be found in [15, Section 1.3.1]. In this section, we characterize the approximation numbers of (2.7) with (2.8).

First, we recall some lemmas which we shall need on the sequence space level. Lemma 3.2 is taken from [22] and Lemma 3.3 in the case $1 \le p_2 \le p_1 \le \infty$ may be found in [43, Section 11.11.5]. The proof may be directly generalised to the quasi-Banach setting $0 < p_2 \le p_1 \le \infty$.

For $0 < p \le \infty$, we set

$$p' = \begin{cases} \frac{p}{p-1} & \text{if} \quad 1 < p < \infty, \\ 1 & \text{if} \quad p = \infty, \\ \infty & \text{if} \quad 0 < p \le 1. \end{cases}$$

Lemma 3.2. *For $1 \le n \le m < \infty$ and $1 \le p_1 < p_2 \le \infty$, we define*

$$\Phi(m,n,p_1,p_2) := \begin{cases} \left(\min\{1, m^{\frac{1}{p_2}} n^{-\frac{1}{2}}\}\right)^{\frac{\frac{1}{p_1}-\frac{1}{p_2}}{\frac{1}{2}-\frac{1}{p_2}}} & \text{if} \quad 2 \le p_1 < p_2 \le \infty, \\ \max\{m^{\frac{1}{p_2}-\frac{1}{p_1}}, \min\{1, m^{\frac{1}{p_2}} n^{-\frac{1}{2}}\} \cdot \sqrt{1-\frac{n}{m}}\} & \text{if} \quad 1 \le p_1 < 2 \le p_2 \le \infty, \\ \max\{m^{\frac{1}{p_2}-\frac{1}{p_1}}, \sqrt{1-\frac{n}{m}}^{\frac{\frac{1}{p_1}-\frac{1}{p_2}}{\frac{1}{p_1}-\frac{1}{2}}}\} & \text{if} \quad 1 \le p_1 < p_2 \le 2 \end{cases}$$

and

$$\Psi(m,n,p_1,p_2) := \begin{cases} \Phi(m,n,p_1,p_2) & \text{if} \quad 1 \le p_1 < p_2 \le p_1', \\ \Phi(m,n,p_2',p_1') & \text{if} \quad \max(p_1,p_1') < p_2 \le \infty. \end{cases}$$

Then if $1 \le p_1 < p_2 \le \infty$ and $(p_1,p_2) \ne (1,\infty)$

$$a_n(id : \ell_{p_1}^m \to \ell_{p_2}^m) \approx \Phi(m,n,p_1,p_2), \qquad 1 \le n \le m < \infty.$$

The constants of equivalence may depend on p_1 and p_2 but are independent of m and n.

Lemma 3.3. *If $1 \le n \le m < \infty$ and $0 < p_2 \le p_1 \le \infty$, then*

$$a_n(id : \ell_{p_1}^m \to \ell_{p_2}^m) = (m-n+1)^{\frac{1}{p_2}-\frac{1}{p_1}}.$$

Lemma 3.4. *Let $0 < p \le 1$.*

(i) Let $0 < \lambda < 1$. Then there is a number $c_\lambda > 0$ such that

$$a_n(id : \ell_p^m \to \ell_\infty^m) \le \frac{c_\lambda}{\sqrt{n}} \tag{3.1}$$

holds for all natural numbers n and m with $m^\lambda < n \le m$.

(ii) There is a number $c > 0$ such that

$$a_n(id : \ell_p^{2n} \to \ell_\infty^{2n}) \ge \frac{c}{\sqrt{n}}, \quad n \ge 1. \tag{3.2}$$

Proof. Let $A = (a_{i,j})_{i,j=1}^m$ be an $m \times m$ matrix. Then

$$||A|\mathcal{L}(\ell_1^m, \ell_\infty^m)|| = ||A|\mathcal{L}(\ell_p^m, \ell_\infty^m)|| = \max_{i,j=1,\dots,m} |a_{i,j}|$$

for every $0 < p \le 1$. Hence, the approximation numbers of $id : \ell_p^m \to \ell_\infty^m$ do not depend on $0 < p \le 1$ and it is enough, when we prove Lemma 3.4 only for $p = 1$.

The first part follows from a combinatorial result of Kashin, cf. [26, 27] and [43, Section 11.11.11]:

Let $0 < \lambda < 1$ and $m^\lambda \le n \le m$ be natural numbers. Then there are m ℓ_2^n-unit vectors $\{f_i\}_{i=1}^m \subset \mathbb{R}^n$, such that

$$|(f_i, f_j)| \le \frac{c_\lambda}{\sqrt{n}}, \quad \text{if } i \neq j.$$

We set $A = (a_{i,j})_{i,j=1}^m$ with $a_{i,j} = (f_i, f_j)$. Then A is a matrix with rank $A \le n$ and $||I - A|\mathcal{L}(\ell_1^m, \ell_\infty^m)|| \le \frac{c_\lambda}{\sqrt{n}}$.

The proof of the second part follows trivially from the result of Stechkin, cf. [56] and [43, Section 11.11.8]:

$$a_n(id : \ell_1^m \to \ell_2^m) = \left(\frac{m-n+1}{m}\right)^{1/2}$$

and

$$||id : \ell_\infty^m \to \ell_2^m|| = \sqrt{m}.$$

□

Theorem 3.5. *Let $-\infty < s_2 < s_1 < \infty$ and $0 < p_1, p_2, q_1, q_2 \le \infty$ with (2.8). Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then (2.7) is compact and for $n \in \mathbb{N}$*

$$a_n(\mathcal{I}d) \approx n^{-\frac{s_1-s_2}{d}+\left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+} \quad \text{if} \quad \begin{cases} \text{either} & 0 < p_1 \le p_2 \le 2, \\ \text{or} & 2 \le p_1 \le p_2 \le \infty, \\ \text{or} & 0 < p_2 \le p_1 \le \infty, \end{cases} \tag{3.3}$$

$$a_n(\mathcal{I}d) \approx n^{-\frac{s_1-s_2}{d}+\frac{1}{p}-\frac{1}{2}} \quad \text{if} \quad 0 < p_1 < 2 < p_2 < \infty \text{ and } \frac{s_1-s_2}{d} > \frac{1}{p} = \max\Big(1-\frac{1}{p_2}, \frac{1}{p_1}\Big), \tag{3.4}$$

$$a_n(\mathcal{I}d) \approx n^{\left(-\frac{s_1-s_2}{d}+\frac{1}{p_1}-\frac{1}{p_2}\right)\cdot\frac{\min(p_1',p_2)}{2}} \quad \text{if} \quad \frac{s_1-s_2}{d} < \frac{1}{p} = \max\Big(1-\frac{1}{p_2}, \frac{1}{p_1}\Big), \text{ and either } 1 < p_1 < 2 < p_2 = \infty \text{ or } 0 < p_1 < 2 < p_2 < \infty \tag{3.5}$$

$$a_n(\mathcal{I}d) \approx n^{-\frac{s_1-s_2}{d}+\frac{1}{p_1}-\frac{1}{2}} \quad \text{if} \quad 0 < p_1 \le 1 < p_2 = \infty. \tag{3.6}$$

Proof. Approximation numbers form an additive and multiplicative scale of s-numbers. This fact may be verified directly, or the reader may consult [43, Section 11.2] in the Banach space settings and [15, Section 1.3] for the extension to quasi-Banach spaces.

Hence (2.12) applies to approximation numbers and we may restrict ourselves to sequence spaces.

The estimates covered by (3.3)-(3.5) are known. We refer to [15, Section 3.3.4] and [4]. The proof given in [15] is rather complicated, but [4] uses an approach very similar to ours.

It remains to prove the only missing case (3.6). We use Lemma 3.4 to estimate the approximation numbers of

$$id : b^{s_1,M}_{p_1q_1} = \ell_{q_1}(2^{\nu(s_1-\frac{d}{p_1})}\ell^{M_\nu}_{p_1}) \to \ell_{q_2}(2^{\nu s_2}\ell^{M_\nu}_{\infty}) = b^{s_2,M}_{\infty q_2},$$

where $M = \{M_\nu\}_{\nu=0}^{\infty}$ is an admissible sequence. Let

$$id_\nu : 2^{\nu(s_1-\frac{d}{p_1})}\ell^{M_\nu}_{p_1} \to 2^{\nu s_2}\ell^{M_\nu}_{\infty}, \quad \nu = 0,1,2,\ldots$$

denote the identity operator between the finite dimensional building blocks of the considered sequence spaces. With a slight abuse of notation, we get

$$id = \sum_{\nu=0}^{\infty} id_\nu, \tag{3.7}$$

which, combined with the additivity of approximation numbers, leads to

$$a^{\omega}_{n'}(id) \le \sum_{\nu=0}^{N_1} a^{\omega}_{n_\nu}(id_\nu) + \sum_{\nu=N_1+1}^{N_2} a^{\omega}_{n_\nu}(id_\nu) + \sum_{\nu=N_2+1}^{\infty} ||id_\nu||^{\omega},$$

where $N_1 < N_2$ are natural numbers, $n' - 1 = \sum_{\nu=0}^{N_2}(n_\nu - 1)$ and $\omega = \min(1, q_2)$. We set

$$n_\nu = \begin{cases} M_\nu + 1 & \text{if } 0 \le \nu \le N_1, \\ n^{1+\alpha}2^{-\alpha\nu d} & \text{if } N_1 + 1 \le \nu \le N_2, \end{cases}$$

where

$$0 < \alpha < 2\Big(\frac{s}{d} - \frac{1}{p_1}\Big) \tag{3.8}$$

and

$$N_1 = \left[\frac{\log_2 n}{d}\right], \quad N_2 = \left[\frac{\frac{s}{d} - \frac{1}{p} + \frac{1}{2}}{\frac{s}{d} - \frac{1}{p}} \cdot \frac{\log_2 n}{d}\right] \ge N_1.$$

Here, $[a]$ denotes the integer part of a real number a.
For this choice we get

$$n' = \sum_{\nu=0}^{N_2}(n_\nu - 1) + 1 \approx 2^{\nu N_1 d} + N_1^{1+\alpha}2^{-\alpha\nu d} \approx n.$$

A simple calculation shows that there is a number $\lambda > 0$ such that $M^{\lambda}_{\nu} \le n_\nu \le M_\nu$. Hence

$$a_{n_\nu}(id_\nu) \le \begin{cases} 0 & \text{if } 0 \le \nu \le N_1, \\ \frac{c_\lambda}{\sqrt{n_\nu}}2^{-\nu(s-\frac{d}{p_1})} & \text{if } N_1 + 1 \le \nu \le N_2 \end{cases}$$

and

$$\sum_{\nu=0}^{N_1} a^{\omega}_{n_\nu}(id_\nu) = 0,$$

$$\sum_{\nu=N_1+1}^{N_2} a^{\omega}_{n_\nu}(id_\nu) \le \sum_{\nu=N_1+1}^{N_2} \frac{c^{\omega}_{\lambda}}{\sqrt{n^{\omega}_{\nu}}} \le cn^{-\frac{1+\alpha}{2}\omega} \sum_{\nu=N_1+1}^{N_2} 2^{-\nu d\omega(\frac{s}{d}-\frac{1}{p_1}-\frac{\alpha}{2})} \lesssim n^{-\omega\left(\frac{s}{d}-\frac{1}{p_1}+\frac{1}{2}\right)},$$

$$\sum_{\nu=N_2+1}^{\infty} ||id_\nu||^{\omega} \le \sum_{\nu=N_2+1}^{\infty} 2^{-\nu\omega(s-\frac{d}{p_1})} \lesssim n^{-\omega\left(\frac{s}{d}-\frac{1}{p_1}+\frac{1}{2}\right)}.$$

It follows, that there is a constant $c > 0$ such that

$$a_{cn}(id) \lesssim n^{-\left(\frac{s}{d}-\frac{1}{p_1}+\frac{1}{2}\right)}, \quad n \geq 1,$$

which is equivalent to

$$a_n(id) \lesssim n^{-\left(\frac{s}{d}-\frac{1}{p_1}+\frac{1}{2}\right)}, \quad n \geq 1. \tag{3.9}$$

The proof of the reverse inequality to (3.9) follows easily from the second part of Lemma 3.4.

Let $M' = \{M'_\nu\}_{\nu=0}^{\infty}$ be an admissible sequence. Then, for $\nu \geq \nu_0$

$$a_n(id) \geq a_n(id_\nu) \gtrsim 2^{-\nu(s-\frac{d}{p_1})} \cdot \frac{1}{\sqrt{n}}$$

if $n = \left[\frac{M_\nu}{2}\right]$. This leads to

$$a_n(id) \gtrsim n^{-\left(\frac{s}{d}-\frac{1}{p_1}+\frac{1}{2}\right)}, \quad n = \left[\frac{M_\nu}{2}\right], \quad \nu \geq \nu_0$$

and by means of the monotonicity of the approximation numbers the result follows. $\square$

Remark 3.6. We have used the open case (3.6) to demonstrate the typical use of the wavelet decomposition method and (2.12). Also (3.3)–(3.5) could be proven exactly in the same manner. For example, the proof of (3.5) in [4] follows along this line.

Remark 3.7. Although the results were stated only for Besov spaces, with the aid of (2.4) and (2.5) we may extend them also to Triebel-Lizorkin spaces, Sobolev and Lebesgue spaces and $C(\Omega), L_1(\Omega)$ and $L_\infty(\Omega)$. We return to this point later on.

Remark 3.8. The first estimates on approximation numbers of Sobolev embeddings of function spaces were obtained by Kolmogorov [30], who dealt with the Hilbert space case $p_1 = q_1 = p_2 = q_2 = 2$. Later on, Birman and Solomyak [3] studied the embeddings of Sobolev spaces. Finally, Kashin [29] observed the effect of "small smoothness" expressed by (3.5). In the framework of Besov spaces the results are contained in [15, 4]. Nowadays, the proof of (3.3)–(3.5) could be done very similar to the proof of (3.6), only using Lemmas 3.2 and 3.3 instead of Lemma 3.4.

4 Kolmogorov and Gelfand numbers

In this chapter we deal with Kolmogorov and Gelfand numbers. To begin with we recall their definition and describe their decay in connection with Sobolev embeddings of Besov spaces. We use the symbol $A \subset\subset B$ if A is a closed subspace of a topological vector space B.

Definition 4.1. Let X, Y be two quasi-Banach spaces and let $T \in \mathcal{L}(X, Y)$.

(i) For $n \in \mathbb{N}$, we define the nth Kolmogorov number by

$$d_n(T) = \inf\{||Q_N^Y T|| : N \subset\subset Y, \dim(N) < n\}.$$

Here, Q_N^Y stands for the natural surjection of Y onto the quotient space Y/N.

(ii) For $n \in \mathbb{N}$, we define the nth Gelfand number by

$$c_n(T) = \inf\{||TJ_M^X|| : M \subset\subset X, \operatorname{codim}(M) < n\}.$$

Here, J_M^X stands for the natural injection of M into X.

Clearly, the notion *dimension of a subspace* is purely algebraic and may be freely used also in the setting of quasi-Banach spaces. We refer to [50, Section 1.40] for the definition of a quotient subspace in the framework of general topological vector spaces (including quasi-Banach spaces as a special case). Finally, the codimension of a subspace may be defined as the dimension of the quotient space.

Both, Gelfand and Kolmogorov numbers, are additive and multiplicative s-scales. This follows directly from Definition 4.1, but the reader may wish to consult [44, Sections 2.4, 2.5] for the proof in the Banach space case. The generalisation to p-Banach spaces is obvious and causes no complications. Also the following relations are trivial:

$$c_n(T) \le a_n(T), \qquad d_n(T) \le a_n(T), \qquad n \in \mathbb{N}. \tag{4.1}$$

The Gelfand and Kolmogorov numbers are dual to each other in the following sense, cf. [44, Section 11.7.6-7]: If X and Y are Banach spaces, then

$$c_n(T^*) = d_n(T) \tag{4.2}$$

for all compact operators $T \in \mathcal{L}(X, Y)$ and

$$d_n(T^*) = c_n(T) \tag{4.3}$$

for all $T \in \mathcal{L}(X, Y)$.

The following result is due to Gluskin, cf. [21, 22] with [56, 24, 26, 27] as forerunners. It gives a very precise information on the behaviour of $d_n(id : \ell_{p_1}^m \to \ell_{p_2}^m)$ in the Banach space setting.

Lemma 4.2. *For $1 \le n \le m < \infty$ and $1 \le p_1, p_2 \le \infty$, we define*

$$\Phi(m,n,p_1,p_2) := \begin{cases} (m-n+1)^{\frac{1}{p_2}-\frac{1}{p_1}} & \text{if} \quad 1 \le p_2 \le p_1 \le \infty, \\ \left(\min\{1, m^{\frac{1}{p_2}} n^{-\frac{1}{2}}\}\right)^{\frac{\frac{1}{p_1}-\frac{1}{p_2}}{\frac{1}{2}-\frac{1}{p_2}}} & \text{if} \quad 2 \le p_1 < p_2 \le \infty, \\ \max\{m^{\frac{1}{p_2}-\frac{1}{p_1}}, \sqrt{1-\frac{n}{m}}^{\frac{\frac{1}{p_1}-\frac{1}{p_2}}{\frac{1}{p_1}-\frac{1}{2}}}\} & \text{if} \quad 1 \le p_1 < p_2 \le 2, \\ \max\{m^{\frac{1}{p_2}-\frac{1}{p_1}}, \min\{1, m^{\frac{1}{p_2}} n^{-\frac{1}{2}}\} \cdot \sqrt{1-\frac{n}{m}}\} & \text{if} \quad 1 \le p_1 < 2 < p_2 \le \infty. \end{cases}$$

Then

$$d_n(id : \ell_{p_1}^m \to \ell_{p_2}^m) \approx \Phi(m,n,p_1,p_2), \qquad 1 \le n \le m < \infty,$$

if $p_2 < \infty$. The constants of equivalence may depend on p_1 and p_2 but are independent of m and n.

Furthermore, there are two constants c_{p_1} and C_{p_1} such that

$$c_{p_1}\Phi(m,n,p_1,\infty) \le d_n(id : \ell_{p_1}^m \to \ell_\infty^m) \le C_{p_2}\Phi(m,n,p_1,\infty)\left(\log\left(\frac{em}{n}\right)\right)^{3/2},$$

for $1 \le p_1 \le \infty$.

Again we shall add some estimates which apply to quasi-Banach spaces.

Lemma 4.3. *If $0 < p_2 \le p_1 \le \infty$, then there is a constant $c > 0$ such that*

$$d_{[cn]+1}(\ell_{p_1}^{2n}, \ell_{p_2}^{2n}) \gtrsim n^{\frac{1}{p_2}-\frac{1}{p_1}}, \quad n \in \mathbb{N},$$

where $[cn]$ denotes the upper integer part of cn.

Proof. If $p_2 \ge 1$, then the result is a special case of [43, Section 11.11.4], which states that

$$d_n(\ell_{p_1}^m, \ell_{p_2}^m) = (m-n+1)^{\frac{1}{p_2}-\frac{1}{p_1}}, \quad 1 \le n \le m.$$

Let us mention, that (in contrast to Lemma 3.3 and Lemma 4.8) the estimate

$$d_n(\ell_{p_1}^m, \ell_{p_2}^m) = (m-n+1)^{\frac{1}{p_2}-\frac{1}{p_1}}, \quad 1 \le n \le m \le \infty,$$

is *not* true for Kolmogorov numbers if $0 < p_2 \le p_1 \le \infty$ and $p_2 < 1$. Simple counterexamples can be constructed directly.

If $p_2 < 1$ the proof is based on an inequality between entropy numbers and Kolmogorov numbers. First, we recall the basic facts about entropy numbers. Let $T : X \to Y$ be a bounded linear operator between two quasi-Banach spaces X and Y and let U_X and U_Y be the unit ball of X and Y, respectively. If $k \in \mathbb{N}$, we define the kth entropy number $e_k(T)$ as the infimum of all $\epsilon > 0$ such that

$$T(U_X) \subset \bigcup_{j=1}^{2^{k-1}} (y_j + \epsilon U_Y) \quad \text{for some} \quad y_1, \dots, y^{2^{k-1}} \in Y.$$

We refer to [43] and [15] for detailed discussions of this concept, its history and further references.

The following Lemma may be found in [1], cf. also [5] and [47, Section 5].

Lemma 4.4. *If $\alpha > 0$ and $0 < p < 1$, then there is a constant $c_{\alpha,p} > 0$ such that for all p-Banach spaces X and Y, all linear mappings $T : X \to Y$ and all $n \in \mathbb{N}$ we have*

$$\sup_{k \le n} k^\alpha e_k(T) \le c_{\alpha,p} \sup_{k \le n} k^\alpha d_k(T).$$

We apply this lemma to $T = id : \ell_{p_1}^{2n} \to \ell_{p_2}^{2n}$ and combine it with the estimate (cf. [53])

$$e_k(T) \gtrsim 2^{-\frac{k}{4n}} (2n)^{\frac{1}{p_2}-\frac{1}{p_1}}, \quad k, n \in \mathbb{N}.$$

This leads to

$$n^\alpha n^{\frac{1}{p_2}-\frac{1}{p_1}} \lesssim \sup_{k \le n} k^\alpha d_k(T).$$

Hence, for every $n \in \mathbb{N}$ there is a $k_n \le n$ such that

$$n^\alpha n^{\frac{1}{p_2}-\frac{1}{p_1}} \lesssim k_n^\alpha d_{k_n}(T) \le k_n^\alpha (2n)^{\frac{1}{p_2}-\frac{1}{p_1}}. \tag{4.4}$$

We conclude, that there is a constant $1 \ge c > 0$ such that $n \ge k_n \ge cn$ for all $n \in \mathbb{N}$. Finally, we insert this estimate into (4.4) and the result follows. $\square$

It is an obvious fact that the convex hull of the unit ball of $\ell_p^m, 0 < p < 1$, is the unit ball of ℓ_1^m. This can be combined with the following simple observation, cf. [35, Section 13.1].

Lemma 4.5. *Let X be a Banach space and let $K \subset X$. We define by*

$$d_n(K,X) = \inf\{\sup_{x\in K}\inf_{y\in N}||x-y|| : N \subset\subset Y, \dim(N) < n\}$$

the nth Kolmogorov number of the set K.
Then

$$d_n(K,X) = d_n(\mathrm{conv}K, X),$$

where conv$K$ *is the convex hull of K.*

Theorem 4.6. *Let $-\infty < s_2 < s_1 < \infty$ and $0 < p_1, p_2, q_1, q_2 \le \infty$ with (2.8). Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then (2.7) is compact and for $n \in \mathbb{N}$*

$$d_n(\mathcal{I}d) \approx n^{-\frac{s_1-s_2}{d}+\left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+} \qquad \text{if} \quad \begin{cases} \text{either} & 0 < p_1 \le p_2 \le 2, \\ \text{or} & 0 < p_2 \le p_1 \le \infty, \end{cases} \tag{4.5}$$

$$d_n(\mathcal{I}d) \approx n^{-\frac{s_1-s_2}{d}} \qquad \text{if} \quad 2 < p_1 \le p_2 \le \infty \ \text{ and } \ \frac{s_1-s_2}{d} > \frac{1}{2}\frac{\frac{1}{p_1}-\frac{1}{p_2}}{\frac{1}{2}-\frac{1}{p_2}}, \tag{4.6}$$

$$d_n(\mathcal{I}d) \approx n^{\frac{p_2}{2}\left(-\frac{s_1-s_2}{d}+\frac{1}{p_1}-\frac{1}{p_2}\right)} \qquad \text{if} \quad 2 < p_1 \le p_2 \le \infty \ \text{ and } \ \frac{s_1-s_2}{d} < \frac{1}{2}\frac{\frac{1}{p_1}-\frac{1}{p_2}}{\frac{1}{2}-\frac{1}{p_2}}, \tag{4.7}$$

$$d_n(\mathcal{I}d) \approx n^{\left(-\frac{s_1-s_2}{d}+\frac{1}{p_1}-\frac{1}{2}\right)} \qquad \text{if} \quad 0 < p_1 < 2 < p_2 \le \infty \ \text{ and } \ \frac{s_1-s_2}{d} > \frac{1}{p_1}, \tag{4.8}$$

$$d_n(\mathcal{I}d) \approx n^{\frac{p_2}{2}\left(-\frac{s_1-s_2}{d}+\frac{1}{p_1}-\frac{1}{p_2}\right)} \qquad \text{if} \quad 0 < p_1 < 2 < p_2 < \infty \ \text{ and } \ \frac{1}{p_1}-\frac{1}{p_2} < \frac{s_1-s_2}{d} < \frac{1}{p_1}. \tag{4.9}$$

Proof. Lubitz [36] used the results of [21] and was able to prove (4.5)–(4.9) if $1 \le p_1, p_2 \le \infty$ up to a certain logarithmic gap. This gap originates from using only the weaker results of [21] instead of the sharp inequalities in [22]. Using [22] and the method of Lubitz (which is very similar to the discretization method presented above), the proof of (4.5)–(4.9) in the Banach space setting follows immediately.

Hence, we concentrate on the proof of

(♣) (4.5) if $0 < p_2 \le p_1 \le \infty$ and $0 < p_2 < 1$,

(♡) (4.5) if $0 < p_1 < p_2 \le 2$ and $0 < p_1 < 1$,

(♠) (4.8) if $0 < p_1 < 1$, $2 < p_2 \le \infty$ and $\dfrac{s_1 - s_2}{d} > \dfrac{1}{p_1}$,

(♢) (4.8) if $0 < p_1 < 1$, $2 < p_2 < \infty$ and $\dfrac{1}{p_1} - \dfrac{1}{p_2} < \dfrac{s_1 - s_2}{d} < \dfrac{1}{p_1}$.

Let us mention that all the estimates from above follow from the estimates given in Theorem 3.5 and (4.1). We shall give the proof of the estimates from below in following three steps.

Step 1. - Proof of (♣)

The proof of (4.5) can be finished in the same manner as in the proof of Theorem 3.5. Namely, if $M' = \{M'_\nu\}_{\nu=0}^\infty$ is an admissible sequence, we get for $\nu \ge \nu_0$

$$d_n(id) \ge d_n(id_\nu) \gtrsim 2^{-\nu(s_1 - s_2 - \frac{d}{p_1} + \frac{d}{p_2})} \cdot M_\nu^{\frac{1}{p_2} - \frac{1}{p_1}}$$

for $n = \left[\frac{c}{2} \cdot M'_\nu\right]$, where c is the constant from Lemma 4.3. This leads to

$$d_n(id) \gtrsim n^{-\frac{s_1 - s_2}{d}}, \quad n = \left[\frac{c}{2} \cdot M'_\nu\right], \quad \nu \ge \nu_0$$

Again the monotonicity of the Kolmogorov numbers completes the proof.

Step 2. - Proof of (♠) *and* (♢)

It follows from Lemma 4.5, that if $0 < p_1 < 1$ and $2 < p_2 \le \infty$

$$d_n(\ell_{p_1}^m, \ell_{p_2}^m) = d_n(\ell_1^m, \ell_{p_2}^m), \quad 1 \le n \le m < \infty. \tag{4.10}$$

The proof of (♠) follows from (4.10), (4.2), Lemma 4.2 and the choice $n = \left[\frac{M'_\nu}{2}\right]$.

The proof of (♢) follows in the same way, but with $n = \left[(M'_\nu)^{\frac{2}{p_2}}\right]$.

Step 3. - Proof of (♡)

We generalise the idea of Lemma 4.5 to p-Banach spaces, namely we show that for $0 < p_1 < p_2 \le 2$

$$d_n(\ell_{p_1}^m, \ell_{p_2}^m) = d_n(\ell_{\min(1,p_2)}^m, \ell_{p_2}^m), \quad 1 \le n \le m < \infty. \tag{4.11}$$

If $p_2 \ge 1$, this follows immediately from Lemma 4.5. If $p_2 \le 1$, we show that

$$d_n(\ell_{p_1}^m, \ell_{p_2}^m) \ge d_n(E_m, \ell_{p_2}^m) \ge d_n(\ell_{p_2}^m, \ell_{p_2}^m). \tag{4.12}$$

Here, $E_m = \{e_i\}_{i=1}^m \subset \mathbb{R}^m$ and $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ are the canonical unit vectors having all but one components 0 and the i^{th} component 1.

Of course, (4.12) implies one half of (4.11), the second one being obvious. From (4.12), only the second inequality needs a proof. Let $N \subset\subset \ell_{p_2}^m = Y$ be such that

$$\sup_{i=1,\ldots,n} \inf_{y \in N} ||e_i - y||_{p_2} \le (1+\varepsilon) d_n(E_m, \ell_{p_2}^m)$$

with dim $N < n$. Hence, to every $e_i \in E_m$ there is a $f_i \in N$ such that

$$||e_i - f_i||_Y \le (1+\varepsilon)^2 d_n(E_m, \ell_{p_2}^m).$$

To every $x \in \ell_{p_2}^m, x = \sum_{i=1}^m x_i e_i$ with $\sum_{i=1}^m |x_i|^{p_2} \leq 1$ we associate $\tilde{x}(x) = \sum_{i=1}^m x_i f_i \in N$. The estimate

$$\begin{aligned} d_n(id : \ell_{p_2}^m \to \ell_{p_2}^m)^{p_2} &\leq \sup_{||x||_{p_2} \leq 1} \inf_{y \in N} ||x-y||_{p_2}^{p_2} \\ &\leq \sup_{||x||_{p_2} \leq 1} ||x - \tilde{x}(x)||_{p_2}^{p_2} = \sup_{||x||_{p_2} \leq 1} \left|\left| \sum_{i=1}^m x_i (e_i - f_i) \right|\right|_{p_2}^{p_2} \\ &\leq \sup_{||x||_{p_2} \leq 1} \sum_{i=1}^m ||x_i(e_i - f_i)||_{p_2}^{p_2} = \sup_{||x||_{p_2} \leq 1} \sum_{i=1}^m |x_i|^{p_2} ||e_i - f_i||_{p_2}^{p_2} \\ &\leq \sup_{||x||_{p_2} \leq 1} \sum_{i=1}^m |x_i|^{p_2} (1+\varepsilon)^{2p_2} d_n(E_m, \ell_{p_2}^m)^{p_2} \\ &\leq (1+\varepsilon)^{2p_2} d_n(E_m, \ell_{p_2}^m)^{p_2} \end{aligned}$$

finishes the proof of (4.12).
The proof of (♡) follows in the same way as in the first and the second step. □

Now, we turn our attention to Gelfand numbers. First, we collect some information about $c_n(id : \ell_{p_1}^m \to \ell_{p_2}^m)$, cf. [22], (4.2) and (4.3).

Lemma 4.7. *For $1 \leq n \leq m < \infty$ and $1 \leq p_1, p_2 \leq \infty$, we define*

$$\Phi(m,n,p_1,p_2) := \begin{cases} (m-n+1)^{\frac{1}{p_2} - \frac{1}{p_1}} & \text{if} \quad 1 \leq p_2 \leq p_1 \leq \infty, \\ \left(\min\{1, m^{1-\frac{1}{p_1}} n^{-\frac{1}{2}}\}\right)^{\frac{\frac{1}{p_1} - \frac{1}{p_2}}{\frac{1}{p_1} - \frac{1}{2}}} & \text{if} \quad 1 < p_1 < p_2 \leq 2, \\ \max\{m^{\frac{1}{p_2} - \frac{1}{p_1}}, \sqrt{1 - \frac{n}{m}}^{\frac{\frac{1}{p_1} - \frac{1}{p_2}}{\frac{1}{2} - \frac{1}{p_2}}}\} & \text{if} \quad 2 \leq p_1 < p_2 \leq \infty, \\ \max\{m^{\frac{1}{p_2} - \frac{1}{p_1}}, \min\{1, m^{1-\frac{1}{p_1}} n^{-\frac{1}{2}}\} \cdot \sqrt{1 - \frac{n}{m}}\} & \text{if} \quad 1 < p_1 \leq 2 < p_2 \leq \infty. \end{cases}$$

Then, if $p_1 > 1$,

$$c_n(id : \ell_{p_1}^m \to \ell_{p_2}^m) \approx \Phi(m,n,p_1,p_2), \qquad 1 \leq n \leq m < \infty.$$

Furthermore, there are two constants c_{p_2} and C_{p_2} such that

$$c_{p_2} \Psi(m,n,p_2) \leq c_n(id : \ell_1^m \to \ell_{p_2}^m) \leq C_{p_2} \Psi(m,n,p_2) \left(\log\left(\frac{em}{n}\right)\right)^{3/2},$$

where

$$\Psi(m,n,p_2) := \begin{cases} n^{1-\frac{1}{p_2}} & \text{if} \quad 1 < p_2 \leq 2, \\ \min\{1, \max\{m^{1-\frac{1}{p_2}}, m^{-\frac{1}{2}} \sqrt{\frac{m}{n} - 1}\}\} & \text{if} \quad 2 \leq p_2 \leq \infty. \end{cases}$$

The proof of this lemma follows by (4.2) or (4.3) and Lemma 4.2.

Lemma 4.8. *If* $0 < p_2 \le p_1 \le \infty$, *then*

$$c_n(\ell_{p_1}^m, \ell_{p_2}^m) = (m-n+1)^{\frac{1}{p_2}-\frac{1}{p_1}}.$$

The proof of this lemma follows literally [44, Section 11.11.4].

Lemma 4.9. *Let* $0 < p < 1$. *Then there is a real constant* $c > 0$ *such that*

$$c_n(id : \ell_p^m \to \ell_2^m) \le c \left[\frac{n}{\log\left(1+\frac{m}{n}\right)} \right]^{\frac{1}{2}-\frac{1}{p}}, \quad 1 \le n \le m < \infty.$$

Proof. This lemma slightly generalises a result of Kashin [28], which was later improved by Gluskin [22] and Garnaev and Gluskin [20]. We closely follow the presentation given in [35, Chapter 14].

Let $\mathbf{y} = (y_1, \ldots, y_n)$ be a multivector, with $y_1, \ldots, y_n \in S^{m-1}$, the unit sphere of $\mathbb{R}^m$. We set

$$F_{m,n}(x, \mathbf{y}) = \frac{|(x, y_1)| + \cdots + |(x, y_n)|}{n}, \quad x \in \mathbb{R}^m.$$

We equip the space

$$\Sigma_{m,n} = \underbrace{S^{m-1} \times \cdots \times S^{m-1}}_{n \text{ times}}$$

with the natural rotation invariant probability measure P. Then (cf. [35, Lemma 4.1, Chapter 14]) we have the following

Lemma 4.10. *For any* $x \in S^{m-1}$ *and* $m, n \ge 2$

$$P\left\{ \mathbf{y} \in \Sigma_{m,n} : \frac{1}{8\sqrt{m}} \le F(x, \mathbf{y}) \le \frac{3}{\sqrt{m}} \right\} > \begin{cases} 1 - e^{-n}, & n > 2, \\ \frac{1}{2}, & n = 2. \end{cases}$$

Let l and m be natural numbers with $1 \le l \le m$. Let b_p^m denote the unit ball of ℓ_p^m. We denote by $b_p^{m,l}$ the subset of all vectors from b_p^m whose coordinates are of the form $\frac{k}{l}$, $k \in \mathbb{Z}$. Then there is a real constant $\tilde{c} > 0$ such that for any natural number $n \le m$ with

$$l = \left[\frac{1}{2\tilde{c}} \left(\frac{n}{\log\left(1+\frac{m}{n}\right)} \right)^{1/p} \right] \ge 1$$

there exists a multivector $\mathbf{y} = (y_1, \ldots, y_n)$ such that for all $x \in b_p^{m,l}$

$$\frac{1}{8\sqrt{m}} ||x||_2 \le F(x, \mathbf{y}) \le \frac{3}{\sqrt{m}} ||x||_2. \tag{4.13}$$

To prove it, we need to estimate the number of the elements of $b_p^{m,l}$ from above. It could be done directly, but we prefer to use known results. Observe that the mutual ℓ_∞^m distance of the points in $b_p^{m,l}$ is at least $\frac{1}{l}$. Hence, if $M_p^{m,l} = \# b_p^{m,l}$ (i.e. the number of elements of $b_p^{m,l}$) is greater than 2^n for some natural number n, then

$$e_n(id : \ell_p^m \to \ell_\infty^m) \ge \frac{1}{2l}. \tag{4.14}$$

But, according to [53] and [15, Section 3.2.2], there is a constant $\tilde{c}$ such that

$$e_n(id:\ell_p^m \to \ell_\infty^m) \le \tilde{c}\left(\frac{\log(1+\frac{m}{n})}{n}\right)^{1/p}, \quad 1 \le n \le m. \tag{4.15}$$

From (4.14) and (4.15), it follows that if

$$\frac{1}{2l} > \tilde{c}\left(\frac{\log(1+\frac{m}{n})}{n}\right)^{1/p},$$

then $M_p^{m,l} \le 2^n < e^n$. This, combined with Lemma 4.10 ensures the existence of the multivector $\mathbf{y}$.

Let $b_p^{m,l}$ be as above and let b_∞^m be a unit ball of ℓ_∞^m. Let $V_p^{m,l} = b_p^{m,l} \cap (\frac{1}{l} b_\infty^m)$ be the set of all vectors in $\mathbb{R}^m$ with the ℓ_p^m-quasinorm at most one and with components in $\{0, \pm\frac{1}{l}\}$. Then we claim that

$$b_p^m \cap \left(\frac{1}{l} b_\infty^m\right) = \mathrm{conv}_p(V_p^{m,l}) \subset \mathrm{conv}(V_p^{m,l}), \tag{4.16}$$

where $\mathrm{conv}_p(V_p^{m,l})$ is the so-called p-convex hull of $V_p^{m,l}$. We refer to [18, 19, 25] for the notion of p-convexity, p-extreme points and the quasi-convex variant of the Krein-Milman theorem, which gives the identity in (4.16). The inclusion is a simple consequence of the fact that $p < 1$.

To prove Lemma 4.9, we need to find $N \subset\subset \mathbb{R}^m$ of codimension at most n such that for each point $x \in N \cap b_p^m$ we have $||x||_2 \le \frac{c}{\sqrt{l}}$.

Let $\mathbf{y}$ be one multivector with (4.13). We set

$$N = \{x \in \mathbb{R}^m : F(x, \mathbf{y}) = 0\}.$$

Let $x \in N \cap b_p^m$ and let $x' \in b_p^{m,l}$ be the closest point to x, hence $||x - x'||_\infty \le \frac{1}{l}$. We set $x'' = x - x'$. Then

$$||x''||_2 \le ||x''||_p^{\frac{p}{2}} \cdot ||x''||_\infty^{1-\frac{p}{2}} \le l^{\frac{p}{2}-1}. \tag{4.17}$$

It remains to estimate $||x'||_2$. This will be done by estimating the value of $F(x', \mathbf{y})$. The estimate

$$F(x', \mathbf{y}) \ge \frac{1}{8\sqrt{m}} ||x'||_2 \tag{4.18}$$

follows from (4.13) and the fact that $x' \in b_p^{m,l}$. On the other hand, because of $x \in N$ and F is subadditive,

$$F(x', \mathbf{y}) \le F(x, \mathbf{y}) + F(x'', \mathbf{y}) = F(x'', \mathbf{y}). \tag{4.19}$$

For all $\tilde{x} \in V_p^{m,l} \subset b_p^{m,l}$, we have

$$F(\tilde{x}, \mathbf{y}) \le \frac{3}{\sqrt{m}} ||\tilde{x}||_2 \le 3m^{-\frac{1}{2}} l^{\frac{p}{2}-1} \tag{4.20}$$

and by subadditivity of F and (4.16), the same holds also for $x'' \in b_p^m \cap \left(\frac{1}{l} b_\infty^m\right)$.

We insert (4.20) into (4.19) and (4.18) and get $||x'||_2 \le 24 l^{\frac{p}{2}-1}$, and together with (4.17), $||x|| \le \frac{25}{\sqrt{l}}$. □

Lemma 4.11. *Let $0 < p_1 < 1$ and $p_1 < p_2 \le \infty$. Then there is a real constant $c > 0$ such that*

$$c_n(id : \ell_{p_1}^m \to \ell_{p_2}^m) \le c \left[\frac{n}{\log\left(1+\frac{m}{n}\right)}\right]^{\frac{1}{\min(p_2,2)}-\frac{1}{p_1}}, \quad 1 \le n \le m < \infty.$$

Theorem 4.12. *Let $-\infty < s_2 < s_1 < \infty$ and $0 < p_1, p_2, q_1, q_2 \le \infty$ with (2.8). Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then (2.7) is compact and for $n \in \mathbb{N}$*

$$c_n(\mathcal{I}d) \approx n^{-\frac{s_1-s_2}{d}+\left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+} \qquad \text{if} \quad \begin{cases} \text{either} & 2 \le p_1 < p_2 \le \infty, \\ \text{or} & 0 < p_2 \le p_1 \le \infty, \end{cases} \tag{4.21}$$

$$c_n(\mathcal{I}d) \approx n^{-\frac{s_1-s_2}{d}} \qquad \text{if} \quad 0 < p_1 < p_2 \le 2 \quad \text{and} \quad \frac{s_1-s_2}{d} > \frac{1}{2}\frac{\frac{1}{p_1}-\frac{1}{p_2}}{\frac{1}{p_1}-\frac{1}{2}}, \tag{4.22}$$

$$c_n(\mathcal{I}d) \approx n^{\frac{p_1'}{2}\left(-\frac{s_1-s_2}{d}+\frac{1}{p_1}-\frac{1}{p_2}\right)} \qquad \text{if} \quad 1 < p_1 < p_2 \le 2 \quad \text{and} \quad \frac{s_1-s_2}{d} < \frac{1}{2}\frac{\frac{1}{p_1}-\frac{1}{p_2}}{\frac{1}{p_1}-\frac{1}{2}}, \tag{4.23}$$

$$c_n(\mathcal{I}d) \approx n^{\left(-\frac{s_1-s_2}{d}+\frac{1}{2}-\frac{1}{p_2}\right)} \qquad \text{if} \quad 0 < p_1 < 2 < p_2 \le \infty \quad \text{and} \quad \frac{s_1-s_2}{d} > 1-\frac{1}{p_2}, \tag{4.24}$$

$$c_n(\mathcal{I}d) \approx n^{\frac{p_1'}{2}\left(-\frac{s_1-s_2}{d}+\frac{1}{p_1}-\frac{1}{p_2}\right)} \qquad \text{if} \quad 1 < p_1 < 2 < p_2 \le \infty \quad \text{and} \quad \frac{1}{p_1}-\frac{1}{p_2} < \frac{s_1-s_2}{d} < 1-\frac{1}{p_2}. \tag{4.25}$$

Proof. As Gelfand numbers are multiplicative and additive s-numbers, we may invoke (2.12) and restrict again to sequence spaces. Then, the method of the proof of Theorem 3.5 applies. The estimates on the sequence space side are given by Lemma 4.2 and (4.2). This approach finishes the proof in case $1 \le p_1, p_2 \le \infty$.

In the cases, when $p_1 < 1$ and/or $p_2 < 1$, (4.2) and (4.3) fail and Lemma 4.2 does not provide suitable estimates for $c_n(id : \ell_{p_1}^m \to \ell_{p_2}^m)$. Hence, we are forced to treat these cases separately.

- (♣) (4.21) if $0 < p_2 \le p_1 \le \infty$ and $0 < p_2 < 1$,
- (♡) (4.22) if $0 < p_1 < p_2 \le 2$ and $0 < p_1 < 1$,
- (♠) (4.24) if $0 < p_1 < 1$ and $2 < p_2 \le \infty$.

Step 1. - Proof of (♣)

The proof of the estimate from below in (♣) follows exactly as in the proof of Theorem 4.6 with Lemma 4.3 replaced by Lemma 4.8.

The estimate from above in (♣) is provided by the corresponding statement about approximation numbers, cf. Theorem 3.5 and (4.1).

Step 2. - Proof of the estimates from below in (♡) *and* (♠)

If $1 \le p_2 \le \infty$, we use the estimate

$$c_n(id : \ell_1^m \to \ell_{p_2}^m) \le ||id : \ell_1^m \to \ell_{p_1}^m|| \cdot c_n(id : \ell_{p_1}^m \to \ell_{p_2}^m) \tag{4.26}$$

and if $p_2 < 1$, we use the estimate

$$c_n(id : \ell_{p_2}^m \to \ell_{p_2}^m) \le ||id : \ell_{p_2}^m \to \ell_{p_1}^m|| \cdot c_n(id : \ell_{p_1}^m \to \ell_{p_2}^m). \tag{4.27}$$

This leads to

$$c_n(id : \ell_{p_1}^{2n} \to \ell_{p_2}^{2n}) \gtrsim \begin{cases} n^{\frac{1}{2}-\frac{1}{p_1}} & \text{if} \quad 2 \le p_2 \le \infty, \\ n^{\frac{1}{p_2}-\frac{1}{p_1}} & \text{if} \quad 0 < p_2 \le 2 \end{cases} \tag{4.28}$$

and the proof of the estimates from below included in (♡) and (♠) may be again finished as in the proof of Theorem 4.6.

Step 3. - Proof of the estimates from above in (♡) *and* (♠)

Again, the knowledge of the behaviour of $c_n(id : \ell_{p_1}^m \to \ell_{p_2}^m)$ is of a crucial importance. Lemma 4.11 contains already the necessary information and the proof can be finished using the standard discretization method. □

5 Conclusion

In Theorems 3.5, 4.6 and 4.12 we gave an overview of the behaviour of approximation, Kolmogorov and Gelfand numbers of

$$\mathcal{I}d : B^{s_1}_{p_1q_1}(\Omega) \to B^{s_2}_{p_2q_2}(\Omega),$$

where Ω is a bounded domain in $\mathbb{R}^d$ with smooth (i.e. Lipschitz) boundary and the parameters satisfy

$$s_1 - s_2 > d\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big)_+.$$

The reader has surely noticed, that all the obtained results about the asymptotic decay of $a_n(\mathcal{I}d)$, $d_n(\mathcal{I}d)$ and $c_n(\mathcal{I}d)$ do not depend on the fine parameters $0 < q_1, q_2 \le \infty$. This is of course no coincidence. The reason lies in the roots of the method we have used, namely in (3.7).

Nevertheless, the presented bounds from above and from below coincide in all "non-limiting" cases. Unfortunately, this method has also its natural bounds. For example, if $0 < p_1 < 2 < p_2 \le \infty$ and $s_1 - s_2 = d \max(1 - \frac{1}{p_2}, \frac{1}{p_1})$, then Theorem 3.5 fails to characterize the decay of $a_n(\mathcal{I}d)$. One observes, that in this case both (3.4) and (3.5) meet at $n^{-\frac{1}{2}}$, but (in general) this is not the exact speed of the decay of $a_n(\mathcal{I}d)$. It was shown by Kulanin [33], that additional logarithmic factors come into play. Their exact order seems to be unknown, but we believe that it depends on q_1 and q_2. So, for principle reasons, the decomposition method can not be extended to this "limiting" case.

Using the elementary embeddings (2.4), we conclude, that all the results hold for Triebel-Lizorkin spaces, Lebesgue spaces, Sobolev spaces, Bessel potential spaces and Hölder-Zygmund spaces as well.

For example, Theorem 3.5 may be stated in the framework of Bessel potential spaces and their embeddings into $C(\Omega)$ and $L_\infty(\Omega)$.

Theorem 5.1. *Let $1 \le p \le \infty$, $s > \frac{d}{p}$ and let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then the embeddings*

$$\mathcal{I}d_1 : H^s_p(\Omega) \to \mathcal{C}(\Omega) \tag{5.1}$$

$$\mathcal{I}d_2 : H^s_p(\Omega) \to L_\infty(\Omega) \tag{5.2}$$

are compact and

$$\begin{array}{lll} a_n(\mathcal{I}d_1) \approx a_n(\mathcal{I}d_2) \approx n^{-\frac{s}{d}+\frac{1}{p}} & \textit{if} \quad 2 \le p \le \infty, & \\ a_n(\mathcal{I}d_1) \approx a_n(\mathcal{I}d_2) \approx n^{-\frac{s}{d}+\frac{1}{\tilde{p}}-\frac{1}{2}} & \textit{if} \quad 0 < p < 2 \quad \textit{and} & \dfrac{s}{d} > \dfrac{1}{\tilde{p}} = \max\Big(1, \dfrac{1}{p}\Big), \\ a_n(\mathcal{I}d_1) \approx a_n(\mathcal{I}d_2) \approx n^{\left(-\frac{s}{d}+\frac{1}{p}\right)\cdot\frac{p'}{2}} & \textit{if} \quad 1 < p < 2 \quad \textit{and} & \dfrac{1}{p} < \dfrac{s}{d} < 1. \end{array}$$

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On sharp embeddings of Besov and Triebel-Lizorkin spaces in the subcritical case

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Abstract

We discuss the growth envelopes of Fourier-analytically defined Besov and Triebel-Lizorkin spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ in the limiting case $s = \sigma_p := n\max(\frac{1}{p} - 1, 0)$. These results may be also reformulated as optimal embeddings into the scale of Lorentz spaces $L_{p,q}(\mathbb{R}^n)$. We close several open problems outlined already in [H. Triebel, *The structure of functions*, Birkhäuser, Basel, 2001] and explicitly stated in [D. D. Haroske, *Envelopes and sharp embeddings of function spaces*, Chapman & Hall / CRC, Boca Raton, 2007].

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1 Introduction and main results

In this paper we prove sharp embedding theorems for Besov and Triebel-Lizorkin spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ in some limiting cases of the range guaranteeing that these spaces consist of locally integrable functions. As proven in [12, Theorem 3.3.2],

$$B^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_1^{\mathrm{loc}}(\mathbb{R}^n) \Leftrightarrow \begin{cases} \text{either} & s > \sigma_p := n\max(\frac{1}{p} - 1, 0), \\ \text{or} & s = \sigma_p, 1 < p \le \infty, 0 < q \le \min(p, 2), \\ \text{or} & s = \sigma_p, 0 < p \le 1, 0 < q \le 1 \end{cases} \quad (1)$$

and

$$F^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_1^{\mathrm{loc}}(\mathbb{R}^n) \Leftrightarrow \begin{cases} \text{either} & s > \sigma_p, \\ \text{or} & s = \sigma_p, 1 \le p < \infty, 0 < q \le 2, \\ \text{or} & s = \sigma_p, 0 < p < 1, 0 < q \le \infty. \end{cases} \quad (2)$$

The embeddings can be measured quantitavily by the *growth envelope function* of X as defined by D. D. Haroske and H. Triebel (see [3], [4], [16] and references given there) by

$$\mathcal{E}_G^X(t) := \sup_{||f|X||\le 1} f^*(t), \quad 0 < t < 1,$$

where f^* denotes the non-increasing rearrangement of f.

In the case where $\mathcal{E}_G^X(t) \approx t^{-\alpha}$ for $0 < t < 1$ and some $\alpha > 0$ the *growth envelope index* u_X is given as the infimum of all numbers v, $0 < v \le \infty$, such that

$$\left(\int_0^\epsilon \left[\frac{f^*(t)}{\mathcal{E}_G^X(t)}\right]^v \frac{dt}{t}\right)^{1/v} \le c\,||f|X|| \tag{3}$$

(with the usual modification for $v = \infty$) holds for some $\epsilon > 0, c > 0$ and all $f \in X$. The pair $\mathfrak{E}_G(X) = (\mathcal{E}_G^X, u_X)$ is called *growth envelope* for the function space X.

In the case $\sigma_p < s$, the growth envelopes of $A^s_{p,q}(\mathbb{R}^n)$ are known, cf. [16, Theorem 15.2] and [4, Theorem 8.1]. If $s = \sigma_p$ and (1) or (2) is fulfilled in the B or F case, respectively, then the growth function is given by $t^{-\frac{1}{\max(p,1)}}$, but the known information about the growth index u is not complete, cf. [16, Remarks 12.5, 15.1] and [4, Prop. 8.12, 8.14 and Remark 8.15].

The growth index of $B^{\sigma_p}_{p,q}(\mathbb{R}^n)$ satisfies

$$\begin{cases} q \le u \le p & \text{if } 1 \le p < \infty \text{ and } 0 < q \le \min(p,2), \\ q \le u \le 1 & \text{if } 0 < p < 1 \text{ and } 0 < q \le 1. \end{cases} \tag{4}$$

The growth index of $F^{\sigma_p}_{p,q}(\mathbb{R}^n)$ satisfies $p \le u \le 1$ if $0 < p < 1$ and $0 < q \le \infty$ and is equal to p, if $1 \le p < \infty$ and $0 < q \le 2$.

The growth envelopes of $B^0_{\infty,q}$ defined on torus $\mathbb{T}^n = (\mathbb{R}/\mathbb{Z})^n$ with $1 \le q \le 2$ were identified recently by Seeger and Trebels in [10] and are equivalent to $|\log t|^{1/q'}$ for $0 < t \le 1/2$. We fill the remaining gaps for the range $p < \infty$.

Theorem 1.1. *(i) Let* $1 \le p < \infty$ *and* $0 < q \le \min(p,2)$. *Then*

$$\mathfrak{E}_G(B^0_{p,q}) = (t^{-\frac{1}{p}}, p).$$

(ii) Let $0 < p < 1$ *and* $0 < q \le 1$. *Then*

$$\mathfrak{E}_G(B^{\sigma_p}_{p,q}) = (t^{-1}, q).$$

(iii) Let $0 < p < 1$ *and* $0 < q \le \infty$. *Then*

$$\mathfrak{E}_G(F^{\sigma_p}_{p,q}) = (t^{-1}, p).$$

These results are closely related to optimal embeddings into the scale of Lorentz spaces. In this context, we prove the following

Theorem 1.2. *(i) Let $1 \le p < \infty$ and $0 < q \le \min(p, 2)$. Then*

$$B^0_{p,q}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n).$$

(ii) Let $0 < p < 1$ and $0 < q \le 1$. Then

$$B^{\sigma_p}_{p,q}(\mathbb{R}^n) \hookrightarrow L_{1,q}(\mathbb{R}^n). \tag{5}$$

(iii) Let $0 < p < 1$ and $0 < q \le \infty$. Then

$$F^{\sigma_p}_{p,q}(\mathbb{R}^n) \hookrightarrow L_{1,p}(\mathbb{R}^n)$$

and all these embeddings are optimal with respect to the second fine parameter of the scale of the Lorentz spaces.

Remark 1.3. (i) Let us observe, that (5) improves [12, Theorem 3.2.1] and [11, Theorem 2.2.3], where the embedding $B^{n(\frac{1}{p}-1)}_{p,q}(\mathbb{R}^n) \hookrightarrow L_1(\mathbb{R}^n)$ is proved for all $0 < p < 1$ and $0 < q \le 1$.

(ii) We also mention, that growth envelopes for function spaces with minimal smoothness were recently studied in [2]. These authors worked with spaces defined by differences and their results differ from ours in logarithmic factors. This shows indirectly, that the Fourier-analytical definition and the classical definition of Besov spaces do not coincide for $s = 0$ - an effect observed in detail recently by Schneider [9].

We denote the Lebesgue and Lorentz spaces by $L_p(\mathbb{R}^n)$ and $L_{p,q}(\mathbb{R}^n)$, respectively. The reader may consult [13, Chapter 5, Section 3] or [1, Chapter 4, Section 4]. We shall use the following well known property of Lorentz spaces $L_{1,q}$. It's proof follows immediately from Hardy's lemma (cf. [1, Chapter 2, Proposition 3.6]).

Lemma 1.4. *Let $0 < q < 1$. Then the $|| \cdot |L_{1,q}(\mathbb{R}^n)||$ is the q-norm, it means*

$$||f_1 + f_2|L_{1,q}(\mathbb{R}^n)||^q \le ||f_1|L_{1,q}(\mathbb{R}^n)||^q + ||f_2|L_{1,q}(\mathbb{R}^n)||^q$$

holds for all $f_1, f_2 \in L_{1,q}(\mathbb{R}^n)$.

We work with Fourier-analytically defined Besov and Triebel-Lizorkin spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ as studied for example in [8], [14], [15] and [17]. We shall also use the sequence spaces b^s_{pq} associated to $B^s_{p,q}(\mathbb{R}^n)$ in a way described in [17, Chapters 2 and 3]. This approach goes back to [5] and [6].

All the unimportant constants are denoted by the letter c, whose meaning may differ from one occurrence to another. If $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ are two sequences of positive real numbers, we write $a_n \lesssim b_n$ if, and only if, there is a positive real number $c > 0$ such that $a_n \le c\, b_n, n \in \mathbb{N}$. Furthermore, $a_n \approx b_n$ means that $a_n \lesssim b_n$ and simultaneously $b_n \lesssim a_n$.

2 Proofs of the main results

2.1 Proof of Theorem 1.1 (i)

In view of (4), it is enough to prove, that for $1 \le p < \infty$ and $0 < q \le \min(p, 2)$ the index u associated to $B^0_{p,q}(\mathbb{R}^n)$ is greater or equal to p.

We assume in contrary that (3) is fulfilled for some $0 < v < p$, $\epsilon > 0$, $c > 0$ and all $f \in B^0_{p,q}(\mathbb{R}^n)$. Let ψ be a non-vanishing C^∞ function in $\mathbb{R}^n$ supported in $[0,1]^n$ with $\int_{\mathbb{R}^n} \psi(x)dx = 0$.

Let $J \in \mathbb{N}$ be such that $2^{-Jn} < \epsilon$ and consider the function

$$f_j = \sum_{m=1}^{2^{(j-J)n}} \lambda_{jm} \psi(2^j(x - (m, 0, \dots, 0))), \quad j > J, \tag{6}$$

where

$$\lambda_{jm} = \frac{1}{m^{\frac{1}{p}} \log^{\frac{1}{v}}(m+1)}, \quad m = 1, \dots, 2^{(j-J)n}.$$

Then (6) represents an atomic decomposition of f in the space $B^0_{p,q}(\mathbb{R}^n)$ according to [17, Chapter 1.5] and we obtain (recall that $v < p$)

$$\begin{aligned} ||f_j | B^0_{p,q}(\mathbb{R}^n)|| &\lesssim 2^{-j\frac{n}{p}} \left(\sum_{m=1}^{2^{(j-J)n}} \lambda^p_{jm} \right)^{1/p} \\ &\le 2^{-j\frac{n}{p}} \left(\sum_{m=1}^{\infty} m^{-1} (\log(m+1))^{-\frac{p}{v}} \right)^{1/p} \lesssim 2^{-j\frac{n}{p}}. \end{aligned} \tag{7}$$

On the other hand,

$$\begin{aligned} &\left(\int_0^\epsilon \left[f^*_j(t) t^{\frac{1}{p}} \right]^v \frac{dt}{t} \right)^{1/v} \ge \left(\int_0^{2^{-Jn}} f^*_j(t)^v t^{v/p-1} dt \right)^{1/v} \\ &\gtrsim \left(\sum_{m=1}^{2^{(j-J)n}} \lambda^v_{jm} \int_{c\,2^{-jn}(m-1)}^{c\,2^{-jn}m} t^{v/p-1} dt \right)^{1/v} \gtrsim \left(\sum_{m=1}^{2^{(j-J)n}} \lambda^v_{jm} 2^{-jnv/p} m^{v/p-1} \right)^{1/v} \\ &= 2^{-j\frac{n}{p}} \left(\sum_{m=1}^{2^{(j-J)n}} \frac{1}{m \log(m+1)} \right)^{1/v}. \end{aligned}$$

As the last series is divergent for $j \to \infty$, this is in a contradiction with (7) and (3) cannot hold for all $f_j, j > J$.

Remark 2.1. Observe, that Theorem 1.2 (i) is a direct consequence of Theorem 1.1 (i). The embeddings $B^0_{1,q}(\mathbb{R}^n) \hookrightarrow B^0_{1,1}(\mathbb{R}^n) \hookrightarrow L_1(\mathbb{R}^n)$ if $p = 1$ and $B^0_{p,q}(\mathbb{R}^n) \hookrightarrow F^0_{p,2}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ if $1 < p < \infty$ show, that $B^0_{p,q}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n)$. And Theorem 1.1 (i) implies that if $B^0_{p,q}(\mathbb{R}^n) \hookrightarrow L_{p,v}(\mathbb{R}^n)$ for some $0 < v < \infty$, then $p \le v$. This proves the optimality of Theorem 1.2 (i) in the frame of the scale of Lorentz spaces.

2.2 Proof of Theorem 1.1 (ii) and Theorem 1.2 (ii)

Let $0 < p < 1$, $0 < q \le 1$ and $s = \sigma_p = n\left(\frac{1}{p} - 1\right)$. We prove first Theorem 1.2 (ii), i.e. we show that

$$B^{\frac{n}{p}-n}_{p,q}(\mathbb{R}^n) \hookrightarrow L_{1,q}(\mathbb{R}^n),$$

or, equivalently,

$$\left(\int_0^\infty [tf^*(t)]^q \frac{dt}{t}\right)^{1/q} \le c\,||f|B^{\frac{n}{p}-n}_{p,q}(\mathbb{R}^n)||, \qquad f \in B^{\frac{n}{p}-n}_{pq}(\mathbb{R}^n).$$

Let

$$f = \sum_{j=0}^\infty f_j = \sum_{j=0}^\infty \sum_{m\in\mathbb{Z}^n} \lambda_{jm} a_{jm}$$

be the optimal atomic decomposition of an $f \in B^{\frac{n}{p}-n}_{p,q}(\mathbb{R}^n)$, again in the sense of [17, Chapter 1.5]. Then

$$||f|B^{\frac{n}{p}-n}_{p,q}(\mathbb{R}^n)|| \approx \left(\sum_{j=0}^\infty 2^{-jqn}\left(\sum_{m\in\mathbb{Z}^n} |\lambda_{jm}|^p\right)^{q/p}\right)^{1/q} \tag{8}$$

and by Lemma 1.4

$$||f|L_{1,q}(\mathbb{R}^n)|| = ||\sum_{j=0}^\infty f_j|L_{1,q}(\mathbb{R}^n)|| \le \left(\sum_{j=0}^\infty ||f_j|L_{1,q}(\mathbb{R}^n)||^q\right)^{1/q}. \tag{9}$$

We shall need only one property of the atoms a_{jm}, namely that their support is contained in the cube $\tilde{Q}_{jm}$ - a cube centred at the point $2^{-j}m$ with sides parallel to the coordinate axes and side length $\alpha 2^{-j}$, where $\alpha > 1$ is fixed and independent of f. We denote by $\tilde{\chi}_{jm}(x)$ the characteristic functions of $\tilde{Q}_{jm}$ and by χ_{jl} the characteristic function of the interval $(l2^{-jn}, (l+1)2^{-jn})$. Hence

$$f_j(x) \le c \sum_{m\in\mathbb{Z}^n} |\lambda_{jm}|\tilde{\chi}_{jm}(x), \quad x \in \mathbb{R}^n$$

and

$$\begin{aligned}
||f_j|L_{1,q}(\mathbb{R}^n)|| &\lesssim \left(\int_0^\infty \sum_{l=0}^\infty [(\lambda_j)^*_l \chi_{jl}(t)]^q\, t^{q-1} dt\right)^{1/q} \\
&\le \left(\sum_{l=0}^\infty [(\lambda_j)^*_l]^q \int_{2^{-jn}l}^{2^{-jn}(l+1)} t^{q-1} dt\right)^{1/q} \\
&\lesssim 2^{-jn}\left(\sum_{l=0}^\infty [(\lambda_j)^*_l]^q\,(l+1)^{q-1}\right)^{1/q} \lesssim 2^{-jn}||\lambda_j|\ell_p||.
\end{aligned} \tag{10}$$

The last inequality follows by $(l+1)^{q-1} \leq 1$ and $\ell_p \hookrightarrow \ell_q$ if $p \leq q$. If $p > q$, the same follows by Hölder's inequality with respect to indices $\alpha = \frac{p}{q}$ and $\alpha' = \frac{p}{p-q}$:

$$\left(\sum_{l=0}^{\infty} [(\lambda_j)_l^*]^q \, (l+1)^{q-1}\right)^{1/q}$$
$$\leq \left(\sum_{l=0}^{\infty} [(\lambda_j)_l^*]^{q\cdot\frac{p}{q}}\right)^{\frac{1}{q}\cdot\frac{q}{p}} \cdot \left(\sum_{l=0}^{\infty} (l+1)^{(q-1)\cdot\frac{p}{p-q}}\right)^{\frac{1}{q}\cdot\frac{p-q}{p}} \leq c\,||\lambda_j|\ell_p||.$$

Here, we used that for $0 < q < p < 1$ the exponent $\frac{(q-1)p}{p-q} = -1 + \frac{(p-1)q}{p-q}$ is strictly smaller than -1.

The proof now follows by (8), (9) and (10).

$$||f|L_{1,q}(\mathbb{R}^n)|| \leq \left(\sum_{j=0}^{\infty} ||f_j|L_{1,q}(\mathbb{R}^n)||^q\right)^{1/q} \leq c\left(\sum_{j=0}^{\infty} 2^{-jnq}||\lambda_j|\ell_p||^q\right)^{1/q}$$
$$\leq c\,||f|B^{\sigma_p}_{p,q}(\mathbb{R}^n)||.$$

Remark 2.2. We actually proved, that (3) holds for $X = B^{\frac{n}{p}-n}_{pq}(\mathbb{R}^n)$, $v = q$ and $\epsilon = \infty$. This, together with (4) implies immediately Theorem 1.1 (ii).

2.3 Proof of Theorem 1.1 (iii) and Theorem 1.2 (iii)

Let $0 < p < 1$ and $0 < q \leq \infty$. By the Jawerth embedding (cf. [7] or [18]) and Theorem 1.1 (ii) we get for any $0 < p < \tilde{p} < 1$

$$F^{\sigma_p}_{p,q}(\mathbb{R}^n) \hookrightarrow B^{\sigma_{\tilde{p}}}_{\tilde{p},p}(\mathbb{R}^n) \hookrightarrow L_{1,p}(\mathbb{R}^n).$$

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The Jawerth-Franke embedding of spaces with dominating mixed smoothness

Markus Hansen and Jan Vybíral

Abstract

We give a proof of the Jawerth embedding for function spaces with dominating mixed smoothness of Besov and Triebel-Lizorkin type

$$S^{\overline{r}^0}_{p_0,q_0}F(\mathbb{R}^{d_1}\times\cdots\times\mathbb{R}^{d_N}) \hookrightarrow S^{\overline{r}^1}_{p_1,p_0}B(\mathbb{R}^{d_1}\times\cdots\times\mathbb{R}^{d_N}),$$

where

$$0<p_0<p_1\le\infty \text{ and } 0<q_0,q_1\le\infty$$

and

$$\overline{d}=(d_1,\dots,d_N)\in\mathbb{N}^N, \overline{r}^i=(r^i_1,\dots,r^i_N)\in\mathbb{R}^N, i=0,1$$

with

$$r^0_i-\frac{d_i}{p_0}=r^1_i-\frac{d_i}{p_1},\quad i=1,\dots,N.$$

If $p_1<\infty$, we prove also the Franke embedding

$$S^{\overline{r}^0}_{p_0,p_1}B(\mathbb{R}^{d_1}\times\cdots\times\mathbb{R}^{d_N}) \hookrightarrow S^{\overline{r}^1}_{p_1,q_1}F(\mathbb{R}^{d_1}\times\cdots\times\mathbb{R}^{d_N}).$$

Our main tools are discretization by a wavelet isomorphism and multivariate rearrangements.

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1 Introduction and main results

1.1 Introduction

Our aim is to study function spaces with dominating mixed smoothness properties. These spaces were first defined by S. M. Nikol'skij in [18] and [19]. He introduced the spaces of Sobolev type

$$S_p^{\overline{r}}W(\mathbb{R}^2) = \Big\{ f \in L_p(\mathbb{R}^2) : ||f|S_p^{\overline{r}}W(\mathbb{R}^2)|| = ||f|L_p(\mathbb{R}^2)|| + \Big|\Big| \frac{\partial^{r_1} f}{\partial x_1^{r_1}} \mid L_p(\mathbb{R}^2) \Big|\Big| + \\ + \Big|\Big| \frac{\partial^{r_2} f}{\partial x_2^{r_2}} \mid L_p(\mathbb{R}^2) \Big|\Big| + \Big|\Big| \frac{\partial^{r_1+r_2} f}{\partial x_1^{r_1} \partial x_2^{r_2}} \mid L_p(\mathbb{R}^2) \Big|\Big| < \infty \Big\},$$

where $1 < p < \infty, r = (r_1, r_2) \in \mathbb{N}_0^2$. The mixed derivative $\frac{\partial^{r_1+r_2} f}{\partial x_1^{r_1} \partial x_2^{r_2}}$ plays the dominant part here and gave the name to this class of spaces.

We prefer to work with the following more general version. Namely, let $N \geq 2$ be a natural number and let $d_1, \ldots, d_N$ be natural numbers. We set $\overline{d} = (d_1, \ldots, d_N)$ and $d = d_1 + \cdots + d_N$. Let further $\overline{r} = (r_1, \ldots, r_N) \in \mathbb{N}_0^N$ and $1 < p < \infty$. Then

$$S_p^{\overline{r}}W(\mathbb{R}^{\overline{d}}) = S_p^{\overline{r}}W(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) = \Big\{ f \in L_p(\mathbb{R}^d) : ||D^\alpha f|L_p(\mathbb{R}^d)|| < \infty \\ \text{for all} \quad \alpha = (\alpha_1, \ldots, \alpha_N), \alpha_i \in \mathbb{N}_0^{d_i} \quad \text{and} \quad |\alpha_i| \leq r_i \quad \text{for} \quad i = 1, \ldots, N \Big\}.$$

The spaces of this type found many applications in connection with partial differential equations ([18], [19], [37], [17], [38]), approximation theory ([28], [29], [33], [27]), information based complexity ([36], [20]) and other areas of mathematics. The reader may consult the survey [22] for more references.

The Fourier-analytic approach to these function spaces is based on the so-called *decomposition of unity.*

Let $\varphi \in S(\mathbb{R}^n)$ be from the Schwartz-space of smooth rapidly decreasing functions with

$$\varphi(x) = 1 \qquad \text{if } |x| \leq 4/3 \quad \text{and} \quad \varphi(x) = 0 \qquad \text{if } |x| \geq 3/2.$$

We put $\varphi_0 = \varphi$, $\varphi_1 = \varphi(\cdot/2) - \varphi$ and

$$\varphi_j(x) = \varphi_1(2^{-j+1}x), \qquad x \in \mathbb{R}^n, j \in \mathbb{N}.$$

We observe, that the system $\{\varphi_j\}_{j=0}^\infty$ satisfies

$$\sum_{j=0}^{\infty} \varphi_j(t) = 1 \quad \text{for all } t \in \mathbb{R}^n. \tag{1.1}$$

Let $N \geq 2$ be again a natural number and let $d_1, \ldots d_N$ be natural numbers. We define d and $\overline{d}$ as above. For $i = 1, \ldots, N$ we define $\{\varphi_j^i\}_{j=0}^\infty \subset S(\mathbb{R}^{d_i})$ as described above and put for $\overline{k} = (k_1, \ldots, k_N) \in \mathbb{N}_0^N$ and $x = (x^1, \ldots, x^N) \in \mathbb{R}^d$

$$\varphi_{\overline{k}}(x) := \varphi^1_{k_1}(x^1) \cdots \varphi^N_{k_N}(x^N). \tag{1.2}$$

As

$$\sum_{\overline{k}\in\mathbb{N}_0^N} \varphi_{\overline{k}}(x) = \left(\sum_{k_1=0}^{\infty} \varphi_{k_1}(x^1)\right) \cdots \left(\sum_{k_N=0}^{\infty} \varphi_{k_N}(x^N)\right) = 1$$

for all $x = (x^1, \ldots, x^N) \in \mathbb{R}^d$, we see that $\{\varphi_{\overline{k}}\}_{\overline{k}\in\mathbb{N}_0^N}$ forms also a decomposition of unity on $\mathbb{R}^d$ with the tensor product structure.

We denote by $\widehat{f}$ the Fourier transform of a distribution $f \in S'(\mathbb{R}^d)$ and by $f^\vee$ its inverse transform.

Definition 1.1. Let $\overline{r} \in \mathbb{R}^N$, $0 < q \le \infty$ and $\varphi = \{\varphi_{\overline{k}}\}_{\overline{k}\in\mathbb{N}_0^N}$ be as above.

1. Let $0 < p \le \infty$. Then $S^{\overline{r}}_{p,q}B(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N})$ is the set of all $f \in S'(\mathbb{R}^d)$, such that

$$\left\| f \middle| S^{\overline{r}}_{p,q}B(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) \right\|_{\varphi} := \left(\sum_{\overline{k}\in\mathbb{N}_0^N} 2^{\overline{k}\cdot\overline{r}q} \left\| \left(\varphi_{\overline{k}}\widehat{f}\right)^\vee \middle| L_p(\mathbb{R}^d) \right\|^q \right)^{1/q} \tag{1.3}$$

is finite.

2. Let $0 < p < \infty$. Then $S^{\overline{r}}_{p,q}F(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N})$ is the set of all $f \in S'(\mathbb{R}^d)$, such that

$$\left\| f \middle| S^{\overline{r}}_{p,q}F(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) \right\|_{\varphi} := \left\| \left(\sum_{\overline{k}\in\mathbb{N}_0^N} 2^{\overline{k}\cdot\overline{r}q} \left| \left(\varphi_{\overline{k}}\widehat{f}\right)^\vee(\cdot) \right|^q \right)^{1/q} \middle| L_p(\mathbb{R}^d) \right\| \tag{1.4}$$

is finite.

Let us mention, that (1.3) and (1.4) lead to equivalent quasi-norms for different choices of $\{\varphi_{\overline{k}}\}$. If $d_1 = d_2 = \cdots = d_N$, then this and other basic aspects of the theory of function spaces with dominating mixed smoothness may be found in [1], [24], [2] or [34]. We refer to [10] for the general case. To shorten the notation, we write sometimes $S^{\overline{r}}_{p,q}B(\mathbb{R}^{\overline{d}})$ instead of $S^{\overline{r}}_{p,q}B(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N})$ and similar in the $F-$case.

One of the main features of the classes $S^{\overline{r}}_{p,q}B(\mathbb{R}^{\overline{d}})$ and $S^{\overline{r}}_{p,q}F(\mathbb{R}^{\overline{d}})$ consists in the fact, that their quasi-norms are cross-quasi-norms, i.e. if

$$f = (f_1 \otimes \cdots \otimes f_N),$$

where $f_i \in S'(\mathbb{R}^{d_i}), i = 1, \ldots, N$ and f is a tensor product of tempered distributions in the sense of [25, Chapters IV and VII] or [12, Chapter X], then

$$||f_1 \otimes \cdots \otimes f_N | S^{\overline{r}}_{p,q}B(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N})|| = \prod_{i=1}^{N} ||f_i | B^{r_i}_{p,q}(\mathbb{R}^{d_i})|| \tag{1.5}$$

and

$$||f_1 \otimes \cdots \otimes f_N | S^{\overline{r}}_{p,q} F(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N})|| = \prod_{i=1}^{N} ||f_i | F^{r_i}_{p,q}(\mathbb{R}^{d_i})|| \tag{1.6}$$

where $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ are the Fourier-analytic Besov spaces and Triebel-Lizorkin spaces, respectively.

1.2 Main result

Our main result is the following theorem:

Theorem 1.2. *Let $\overline{r}^0, \overline{r}^1 \in \mathbb{R}^N$, $0 < p_0 < p_1 \le \infty$ and $0 < q_0, q_1 \le \infty$ with*

$$r^0_j - \frac{d_j}{p_0} = r^1_j - \frac{d_j}{p_1}, \quad j = 1, \ldots, N. \tag{1.7}$$

1. *Then*
$$S^{\overline{r}^0}_{p_0,q_0} F(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) \hookrightarrow S^{\overline{r}^1}_{p_1,q_1} B(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) \tag{1.8}$$
if, and only if, $p_0 \le q_1$.

2. *If $p_1 < \infty$, then*
$$S^{\overline{r}^0}_{p_0,q_0} B(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) \hookrightarrow S^{\overline{r}^1}_{p_1,q_1} F(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) \tag{1.9}$$
if, and only if, $q_0 \le p_1$.

Remark 1.3. (i) The original proofs in the isotropic case, cf. [13] and [9], use interpolation techniques. This approach was applied in [23] also to function spaces with dominating mixed smoothness but (although these authors succeeded to overcome numerous obstacles) led only to partial results.

Here, we shall use a different method of proof, originally introduced in [35] to prove Theorem 1.2 in the isotropic situation.

(ii) Embeddings of Jawerth-Franke type have been proved already for several other scales of function spaces of Besov and Triebel-Lizorkin type. We refer to [8, Appendix C.3] for anisotropic case, to [6] and [11] for weighted function spaces and to [5] and [7] for spaces with generalised smoothness. In general, all these authors used the method of Jawerth and Franke and we believe that in all these cases one could apply our approach as well.

(iii) Embedding (1.8) was already obtained by Krbec and Schmeisser (cf. [15, Lemma 4.7]) in the special case $N = 2$ and $p_1 = \infty$. Furthermore, Schmeisser and Sickel (cf. [23, Theorem 3]) proved (1.9) in the Banach space setting, i.e. $1 \le p_0 < p_1 < \infty$ and $1 \le q_0, q_1 \le \infty$. The use of duality arguments allowed to prove also (1.8) but only for $1 < p_0 < \infty$. Our approach yields the proof of Theorem 1.2 without any further restrictions on the parameters.

(iv) For applications the following reformulation of Theorem 1.2 might be useful.

Theorem 1.4. *Let* $\overline{r}^0, \overline{r}, \overline{r}^1 \in \mathbb{R}^N$, $0 < p_0 < p < p_1 \leq \infty$ *with*

$$r_j^0 - \frac{d_j}{p_0} = r_j - \frac{d_j}{p} = r_j^1 - \frac{d_j}{p_1}, \quad j = 1, \ldots, N.$$

Let $0 < q, u, v \leq \infty$. *Then*

$$S^{\overline{r}^0}_{p_0,u}B(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) \hookrightarrow S^{\overline{r}}_{p,q}F(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) \hookrightarrow S^{\overline{r}^1}_{p_1,v}B(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N})$$

if, and only if, $0 < u \leq p \leq v \leq \infty$.

1.3 Further consequences

Let $C(\mathbb{R}^d)$ be the space of all complex-valued bounded and uniformly continuous functions on $\mathbb{R}^d$. One of the well studied problems in the isotropic case is the embedding of Besov and Triebel-Lizorkin spaces into $C(\mathbb{R}^d)$ or $L_r(\mathbb{R}^d)$ with $1 \leq r \leq \infty$. This problem is connected with the works of Grisvard, Peetre, Golovkin, Stein, Zygmund, Besov or Iljin. We refer to [4] and [26] for details.

We use (1.8) to characterize those spaces $S^{\overline{r}}_{p,q}B(\mathbb{R}^{\overline{d}})$ and $S^{\overline{r}}_{p,q}F(\mathbb{R}^{\overline{d}})$ which are embedded in $C(\mathbb{R}^d)$ and $L_u(\mathbb{R}^d)$, $1 < u \leq \infty$. This approach was applied already in [16], cf. also [22]. Unfortunately, there was a flaw in the arguments used in [16].

Theorem 1.5. *(i) Let* $\overline{r} \in \mathbb{R}^N$, $0 < p \leq \infty$ *and* $0 < q \leq \infty$. *Then the following three assertions are equivalent.*

(a) $\quad S^{\overline{r}}_{p,q}B(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) \hookrightarrow C(\mathbb{R}^d)$,

(b) $\quad S^{\overline{r}}_{p,q}B(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) \hookrightarrow L_\infty(\mathbb{R}^d)$,

$$(c) \quad \begin{cases} r_i - \dfrac{d_i}{p} > 0 & \textit{for all} \quad i = 1, \ldots, N \quad \textit{or} \\ r_i - \dfrac{d_i}{p} \geq 0 & \textit{for all} \quad i = 1, \ldots, N \quad \textit{and} \quad 0 < q \leq 1. \end{cases}$$

(ii) Let $\overline{r} \in \mathbb{R}^N$, $0 < p < \infty$ *and* $0 < q \leq \infty$. *Then the following three assertions are equivalent.*

(a') $\quad S^{\overline{r}}_{p,q}F(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) \hookrightarrow C(\mathbb{R}^d)$,

(b') $\quad S^{\overline{r}}_{p,q}F(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) \hookrightarrow L_\infty(\mathbb{R}^d)$,

$$(c') \quad \begin{cases} r_i - \dfrac{d_i}{p} > 0 & \textit{for all} \quad i = 1, \ldots, N \quad \textit{or} \\ r_i - \dfrac{d_i}{p} \geq 0 & \textit{for all} \quad i = 1, \ldots, N \quad \textit{and} \quad 0 < p \leq 1. \end{cases}$$

We consider a similar problem also for L_u, $1 < u < \infty$. Due to the Littlewood-Paley theory the number 2 plays an exceptional role if $1 < u < \infty$.

Theorem 1.6. *(i) Let* $\overline{r} \in \mathbb{R}^N$, $1 < u < \infty$, $0 < p \leq \infty$ *and* $0 < q \leq \infty$. *Then* $S^{\overline{r}}_{p,q}B(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) \hookrightarrow L_u(\mathbb{R}^d)$ *if, and only if,* $p \leq u$ *and*

$$\begin{cases} r_i > \dfrac{d_i}{p} - \dfrac{d_i}{u} \quad \textit{for all} \quad i = 1, \ldots, N \quad \textit{or} \\ r_i \geq \dfrac{d_i}{p} - \dfrac{d_i}{u} \quad \textit{for all} \quad i = 1, \ldots, N, \quad 0 < p < u \quad \textit{and} \quad 0 < q \leq u \quad \textit{or} \\ r_i \geq 0 \quad \textit{for all} \quad i = 1, \ldots, N, \quad p = u \quad \textit{and} \quad 0 < q \leq \min(u, 2). \end{cases}$$

(ii) Let $\overline{r} \in \mathbb{R}^N$, $1 < u < \infty$, $0 < p < \infty$ *and* $0 < q \leq \infty$. *Then* $S^{\overline{r}}_{p,q}F(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) \hookrightarrow L_u(\mathbb{R}^d)$ *if, and only if,* $p \leq u$ *and*

$$\begin{cases} r_i > \dfrac{d_i}{p} - \dfrac{d_i}{u} \quad \textit{for all} \quad i = 1, \ldots, N \quad \textit{or} \\ r_i \geq \dfrac{d_i}{p} - \dfrac{d_i}{u} \quad \textit{for all} \quad i = 1, \ldots, N \quad \textit{and} \quad 0 < p < u \quad \textit{or} \\ r_i \geq 0 \quad \textit{for all} \quad i = 1, \ldots, N, \quad p = u \quad \textit{and} \quad 0 < q \leq 2. \end{cases}$$

Remark 1.7. Let $1 < u \leq \infty$. Direct comparison of Theorems 1.5-1.6 with similar assertions for isotropic Besov and Triebel-Lizorkin spaces (cf. [26]) shows, that $S^{\overline{r}}_{p,q}B(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) \hookrightarrow L_u(\mathbb{R}^d)$ if, and only if, $B^{r_i}_{p,q}(\mathbb{R}^{d_i}) \hookrightarrow L_u(\mathbb{R}^{d_i})$ for all $i = 1, \ldots, N$. The same statement holds true, if $L_\infty(\mathbb{R}^d)$ is replaced by $C(\mathbb{R}^d)$ and also for the Triebel-Lizorkin spaces.

Remark 1.8. Also the optimal embeddings into $L_1(\mathbb{R}^n)$ and $L_1^{\mathrm{loc}}(\mathbb{R}^n)$ - the space of locally integrable functions - are very well known in the isotropic case. To extend these results to function spaces with dominating mixed smoothness, it would be probably necessary to consider the analog of the Hardy space H^1 and of the space of bounded mean oscillation BMO in the framework of dominating mixed smoothness. But this goes beyond the scope of this work.

2 Proofs

2.1 Preliminaries

Our approach is based on two classical techniques - decomposition theorems and multivariate rearrangements.

First, we describe the sequence spaces associated to $S^{\overline{r}}_{p,q}B(\mathbb{R}^{\overline{d}})$ and $S^{\overline{r}}_{p,q}F(\mathbb{R}^{\overline{d}})$. Let $m \in \mathbb{Z}^d$, $m = (m^1, \ldots, m^N)$ with $m^i \in \mathbb{Z}^{d_i}$, and $\overline{\nu} \in \mathbb{N}_0^N$. Then $Q_{\overline{\nu},m}$ denotes the closed cube in $\mathbb{R}^d$ with sides parallel to the coordinate axes, centred at the point $2^{-\overline{\nu}}m = (2^{-\nu_1}m^1, \ldots, 2^{-\nu_N}m^N)$, and with sides of the lengths $2^{-\nu_1}, \ldots, 2^{-\nu_N}$. Explicitly,

$$Q_{\overline{\nu},m} = \{x \in \mathbb{R}^d : |x^i - 2^{-\nu_i}m^i|_\infty \leq 2^{-\nu_i - 1}, i = 1, \ldots, N\}, \tag{2.1}$$

where $x = (x^1, \ldots, x^N), x^i \in \mathbb{R}^{d_i}$, and $|t|_\infty = \max_{i=1,\ldots,n} |t_i|$, $t \in \mathbb{R}^n$. By $\chi_{\overline{\nu},m} = \chi_{Q_{\overline{\nu},m}}$ we denote the characteristic function of $Q_{\overline{\nu},m}$. If

$$\lambda = \{\lambda_{\overline{\nu},m} \in \mathbb{C} : \overline{\nu} \in \mathbb{N}_0^N, m \in \mathbb{Z}^d\},$$

$\overline{r} \in \mathbb{R}^N$ and $0 < p, q \leq \infty$, we set

$$\|\lambda|s^{\overline{r}}_{p,q}b\| = \left(\sum_{\overline{\nu}\in\mathbb{N}_0^N} 2^{\overline{\nu}\cdot(\overline{r}-\overline{d}/p)q}\left(\sum_{m\in\mathbb{Z}^d}|\lambda_{\overline{\nu},m}|^p\right)^{q/p}\right)^{1/q}, \tag{2.2}$$

appropriately modified if $p = \infty$ and/or $q = \infty$. If $p < \infty$, we define also

$$\|\lambda|s^{\overline{r}}_{p,q}f\| = \left\|\left(\sum_{\overline{\nu}\in\mathbb{N}_0^N}\sum_{m\in\mathbb{Z}^d}|2^{\overline{\nu}\cdot\overline{r}}\lambda_{\overline{\nu},m}|^q\chi_{\overline{\nu},m}(\cdot)\right)^{1/q}\Bigg|L_p(\mathbb{R}^d)\right\|. \tag{2.3}$$

Using the wavelet decomposition techniques, one may give linear isomorphisms between function spaces with dominating mixed smoothness properties and corresponding sequence spaces. We refer to [35] if $d_1 = d_2 = \cdots = d_N = 1$ and to [10] in the general case.

This allows to reduce the proof of Theorem 1.2 to the embeddings of sequence spaces. Hence, it is enough to prove that under condition (1.7)

$$s^{\overline{r}_0}_{p_0,q_0}f \hookrightarrow s^{\overline{r}_1}_{p_1,q_1}b \tag{2.4}$$

if, and only if, $p_0 \leq q_1$ and

$$s^{\overline{r}_0}_{p_0,q_0}b \hookrightarrow s^{\overline{r}_1}_{p_1,q_1}f, \tag{2.5}$$

if, and only if, $q_0 \leq p_1$.

Now, we present briefly the concept of non-increasing rearrangement. We refer to [3, Chapter 2] for details.

Definition 2.1. Let μ be the Lebesgue measure in $\mathbb{R}^n$. If h is a measurable function on $\mathbb{R}^n$, we define the *non-increasing rearrangement* of h through

$$h^*(t) = \sup\{\lambda > 0 : \mu\{x \in \mathbb{R}^n : |h(x)| > \lambda\} > t\}, \qquad t \in (0, \infty). \tag{2.6}$$

We shall need also the so-called multivariate rearrangements.

Let $f : (0,\infty)^{k-1} \times \mathbb{R}^{d_k} \times \cdots \times \mathbb{R}^{d_N} \to \mathbb{C}$, $k \leq N$, be a measurable function. We set

$$(R_k f)(t_1,\ldots,t_{k-1},s,y^{k+1},\ldots,y^N) = [f(t_1,\ldots,t_{k-1},\cdot,y^{k+1},\ldots,y^N)]^*(s),$$
$$s > 0, \quad t_1,\ldots,t_{k-1} \in (0,\infty), \quad y^i \in \mathbb{R}^{d_i}, i = k+1,\ldots,N.$$

We define the *multivariate non-increasing rearrangement* of $f : \mathbb{R}^d \to \mathbb{C}$ by

$$(Rf)(\overline{s}) = (R_N \circ \cdots \circ R_1 f)(\overline{s}), \quad \overline{s} = (s_1,\ldots,s_N) \in (0,\infty)^N.$$

The utility of multivariate rearrangements in connection with embeddings of Sobolev type has been discovered by Kolyada [14]. Later on, it was used by Krbec and Schmeisser [16] in connection with function spaces with dominating mixed smoothness.

We shall use the following two properties. They are well known in the scalar case $N = 1$ (cf. [3]) and may be easily generalised to $N > 1$.

Lemma 2.2. *If $0 < p \le \infty$, then*

$$||h|L_p(\mathbb{R}^d)|| = ||Rh|L_p((0,\infty)^N)||$$

for every measurable function h.

Lemma 2.3. *Let h_1 and h_2 be two non-negative measurable functions on $\mathbb{R}^d$. If $1 \le p \le \infty$, then*

$$||h_1 + h_2|L_p(\mathbb{R}^d)|| \le ||Rh_1 + Rh_2|L_p((0,\infty)^N)||.$$

If $g : (0,\infty)^N \to \mathbb{R}$ is measurable, we define the average operator $\mathcal{A}g$ by

$$(\mathcal{A}g)(\overline{t}) := \left(\prod_{i=1}^N t_i\right)^{-1} \int_{[0,\overline{t}]} |g(x)|dx \quad \text{for} \quad \overline{t} \in (0,\infty)^N.$$

The following property is also well known if $N = 1$, the generalisation to $N > 1$ follows by iteration.

Lemma 2.4. *If $1 < p \le \infty$, then there is a constant c_p such that*

$$||\mathcal{A}h|L_p((0,\infty)^N)|| \le c_p||h|L_p((0,\infty)^N)||$$

for every measurable function h defined on $(0,\infty)^N$.

2.2 Proof of Theorem 1.2

Step 1. Proof of (2.4)

We observe, that the operator

$$I_{\overline{r}} : \lambda_{\overline{\nu},m} \to \tilde{\lambda}_{\overline{\nu},m} = 2^{\overline{\nu}\cdot\overline{r}}\lambda_{\overline{\nu},m}, \quad \overline{\nu} \in \mathbb{N}_0^N, \quad m \in \mathbb{Z}^d$$

forms a linear isomorphism from $s^{\overline{r}^0}_{p,q}b$ onto $s^{\overline{r}_0-\overline{r}}_{p,q}b$, where $\overline{r}, \overline{r}^0 \in \mathbb{R}^N$ are arbitrary. The same statement holds for the f-spaces as well.

We combine this with the simple embedding

$$s^{\overline{r}}_{p,q_0}f \hookrightarrow s^{\overline{r}}_{p,q_1}f \quad \text{if} \quad 0 < q_0 \le q_1 \le \infty,$$

and hence it is enough to prove that

$$s^{\overline{r}}_{p_0,\infty}f \hookrightarrow s^0_{p_1,p_0}b, \tag{2.7}$$

where

$$r_i = d_i\left(\frac{1}{p_0} - \frac{1}{p_1}\right), \quad i = 1,\dots,N. \tag{2.8}$$

Let $\lambda \in s^{\overline{r}}_{p_0,\infty}f$. We set

$$h(x) = \sup_{\overline{\nu}\in\mathbb{N}_0^N} 2^{\overline{\nu}\cdot\overline{r}} \sum_{m\in\mathbb{Z}^d} |\lambda_{\overline{\nu},m}|\chi_{\overline{\nu},m}(x), \quad x \in \mathbb{R}^d. \tag{2.9}$$

Using this notation, we get

$$|\lambda_{\overline{\nu},m}| \leq 2^{-\overline{\nu}\cdot\overline{r}} \inf_{x\in Q_{\overline{\nu},m}} h(x), \quad \overline{\nu} \in \mathbb{N}_0^N, \quad m \in \mathbb{Z}^d$$

and

$$\big\|h\big|L_{p_0}(\mathbb{R}^d)\big\| = \big\|\lambda\big|s^{\overline{r}}_{p_0,\infty}f\big\| < \infty. \tag{2.10}$$

The main step of our calculation is the following estimate:

$$\sum_{m\in\mathbb{Z}^d} \inf_{x\in Q_{\overline{\nu},m}} h(x)^{p_1} \leq \sum_{\overline{k}\in\mathbb{N}^N} (Rh)(2^{-(\nu_1+1)d_1}k_1,\ldots,2^{-(\nu_N+1)d_N}k_N)^{p_1}, \quad \overline{\nu}\in\mathbb{N}_0^N, \tag{2.11}$$

for $0 < p_1 < \infty$ and

$$\sup_{m\in\mathbb{Z}^d} \inf_{x\in Q_{\overline{\nu},m}} h(x) \leq (Rh)(2^{-(\nu_1+1)d_1},\ldots,2^{-(\nu_N+1)d_N}) \tag{2.12}$$

for $p_1 = \infty$ and all $\overline{\nu}\in\mathbb{N}_0^N$.
We start with the case $p_1 = \infty$. To prove (2.12) we fix some $\overline{\nu}\in\mathbb{N}_0^N$. Then we make use of the fact, that the sets $Q_{\overline{\nu},m}$ have a product structure. Hence, they may be rewritten as

$$Q_{\overline{\nu},m} = Q_{\nu_1,m^1}\times\cdots\times Q_{\nu_N,m^N}.$$

Let $\varepsilon > 0$, and fix $x^2\in\mathbb{R}^{d_2},\ldots,x^N\in\mathbb{R}^{d_N}$. Then there is some $m_0^1\in\mathbb{Z}^{d_1}$, such that

$$\sup_{m^1\in\mathbb{Z}^{d_1}} \inf_{y\in Q_{\nu_1,m^1}} h(y,x^2,\ldots,x^N) < \inf_{y\in Q_{\nu_1,m_0^1}} h(y,x^2,\ldots,x^N)+\varepsilon. \tag{2.13}$$

Let us point out, that (2.10) implies that the supremum on the left-hand side of (2.13) is finite for almost every $(x^2,\ldots,x^N)\in\mathbb{R}^{d_2+\cdots+d_N}$.
Obviously,

$$h(x^1,x^2,\ldots,x^N) > \inf_{y\in Q_{\nu_1,m_0^1}} h(y,x^2,\ldots,x^N)-\varepsilon$$

holds for all $x^1\in Q_{\nu_1,m_0^1}$. This is a set of Lebesgue-measure $2^{-\nu_1d_1} > 2^{-(\nu_1+1)d_1}$. From this, it follows

$$\begin{aligned}(R_1h)(2^{-(\nu_1+1)d_1},x^2,\ldots,x^N) &\geq \inf_{y\in Q_{\nu_1,m_0^1}} h(y,x^2,\ldots,x^N)-\varepsilon\\ &\geq \sup_{m^1\in\mathbb{Z}^{d_1}} \inf_{y\in Q_{\nu_1,m^1}} h(y,x^2,\ldots,x^N)-2\varepsilon.\end{aligned}$$

With $\varepsilon\to 0$ we get

$$\begin{aligned}\sup_{k_1\in\mathbb{N}}(R_1h)(2^{-(\nu_1+1)d_1}k_1,x^2,\ldots,x^N) &= (R_1h)(2^{-(\nu_1+1)d_1},x^2,\ldots,x^N)\\ &\geq \sup_{m^1\in\mathbb{Z}^{d_1}} \inf_{y\in Q_{\nu_1,m^1}} h(y,x^2,\ldots,x^N).\end{aligned}$$

If we use the same argument for the function $(R_1h)(2^{-(\nu_1+1)d_1}, \cdot, x^3, \dots, x^N)$, we get

$$\begin{aligned}
&\sup_{k_1,k_2\in\mathbb{N}} (R_2\circ R_1h)(2^{-(\nu_1+1)d_1}k_1, 2^{-(\nu_2+1)d_2}k_2, x^3,\dots,x^N)\\
&\qquad\geq \sup_{m^2\in\mathbb{Z}^{d_2}} \inf_{y^2\in Q_{\nu_2,m^2}} (R_1h)(2^{-(\nu_1+1)d_1}, y^2, x^3,\dots,x^N)\\
&\qquad\geq \sup_{m^2\in\mathbb{Z}^{d_2}} \inf_{y^2\in Q_{\nu_2,m^2}} \sup_{m^1\in\mathbb{Z}^{d_1}} \inf_{y^1\in Q_{\nu_1,m^1}} h(y^1,y^2,x^3,\dots,x^N)\\
&\qquad\geq \sup_{m^2\in\mathbb{Z}^{d_2},m^1\in\mathbb{Z}^{d_1}} \inf_{y^2\in Q_{\nu_2,m^2},y^1\in Q_{\nu_1,m^1}} h(y^1,y^2,x^3,\dots,x^N).
\end{aligned}$$

Further iteration yields

$$\begin{aligned}
&\sup_{\overline{k}\in\mathbb{N}^N} (Rh)(2^{-(\nu_1+1)d_1}k_1,\dots,2^{-(\nu_N+1)d_N}k_N)\\
&\quad\geq \sup_{m^N\in\mathbb{Z}^{d_N},\dots,m^1\in\mathbb{Z}^{d_1}} \inf_{y^N\in Q_{\nu_N,m^N},\dots,y^1\in Q_{\nu_1,m^1}} h(y^1,\dots,y^N) = \sup_{m\in\mathbb{Z}^d} \inf_{y\in Q_{\overline{\nu},m}} h(y),
\end{aligned}$$

and (2.12) is proven.

If $0<p_1<\infty$ then (2.11) may be proved by similar arguments, but we prefer to present an alternative way. We shall use the abbreviation $\eta_m := \inf_{x\in Q_{\overline{\nu},m}} h(x)$.

By (2.9) and (2.10) we have

$$0\leq \eta_m<\infty, \quad m\in\mathbb{Z}^d.$$

As the interiors of $Q_{\overline{\nu},m}$ are mutually disjoint, we may define a new function $\tilde{h}$ by

$$\tilde{h}(y)=\eta_m, \quad y\in interior(Q_{\overline{\nu},m}) \quad \text{and} \quad \tilde{h}(y)=0 \quad \text{if} \quad y\in boundary(Q_{\overline{\nu},m}).$$

One observes immediately, that $0\leq\tilde{h}(x)\leq h(x)$ for $x\in\mathbb{R}^{\overline{d}}$ and therefore $(R\tilde{h})(t)\leq(Rh)(t)$ for $t\in(0,\infty)^N$.

It follows, that

$$\left(\sum_{m\in\mathbb{Z}^d} \inf_{x\in Q_{\overline{\nu},m}} h(x)^{p_1}\right)^{1/p_1} = 2^{\overline{\nu}\cdot\overline{d}/p_1}||\tilde{h}|L_{p_1}(\mathbb{R}^d)|| = 2^{\overline{\nu}\cdot\overline{d}/p_1}||R\tilde{h}|L_{p_1}((0,\infty)^N)||.$$

As $\tilde{h}$ is constant on the cubes $Q_{\overline{\nu},m}$, $R\tilde{h}$ is constant on the cubes $Q'_{\overline{\nu},\overline{m}}$ with vertices in $(2^{-\nu_1d_1}m_1,\dots,$
$2^{-\nu_Nd_N}m_N)$ and $(2^{-\nu_1d_1}(m_1+1),\dots,2^{-\nu_Nd_N}(m_N+1))$ and sides parallel to the coordinate axes. Here, $\overline{m}=(m_1,\dots,m_N)\in\mathbb{N}_0^N$.

Hence

$$\begin{aligned}
2^{\overline{\nu}\cdot\overline{d}/p_1}||R\tilde{h}|L_{p_1}((0,\infty)^N)|| &\leq \left(\sum_{\overline{k}\in\mathbb{N}^N}(R\tilde{h})(2^{-(\nu_1+1)d_1}k_1,\dots,2^{-(\nu_N+1)d_N}k_N)^{p_1}\right)^{1/p_1}\\
&\leq \left(\sum_{\overline{k}\in\mathbb{N}^N}(Rh)(2^{-(\nu_1+1)d_1}k_1,\dots,2^{-(\nu_N+1)d_N}k_N)^{p_1}\right)^{1/p_1}
\end{aligned}$$

and (2.11) follows.

Now we are ready to give the proof of (2.4). Using condition (1.7) we obtain $\overline{r}p_0 + \overline{d}p_0/p_1 = \overline{d}$ and hence

$$
\begin{aligned}
\big\|\lambda\big|s^0_{p_1,p_0}b\big\|^{p_0} &\leq \sum_{\overline{\nu}\in\mathbb{N}_0^N} 2^{-\overline{\nu}\cdot\overline{d}} \left(\sum_{\overline{m}\in\mathbb{Z}^d} \inf_{x\in Q_{\overline{\nu}\,\overline{m}}} h(x)^{p_1}\right)^{\frac{p_0}{p_1}} \\
&\leq \sum_{\overline{\nu}\in\mathbb{N}_0^N} 2^{-\overline{\nu}\cdot\overline{d}} \left(\sum_{\overline{k}\in\mathbb{N}^N} (Rh)(2^{-(\nu_1+1)d_1}k_1,\dots,2^{-(\nu_N+1)d_N}k_N)^{p_1}\right)^{\frac{p_0}{p_1}} \\
&\leq \sum_{\overline{\nu}\in\mathbb{N}_0^N} 2^{-\overline{\nu}\cdot\overline{d}} \left(\sum_{\overline{l}\in\mathbb{N}_0^N} \sum_{\substack{\overline{k}\in\mathbb{N}^N:\\ \forall i: 2^{l_i d_i}\leq k_i < 2^{(l_i+1)d_i}}} (Rh)(2^{-(\nu_1+1)d_1}k_1,\dots,2^{-(\nu_N+1)d_N}k_N)^{p_1}\right)^{\frac{p_0}{p_1}} \\
&\lesssim \sum_{\overline{\nu}\in\mathbb{N}_0^N} 2^{-\overline{\nu}\cdot\overline{d}} \left(\sum_{\overline{l}\in\mathbb{N}_0^N} 2^{\overline{l}\cdot\overline{d}} (Rh)(2^{(l_1-\nu_1-1)d_1},\dots,2^{(l_N-\nu_N-1)d_N})^{p_1}\right)^{\frac{p_0}{p_1}} \\
&\leq \sum_{\overline{\nu}\in\mathbb{N}_0^N} 2^{-\overline{\nu}\cdot\overline{d}} \sum_{\overline{l}\in\mathbb{N}_0^N} 2^{\overline{l}\cdot\overline{d}\frac{p_0}{p_1}} (Rh)(2^{(l_1-\nu_1-1)d_1},\dots,2^{(l_N-\nu_N-1)d_N})^{p_0}.
\end{aligned}
$$

We substitute $\overline{n} = \overline{l} - \overline{\nu} - 1$ and find

$$
\begin{aligned}
\big\|\lambda\big|s^0_{p_1,p_0}b\big\|^{p_0} &\leq \sum_{\overline{\nu}\in\mathbb{N}_0^N} 2^{-\overline{\nu}\cdot\overline{d}} \sum_{\overline{n}\in\mathbb{Z}^N:\overline{n}+\overline{\nu}+1\in\mathbb{N}_0^N} 2^{(\overline{n}+\overline{\nu}+1)\cdot\overline{d}\frac{p_0}{p_1}} (Rh)(2^{n_1d_1},\dots,2^{n_Nd_N})^{p_0} \\
&\leq 2^{d\frac{p_0}{p_1}} \sum_{\overline{n}\in\mathbb{Z}^N} 2^{\overline{n}\cdot\overline{d}\frac{p_0}{p_1}} (Rh)(2^{n_1d_1},\dots,2^{n_Nd_N})^{p_0} \sum_{\overline{\nu}\in\mathbb{Z}^N:\overline{\nu}+1\geq-\overline{n}} 2^{\overline{\nu}\cdot\overline{d}(\frac{p_0}{p_1}-1)} \\
&\lesssim \sum_{\overline{n}\in\mathbb{Z}^N} 2^{\overline{n}\cdot\overline{d}\frac{p_0}{p_1}} (Rh)(2^{n_1d_1},\dots,2^{n_Nd_N})^{p_0} 2^{-\overline{n}\cdot\overline{d}(\frac{p_0}{p_1}-1)} \\
&= \sum_{\overline{n}\in\mathbb{Z}^N} 2^{\overline{n}\cdot\overline{d}} (Rh)(2^{n_1d_1},\dots,2^{n_Nd_N})^{p_0} \sim \big\|Rh\big|L_{p_0}((0,\infty)^N)\big\|^{p_0} \\
&= \big\|h\big|L_{p_0}(\mathbb{R}^d)\big\|^{p_0}.
\end{aligned}
$$

This finishes the proof of (2.4) under the condition (1.7) and $p_1 < \infty$. In case $p_1 = \infty$ one can estimate more directly

$$
\begin{aligned}
\big\|\lambda\big|s^0_{\infty,p_0}b\big\|^{p_0} &\leq \sum_{\overline{\nu}\in\mathbb{N}_0^N} 2^{-\overline{\nu}\cdot\overline{d}} \sup_{\overline{m}\in\mathbb{Z}^d} \inf_{x\in Q_{\overline{\nu}\,\overline{m}}} h(x)^{p_0} \\
&\leq \sum_{\overline{\nu}\in\mathbb{N}_0^N} 2^{-\overline{\nu}\cdot\overline{d}} (Rh)(2^{-(\nu_1+1)d_1},\dots,2^{-(\nu_N+1)d_N})^{p_0} \\
&\leq 2^d \sum_{\overline{\nu}\in\mathbb{Z}^N} 2^{-(\overline{\nu}+1)\cdot\overline{d}} (Rh)(2^{-(\nu_1+1)d_1},\dots,2^{-(\nu_N+1)d_N})^{p_0} \\
&\sim \big\|Rh\big|L_{p_0}((0,\infty)^N)\big\|^{p_0} = \big\|h\big|L_{p_0}(\mathbb{R}^d)\big\|^{p_0}.
\end{aligned}
$$

Step 2. Proof of (2.5)

We use similar arguments as in Step 1, this time combined with duality.

Using lifting properties and trivial embeddings, we may again restrict the proof to

$$s^{\overline{r}}_{p_0,p_1}b \hookrightarrow s^0_{p_1,q}f,$$

where

$$r_i = d_i\left(\frac{1}{p_0}-\frac{1}{p_1}\right), \quad i=1,\ldots,N$$

and $0<q<p_0$.

Let $\lambda = \{\lambda_{\overline{\nu}}\}_{\overline{\nu}\in\mathbb{N}_0^N} = \{\lambda_{\overline{\nu},m}\}_{\overline{\nu}\in\mathbb{N}_0^N, m\in\mathbb{Z}^d}$ be in $s^{\overline{r}}_{p_0,p_1}b$. The multivariate non-increasing rearrangement of $\lambda_{\overline{\nu}} = \{\lambda_{\overline{\nu},m}\}_{m\in\mathbb{Z}^d}$ is defined similar to Definition 2.1 and denoted by $\widetilde{\lambda}_{\overline{\nu}} = \{\widetilde{\lambda}_{\overline{\nu},\overline{m}}\}_{\overline{m}\in\mathbb{N}_0^N}$. As $\lambda_{\overline{\nu}}\in\ell_{p_0}(\mathbb{Z}^d)$, this rearrangement is also a rearrangement of a sequence in the classical sense. Furthermore, we write $\widetilde{\chi}_{\overline{\nu}\,\overline{m}}$ for characteristic functions of cubes $Q'_{\overline{\nu},\overline{m}}\subset(0,\infty)^N$, which were used already in the Step 1.

Then, using $q<p_1$ and Lemma 2.3,

$$\begin{aligned}\left\|\lambda\middle|s^0_{p_1q}f\right\| &= \left\|\left(\sum_{\overline{\nu}\in\mathbb{N}_0^N}\sum_{m\in\mathbb{Z}^d}|\lambda_{\overline{\nu}\,m}|^q\chi_{\overline{\nu},m}(x)\right)^{1/q}\middle|L_{p_1}(\mathbb{R}^d)\right\| \\ &\le \left\|\sum_{\overline{\nu}\in\mathbb{N}_0^N}\sum_{\overline{m}\in\mathbb{N}_0^N}\widetilde{\lambda}^q_{\overline{\nu}\,\overline{m}}\widetilde{\chi}_{\overline{\nu}\,\overline{m}}(x)\middle|L_{\frac{p_1}{q}}((0,\infty)^N)\right\|^{1/q}. \qquad (2.14)\end{aligned}$$

Let α and β be the conjugate exponents of $\frac{p_0}{q}$ and of $\frac{p_1}{q}$, respectively. Using duality, (2.14) may be rewritten as

$$\begin{aligned}\left\|\lambda\middle|s^0_{p_1q}f\right\| &\le \sup_g\left(\int\limits_{(0,\infty)^N} g(x)\left(\sum_{\overline{\nu}\in\mathbb{N}_0^N}\sum_{\overline{m}\in\mathbb{N}_0^N}\widetilde{\lambda}^q_{\overline{\nu},\overline{m}}\widetilde{\chi}_{\overline{\nu},\overline{m}}(x)\right)dx\right)^{1/q} \\ &= \sup_g\left(\sum_{\nu\in\mathbb{N}_0^N}\sum_{\overline{m}\in\mathbb{N}_0^N}2^{-\overline{\nu}\cdot\overline{d}}\widetilde{\lambda}^q_{\overline{\nu},\overline{m}}g_{\overline{\nu},\overline{m}}\right)^{1/q}, \qquad (2.15)\end{aligned}$$

where the supremum is taken over all non-negative functions $g:(0,\infty)^N\to[0,\infty]$, which are non-increasing in each variable, $\|g|L_\beta((0,\infty)^N)\|\le 1$ and $g_{\overline{\nu},\overline{m}} = 2^{\overline{\nu}\cdot\overline{d}}\int g(x)\widetilde{\chi}_{\overline{\nu},\overline{m}}(x)dx$.

We use twice Hölder's inequality and estimate (2.15) from above by

$$\left(\sum_{\nu\in\mathbb{N}_0^N}2^{-\overline{\nu}\cdot\overline{d}}\left(\sum_{\overline{m}\in\mathbb{N}_0^N}\widetilde{\lambda}^{p_0}_{\overline{\nu},\overline{m}}\right)^{\frac{p_1}{p_0}}\right)^{1/p_1}\cdot\sup_g\left(\sum_{\overline{\nu}\in\mathbb{N}_0^N}2^{-\overline{\nu}\cdot\overline{d}}\left(\sum_{\overline{m}\in\mathbb{N}_0^N}g^\alpha_{\overline{\nu}\,\overline{m}}\right)^{\beta/\alpha}\right)^{\frac{1}{\beta q}}. \qquad (2.16)$$

The first factor in (2.16) is equal to $\|\lambda|s^{\overline{r}}_{p_0,p_1}b\|$ due to condition (1.7). Hence it is enough to prove that there is a constant $c>0$, such that

$$\left(\sum_{\overline{\nu}\in\mathbb{N}_0^N} 2^{-\overline{\nu}\cdot\overline{d}}\left(\sum_{\overline{m}\in\mathbb{N}_0^N} g^{\alpha}_{\overline{\nu},\overline{m}}\right)^{\beta/\alpha}\right)^{\frac{1}{\beta q}} \leq c$$

for every non-negative measurable function g, which is non-increasing in each component and with $\|g|L_\beta((0,\infty)^N)\|\leq 1$.

First, we use the monotonicity of g and obtain

$$\begin{aligned}\sum_{\overline{m}\in\mathbb{N}_0^N} g^{\alpha}_{\overline{\nu}\,\overline{m}} &= \sum_{\overline{l}\in\mathbb{N}_0^N}\ \sum_{\substack{\overline{m}\in\mathbb{N}_0^N:\\ \forall i: 2^{l_i d_i}-1\leq m_i<2^{(l_i+1)d_i}-1}} g^{\alpha}_{\overline{\nu},\overline{m}}\\
&\lesssim \sum_{\overline{l}\in\mathbb{N}_0^N} 2^{\overline{l}\cdot\overline{d}}\left(2^{\overline{\nu}\cdot\overline{d}}\int\limits_{W_{\overline{\nu},(2^{l_1d_1},\dots,2^{l_Nd_N})}} g(x)dx\right)^{\alpha}\\
&\lesssim \sum_{\overline{l}\in\mathbb{N}_0^N} 2^{\overline{l}\cdot\overline{d}}(\mathcal{A}g)(2^{(l_1-\nu_1)d_1},\dots,2^{(l_N-\nu_N)d_N})^{\alpha},\end{aligned}$$

where $W_{\overline{\nu},\overline{k}}=[2^{-\nu_1d_1}(k_1-1),2^{-\nu_1d_1}k_1]\times\cdots\times[2^{-\nu_Nd_N}(k_N-1),2^{-\nu_Nd_N}k_N]$.

Using $1<\beta<\alpha<\infty$, this leads to

$$\begin{aligned}\sum_{\overline{\nu}\in\mathbb{N}_0^N} 2^{-\overline{\nu}\cdot\overline{d}}\left(\sum_{\overline{m}\in\mathbb{N}_0^N} g^{\alpha}_{\overline{\nu}\,\overline{m}}\right)^{\beta/\alpha} &\leq \sum_{\overline{\nu}\in\mathbb{N}_0^N} 2^{-\overline{\nu}\cdot\overline{d}}\sum_{\overline{l}\in\mathbb{N}_0^N} 2^{\overline{l}\cdot\overline{d}\frac{\beta}{\alpha}}(\mathcal{A}g)(2^{(l_1-\nu_1)d_1},\dots,2^{(l_N-\nu_N)d_N})^{\beta}\\
&= \sum_{\overline{k}\in\mathbb{Z}^N} 2^{\overline{k}\cdot\overline{d}\frac{\beta}{\alpha}}\sum_{\overline{\nu}\in\mathbb{N}_0^N:\overline{\nu}\geq-\overline{k}} 2^{-\overline{\nu}\cdot\overline{d}}2^{\overline{\nu}\cdot\overline{d}\frac{\beta}{\alpha}}(\mathcal{A}g)(2^{k_1d_1},\dots,2^{k_Nd_N})^{\beta}\\
&\leq \sum_{\overline{k}\in\mathbb{Z}^N} 2^{\overline{k}\cdot\overline{d}\frac{\beta}{\alpha}}(\mathcal{A}g)(2^{k_1d_1},\dots,2^{k_Nd_N})^{\beta}\sum_{\overline{\nu}\in\mathbb{Z}^N:\overline{\nu}\geq-\overline{k}} 2^{\overline{\nu}\cdot\overline{d}(\frac{\beta}{\alpha}-1)}\\
&\lesssim \sum_{\overline{k}\in\mathbb{Z}^N} 2^{\overline{k}\cdot\overline{d}\frac{\beta}{\alpha}}(\mathcal{A}g)(2^{k_1d_1},\dots,2^{k_Nd_N})^{\beta}2^{-\overline{k}\cdot\overline{d}(\frac{\beta}{\alpha}-1)}\\
&\sim \left\|\mathcal{A}g\middle|L_\beta((0,\infty)^N)\right\|^{\beta}\sim\left\|g\middle|L_\beta((0,\infty)^N)\right\|^{\beta}\leq 1.\end{aligned}$$

This finishes the proof of (2.5).

Step 3.

We show, that if (1.7) and (2.4) hold, then $p_0\leq q_1$. Suppose, that $0<q_1<p_0<\infty$ and set

$$\lambda_{\overline{\nu},m}=\begin{cases}\nu_1^{-1/q_1}2^{\nu_1(d_1/p_1-r_1^1)} & \text{if}\quad \overline{\nu}=(\nu_1,0,\dots,0),\ \nu_1\in\mathbb{N}\\ & \text{and}\quad m=(0,\dots,0)\in\mathbb{Z}^d,\\ 0, & \text{otherwise.}\end{cases}$$

Direct calculations show that $||\lambda|s^{\overline{r}_1}_{p_1q_1}b|| = \infty$ and $||\lambda|s^{\overline{r}_0}_{p_0q_0}f|| < \infty$. Hence (2.4) does not hold.

Step 4.

We show, that (2.5) implies $q_0 \le p_1$. To this end we assume that $0 < p_1 < q_0 \le \infty$ and set

$$\lambda_{\overline{\nu},m} = \begin{cases} \nu_1^{-1/p_1} 2^{\nu_1(d_1/p_1 - r_1^1)} & \text{if} \quad \overline{\nu} = (\nu_1, 0, \ldots, 0),\ \nu_1 \in \mathbb{N} \\ & \text{and} \quad m = (0, \ldots, 0) \in \mathbb{Z}^d, \\ 0, & \text{otherwise.} \end{cases}$$

This leads to $||\lambda|s^{\overline{r}_1}_{p_1q_1}f|| = \infty$ and $||\lambda|s^{\overline{r}_0}_{p_0q_0}b|| < \infty$. Hence (2.5) does not hold.

2.3 Proof of Theorem 1.5

If (c) is satisfied, then we use the embedding

$$S^0_{\infty,1}B(\mathbb{R}^{\overline{d}}) \hookrightarrow C(\mathbb{R}^d), \tag{2.17}$$

which follows directly from Definition 1.1, and the Sobolev embedding (cf. [24, Theorem 2.4.1])

$$S^{\overline{r}^0}_{p_0,q_0}B(\mathbb{R}^{\overline{d}}) \hookrightarrow S^{\overline{r}^1}_{p_1,q_1}B(\mathbb{R}^{\overline{d}})$$

if

$$r_j^0 - \frac{d_j}{p_0} = r_j^1 - \frac{d_j}{p_1}, \quad j = 1, \ldots, N, \quad 0 < p_0 < p_1 \le \infty \quad \text{and} \quad 0 < q_0 \le q_1 \le \infty.$$

Hence, $S^{\overline{r}}_{p,q}B(\mathbb{R}^{\overline{d}}) \hookrightarrow C(\mathbb{R}^d) \hookrightarrow L_\infty(\mathbb{R}^d)$. This proves $(c) \implies (a) \implies (b)$.

If (c) is not satisfied, we look for a distribution $f \in S^{\overline{r}}_{pq}B(\mathbb{R}^{\overline{d}})$, which may *not* be represented by a bounded measurable function in the usual sense. The counterexamples may be given directly using the wavelet expansions as presented in [10]. But one may proceed also indirectly:

Let us assume that $r_j - \dfrac{d_j}{p} < 0$ for some $1 \le j \le N$ or $r_j - \dfrac{d_j}{p} \le 0$ for some $1 \le j \le N$ and $q > 1$. In both cases, it is known that there is a distribution $\psi_j \in B^{r_j}_{pq}(\mathbb{R}^{d_j})$, such that $\psi_j \notin L_\infty(\mathbb{R}^{d_j})$, cf. [26, Theorem 3.3.1]. Now it is enough to consider

$$f = \psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_N,$$

where $\psi_i \in S(\mathbb{R}^{d_i}), i \ne j$, are suitably chosen smooth functions. The proof of (ii) uses similar arguments, this time combined with (1.8).

2.4 Proof of Theorem 1.6

The proof of Theorem 1.6 follows by similarly with (2.17) replaced by

$$S^0_{u,2}F(\mathbb{R}^{\overline{d}}) = L_u(\mathbb{R}^d), \quad 1 < u < \infty.$$

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Sobolev and Jawerth embeddings for spaces with variable smoothness and integrability

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Abstract

We consider the Triebel-Lizorkin spaces $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ of variable smoothness and integrability as introduced recently by Diening, Hästö and Roudenko in [9]. Under certain regularity conditions on the function parameters involved we show that

$$F^{s_0(\cdot)}_{p_0(\cdot),q(\cdot)}(\mathbb{R}^n) \hookrightarrow F^{s_1(\cdot)}_{p_1(\cdot),q(\cdot)}(\mathbb{R}^n)$$

if

$$s_0(x) \geq s_1(x) \quad \text{and} \quad s_0(x) - \frac{n}{p_0(x)} = s_1(x) - \frac{n}{p_1(x)} \quad \text{for all} \quad x \in \mathbb{R}^n$$

with embeddings of Sobolev and Bessel potential spaces included as special cases.

If $\inf\limits_{x\in\mathbb{R}^n}(s_0(x) - s_1(x)) > 0$ we recover also the analogue of the Jawerth embedding

$$F^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)}(\mathbb{R}^n) \hookrightarrow F^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)$$

for any q_0, q_1.

The proofs are based on the decomposition techniques of [9] and work exclusively with the associated sequence spaces $f^{s(\cdot)}_{p(\cdot),q(\cdot)}$.

AMS Classification: 46E35, 46E30

Keywords and phrases: Triebel-Lizorkin spaces, variable smoothness, variable integrability, Jawerth embedding, Sobolev embedding

1 Introduction

The interplay between smoothness and integrability constitutes one of the corner stones of the theory of function spaces. It can be traced back as far as to Hardy and Littlewood [13, 14], but the decisive breakthrough was achieved by Sobolev [33], who proved the famous embedding

$$W_p^m(\Omega) \hookrightarrow L_q(\Omega), \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary, $L_q(\Omega)$ stands for the usual Lebesgue space and $W_p^m(\Omega)$ denotes the Sobolev space of functions with all distributive derivatives of order smaller or equal to m bounded in the $L_p(\Omega)$ norm. The crucial relation between the involved parameters $m \in \mathbb{N}$, $1 < p < n/m$ and $1 < q < \infty$ is

$$\frac{1}{q} = \frac{1}{p} - \frac{m}{n}. \tag{1.2}$$

During the last seventy years, many scales of spaces of smooth functions were defined using various techniques (e.g. derivatives, differences, Fourier coefficients or Fourier transform) with the corresponding analogues of (1.1) and (1.2) playing usually an important role in most of the applications. Actually, it seems that any new scale of spaces of smooth functions needs to exhibit some kind of interaction between smoothness and integrability to be accepted by the mathematical audience.

In recent years there has been a growing interest in function spaces describing local regularity properties of functions. The first spaces of this type are the spaces of variable integrability, which were introduced by Orlicz [27] already in 1931 and studied in detail by Kováčik and Rákosník [24] in 1991 together with the corresponding Sobolev spaces of variable integrability. During 1990's these spaces found applications in the study of variational integrals with non-standard growth, but it was probably the work of Růžička [29, 30, 31] on electrorheological fluids what promoted an enormous interest in these spaces. Since then, more than one hundred papers on this topic appeared. We refer to [8] for a brief overview and an extensive collection of references.

Another way how to describe the local properties of a function was outlined already by Peetre in [28, page 266] in Chapter 12 named "Some strange new spaces" and resulted in the concept of 2-microlocal spaces, cf. [5] and [20]. Along a different line of study, Leopold [25] introduced spaces of Besov-type with variable smoothness, but constant integrability. This approach was further developed by Besov [3, 4].

The Sobolev embedding for the spaces with variable integrability was addressed already by Kováčik and Rákosník [24] and later on by Růžička [31]. But their results failed to cover the optimal exponent according to (1.1). Edmunds and Rákosník [10, 11] proved the optimal Sobolev embedding theorem under Lipschitz and Hölder continuity of the exponents, cf. also [16]. Finally, Diening [7] and Samko [32] showed, that log-Hölder continuity is sufficient.

The embeddings of Besov and Triebel-Lizorkin spaces of variable smoothness were obtained by Besov [4] in a fairly general form. It seems that Leopold [26] was the only one up to now who tried to connect the function spaces with variable smoothness with spaces of variable integrability. Unfortunately, he also failed to recover the optimal exponent.

The last step (up to now) was done by Diening, Hästö and Roudenko in [9]. These authors combined the concept of spaces with variable integrability of Orlicz, Kováčik and Rákosník with the concept of variable smoothness of Leopold and Besov (which is in some sense very similar to the ideas of Peetre, Bony and Jaffard) and proposed the function spaces of Triebel-Lizorkin type of variable smoothness *and* integrability, cf. Definition 2.5. They proved (under some restrictions on the function parameters involved), that these spaces include the Lebesgue and Sobolev spaces of variable integrability and the spaces of variable smoothness as special cases. They proved also a certain version of the atomic decomposition theorem, which is a well known tool in the theory of function spaces of Besov and Triebel-Lizorkin type. Finally, they proved an analogue of the usual trace theorem, which exhibits the interplay between smoothness and integrability. The reader may consult also [17], [15] and references given there for other versions of the trace embedding theorem for Sobolev spaces with varying integrability.

Although mentioned on several places in [9] (and even in the abstract), the authors have not presented any version of Sobolev embedding, which would not only result in a generalization of (1.1) with (1.2) holding pointwise, but would (in the sense described above) help to justify the existence of this scale of function spaces - at least until this promising line of research finds any applications.

Our aim is to fill this gap. In the frame of Triebel-Lizorkin spaces with constant parameters, the following analogue of Sobolev embedding is true.

Theorem 1.1. *(Jawerth, [21]). Let*

$$-\infty < s_1 < s_0 < \infty, \quad 0 < p_0 < p_1 < \infty, \quad 0 < q \le \infty \tag{1.3}$$

with

$$s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}. \tag{1.4}$$

Then

$$F^{s_0}_{p_0,\infty}(\mathbb{R}^n) \hookrightarrow F^{s_1}_{p_1,q}(\mathbb{R}^n). \tag{1.5}$$

The remarkable effect, which was first observed by Jawerth and which is in some sense unique to the Triebel-Lizorkin spaces, is the improvement in the third fine parameter $q > 0$, which may be chosen arbitrarily small. Of course, (1.5) holds only for $q = \infty$ if $s_0 = s_1$ (or, equivalently, $p_0 = p_1$). If the smoothness and integrability parameters s and p become functions of $x \in \mathbb{R}^n$, then it seems to be appropriate to assume that (1.4) holds pointwise, i.e.

$$s_0(x) - \frac{n}{p_0(x)} = s_1(x) - \frac{n}{p_1(x)}, \quad x \in \mathbb{R}^n \tag{1.6}$$

and if the improvement in the fine parameter is to be achieved, that also

$$\inf_{x\in\mathbb{R}^n} (s_0(x) - s_1(x)) = n \inf_{x\in\mathbb{R}^n} \Big(\frac{1}{p_0(x)} - \frac{1}{p_1(x)}\Big) > 0. \tag{1.7}$$

We prove that these "natural" assumptions (combined with appropriate regularity conditions) are really sufficient. We show, that if $s_1(x) \le s_0(x)$ and $p_0(x) \le p_1(x)$ with (1.6) and $0 < q(x) \le \infty$ for all $x \in \mathbb{R}^n$, then

$$F^{s_0(\cdot)}_{p_0(\cdot),q(\cdot)}(\mathbb{R}^n) \hookrightarrow F^{s_1(\cdot)}_{p_1(\cdot),q(\cdot)}(\mathbb{R}^n). \tag{1.8}$$

If also (1.7) is satisfied, then even

$$F^{s_0(\cdot)}_{p_0(\cdot),\infty}(\mathbb{R}^n) \hookrightarrow F^{s_1(\cdot)}_{p_1(\cdot),q(\cdot)}(\mathbb{R}^n)$$

holds.

2 Preliminaries

Let $S(\mathbb{R}^n)$ be the Schwartz space of all complex-valued rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^n$ and let $S'(\mathbb{R}^n)$ be its dual - the space of all tempered distributions. For $f \in S'(\mathbb{R}^n)$ we denote by $\widehat{f} = Ff$ its Fourier transform and by $f^\vee$ or $F^{-1}f$ its inverse Fourier transform. We give a Fourier-analytic definition of Triebel-Lizorkin spaces, which relies on the so-called *dyadic resolution of unity.* Let $\varphi \in S(\mathbb{R}^n)$ with

$$\varphi(x) = 1 \quad \text{if} \quad |x| \le 1 \quad \text{and} \quad \varphi(x) = 0 \quad \text{if} \quad |x| \ge \frac{3}{2}. \tag{2.1}$$

We put $\varphi_0 = \varphi$ and $\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$ for $j \in \mathbb{N}$ and $x \in \mathbb{R}^n$. This leads to the identity

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1, \qquad x \in \mathbb{R}^n.$$

Definition 2.1. Let $s \in \mathbb{R}, 0 < p < \infty, 0 < q \le \infty$. Then $F^s_{pq}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$||f|F^s_{pq}(\mathbb{R}^n)|| = \left|\left| \left(\sum_{j=0}^{\infty} 2^{jsq} |(\varphi_j \widehat{f})^\vee(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n) \right|\right| < \infty \tag{2.2}$$

(with the usual modification for $q = \infty$).

Remark 2.2. (i) These spaces have a long history. In this context we recommend [28, 34, 35, 37] as standard references. We point out that the spaces $F^s_{pq}(\mathbb{R}^n)$ are independent of the choice of φ in the sense of equivalent (quasi-)norms. Special cases of this scale include Lebesgue spaces, Sobolev spaces and inhomogeneous Hardy spaces.

(ii) Interchanging the order of L_p and ℓ_q norm in (2.2) would lead to the Fourier-analytic definition of Besov spaces. Unfortunately, they seem to be less convenient for describing local regularity properties of distributions, because they lack the so-called *localization principle*, cf. [35, Theorem 2.4.7]. Hence (also in correspondence with [9]) we study only the F-scale.

Next, we introduce the Lebesgue spaces of variable integrability.

Definition 2.3. Let $p : \mathbb{R}^n \to (0,\infty)$ be a measurable function. Then the space $L_{p(\cdot)}(\mathbb{R}^n)$ consists of all measurable functions $f : \mathbb{R}^n \to [-\infty,\infty]$ such that $||f|L_{p(\cdot)}(\mathbb{R}^n)|| < \infty$, where

$$||f|L_{p(\cdot)}(\mathbb{R}^n)|| = \inf\{\lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1\}$$

is the Minkowski functional of the absolutely convex set $\{f : \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx \le 1\}$.

Remark 2.4. (i) One could also consider (and it was done so already by Kováčik and Rákosník in [24]) that $p(x) = \infty$ on a set of a positive measure. But Definition 2.3 is already sufficient for our purpose, cf. also Remark 2.6.

(ii) If $p(x) \ge 1$ for all $x \in \mathbb{R}^n$, then $L_{p(\cdot)}(\mathbb{R}^n)$ are Banach spaces. To ensure, that $L_{p(\cdot)}(\mathbb{R}^n)$ are at least quasi-Banach spaces, we assume that

$$p^- := \inf_{x\in\mathbb{R}^n} p(x) > 0.$$

The generalization of Definition 2.1 to the setting of variable smoothness and integrability as it was given by [9] is surprisingly simple.

Definition 2.5. Let $-\infty < s(x) < +\infty, 0 < p(x) < \infty, 0 < q(x) \le \infty$. Then $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$||f|F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)|| = \left|\left|\left(\sum_{j=0}^{\infty} 2^{js(\cdot)q(\cdot)}|(\varphi_j \widehat{f})^\vee(\cdot)|^{q(\cdot)}\right)^{1/q(\cdot)} |L_{p(\cdot)}(\mathbb{R}^n)\right|\right| < \infty \tag{2.3}$$

(with the usual modification for $q(x) = \infty$).

Remark 2.6. This definition introduces the Triebel-Lizorkin spaces of variable smoothness, integrability and summability under almost no conditions on $s(\cdot), p(\cdot)$ and $q(\cdot)$. Unfortunately, these spaces may depend on the choice of the function φ as described in (2.1). This is the case already when s and $q < \infty$ are constant and $p = \infty$. We refer to [34, Chapter 2.3.4] for a detailed discussion of related aspects. So, a first natural restriction seems to be the condition

$$p^+ = \sup_{x\in\mathbb{R}^n} p(x) < \infty.$$

Together with Remark 2.4(ii) this leads to

$$0 < p^- := \inf_{z\in\mathbb{R}^n} p(z) \le p(x) \le \sup_{z\in\mathbb{R}^n} p(z) =: p^+ < \infty, \quad x \in \mathbb{R}^n. \tag{2.4}$$

Next we present the regularity assumptions of [9].

Definition 2.7. Let g be a continuous function on $\mathbb{R}^n$.

(i) We say, that g is *1-locally* log-*Hölder continuous*, abbreviated $g \in C^{\log}_{1-\mathrm{loc}}(\mathbb{R}^n)$, if there exists $c > 0$ such that

$$|g(x) - g(y)| \le \frac{c}{\log(e + 1/||x-y||_\infty)} \quad \text{for all} \quad x, y \in \mathbb{R}^n \quad \text{with} \quad ||x-y||_\infty \le 1.$$

Here, $||z||_\infty = \max\{|z_1|, \dots, |z_n|\}$ denotes the maximum norm of $z \in \mathbb{R}^n$.

(ii) We say, that g is *locally* log-*Hölder continuous*, abbreviated $g \in C^{\log}_{\text{loc}}(\mathbb{R}^n)$, if there exists $c > 0$ such that

$$|g(x) - g(y)| \le \frac{c}{\log(e + 1/|x-y|)}, \quad x, y \in \mathbb{R}^n.$$

(iii) We say, that g is *globally* log-*Hölder continuous*, abbreviated $g \in C^{\log}(\mathbb{R}^n)$, if it is locally log-Hölder continuous and there exists $c > 0$ and $g_\infty \in \mathbb{R}$ such that

$$|g(x) - g_\infty| \le \frac{c}{\log(e + |x|)}, \quad x \in \mathbb{R}^n.$$

Remark 2.8. (i) The conditions (ii) and (iii) are overtaken literally from [9] and we shall need them only for the transference of our results from sequence spaces to function spaces. It is the less restrictive condition (i), which we shall involve in our proofs.

(ii) The condition (i) is very similar to the original condition of Diening used in [6] to show the boundedness of the maximal operator.

We shall use the property (i) in the form formulated in next Lemma. We leave out the trivial proof.

Lemma 2.9. *Let $g \in C^{\log}_{1-\text{loc}}(\mathbb{R}^n)$. Then there exists a constant $c > 0$ such that for every $j \in \mathbb{N}_0$ and every $x, y \in \mathbb{R}^n$ with $||x - y||_\infty \le 2^{-j}$ the following inequalities hold:*

$$\frac{1}{c} \le 2^{-j|g(x)-g(y)|} \le 2^{j(g(x)-g(y))} \le 2^{j|g(x)-g(y)|} \le c.$$

Definition 2.10. (Standing assumptions of [9]). Let p and q be positive functions on $\mathbb{R}^n$ such that $\frac{1}{p}, \frac{1}{q} \in C^{\log}(\mathbb{R}^n)$ and let $s \in C^{\log}_{\text{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with $s(x) \ge 0$ and let $s(x)$ have a limit at infinity.

Remark 2.11. (i) Let us note, that the *Standing assumptions* imply in particular (2.4) and a similar chain of inequalities for $q(x)$.

We introduce the sequence spaces associated with the Triebel-Lizorkin spaces of variable smoothness and integrability. Let $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$. Then Q_{jm} denotes the closed cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes, centered at $2^{-j}m$, and with side length 2^{-j}. By $\chi_{jm} = \chi_{Q_{jm}}$ we denote the characteristic function of Q_{jm}. If

$$\gamma = \{\gamma_{jm} \in \mathbb{C} : j \in \mathbb{N}_0, m \in \mathbb{Z}^n\},$$

$-\infty < s(x) < \infty$, $0 < p(x) < \infty$ and $0 < q(x) \le \infty$ for all $x \in \mathbb{R}^n$, we define

$$\begin{aligned} ||\gamma|f^{s(\cdot)}_{p(\cdot),q(\cdot)}|| &= \left|\left| \left(\sum_{j=0}^\infty \sum_{m\in\mathbb{Z}^n} 2^{js(\cdot)q(\cdot)} |\gamma_{jm}|^{q(\cdot)} \chi_{jm}(\cdot) \right)^{1/q(\cdot)} \Big| L_{p(\cdot)}(\mathbb{R}^n) \right|\right| \qquad (2.5) \\ &= \left|\left| \sum_{j=0}^\infty \sum_{m\in\mathbb{Z}^n} 2^{js(\cdot)} |\gamma_{jm}| \chi_{jm}(\cdot) \Big| L_{p(\cdot)}(\ell_{q(\cdot)}) \right|\right|. \end{aligned}$$

Establishing the connection between the function spaces $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and the sequence spaces $f^{s(\cdot)}_{p(\cdot),q(\cdot)}$ was the main aim of [9]. Following [18] and [19], these authors investigated the properties of the so-called φ-transform (denoted by S_φ) and obtained the following result.

Theorem 2.12. *Under the Standing assumptions of [9]*

$$||f|F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)|| \approx ||S_\varphi f|f^{s(\cdot)}_{p(\cdot),q(\cdot)}||$$

with constants independent of $f \in F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Remark 2.13. (i) The assumptions on s in the Theorem 2.12 seem to be too restrictive. It seems, that several authors now try to prove similar results also for $s(x)$, which are not necessarily positive or convergent at infinity. We refer at least to [23] and [39].

From this reason we formulate the theorems of embeddings of sequence spaces under minimal assumptions, which shall really be needed in the proof. If later on any improved version of Theorem 2.12 should appear, the results may then be easily taken over.

(ii) We shall use only a corollary of Theorem 2.12, namely that (under the *Standing assumptions*) the space $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is isomorphic to a subspace of $f^{s(\cdot)}_{p(\cdot),q(\cdot)}$ via the S_φ transform.

3 Main results

First, we state the results in the form of embeddings of sequence spaces under those assumptions really needed in the proof. Later on, we combine those with the *Standing assumptions of [9]* and obtain similar results also for the embeddings of function spaces. Finally, we state separately the embeddings of Sobolev and Bessel potential spaces.

Theorem 3.1. *Let* $-\infty < s_1(x) \le s_0(x) < \infty$, $0 < p_0(x) \le p_1(x) < \infty$ *for all* $x \in \mathbb{R}^n$ *with* $0 < p_0^- \le p_1^- \le p_1^+ < \infty$ *and*

$$s_0(x) - \frac{n}{p_0(x)} = s_1(x) - \frac{n}{p_1(x)}, \quad x \in \mathbb{R}^n.$$

Let $q(x) = \infty$ *for all* $x \in \mathbb{R}^n$ *or* $0 < q^- \le q(x) < \infty$ *for all* $x \in \mathbb{R}^n$ *and* $s_0, \frac{1}{p_0} \in C^{\log}_{1-\mathrm{loc}}(\mathbb{R}^n)$. *Then*

$$f^{s_0(\cdot)}_{p_0(\cdot),q(\cdot)} \hookrightarrow f^{s_1(\cdot)}_{p_1(\cdot),q(\cdot)}.$$

Proof. Step 1. $q(x) = \infty$ *for all* $x \in \mathbb{R}^n$.
We set

$$h(x) = \sup_{j,m} 2^{js_0(x)}|\gamma_{jm}|\chi_{jm}(x), \quad x \in \mathbb{R}^n. \tag{3.1}$$

Here, and later on, the supremum is taken over all $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$. Then by (2.5)

$$||\gamma|f^{s_0(\cdot)}_{p_0(\cdot),\infty}|| = ||h|L_{p_0(\cdot)}(\mathbb{R}^n)|| \tag{3.2}$$

and trivially

$$2^{js_0(x)}|\gamma_{jm}| \le h(x), \quad x \in Q_{jm}, \tag{3.3}$$

which leads to

$$|\gamma_{jm}| \le \inf_{x\in Q_{jm}} 2^{-js_0(x)}h(x), \quad j \in \mathbb{N}_0, \quad m \in \mathbb{Z}_n. \tag{3.4}$$

Using consequently (2.5), (3.4) and Lemma 2.9 for s_0 we may estimate

$$\begin{aligned}
\|\gamma|f^{s_1(\cdot)}_{p_1(\cdot),\infty}\| &= \left\|\sup_{j,m} 2^{js_1(x)}|\gamma_{jm}|\chi_{jm}(x)|L_{p_1(\cdot)}(\mathbb{R}^n)\right\| \\
&\le \left\|\sup_{j,m} 2^{js_1(x)}\Big(\inf_{y\in Q_{jm}} 2^{-js_0(y)}h(y)\Big)\chi_{jm}(x)|L_{p_1(\cdot)}(\mathbb{R}^n)\right\| \\
&= \left\|\sup_{j,m} 2^{j(s_1(x)-s_0(x))}\Big(\inf_{y\in Q_{jm}} 2^{j(s_0(x)-s_0(y))}h(y)\Big)\chi_{jm}(x)|L_{p_1(\cdot)}(\mathbb{R}^n)\right\| \\
&\le c\left\|\sup_{j,m} 2^{jn\left(\frac{1}{p_1(x)}-\frac{1}{p_0(x)}\right)}\Big(\inf_{y\in Q_{jm}} h(y)\Big)\chi_{jm}(x)|L_{p_1(\cdot)}(\mathbb{R}^n)\right\|.
\end{aligned}$$

Let $A_{-1}\subset\mathbb{R}^n$ stand for those x, where

$$\sup_{j,m} 2^{jn\left(\frac{1}{p_1(x)}-\frac{1}{p_0(x)}\right)}\Big(\inf_{y\in Q_{jm}} h(y)\Big)\chi_{jm}(x)=0. \tag{3.5}$$

For each $x\in\mathbb{R}^n$ we denote by $J=J_x\in\mathbb{N}_0$ the smallest non-negative integer, such that

$$\sup_{j,m} 2^{jn\left(\frac{1}{p_1(x)}-\frac{1}{p_0(x)}\right)}\Big(\inf_{y\in Q_{jm}} h(y)\Big)\chi_{jm}(x)\le 2\cdot 2^{Jn\left(\frac{1}{p_1(x)}-\frac{1}{p_0(x)}\right)}\sum_{m\in\mathbb{Z}^n}\Big(\inf_{y\in Q_{Jm}} h(y)\Big)\chi_{Jm}(x). \tag{3.6}$$

We may assume, that for almost all $x\in\mathbb{R}^n$ (3.5) is finite. Otherwise $h(x)=\infty$ on a set of positive measure and there is nothing to prove. Furthermore, we denote by $A_J\subset\mathbb{R}^n$ those x with $J_x=J\in\mathbb{N}_0$.

Let $\lambda>0$ be a positive real number such that

$$\begin{aligned}
1 &\ge \int_{\mathbb{R}^n}\left(\frac{h(x)}{\lambda}\right)^{p_0(x)}dx=\sum_{J=-1}^{\infty}\int_{A_J}\left(\frac{h(x)}{\lambda}\right)^{p_0(x)}dx \qquad (3.7)\\
&\ge \sum_{J=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\int_{A_J\cap Q_{Jm}}\left(\frac{h(x)}{\lambda}\right)^{p_0(x)}dx.
\end{aligned}$$

We set

$$h_{jm}:=\frac{\inf_{y\in Q_{jm}} h(y)}{\lambda},\quad j\in\mathbb{N}_0,\quad m\in\mathbb{Z}^n$$

and show, that there is a constant $C>0$ such that

$$\int_{\mathbb{R}^n}\left(C^{-1}\sup_{j,m}2^{jn\left(\frac{1}{p_1(x)}-\frac{1}{p_0(x)}\right)}h_{jm}\chi_{jm}(x)\right)^{p_1(x)}dx\le 1.$$

We split the integration over $\mathbb{R}^n$ into integrals over A_J and use (3.6).

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left(C^{-1}\sup_{j,m}2^{jn\left(\frac{1}{p_1(x)}-\frac{1}{p_0(x)}\right)}h_{jm}\chi_{jm}(x)\right)^{p_1(x)}dx\\
&\le\sum_{J=0}^{\infty}\int_{A_J}\left((C/2)^{-1}\sum_{m\in\mathbb{Z}^n}2^{Jn\left(\frac{1}{p_1(x)}-\frac{1}{p_0(x)}\right)}h_{Jm}\chi_{Jm}(x)\right)^{p_1(x)}dx\\
&=\sum_{J=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\int_{A_J}\left((C/2)^{-1}2^{Jn\left(\frac{1}{p_1(x)}-\frac{1}{p_0(x)}\right)}h_{Jm}\right)^{p_1(x)}\chi_{Jm}(x)dx\\
&=\sum_{J=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\int_{A_J\cap Q_{Jm}}(C/2)^{-p_1(x)}2^{Jn\left(1-\frac{p_1(x)}{p_0(x)}\right)}h_{Jm}^{p_1(x)}dx \qquad (3.8)
\end{aligned}$$

Let us fix $(J,m) \in \mathbb{N}_0 \times \mathbb{Z}^n$. We shall distinguish two cases.

1. case: $h_{Jm} \le 1$.

Then (as $p_0(x) \le p_1(x)$)

$$2^{Jn\left(1-\frac{p_1(x)}{p_0(x)}\right)} \le 1$$

and

$$h_{Jm}^{p_1(x)} \le h_{Jm}^{p_0(x)}.$$

Hence for $C \ge 2$ we obtain

$$\begin{aligned} \int_{A_J \cap Q_{Jm}} (C/2)^{-p_1(x)} 2^{Jn\left(1-\frac{p_1(x)}{p_0(x)}\right)} h_{Jm}^{p_1(x)} dx &\le \int_{A_J \cap Q_{Jm}} h_{Jm}^{p_0(x)} dx \\ &\le \int_{A_J \cap Q_{Jm}} \left(\frac{h(x)}{\lambda}\right)^{p_0(x)} dx. \end{aligned} \tag{3.9}$$

2. case: $h_{Jm} > 1$.

Then

$$1 \ge \int_{Q_{Jm}} \left(\frac{h(x)}{\lambda}\right)^{p_0(x)} dx \ge \int_{Q_{Jm}} h_{Jm}^{p_0(x)} dx \ge 2^{-Jn} h_{Jm}^{p_0^{Jm}},$$

where $p_0^{Jm} = \inf_{x \in Q_{Jm}} p_0(x) > 0$. Hence

$$1 < h_{Jm} \le 2^{Jn/p_0^{Jm}}. \tag{3.10}$$

We rewrite the integrals in (3.8) as

$$\int_{A_J \cap Q_{Jm}} (C/2)^{-p_1(x)} 2^{Jn\left(1-\frac{p_1(x)}{p_0(x)}\right)} h_{Jm}^{p_1(x)} dx = \int_{A_J \cap Q_{Jm}} \underbrace{(C/2)^{-p_1(x)} 2^{Jn\left(1-\frac{p_1(x)}{p_0(x)}\right)} h_{Jm}^{p_1(x)-p_0(x)}}_{(\star)} h_{Jm}^{p_0(x)} dx \tag{3.11}$$

and show that the estimate $(\star) \le 1$ for $C \ge 2$ large enough and $x \in Q_{Jm}$ finishes immediately the proof. By (3.9) and (3.11) combined with $(\star) \le 1$ and (3.7)

$$\begin{aligned} \sum_{J=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \int_{A_J \cap Q_{Jm}} (C/2)^{-p_1(x)} 2^{Jn\left(1-\frac{p_1(x)}{p_0(x)}\right)} h_{Jm}^{p_1(x)} dx &= \sum_{(J,m): h_{Jm} \le 1} \dots + \sum_{(J,m): h_{Jm} > 1} \dots \\ &\le \sum_{(J,m): h_{Jm} \le 1} \int_{A_J \cap Q_{Jm}} \left(\frac{h(x)}{\lambda}\right)^{p_0(x)} dx + \sum_{(J,m): h_{Jm} > 1} \int_{A_J \cap Q_{Jm}} h_{Jm}^{p_0(x)} dx \\ &\le \sum_{J=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \int_{A_J \cap Q_{Jm}} \left(\frac{h(x)}{\lambda}\right)^{p_0(x)} dx \le 1. \end{aligned}$$

Hence, it remains to prove that $(\star) \le 1$ for all $x \in Q_{Jm}$. By (3.10), it is enough to show that

$$(C/2)^{-p_1(x)} 2^{Jn\left(1-\frac{p_1(x)}{p_0(x)}\right)} \cdot 2^{Jn \cdot \frac{p_1(x)-p_0(x)}{p_0^{Jm}}} \le 1$$

or, equivalently,

$$2^{Jn[p_1(x)-p_0(x)] \cdot [\frac{1}{p_0^{Jm}} - \frac{1}{p_0(x)}]} \le (C/2)^{p_1(x)}.$$

Using Lemma 2.9 for $\frac{1}{p_0}$ (with constant $2^{c_{\log}}$), this follows from

$$2^{n[1-\frac{p_0(x)}{p_1(x)}]\cdot c_{\log}} \leq C/2.$$

As $0 \leq 1 - \frac{p_0(x)}{p_1(x)} \leq 1$, we may choose $C = 2^{nc_{\log}+1} \geq 2$.
Step 2. $0 < q(x) < \infty$ *for all* $x \in \mathbb{R}^n$.
Let $\lambda > 0$ be a positive real number with

$$\int_{\mathbb{R}^n} \Big(\sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} 2^{js_0(x)q(x)}|\gamma_{jm}|^{q(x)}\lambda^{-q(x)}\chi_{jm}(x)\Big)^{p_0(x)/q(x)} dx \leq 1. \tag{3.12}$$

We have to show that there is a constant $C > 0$ independent of $\{\gamma_{jm}\}$, such that

$$\int_{\mathbb{R}^n} \Big(\sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} 2^{js_1(x)q(x)}|\gamma_{jm}|^{q(x)}(C\lambda)^{-q(x)}\chi_{jm}(x)\Big)^{p_1(x)/q(x)} dx \leq 1. \tag{3.13}$$

We show, that under (3.12) the following inequality holds for almost all $x \in \mathbb{R}^n$

$$\Big(\sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} 2^{js_1(x)q(x)}\frac{|\gamma_{jm}|^{q(x)}}{(C\lambda)^{q(x)}}\chi_{jm}(x)\Big)^{p_1(x)} \leq \Big(\sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} 2^{js_0(x)q(x)}\frac{|\gamma_{jm}|^{q(x)}}{\lambda^{q(x)}}\chi_{jm}(x)\Big)^{p_0(x)}. \tag{3.14}$$

Obviously, (3.14) implies (3.13).

For almost every $x \in \mathbb{R}^n$ and every $j \in \mathbb{N}_0$, there is exactly one $m = m(j) \in \mathbb{Z}^n$ such that $x \in Q_{j,m(j)}$. We fix one such an x. Then (3.14) reads like

$$\sum_{j=0}^{\infty} 2^{js_1(x)q(x)}|\gamma_{j,m(j)}|^{q(x)}(C\lambda)^{-q(x)} \leq \Big(\sum_{j=0}^{\infty} 2^{js_0(x)q(x)}|\gamma_{j,m(j)}|^{q(x)}\lambda^{-q(x)}\Big)^{p_0(x)/p_1(x)}. \tag{3.15}$$

We set

$$\alpha_j := 2^{js_0(x)}\frac{|\gamma_{j,m(j)}|}{\lambda}, \quad j \in \mathbb{N}_0$$

and rewrite (3.15) once again. It now becomes

$$\sum_{j=0}^{\infty} 2^{jn\big(\frac{1}{p_1(x)}-\frac{1}{p_0(x)}\big)q(x)}(\alpha_j/C)^{q(x)} \leq \Big(\sum_{j=0}^{\infty} \alpha_j^{q(x)}\Big)^{p_0(x)/p_1(x)}. \tag{3.16}$$

Using (3.12) and Lemma 2.9 for s_0, we get

$$\begin{aligned}
1 &\geq \int_{Q_{j,m(j)}} \Big(2^{js_0(y)q(y)}|\gamma_{j,m(j)}|^{q(y)}\lambda^{-q(y)}\Big)^{p_0(y)/q(y)} dy = \int_{Q_{j,m(j)}} \Big(2^{js_0(y)}|\gamma_{j,m(j)}|\lambda^{-1}\Big)^{p_0(y)} dy \\
&= \int_{Q_{j,m(j)}} \Big(2^{j(s_0(y)-s_0(x))}2^{js_0(x)}|\gamma_{j,m(j)}|\lambda^{-1}\Big)^{p_0(y)} dy \geq \int_{Q_{j,m(j)}} \Big(c\,2^{js_0(x)}|\gamma_{j,m(j)}|\lambda^{-1}\Big)^{p_0(y)} dy \\
&= \int_{Q_{j,m(j)}} \big(c\,\alpha_j\big)^{p_0(y)} dy.
\end{aligned}$$

If $c\alpha_j > 1$, we may further estimate

$$1 \geq 2^{-jn}\big(c\,\alpha_j\big)^{\inf_{z\in Q_{j,m(j)}} p_0(z)},$$

or, equivalently,

$$c\,\alpha_j \leq 2^{\frac{jn}{\inf_{z\in Q_{j,m(j)}} p_0(z)}} = 2^{\frac{jn}{p_0(x)}} 2^{\frac{jn}{\inf_{z\in Q_{j,m(j)}} p_0(z)} - \frac{jn}{p_0(x)}} \leq c' 2^{\frac{jn}{p_0(x)}} \tag{3.17}$$

and this estimate holds true also if $c\,\alpha_j \leq 1$.

If $\sum_{j=0}^{\infty} \alpha_j^{q(x)} \leq 1$, then (3.16) follows by monotonicity and $p_0(x) \leq p_1(x)$ for any $C \geq 1$. If $\sum_{j=0}^{\infty} \alpha_j^{q(x)} = \infty$, then there is nothing to prove. In the remaining case $1 < \sum_{j=0}^{\infty} \alpha_j^{q(x)} < \infty$ we find a non-negative integer $J \in \mathbb{N}_0$ such that

$$2^{\frac{Jnq(x)}{p_0(x)}} < \sum_{j=0}^{\infty} \alpha_j^{q(x)} \leq 2^{\frac{(J+1)nq(x)}{p_0(x)}}. \tag{3.18}$$

We split the sum over $j \in \mathbb{N}_0$ into two parts, apply (3.17) in the first part and use the inequality $p_0(x) \leq p_1(x)$ together with (3.18) in the second part.

$$\begin{aligned}
\sum_{j=0}^{\infty} 2^{jn\left(\frac{1}{p_1(x)}-\frac{1}{p_0(x)}\right)q(x)} \alpha_j^{q(x)} &= \sum_{j=0}^{J} 2^{jn\left(\frac{1}{p_1(x)}-\frac{1}{p_0(x)}\right)q(x)} \alpha_j^{q(x)} + \sum_{j=J+1}^{\infty} 2^{jn\left(\frac{1}{p_1(x)}-\frac{1}{p_0(x)}\right)q(x)} \alpha_j^{q(x)} \\
&\leq c^{q(x)} \sum_{j=0}^{J} 2^{jn\left(\frac{1}{p_1(x)}-\frac{1}{p_0(x)}\right)q(x)} 2^{\frac{jnq(x)}{p_0(x)}} + 2^{(J+1)n\left(\frac{1}{p_1(x)}-\frac{1}{p_0(x)}\right)q(x)} \sum_{j=J+1}^{\infty} \alpha_j^{q(x)} \\
&\leq c^{q(x)} \sum_{j=0}^{J} 2^{\frac{jnq(x)}{p_1(x)}} + 2^{\frac{(J+1)nq(x)}{p_1(x)}} \leq c_1^{q(x)}\, 2^{\frac{(J+1)nq(x)}{p_1(x)}} \\
&\leq c_1^{q(x)}\, 2^{\frac{nq(x)}{p_1(x)}} \Big(\sum_{j=0}^{\infty} \alpha_j^{q(x)}\Big)^{\frac{p_0(x)}{p_1(x)}} \leq C^{q(x)} \Big(\sum_{j=0}^{\infty} \alpha_j^{q(x)}\Big)^{\frac{p_0(x)}{p_1(x)}}.
\end{aligned}$$

In the last line, we used $0 < p_1^- \leq p_1^+ < \infty$ and again (3.18). This finishes the proof of (3.16) and consequently of the whole Step 2. $\square$

Theorem 3.2. *Let* $-\infty < s_1(x) < s_0(x) < \infty$ *and* $0 < p_0(x) < p_1(x) < \infty$ *for all* $x \in \mathbb{R}^n$ *with* $0 < p_0^- < p_1^+ < \infty$,

$$s_0(x) - \frac{n}{p_0(x)} = s_1(x) - \frac{n}{p_1(x)}, \quad x \in \mathbb{R}^n$$

and

$$\varepsilon := \inf_{x\in\mathbb{R}^n} (s_0(x) - s_1(x)) = n \inf_{x\in\mathbb{R}^n} \left(\frac{1}{p_0(x)} - \frac{1}{p_1(x)}\right) > 0. \tag{3.19}$$

Let $s_0, \frac{1}{p_0} \in C^{\log}_{1-\mathrm{loc}}(\mathbb{R}^n)$. *Then, for every* $0 < q \leq \infty$,

$$f^{s_0(\cdot)}_{p_0(\cdot),\infty} \hookrightarrow f^{s_1(\cdot)}_{p_1(\cdot),q}.$$

Proof. We use again the notation of (3.1)-(3.4).

$$
\begin{aligned}
\|\gamma|f^{s_1(\cdot)}_{p_1(\cdot),q}\| &= \Big\|\Big(\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n} 2^{js_1(x)q}|\gamma_{j\,m}|^q\chi_{j\,m}(x)\Big)^{1/q}|L_{p_1(\cdot)}(\mathbb{R}^n)\Big\| \\
&\le \Big\|\Big(\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n} 2^{js_1(x)q}\Big(\inf_{y\in Q_{jm}} 2^{-js_0(y)}h(y)\Big)^q\chi_{jm}(x)\Big)^{1/q}|L_{p_1(\cdot)}(\mathbb{R}^n)\Big\| \qquad (3.20)\\
&\le \Big\|\Big(\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n} 2^{j(s_1(x)-s_0(x))q}\Big(\inf_{y\in Q_{jm}} 2^{j(s_0(x)-s_0(y))}h(y)\Big)^q\chi_{jm}(x)\Big)^{1/q}|L_{p_1(\cdot)}(\mathbb{R}^n)\Big\| \\
&\le c\Big\|\Big(\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n} 2^{jn\big(\frac{1}{p_1(x)}-\frac{1}{p_0(x)}\big)q}\Big(\inf_{y\in Q_{jm}} h(y)\Big)^q\chi_{jm}(x)\Big)^{1/q}|L_{p_1(\cdot)}(\mathbb{R}^n)\Big\|.
\end{aligned}
$$

Let again $\lambda > 0$ be a positive real number, such that

$$\int_{\mathbb{R}^n}\Big(\frac{h(x)}{\lambda}\Big)^{p_0(x)}dx \le 1. \tag{3.21}$$

For almost every $x \in \mathbb{R}^n$ and every $j \in \mathbb{N}_0$ there is exactly one $m = m(j)$ such that $x \in Q_{j,m(j)}$. Fix one such $x \in \mathbb{R}^n$ and set

$$\alpha_j := \frac{\inf\limits_{y\in Q_{j,m(j)}} h(y)}{\lambda}.$$

Then $\{\alpha_j\}_{j=0}^{\infty}$ is a non-decreasing sequence of non-negative real numbers with $\alpha := \lim\limits_{j\to\infty}\alpha_j \le \dfrac{h(x)}{\lambda}$.

Let first $\alpha \le 1$. Then we use the monotonicity of $\{\alpha_j\}$, (3.19) and obtain for $C^q \ge (1-2^{-n\epsilon q})^{-1}$

$$
\begin{aligned}
&\Big(\sum_{j=0}^{\infty} C^{-q}2^{jn\big(\frac{1}{p_1(x)}-\frac{1}{p_0(x)}\big)q}\alpha_j^q\Big)^{p_1(x)/q} \le \Big(\sum_{j=0}^{\infty} C^{-q}2^{jn\big(\frac{1}{p_1(x)}-\frac{1}{p_0(x)}\big)q}\alpha^q\Big)^{p_1(x)/q}\\
&= \Big(\sum_{j=0}^{\infty} C^{-q}2^{jn\big(\frac{1}{p_1(x)}-\frac{1}{p_0(x)}\big)q}\Big)^{p_1(x)/q}\cdot\alpha^{p_1(x)} \le \alpha^{p_0(x)} \le \Big(\frac{h(x)}{\lambda}\Big)^{p_0(x)}. \qquad (3.22)
\end{aligned}
$$

Let us now consider the case $\alpha > 1$. By (3.21), we get

$$1 \ge \int_{\mathbb{R}^n}\Big(\frac{h(x)}{\lambda}\Big)^{p_0(x)}dx \ge \int_{Q_{j,m(j)}}\alpha_j^{p_0(x)}dx.$$

If $\alpha_j > 1$, we may further estimate

$$1 \ge 2^{-jn}\alpha_j^{\inf_{y\in Q_{j,m(j)}} p_0(y)}.$$

We apply Lemma 2.9 for $\frac{1}{p_0}$ to obtain an analogue of (3.17)

$$\alpha_j \le 2^{\frac{jn}{\inf_{y\in Q_{j,m(j)}} p_0(y)}} = 2^{\frac{jn}{p_0(x)}}\cdot 2^{\frac{jn}{\inf_{y\in Q_{j,m(j)}}}-\frac{jn}{p_0(x)}} \le c_{\log}\, 2^{\frac{jn}{p_0(x)}} \tag{3.23}$$

and this estimate holds true also for $\alpha_j \le 1$.

We show, that for $C > 0$ large enough (cf. (3.16))

$$\sum_{j=0}^{\infty} C^{-q}2^{jn\big(\frac{1}{p_1(x)}-\frac{1}{p_0(x)}\big)q}\alpha_j^q \le \alpha^{\frac{qp_0(x)}{p_1(x)}}. \tag{3.24}$$

As $\alpha = \infty$ implies $h(x) = \infty$ and this happens only for a set of $x \in \mathbb{R}^n$ with measure zero, we may choose for almost every $x \in \mathbb{R}^n$ a non-negative integer $J \in \mathbb{N}_0$ such that

$$2^{\frac{Jn}{p_0(x)}} < \alpha \leq 2^{\frac{(J+1)n}{p_0(x)}} \tag{3.25}$$

and split

$$\sum_{j=0}^{\infty} C^{-q} 2^{jn\left(\frac{1}{p_1(x)} - \frac{1}{p_0(x)}\right)q} \alpha_j^q = \underbrace{\sum_{j=0}^{J} \dots}_{I} + \underbrace{\sum_{j=J+1}^{\infty} \dots}_{II}.$$

By (3.23) and (3.25)

$$I = \sum_{j=0}^{J} C^{-q} 2^{\frac{jnq}{p_1(x)}} \cdot 2^{-\frac{jnq}{p_0(x)}} \cdot \alpha_j^q \leq \sum_{j=0}^{J} C^{-q} c_{\log} 2^{\frac{jnq}{p_1(x)}} \leq c^{-1} 2^{\frac{(J+1)nq}{p_1(x)}} \leq 2^{\frac{Jnq}{p_1(x)}} \leq \alpha^{\frac{qp_0(x)}{p_1(x)}}.$$

The monotonicity of $\{\alpha_j\}$ and (3.25) lead to

$$\begin{aligned} II &\leq \sum_{j=J+1}^{\infty} 2^{jn\left(\frac{1}{p_1(x)} - \frac{1}{p_0(x)}\right)q} \alpha_j^q C^{-q} \leq \alpha^q C^{-q} \sum_{j=J+1}^{\infty} 2^{jn\left(\frac{1}{p_1(x)} - \frac{1}{p_0(x)}\right)q} \leq \alpha^q C^{-q} 2^{Jn\left(\frac{1}{p_1(x)} - \frac{1}{p_0(x)}\right)q} \\ &\leq \alpha^q C^{-q} \left(\alpha^{p_0(x)} 2^{-n}\right)^{\left(\frac{1}{p_1(x)} - \frac{1}{p_0(x)}\right)q} = \alpha^{\frac{qp_0(x)}{p_1(x)}} C^{-q} 2^{n\left(\frac{1}{p_0(x)} - \frac{1}{p_1(x)}\right)q} \leq \alpha^{\frac{qp_0(x)}{p_1(x)}} \end{aligned}$$

This finishes the proof of (3.24). Now (3.20), (3.22), (3.24) with (3.21) gives

$$||\gamma|f^{s_1(\cdot)}_{p_1(\cdot),q}|| \leq C||\gamma|f^{s_0(\cdot)}_{p_0(\cdot),\infty}||.$$

□

Remark 3.3. The original proof of Jawerth of Theorem 1.1 used the technique of a distribution function, which fails for $L_{p(\cdot)}(\mathbb{R}^n)$. Another proof was given by Johnsen and Sickel [22] and relied on an inequality of Plancherel-Pólya-Nikol'skij type. Its classical proof [34, Chapter 1.3] is based on dilation arguments and (at least to our knowledge) there is still no analogue of these inequalities for $L_{p(\cdot)}(\mathbb{R}^n)$ up to now.

Our proofs of Theorems 3.1 and 3.2 were motived by [38]. An essential technique used there was the concept of non-increasing rearrangement. Unfortunately, it fails completely in the case of variable integrability exponents $p_0(x)$ and $p_1(x)$. To avoid this obstacle, we had to employ the somehow artificial inequality (3.24) - or its analogue (3.16). To motivate this step, let us consider the interpolation inequality between Lorentz spaces

$$||f|L_{p_1,q}(0,1)|| \leq c\,||f|L_{p_0,\infty}(0,1)||^{\theta} \cdot ||f|L_{\infty}(0,1)||^{1-\theta} \tag{3.26}$$

with

$$0 < p_0 < p_1 < \infty, \quad \frac{1}{p_1} = \frac{\theta}{p_0} + \frac{1-\theta}{\infty}, \quad 0 < \theta < 1$$

and its discrete version

$$\left(\sum_{j=0}^{\infty} 2^{-jnq(\frac{1}{p_0} - \frac{1}{p_1})} f^*(2^{-jn})^q\right)^{1/q} \leq c \left(\sup_{j\in\mathbb{N}_0} 2^{-jn/p_0} f^*(2^{-jn})\right)^{1-\frac{p_0}{p_1}} \cdot \left(\sup_{j\in\mathbb{N}_0} f^*(2^{-jn})\right)^{\frac{p_0}{p_1}}.$$

We refer to [2, Chapter 2] as a standard reference for non-increasing rearrangements and to [2, Chapter 4.4] for the notation connected with Lorentz spaces. We leave the details to the reader. The reader may also observe some similarities between (3.26) and the inequality (4) of [22].

Using Theorem 2.12, we obtain immediately following

Theorem 3.4. *Let s_0, s_1, p_0, p_1 and q be continuous functions satisfying the Standing assumptions of [9]. Let $s_0(x) \geq s_1(x)$ and $p_0(x) \leq p_1(x)$ for all $x \in \mathbb{R}^n$ with*

$$s_0(x) - \frac{n}{p_0(x)} = s_1(x) - \frac{n}{p_1(x)}, \quad x \in \mathbb{R}^n.$$

Then

$$F^{s_0(\cdot)}_{p_0(\cdot),q(\cdot)}(\mathbb{R}^n) \hookrightarrow F^{s_1(\cdot)}_{p_1(\cdot),q(\cdot)}(\mathbb{R}^n).$$

We denote by $W^k_{p(\cdot)}(\mathbb{R}^n)$ the Sobolev space of functions form $L_{p(\cdot)}(\mathbb{R}^n)$, such that all its distributional derivatives of order smaller or equal to k exist and belong to $L_{p(\cdot)}(\mathbb{R}^n)$. Furthermore, we introduce the Bessel potential spaces of variable integrability introduced by Almeida and Samko [1] and by Gurka, Harjulehto and Nekvinda [12]. Let $\sigma \in \mathbb{R}$ and let $B^\sigma = F^{-1}(1+|\xi|^2)^{-\sigma/2}F$ be the Bessel potential operator. We set

$$L^\sigma_{p(\cdot)}(\mathbb{R}^n) = \{B^\sigma f : f \in L_{p(\cdot)}(\mathbb{R}^n)\}$$

and equip this space with norm $||f|L^\sigma_{p(\cdot)}(\mathbb{R}^n)|| = ||B^{-\sigma} f|L_{p(\cdot)}(\mathbb{R}^n)||$.

Let $p \in C^{\log}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$ and $\sigma \in [0,\infty)$. It was shown in [9, Theorem 4.5] that $F^\sigma_{p(\cdot),2}(\mathbb{R}^n) \cong L^\sigma_{p(\cdot)}(\mathbb{R}^n)$ in the sense of equivalent norms. If moreover $\sigma \in \mathbb{N}_0$, then $F^\sigma_{p(\cdot),2}(\mathbb{R}^n) \cong W^\sigma_{p(\cdot)}(\mathbb{R}^n)$.

Hence setting $q = 2$ implies embeddings of Bessel potential spaces.

Theorem 3.5. *Let $0 \leq s_1 \leq s_0 < \infty$ and $p_0, p_1 \in C^{\log}(\mathbb{R}^n)$ with $1 < p_0^- \leq p_0(x) \leq p_1(x) \leq p_1^+ < \infty$ for all $x \in \mathbb{R}^n$. If*

$$s_0 - \frac{n}{p_0(x)} = s_1 - \frac{n}{p_1(x)}, \quad x \in \mathbb{R}^n,$$

then

$$L^{s_0}_{p_0(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{s_1}_{p_1(\cdot)}(\mathbb{R}^n).$$

If $s_1 \in \mathbb{N}_0$, then $L^{s_1}_{p_1(\cdot)}(\mathbb{R}^n)$ may be replaced by $W^{s_1}_{p_1(\cdot)}(\mathbb{R}^n)$ and similarly for s_0.

Remark 3.6. Let us only mention, that if $1 < p^- \leq p^+ < \infty$, then $p \in C^{\log}(\mathbb{R}^n)$ if, and only if, $\frac{1}{p} \in C^{\log}(\mathbb{R}^n)$. So the *Standing assumptions* on p_0 and p_1 are satisfied and the proof becomes trivial.

Theorem 3.7. *Let $s_0, s_1, p_0, p_1, q_0, q_1$ be continuous functions satisfying the Standing assumptions of [9] with*

$$s_0(x) - \frac{n}{p_0(x)} = s_1(x) - \frac{n}{p_1(x)}, \quad x \in \mathbb{R}^n$$

and

$$\inf_{x\in\mathbb{R}^n}(s_0(x) - s_1(x)) = n \inf_{x\in\mathbb{R}^n}\Big(\frac{1}{p_0(x)} - \frac{1}{p_1(x)}\Big) > 0.$$

Then

$$F^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)}(\mathbb{R}^n) \hookrightarrow F^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)}(\mathbb{R}^n).$$

Proof. By monotonicity and using Theorem 3.2, we obtain

$$f^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)} \hookrightarrow f^{s_0(\cdot)}_{p_0(\cdot),\infty} \hookrightarrow f^{s_1(\cdot)}_{p_1(\cdot),q_1^-} \hookrightarrow f^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)}$$

and Theorem 2.12 finishes the proof. □

Finally, we may combine our embedding results with the trace results of [9] and obtain the following Sobolev embeddings for traces. We state it for Sobolev spaces, but a similar assertion holds also for Bessel potential spaces and Triebel-Lizorkin spaces.

Theorem 3.8. *Let* $k \in \mathbb{N}$ *and* $1 < p^- \le p^+ < \frac{n}{k}$ *with* $\frac{1}{p} \in C^{\log}(\mathbb{R}^n)$. *Then*

$$W^k_{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L_{\frac{(n-1)p(\cdot)}{n-kp(\cdot)}}(\mathbb{R}^{n-1}).$$

Proof. By Theorem 3.13. of [9], we have

$$\operatorname{tr} W^k_{p(\cdot)}(\mathbb{R}^n) \to F^{k-\frac{1}{p(\cdot)}}_{p(\cdot),p(\cdot)}(\mathbb{R}^{n-1}),$$

which may be combined with Theorem 3.7

$$F^{k-\frac{1}{p(\cdot)}}_{p(\cdot),p(\cdot)}(\mathbb{R}^{n-1}) \hookrightarrow F^0_{\tilde{p}(\cdot),2}(\mathbb{R}^{n-1}) = L_{\tilde{p}(\cdot)}(\mathbb{R}^{n-1})$$

for $\tilde{p}(\cdot)$ given by

$$k - \frac{1}{p(\cdot)} - \frac{n-1}{p(\cdot)} = -\frac{n-1}{\tilde{p}(\cdot)}.$$

This finishes the proof. □

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A diagonal embedding theorem for function spaces with dominating mixed smoothness

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Abstract

The aim of this paper is to study the diagonal embeddings of function spaces with dominating mixed smoothness. From certain point of view, this paper may be considered as a direct continuation of [8] and [6].

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1 Introduction

Spaces with dominating mixed smoothness were introduced by S. M. Nikol'skii ([4], [5]). The simplest case on the plane $\mathbb{R}^2$ are the spaces of Sobolev type

$$S_p^{\overline{r}}W(\mathbb{R}^2) = \Big\{ f | f \in L_p(\mathbb{R}^2), ||f|S_p^{\overline{r}}W(\mathbb{R}^2)|| = ||f|L_p|| + \Big|\Big| \frac{\partial^{r_1} f}{\partial x_1^{r_1}} \mid L_p \Big|\Big| + \\ + \Big|\Big| \frac{\partial^{r_2} f}{\partial x_2^{r_2}} \mid L_p \Big|\Big| + \Big|\Big| \frac{\partial^{r_1+r_2} f}{\partial x_1^{r_1} \partial x_2^{r_2}} \mid L_p \Big|\Big| < \infty \Big\}, \tag{1.1}$$

where $1 < p < \infty, r_i = 0, 1, 2, \dots; (i = 1, 2)$. The mixed derivative $\frac{\partial^{r_1+r_2} f}{\partial x_1^{r_1} \partial x_2^{r_2}}$ plays the dominant part here and gave the name to this class of spaces.

These spaces were studied extensively by many mathematicians. We quote Amanov ([1]), Schmeisser and Triebel ([7]) to mention at least some of them. We describe some aspects of this theory necessary in the sequel in Section 2. Sections 3 and 4 are devoted to the study of the trace operator

$$T : f(x_1, x_2) \to f(x_1, x_1). \tag{1.2}$$

In [8] Triebel proved that, for $1 \le p \le \infty$, the trace operator (1.2) is a retraction from $S_{p,1}^{(r_1,r_2)}B(\mathbb{R}^2)$ onto $B_{p,1}^{\varrho}(\mathbb{R})$, where $\varrho = \min(r_1, r_2, r_1 + r_2 - \frac{1}{p}) > 0$. The q-dependence was studied in [6]. Rodriguez proved that (1.2) is a retraction from $S_{p,q}^{(r_1,r_2)}B(\mathbb{R}^2)$ onto $B_{p,q}^{\varrho}(\mathbb{R})$, where

$$0 < p \le \infty, 0 < q < \infty, \varrho > \sigma_p = \max\Big(\frac{1}{p} - 1, 0\Big) \text{ and } \min(r_1, r_2) \neq \frac{1}{p}.$$

In the "limiting case" $\min(r_1, r_2) = \frac{1}{p}$ the same result is proven for $q \le \min(1, p)$.

We fill some of the minor gaps left open by Rodriguez in the B-case and study the trace operator in the context of F-spaces. As these include the spaces of dominating mixed smoothness of Sobolev type (1.1), we answer the question of their traces on the diagonal.

I would like to thank to prof. Sickel and prof. Triebel for valuable discussions on this topic.

2 Notation and Definitions

As usual, $\mathbb{R}^d$ denotes the $d-$dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The letter $\mathbb{Z}$ stands for the set of all integer numbers and $\mathbb{C}$ denotes the plain of complex numbers.

If $x, y \in \mathbb{R}^d$, we write $x > y$ if, and only if, $x_i > y_i$ for every $i = 1, \ldots, d$. Similarly, we define the relations $x \geq y, x < y, x \leq y$. Finally, in slight abuse of notation, we write $x > \lambda$ for $x \in \mathbb{R}^d, \lambda \in \mathbb{R}$ if $x_i > \lambda, i = 1, \ldots, d$.

When $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ is a multi-index, we denote its length by $|\alpha| = \sum_{j=1}^d \alpha_j$.

Let $S(\mathbb{R}^d)$ be the Schwartz space of all complex–valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^d$. We denote the $d-$dimensional Fourier transform of a function $\varphi \in S(\mathbb{R}^d)$ by $\hat{\varphi}$. Its inverse is denoted by $\varphi^\vee$. Both $\hat{}$ and $^\vee$ are extended to the dual Schwartz space $S'(\mathbb{R}^d)$ in the usual way.

We recall the basic aspects of the theory of function spaces used in the sequel. We don't mean to give some extensive survey on various decomposition techniques. Especially, as far as the standard Besov ($B^s_{p,q}(\mathbb{R}^d)$) and Triebel-Lizorkin ($F^s_{p,q}(\mathbb{R}^d)$) spaces are considered, we use the references [9] and [10]. Furthermore, we give the definition of function spaces with dominating mixed smoothness in general dimension. Setting $d = 1$, one gets the one-dimensional version $B^s_{p,q}(\mathbb{R})$ or $F^s_{p,q}(\mathbb{R})$, respectively.

Let $\varphi \in S(\mathbb{R})$ with

$$\varphi(t) = 1 \quad \text{if} \quad |t| \leq 1 \quad \text{and} \quad \varphi(t) = 0 \quad \text{if} \quad |t| \geq \frac{3}{2}. \tag{2.1}$$

We put $\varphi_0 = \varphi, \varphi_1(t) = \varphi(t/2) - \varphi(t)$ and

$$\varphi_j(t) = \varphi_1(2^{-j+1}t), \quad t \in \mathbb{R}, \quad j \in \mathbb{N}.$$

For $\overline{k} = (k_1, \ldots, k_d) \in \mathbb{N}_0^d$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we define $\varphi_{\overline{k}}(x) = \varphi_{k_1}(x_1) \cdots \varphi_{k_d}(x_d)$. Then, since

$$\sum_{\overline{k} \in \mathbb{N}_0^d} \varphi_{\overline{k}}(x) = 1 \quad \text{for every} \quad x \in \mathbb{R}^d, \tag{2.2}$$

the system $\{\varphi_{\overline{k}}\}_{\overline{k} \in \mathbb{N}_0^d}$ forms a dyadic resolution of unity with the inner tensor product structure.

Definition 2.1. Let $\overline{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$, $0 < q \leq \infty$.
(i) Let $0 < p \leq \infty$. Then $S^{\overline{r}}_{p,q}B(\mathbb{R}^d)$ is the collection of all $f \in S'(\mathbb{R}^d)$ such that

$$||f|S^{\overline{r}}_{p,q}B(\mathbb{R}^d)||_\varphi = \Big(\sum_{\overline{k} \in \mathbb{N}_0^d} 2^{q\overline{k}\cdot\overline{r}} ||(\varphi_{\overline{k}}\hat{f})^\vee|L_p(\mathbb{R}^d)||^q\Big)^{1/q} = ||2^{\overline{k}\cdot\overline{r}}(\varphi_{\overline{k}}\hat{f})^\vee|\ell_q(L_p)|| \tag{2.3}$$

is finite.
(ii) Let $0 < p < \infty$. Then $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$ is the collection of all $f \in S'(\mathbb{R}^d)$ such that

$$||f|S^{\overline{r}}_{p,q}F(\mathbb{R}^d)||_\varphi = \Big|\Big|\Big(\sum_{\overline{k} \in \mathbb{N}_0^d} |2^{\overline{k}\cdot\overline{r}}(\varphi_{\overline{k}}\hat{f})^\vee(\cdot)|^q\Big)^{1/q}|L_p(\mathbb{R}^d)\Big|\Big| = ||2^{\overline{k}\cdot\overline{r}}(\varphi_{\overline{k}}\hat{f})^\vee|L_p(\ell_q)|| \tag{2.4}$$

is finite.

Remark 2.2. Sometimes, we write $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ meaning one of spaces $S^{\overline{r}}_{p,q}B(\mathbb{R}^d)$ or $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$. As mentioned above, by setting $d=1$, we get $B^s_{p,q}(\mathbb{R}) = S^{(s)}_{p,q}B(\mathbb{R})$ and $F^s_{p,q}(\mathbb{R}) = S^{(s)}_{p,q}F(\mathbb{R})$. If we replace in this case the factor 2^{ks} by $(k+1)^{\alpha}2^{ks}$, $\alpha \in \mathbb{R}$, we get the spaces of generalised smoothness $A^{(s,\alpha)}_{p,q}(\mathbb{R})$. We refer to [3] and references given there for details.

Our approach uses the full power of several decomposition techniques developed for these function spaces in [9], [3] and [12]. They all work with sequence spaces associated to these function spaces.

For $\overline{\nu} \in \mathbb{N}^d_0, \overline{m} \in \mathbb{Z}^d$ we denote by $Q_{\overline{\nu}\,\overline{m}}$ the cube with the centre at the point $2^{-\overline{\nu}}\overline{m} = (2^{-\nu_1}m_1, \ldots, 2^{-\nu_d}m_d)$ with sides parallel to the coordinate axes and of lengths $2^{-\nu_1}, \ldots, 2^{-\nu_d}$. We denote by $\chi_{\overline{\nu}\,\overline{m}} = \chi_{Q_{\overline{\nu}\,\overline{m}}}$ the characteristic function of $Q_{\overline{\nu}\,\overline{m}}$ and by $c\,Q_{\overline{\nu}\,\overline{m}}$ we mean a cube concentric with $Q_{\overline{\nu}\,\overline{m}}$ with sides c times longer.

Definition 2.3. If $0 < p, q \le \infty$, $\overline{r} \in \mathbb{R}^d$ and

$$\lambda = \{\lambda_{\overline{\nu}\,\overline{m}} \in \mathbb{C} : \overline{\nu} \in \mathbb{N}^d_0, \overline{m} \in \mathbb{Z}^d\} \tag{2.5}$$

then we define

$$s^{\overline{r}}_{pq}b = \left\{\lambda : ||\lambda|s^{\overline{r}}_{pq}b|| = \left(\sum_{\overline{\nu}\in\mathbb{N}^d_0} 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})q}\Big(\sum_{\overline{m}\in\mathbb{Z}^d} |\lambda_{\overline{\nu}\,\overline{m}}|^p\Big)^{q/p}\right)^{1/q} < \infty\right\} \tag{2.6}$$

and

$$s^{\overline{r}}_{pq}f = \left\{\lambda : ||\lambda|s^{\overline{r}}_{pq}f|| = \left\|\left(\sum_{\overline{\nu}\in\mathbb{N}^d_0}\sum_{\overline{m}\in\mathbb{Z}^d} |2^{\overline{\nu}\cdot\overline{r}}\lambda_{\overline{\nu}\,\overline{m}}\chi_{\overline{\nu}\,\overline{m}}(\cdot)|^q\right)^{1/q} |L_p(\mathbb{R}^d)\right\| < \infty\right\} \tag{2.7}$$

with the usual modification for p and/or q equal to ∞.

Remark 2.4. We point out that with λ given by (2.5) and $g_{\overline{\nu}}(x) = \sum_{\overline{m}\in\mathbb{Z}^d} \lambda_{\overline{\nu}\,\overline{m}}\chi_{\overline{\nu}\,\overline{m}}(x)$, we obtain

$$||\lambda|s^{\overline{r}}_{pq}b|| = ||2^{\overline{\nu}\cdot\overline{r}}g_{\overline{\nu}}|\ell_q(L_p)||, \qquad ||\lambda|s^{\overline{r}}_{pq}f|| = ||2^{\overline{\nu}\cdot\overline{r}}g_{\overline{\nu}}|L_p(\ell_q)||.$$

Definition 2.5. If $0 < p, q \le \infty$, $r, \alpha \in \mathbb{R}$ and

$$\lambda = \{\lambda_{\mu\,n} \in \mathbb{C} : \mu \in \mathbb{N}_0, n \in \mathbb{Z}\} \tag{2.8}$$

then we define

$$b^{(r,\alpha)}_{pq} = \left\{\lambda : ||\lambda|b^{(r,\alpha)}_{pq}|| = \left(\sum_{\mu\in\mathbb{N}_0} (\mu+1)^{\alpha q}2^{\mu(r-\frac{1}{p})q}\Big(\sum_{n\in\mathbb{Z}} |\lambda_{\mu\,n}|^p\Big)^{q/p}\right)^{1/q} < \infty\right\} \tag{2.9}$$

and

$$f^{(r,\alpha)}_{pq} = \left\{\lambda : ||\lambda|f^{(r,\alpha)}_{pq}|| = \left\|\left(\sum_{\mu\in\mathbb{N}_0}\sum_{n\in\mathbb{Z}} |(\mu+1)^{\alpha}2^{\mu r}\lambda_{\mu\,n}\chi_{\mu\,n}(\cdot)|^q\right)^{1/q} |L_p(\mathbb{R})\right\| < \infty\right\} \tag{2.10}$$

with the usual modification for p and/or q equal to ∞.

Next we briefly describe the atomic and subatomic decomposition. We refer to [11] and [12] for details. Compared to the situation there, we now concentrate on the "regular" case,

$$\overline{r} > \begin{cases} \sigma_p = \max\left(\frac{1}{p} - 1, 0\right) & \text{in the B-case} \\ \sigma_{pq} = \max\left(\frac{1}{\min(p,q)} - 1, 0\right) & \text{in the F-case.} \end{cases} \tag{2.11}$$

Definition 2.6. Let $\overline{K} \in \mathbb{N}_0^d$ and $\gamma > 1$. A $\overline{K}$-times differentiable complex-valued function $a(x)$ is called $\overline{K}$-atom related to $Q_{\overline{\nu}\,\overline{m}}$ if

$$\operatorname{supp} a \subset \gamma Q_{\overline{\nu}\,\overline{m}}, \tag{2.12}$$

and

$$|D^\alpha a(x)| \le 2^{\alpha \cdot \overline{\nu}} \qquad \text{for} \quad 0 \le \alpha \le \overline{K} \tag{2.13}$$

Theorem 2.7. *Let $0 < p, q \le \infty$, ($p < \infty$ in the $F-$case) and $\overline{r} \in \mathbb{R}^d$ with (2.11). Fix $\overline{K} \in \mathbb{N}_0^d$ with*

$$K_i \ge (1 + [r_i])_+ \qquad i = 1, \ldots, d. \tag{2.14}$$

Then $f \in S'(\mathbb{R}^d)$ belongs to $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ if, and only if, it can be represented as

$$f = \sum_{\overline{\nu} \in \mathbb{N}_0^d} \sum_{\overline{m} \in \mathbb{Z}^d} \lambda_{\overline{\nu}\,\overline{m}} a_{\overline{\nu}\,\overline{m}}(x), \quad \textit{convergence being in} \quad S'(\mathbb{R}^d), \tag{2.15}$$

where $\{a_{\overline{\nu}\,\overline{m}}(x)\}_{\overline{\nu} \in \mathbb{N}_0^d, \overline{m} \in \mathbb{Z}^d}$ are $\overline{K}$-atoms related to $Q_{\overline{\nu}\,\overline{m}}$ and $\lambda \in s^{\overline{r}}_{pq}a$. Furthermore,

$$\inf ||\lambda | s^{\overline{r}}_{pq} a||,$$

where the infimum runs over all admissible representations (2.15)*, is an equivalent quasi-norm in $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$.*

Definition 2.8. Let $\psi \in S(\mathbb{R})$ be a non-negative function with

$$\operatorname{supp} \psi \subset \{t \in \mathbb{R} : |t| < 2^\phi\} \tag{2.16}$$

for some $\phi \ge 0$ and

$$\sum_{n \in \mathbb{Z}} \psi(t - n) = 1, \quad t \in \mathbb{R}. \tag{2.17}$$

We define $\Psi(x) = \psi(x_1) \cdot \ldots \cdot \psi(x_d)$ and $\Psi^\beta(x) = x^\beta \Psi(x)$ for $x = (x_1, \ldots, x_d)$ and $\beta \in \mathbb{N}_0^d$. Further let $\overline{r} \in \mathbb{R}^d$ and $0 < p \le \infty$. Then

$$(\beta qu)_{\overline{\nu}\,\overline{m}}(x) = \Psi^\beta(2^{\overline{\nu}} x - \overline{m}), \quad \overline{\nu} \in \mathbb{N}_0^d,\ \overline{m} \in \mathbb{Z}^d \tag{2.18}$$

is called an β-quark related to $Q_{\overline{\nu}\,\overline{m}}$.

Theorem 2.9. *Let $0 < p, q \le \infty$ (with $p < \infty$ in the F-case) and $\overline{r} \in \mathbb{R}^d$ with* (2.11).
(i) *Let*

$$\lambda = \{\lambda^\beta : \beta \in \mathbb{N}_0^d\} \quad \textit{with} \quad \lambda^\beta = \{\lambda^\beta_{\overline{\nu}\,\overline{m}} \in \mathbb{C} : \overline{\nu} \in \mathbb{N}_0^d, \overline{m} \in \mathbb{Z}^d\}$$

and let $\varrho > \phi$, where ϕ is the number from (2.16)*. Then $f \in S'(\mathbb{R}^d)$ belongs to $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ if, and only if, it can be represented as*

$$f = \sum_{\beta \in \mathbb{N}_0^d} \sum_{\overline{\nu} \in \mathbb{N}_0^d} \sum_{\overline{m} \in \mathbb{Z}^d} \lambda^\beta_{\overline{\nu}\,\overline{m}} (\beta qu)_{\overline{\nu}\,\overline{m}}(x), \quad \textit{convergence being in} \quad S'(\mathbb{R}^d), \tag{2.19}$$

where $(\beta qu)_{\overline{\nu}\,\overline{m}}(x)$ *are* β*-quarks related to* $Q_{\overline{\nu}\,\overline{m}}$ *and*

$$\sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|}||\lambda^\beta|s^{\overline{r}}_{pq}a|| < \infty.$$

Furthermore,

$$\inf \sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|}||\lambda^\beta|s^{\overline{r}}_{pq}a|| < \infty,$$

where the infimum runs over all admissible representations (2.19)*, is an equivalent quasi-norm in* $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$*.*

Remark 2.10. According to [9], [10] and [3], similar decomposition theorems are available also for spaces $A^{(s,\alpha)}_{p,q}(\mathbb{R})$. They may be obtained from Theorem 2.7 and Theorem 2.9 by setting $d = 1$ and replacing $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ with $A^{(s,\alpha)}_{p,q}(\mathbb{R})$ and $s^{\overline{r}}_{p,q}a$ with $a^{(s,\alpha)}_{p,q}$.

Lemma 2.11. *Let* $0 < p < \infty, 0 < q \le \infty, \overline{r} \in \mathbb{R}^d$ *and* $\gamma_1, \gamma_2 > 0$*. Let*

$$E_{\overline{\nu}\,\overline{m}} \subset \gamma_1 Q_{\overline{\nu}\,\overline{m}}, \quad \frac{|E_{\overline{\nu}\,\overline{m}}|}{|Q_{\overline{\nu}\,\overline{m}}|} \ge \gamma_2, \quad \overline{\nu} \in \mathbb{N}_0^d, \quad \overline{m} \in \mathbb{Z}^d. \tag{2.20}$$

Then

$$\Big|\Big|\ 2^{\overline{\nu}\cdot\overline{r}}|\lambda_{\overline{\nu}\,\overline{m}}|\chi_{E_{\overline{\nu}\,\overline{m}}}(\cdot)\ \Big|L_p(\ell_q)\Big|\Big| \approx ||\lambda|s^{\overline{r}}_{p,q}f||$$

with constants of equivalence independent of λ*.*

Proof. We follow closely [2]. Namely, from (2.20) we see that

$$\chi_{E_{\overline{\nu}\,\overline{m}}}(x) \le c\,\overline{M}\chi_{Q_{\overline{\nu}\,\overline{m}}}(x), \quad x \in \mathbb{R}^d$$

and

$$\chi_{Q_{\overline{\nu}\,\overline{m}}}(x) \le c\,\overline{M}\chi_{E_{\overline{\nu}\,\overline{m}}}(x), \quad x \in \mathbb{R}^d.$$

Here $\overline{M} = M_2 \circ M_1$, where

$$(M_1 f)(x) = \sup_{s>0} \frac{1}{2s}\int_{x_1-s}^{x_1+s} |f(t, x_2)|\mathrm{d}t, \quad x = (x_1, x_2) \in \mathbb{R}^2, \tag{2.21}$$

and similar for M_2.

Then we take $\omega > 0$ such that $\omega < \min(1, p, q)$ and observe

$$\Big|\Big|\ 2^{\overline{\nu}\cdot\overline{r}}|\lambda_{\overline{\nu}\,\overline{m}}|\chi_{E_{\overline{\nu}\,\overline{m}}}(\cdot)\ \Big|L_p(\ell_q)\Big|\Big| = \Big|\Big|\ 2^{\overline{\nu}\cdot\overline{r}\omega}|\lambda_{\overline{\nu}\,\overline{m}}|^\omega\chi_{E_{\overline{\nu}\,\overline{m}}}(\cdot)\ \Big|L_{\frac{p}{\omega}}(\ell_{\frac{q}{\omega}})\Big|\Big|^{\frac{1}{\omega}}$$

with a direct counterpart for $||\lambda|s^{\overline{r}}_{pq}f||$. This, together with the boundedness of the maximal operator $\overline{M}$ (see [7] or [12] for details) finishes the proof. ☐

By $\Gamma = \{(t, t) \in \mathbb{R}^2 : t \in \mathbb{R}\}$ we denote the diagonal of $\mathbb{R}^2$. As Γ is isomorphic to $\mathbb{R}$, all the function spaces considered so far may be taken over from the real line to Γ. In the natural sense, we get $A^{(r,\alpha)}_{p,q}(\mathbb{R}) = A^{(r,\alpha)}_{p,q}(\Gamma)$ for all admissible α, p, q and r.

Finally, we discuss the notion of the trace. The trace operator Tf, as it is described in (1.2), makes sense only when the function f satisfies some regularity conditions, especially, if it is continuous. This is satisfied for $f \in S^{\overline{r}}_{p,q}A(\mathbb{R}^2)$ with $\overline{r} > \frac{1}{p}$. To avoid this restriction, we use the following general definition of the trace. It is well known that $S^0_{\infty,1}B(\mathbb{R}^2) \hookrightarrow C(\mathbb{R}^2)$. So, for $f \in S^0_{\infty,1}B(\mathbb{R}^2)$, we may define $(\mathrm{tr}_\Gamma f)(t) = f(t,t)$. If $S(\mathbb{R}^2)$ is a dense subspace of $S^{\overline{r}}_{p,q}A(\mathbb{R}^2)$ and tr_Γ satisfies the inequality

$$|| \,\mathrm{tr}_\Gamma f|X(\Gamma)|| \leq c||f|S^{\overline{r}}_{p,q}A(\mathbb{R}^2)||, \quad f \in S(\mathbb{R}^2), \tag{2.22}$$

for some quasi-Banach space $X(\Gamma) \hookrightarrow S'(\mathbb{R})$, then there is a unique extension operator $\mathrm{tr}_\Gamma : S^{\overline{r}}_{p,q}A(\mathbb{R}^2) \to X(\Gamma)$. It turns out that this defines the $\mathrm{tr}_\Gamma f$ for all $f \in S^{\overline{r}}_{p,q}A(\mathbb{R}^2)$ with $\max(p,q) < \infty$ and $\overline{r} = (r_1, r_2)$ with $\overline{r}$ large enough and this definition does not depend on $X(\Gamma)$. In the last case, $q = \infty$, we use the embedding $S^{\overline{r}}_{p,\infty}A(\mathbb{R}^2) \hookrightarrow S^{\overline{r}-\epsilon}_{p,1}A(\mathbb{R}^2)$, with $\epsilon > 0$ small, which defines $\mathrm{tr}_\Gamma f$ as soon as the trace operator is defined on $S^{\overline{r}-\epsilon}_{p,1}A(\mathbb{R}^2)$.

We write $\mathrm{tr}_\Gamma : S^{\overline{r}}_{p,q}A(\mathbb{R}^2) \to X(\Gamma)$, if (2.22) is satisfied for all $f \in S^{\overline{r}}_{p,q}A(\mathbb{R}^2)$. The symbol $\mathrm{tr}_\Gamma\, S^{\overline{r}}_{p,q}A(\mathbb{R}^2) = X(\Gamma)$ is used to denote that $\mathrm{tr}_\Gamma : S^{\overline{r}}_{p,q}A(\mathbb{R}^2) \to X(\Gamma)$ and, moreover, there is an (linear, bounded) extension operator $\mathrm{ext} : X(\Gamma) \to S^{\overline{r}}_{p,q}A(\mathbb{R}^2)$ such that $\mathrm{tr}_\Gamma \circ \mathrm{ext} = id$.

Hence $\mathrm{tr}_\Gamma\, S^{\overline{r}}_{p,q}A(\mathbb{R}^2) = X(\Gamma)$ if, and only if, tr_Γ is a retraction from $S^{\overline{r}}_{p,q}A(\mathbb{R}^2)$ onto $X(\Gamma)$.

3 Traces of B-spaces

Theorem 3.1. *Let $0 < p, q \leq \infty$, and $\overline{r} = (r_1, r_2) \in \mathbb{R}^2$ with*

$$0 < r_1 \leq r_2, \varrho = \min\Big(r_1, r_1 + r_2 - \frac{1}{p}\Big) > \sigma_p.$$

If $r_2 \neq \frac{1}{p}$ or $r_2 = \frac{1}{p}$ and $q \leq \min(1,p)$ then

$$\mathrm{tr}_\Gamma\, S^{\overline{r}}_{p,q}B(\mathbb{R}^2) = B^{\varrho}_{p,q}(\Gamma).$$

If $r_2 = \frac{1}{p}$, $1 \leq \min(p,q)$ then

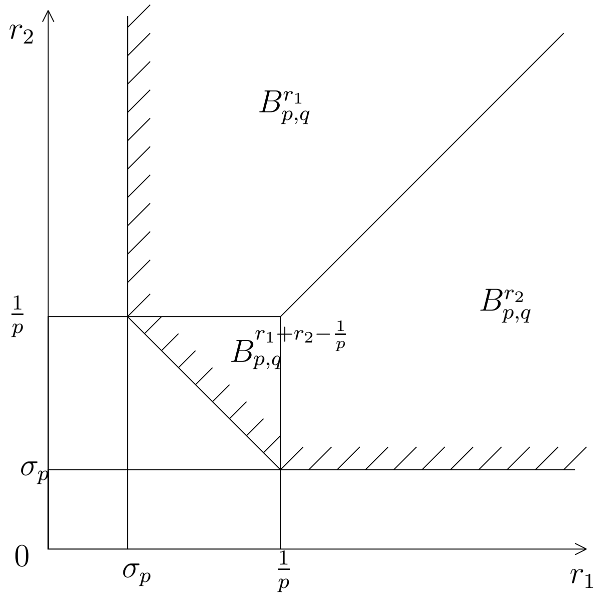
$$\mathrm{tr}_\Gamma\, S^{\overline{r}}_{p,q}B(\mathbb{R}^2) = B^{(r_1, \frac{1}{q}-1)}_{p,q}(\Gamma).$$

Finally, if $r_2 = \frac{1}{p}$, $p \leq \min(1,q)$ then

$$\mathrm{tr}_\Gamma : S^{\overline{r}}_{p,q}B(\mathbb{R}^2) \to B^{(r_1, \frac{1}{q}-\frac{1}{p})}_{p,q}(\Gamma)$$

and

$$\mathrm{ext} : B^{(r_1, \min(\frac{1}{q}-1, 0))}_{p,q}(\Gamma) \to S^{\overline{r}}_{p,q}B(\mathbb{R}^2).$$



Proof. Step 1. - quarkonial decomposition, definition of $\mathrm{tr}_\Gamma f$
Let $f \in S^{\overline{r}}_{p,q}B(\mathbb{R}^2)$. According to Theorem 2.9, f may be decomposed as

$$f = \sum_{\beta \in \mathbb{N}_0^2} f^\beta, \quad f^\beta(x) = \sum_{\overline{\nu} \in \mathbb{N}_0^2} \sum_{\overline{m} \in \mathbb{Z}^2} \lambda^\beta_{\overline{\nu}\,\overline{m}} (\beta q u)_{\overline{\nu}\,\overline{m}}(x) \tag{3.1}$$

with

$$\sup_{\beta \in \mathbb{N}_0^2} 2^{\varrho|\beta|} ||\lambda^\beta | s^{\overline{r}}_{p,q}b|| \approx ||f|S^{\overline{r}}_{p,q}B(\mathbb{R}^2)||. \tag{3.2}$$

We point out that we may assume taht the coefficients λ of the optimal quarkonial decomposition (3.1) depend *linearly* on f. We refer again to [10] and [12] for detailed discussion of this effect.
Naturally, we define

$$\mathrm{tr}_\Gamma f = \sum_{\beta \in \mathbb{N}_0^2} (\mathrm{tr}_\Gamma f)_\beta, \quad (\mathrm{tr}_\Gamma f)_\beta(t) = \sum_{\overline{\nu} \in \mathbb{N}_0^2} \sum_{\overline{m} \in \mathbb{Z}^2} \lambda^\beta_{\overline{\nu}\,\overline{m}} (\beta q u)_{\overline{\nu}\,\overline{m}}(t,t). \tag{3.3}$$

In (3.3) we may restrict to $\overline{m}$ from

$$B_{\overline{\nu}} = \{\overline{m} \in \mathbb{Z}^2 : \mathrm{supp}(\beta q u)_{\overline{\nu}\,\overline{m}} \cap \Gamma \neq \emptyset\}.$$

Next we split

$$B_{\overline{\nu}} = \bigcup_{n \in \mathbb{Z}} B_{\overline{\nu}\,n} \tag{3.4}$$

such that

$$\sup_{\overline{\nu},n} |B_{\overline{\nu}\,n}| < \infty \tag{3.5}$$

and, for $\mu = \max(\nu_1, \nu_2)$,

$$\{t : (\beta q u)_{\overline{\nu}\,\overline{m}}(t,t) \neq 0\} \subset (2^{-\mu}(n-c), 2^{-\mu}(n+c)), \quad \overline{m} \in B_{\overline{\nu}\,n}, \tag{3.6}$$

for some fixed constant $c > 0$.

Using this new notation, we rewrite (3.3).

$$(\mathrm{tr}_\Gamma f)_\beta(t) = \sum_{\mu=0}^{\infty}\sum_{n\in\mathbb{Z}} \sum_{\substack{\overline{\nu}\in\mathbb{N}_0^2 \\ \max(\nu_1,\nu_2)=\mu}} \sum_{\overline{m}\in B_{\overline{\nu}\,n}} \lambda^{\beta}_{\overline{\nu}\,\overline{m}} (\beta q u)_{\overline{\nu}\,\overline{m}}(t,t) = \sum_{\mu=0}^{\infty}\sum_{n\in\mathbb{Z}} \gamma^{\beta}_{\mu\,n} a^{\beta}_{\mu\,n}(t), \tag{3.7}$$

where

$$\gamma^{\beta}_{\mu\,n} = 2^{\phi|\beta|} \sum_{\substack{\overline{\nu}\in\mathbb{N}_0^2 \\ \max(\nu_1,\nu_2)=\mu}} \sum_{\overline{m}\in B_{\overline{\nu}\,n}} |\lambda^{\beta}_{\overline{\nu}\,\overline{m}}|.$$

We have to prove that

1. $a^{\beta}_{\mu\,n}$ are atoms according to Definition 2.6, for $d=1$, related to (μ,n).
2. $||\gamma^\beta|b^{\varrho}_{p,q}|| \le c 2^{\phi|\beta|}||\lambda^\beta|s^{\overline{r}}_{p,q}b||$, resp. $||\gamma^\beta|b^{(r_1,\alpha)}_{p,q}|| \le c 2^{\phi|\beta|}||\lambda^\beta|s^{\overline{r}}_{p,q}b||$ (♣)
3. $\mathrm{tr}_\Gamma f$ defined by (3.3) coincides with the trace operator introduced in Section 2.

It is easy to prove the first statement. The support property (2.12) follows directly from (3.6). Also the second property (2.13) is satisfied (up to some constant which depends only on ψ from Definition 2.8). To prove the third statement, consider $f \in S^0_{\infty,1}B(\mathbb{R}^2)$. Then $\lambda^\beta \in s^0_{\infty 1}b$ for every $\beta\in\mathbb{N}^d_0$ and the series in (3.1) both converge uniformly on $\mathbb{R}^2$. So, for $f\in S^0_{\infty,1}B(\mathbb{R}^2)$, $\mathrm{tr}_\Gamma f$ defined by (3.3) coincides with the trace operator of Section 2. Using density arguments, this may be extended to all $f\in S^{\overline{r}}_{p,q}B(\mathbb{R}^2)$.
So, in the following we concentrate on the proof of (♣).
This will finish the first part of the proof, namely the existence and boundedness of the trace operator $\mathrm{tr}_\Gamma : S^{\overline{r}}_{p,q}B(\mathbb{R}^2) \to B^{\varrho}_{p,q}(\Gamma)$. To see that, denote $\omega=\min(1,p,q)$ and write

$$\begin{aligned} ||\,\mathrm{tr}_\Gamma f|B^{\varrho}_{p,q}(\Gamma)||^{\omega} &\le \sum_{\beta\in\mathbb{N}_0^2} ||(\mathrm{tr}_\Gamma f)_\beta|B^{\varrho}_{p,q}(\Gamma)||^{\omega} \le c\sum_{\beta\in\mathbb{N}_0^2}||\gamma^\beta|b^{\varrho}_{p,q}|| \\ &\le c\sum_{\beta\in\mathbb{N}_0^2} 2^{\phi|\beta|}||\lambda^\beta|s^{\overline{r}}_{p,q}b|| \le c\,\sup_{\beta\in\mathbb{N}_0^2} 2^{\varrho|\beta|}||\lambda^\beta|s^{\overline{r}}_{p,q}b|| \le c\,||f|S^{\overline{r}}_{p,q}B(\mathbb{R}^2)||. \end{aligned}$$

Step 2. - Proof of (♣). We take $\beta\in\mathbb{N}^2_0$ fixed and suppose, that the sequence

$$\lambda^\beta = \lambda = \{\lambda_{\overline{\nu},\overline{m}} : \overline{\nu}\in\mathbb{N}^2_0, \overline{m}\in B_{\overline{\nu}}\}$$

is given. Then we set

$$\gamma_{\mu\,n} = \sum_{\substack{\overline{\nu}\in\mathbb{N}_0^2 \\ \max(\nu_1,\nu_2)=\mu}} \sum_{\overline{m}\in B_{\overline{\nu}\,n}} |\lambda_{\overline{\nu}\,\overline{m}}|, \quad \mu\in\mathbb{N}_0, \quad n\in\mathbb{Z}.$$

We recall (3.4) for the relation of $B_{\overline{\nu}\,n}$ and $B_{\overline{\nu}}$.
Finally, we denote

$$\alpha(\overline{\nu}) = \max(\nu_1,\nu_2)\Big(\varrho - \frac{1}{p}\Big) - \overline{\nu}\cdot\Big(\overline{r} - \frac{1}{p}\Big) \tag{3.8}$$

and

$$\beta = \begin{cases} \frac{1}{q} - \frac{1}{\min(1,p)}, & \text{if} \quad r_2 = \frac{1}{p} \quad \text{and} \quad q \ge \min(1,p), \\ 0 & \text{in other cases.} \end{cases} \tag{3.9}$$

Next, we point out that, if $\varrho = r_1$,

$$\alpha(\overline{\nu}) = \begin{cases} \nu_2(r_1 - r_2) - \nu_1(r_1 - \frac{1}{p}) \leq -\nu_1(r_2 - \frac{1}{p}) & \text{for} \quad \nu_1 \leq \nu_2, \\ -\nu_2(r_2 - \frac{1}{p}) & \text{for} \quad \nu_1 \geq \nu_2. \end{cases} \tag{3.10}$$

and, for $\varrho = r_1 + r_2 - \frac{1}{p}$,

$$\alpha(\overline{\nu}) = \begin{cases} (\nu_2 - \nu_1)(r_1 - \frac{1}{p}) \leq 0 & \text{for} \quad \nu_1 \leq \nu_2, \\ (\nu_1 - \nu_2)(r_2 - \frac{1}{p}) \leq 0 & \text{for} \quad \nu_1 \geq \nu_2. \end{cases} \tag{3.11}$$

The estimates (3.10) and (3.11) play a crucial role in the following calculations.
We need to prove that

$$||\{\gamma_{\mu n}\}|\ell_q((\mu+1)^\beta 2^{\mu(\varrho-\frac{1}{p})}\ell_p)|| \leq c\,||\{\lambda_{\overline{\nu}\,\overline{m}}\}|\ell_q(2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})}\ell_p)||, \tag{3.12}$$

where ℓ_p and ℓ_q on the left-hand side denotes sequence spaces with one-dimensional summation and the same symbols stand for sequence spaces with two-dimensional summation on the right hand side.
If $p \leq 1$, then

$$\sum_{n\in\mathbb{Z}} \gamma^p_{\mu n} \leq \sum_{n\in\mathbb{Z}} \sum_{\substack{\overline{\nu}\in\mathbb{N}_0^2 \\ \max(\nu_1,\nu_2)=\mu}} \sum_{\overline{m}\in B_{\overline{\nu}\,n}} |\lambda^\beta_{\overline{\nu}\,\overline{m}}|^p = \sum_{\substack{\overline{\nu}\in\mathbb{N}_0^2 \\ \max(\nu_1,\nu_2)=\mu}} \sum_{\overline{m}\in B_{\overline{\nu}}} |\lambda^\beta_{\overline{\nu}\,\overline{m}}|^p. \tag{3.13}$$

And if $\frac{q}{p} \leq 1 (\implies \beta = 0)$, we get immediately,

$$\sum_{\mu=0}^\infty 2^{\mu(\varrho-\frac{1}{p})q} \Big(\sum_{n\in\mathbb{Z}} \gamma^p_{\mu n}\Big)^{\frac{q}{p}} \leq \sum_{\mu=0}^\infty 2^{\mu(\varrho-\frac{1}{p})q} \sum_{\substack{\overline{\nu}\in\mathbb{N}_0^2 \\ \max(\nu_1,\nu_2)=\mu}} \Big(\sum_{\overline{m}\in B_{\overline{\nu}}} |\lambda^\beta_{\overline{\nu}\,\overline{m}}|^p\Big)^{\frac{q}{p}}.$$

This, together with (3.8)–(3.11), finishes the proof of (3.12) for $0 < q \leq p \leq 1$.
If $p \leq 1$ and $\frac{q}{p} > 1$, we get by (3.13) and Hölder's inequality

$$\sum_{\mu=0}^\infty (\mu+1)^{\beta q} 2^{\mu(\varrho-\frac{1}{p})q} \Big(\sum_{n\in\mathbb{Z}} \gamma^p_{\mu n}\Big)^{\frac{q}{p}} \leq \sum_{\mu=0}^\infty (\mu+1)^{\beta q} \Big(\sum_{\substack{\overline{\nu}\in\mathbb{N}_0^2 \\ \max(\nu_1,\nu_2)=\mu}} 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})p+\alpha(\overline{\nu})p} \sum_{\overline{m}\in B_{\overline{\nu}}} |\lambda_{\overline{\nu}\,\overline{m}}|^p\Big)^{\frac{q}{p}} \leq$$

$$\leq \sum_{\mu=0}^\infty (\mu+1)^{\beta q} \Big(\sum_{\substack{\overline{\nu}\in\mathbb{N}_0^2 \\ \max(\nu_1,\nu_2)=\mu}} 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})p\frac{q}{p}} \big(\sum_{\overline{m}\in B_{\overline{\nu}}} |\lambda_{\overline{\nu}\,\overline{m}}|^p\big)^{\frac{q}{p}}\Big) \cdot \Big(\sum_{\substack{\overline{\nu}\in\mathbb{N}_0^2 \\ \max(\nu_1,\nu_2)=\mu}} 2^{\alpha(\overline{\nu})p(\frac{q}{p})'}\Big)^{\frac{\frac{q}{p}}{(\frac{q}{p})'}}.$$

Here $(\frac{q}{p})' = \frac{q}{q-p}$ is the conjugated index to $\frac{q}{p}$.
So, if $r_2 \neq \frac{1}{p}$, then $\beta = 0$ and, according to (3.10) and (3.11), the last sum is uniformly bounded and the result follows. If $r_2 = \frac{1}{p}$, the last sum is $\leq c\,(\mu+1)^{\frac{q/p}{(\frac{q}{p})'}} = c\,(\mu+1)^{\frac{q-p}{p}} = (\mu+1)^{-\beta q}$.

Next we consider $p > 1$. From (3.5) we get

$$\sum_{\overline{m}\in B_{\overline{\nu}\,n}} |\lambda_{\overline{\nu}\,\overline{m}}| \le c\,\underbrace{(\sum_{\overline{m}\in B_{\overline{\nu}\,n}} |\lambda_{\overline{\nu}\,\overline{m}}|^p)^{\frac{1}{p}}}_{a_{\overline{\nu}\,n}}, \qquad n\in\mathbb{Z}, \quad \overline{\nu}\in\mathbb{N}_0^2. \tag{3.14}$$

By this notation, we get

$$\begin{aligned}\sum_{\mu=0}^{\infty}(\mu+1)^{\beta q}2^{\mu(\varrho-\frac{1}{p})q}\Big(\sum_{n\in\mathbb{Z}}\gamma_{\mu\,n}^p\Big)^{\frac{q}{p}} &\le \sum_{\mu=0}^{\infty}(\mu+1)^{\beta q}2^{\mu(\varrho-\frac{1}{p})q}\Big(\sum_{n\in\mathbb{Z}}\Big(\sum_{\substack{\overline{\nu}\in\mathbb{N}_0^2\\ \max(\nu_1,\nu_2)=\mu}} a_{\overline{\nu}\,n}\Big)^p\Big)^{\frac{q}{p}} \le \\ &\le \sum_{\mu=0}^{\infty}(\mu+1)^{\beta q}2^{\mu(\varrho-\frac{1}{p})q}\Big(\sum_{\substack{\overline{\nu}\in\mathbb{N}_0^2\\ \max(\nu_1,\nu_2)=\mu}}\big(\sum_{n\in\mathbb{Z}} a_{\overline{\nu}\,n}^p\big)^{\frac{1}{p}}\Big)^q,\end{aligned} \tag{3.15}$$

where in the last step we have used the Minkowski's inequality ($p > 1$).
If $q \le 1(\implies \beta = 0)$, we may estimate the last expression from above by

$$\sum_{\mu=0}^{\infty}2^{\mu(\varrho-\frac{1}{p})q}\sum_{\substack{\overline{\nu}\in\mathbb{N}_0^2\\ \max(\nu_1,\nu_2)=\mu}}\Big(\sum_{n\in\mathbb{Z}} a_{\overline{\nu}\,n}^p\Big)^{\frac{q}{p}} = \sum_{\overline{\nu}\in\mathbb{N}_0^2} 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})}2^{\alpha(\overline{\nu})}\Big(\sum_{\overline{m}\in B_{\overline{\nu}}}|\lambda_{\overline{\nu}\,\overline{m}}|^p\Big)^{\frac{q}{p}}.$$

As $\alpha(\overline{\nu}) \le 0$ for all $\overline{\nu}\in\mathbb{N}_0^2$, this finishes the proof.
If $q > 1$, we continue in (3.15) using Hölder's inequality.

$$\begin{aligned}\mathrm{LHS}(3.15) &\le \sum_{\mu=0}^{\infty}(\mu+1)^{\beta q}\Big(\sum_{\substack{\overline{\nu}\in\mathbb{N}_0^2\\ \max(\nu_1,\nu_2)=\mu}} 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})+\alpha(\overline{\nu})}\big(\sum_{n\in\mathbb{Z}} a_{\overline{\nu}\,n}^p\big)^{\frac{1}{p}}\Big)^q \\ &\le \sum_{\mu=0}^{\infty}(\mu+1)^{\beta q}\Big(\sum_{\substack{\overline{\nu}\in\mathbb{N}_0^2\\ \max(\nu_1,\nu_2)=\mu}} 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})q}\big(\sum_{n\in\mathbb{Z}} a_{\overline{\nu}\,n}^p\big)^{\frac{q}{p}}\Big)\cdot\Big(\sum_{\substack{\overline{\nu}\in\mathbb{N}_0^2\\ \max(\nu_1,\nu_2)=\mu}} 2^{\alpha(\overline{\nu})q'}\Big)^{\frac{q}{q'}}\end{aligned}$$

If now $r_2 \ne \frac{1}{p}$, then the last sum is uniformly bounded for all $\mu\in\mathbb{N}_0$ and we get the desired estimate. If $r_2 = \frac{1}{p}$ we get the same estimate with additional factor $(\mu+1)^{q-1} = (\mu+1)^{-\beta q}$.
Step 2. - extension operators
In this step we prove the boundedness of the corresponding extension operators.
We fix $f \in B^{\varrho}_{p,q}(\Gamma)$ (or $f \in B^{(\varrho,\frac{1}{q}-1)}_{p,q}(\Gamma)$, respectively). Then it may be decomposed into quarks

$$f = \sum_{\beta=0}^{\infty} f^{\beta} = \sum_{\beta=0}^{\infty}\sum_{\mu=0}^{\infty}\sum_{n\in\mathbb{Z}}\lambda^{\beta}_{\mu,n}(\beta q u)_{\mu,n},$$

where the coefficients $\{\lambda^{\beta}_{\mu,n}\}$ depend linearly on f and belong to the corresponding sequence space $b^{\varrho}_{p,q}$ or $b^{(\varrho,\frac{1}{q}-1)}_{p,q}$. Moreover,

$$\sup_{\beta\in\mathbb{N}} 2^{\rho\beta}||\lambda^{\beta}|b^{(\varrho,\alpha)}_{p,q}|| \approx ||f|B^{(\varrho,\alpha)}_{p,q}(\mathbb{R})||$$

with constants independent of f.

We define

$$a^{\beta}_{(\nu_1,\nu_2)(m_1,m_2)}(x_1,x_2) = \begin{cases} (\beta q u)_{\nu_1,m_1}(x_1)h(2^{\nu_2}(x_2-2^{-\nu_1}m_1)), & \nu_2 \le \nu_1,\ m_2 = [2^{\nu_2-\nu_1}m_1+\frac{1}{2}] \\ (\beta q u)_{\nu_2,m_2}(x_2)h(2^{\nu_1}(x_1-2^{-\nu_2}m_2)), & \nu_1 \le \nu_2,\ m_1 = [2^{\nu_1-\nu_2}m_2+\frac{1}{2}], \end{cases}$$

where $h \in S(\mathbb{R})$ with $h(t) = 1$ for $|t| \le 2^{\phi}$ and $h(t) = 0$ for $|t| \ge 2^{\phi+1}$ and ϕ is the constant in (2.16). This definition ensures that $2^{-\phi\beta}a^{\beta}_{\overline{\nu}\,\overline{m}}$ are $\overline{K}$-atoms for every fixed $\overline{K} \in \mathbb{N}_0^2$ up to some constant which depends only on the function ψ involved in the definition of quarks and $\overline{K}$.

If now $r_2 > \frac{1}{p}$ or $r_2 = \frac{1}{p}$ and $q \le \min(1,p)$ then $\{\lambda^{\beta}_{\mu,n}\} \in b^{r_1}_{p,q}$ with $\sup_{\beta\in\mathbb{N}_0} 2^{\rho\beta}||\lambda|b^{r_1}_{p,q}|| \le c||f|B^{r_1}_{p,q}(\mathbb{R})||$. We define

$$\gamma^{\beta}_{(\mu,0)(n,[2^{-\mu}n+\frac{1}{2}])} = \lambda^{\beta}_{\mu,n}, \quad \mu \in \mathbb{N}_0,\ n \in \mathbb{Z} \tag{3.16}$$

and zero otherwise. Finally we set

$$\operatorname{ext} f = \sum_{\beta=0}^{\infty} \operatorname{ext} f^{\beta} = \sum_{\beta=0}^{\infty} \sum_{\overline{\nu}\in\mathbb{N}_0^2} \sum_{\overline{m}\in\mathbb{Z}^2} \gamma^{\beta}_{\overline{\nu}\,\overline{m}} a^{\beta}_{\overline{\nu}\,\overline{m}} \tag{3.17}$$

and observe that for $\omega = \min(1,p,q)$

$$\begin{aligned} ||\operatorname{ext} f|S^{\overline{r}}_{p,q}B(\mathbb{R}^2)||^{\omega} &\le \sum_{\beta=0}^{\infty} ||\operatorname{ext} f^{\beta}|S^{\overline{r}}_{p,q}B(\mathbb{R}^2)||^{\omega} \le c\sum_{\beta=0}^{\infty} 2^{\phi\beta\omega}||\gamma^{\beta}|s^{\overline{r}}_{p,q}b||^{\omega} \\ &\le c \sup_{\beta\in\mathbb{N}_0} 2^{\rho\beta\omega}||\gamma^{\beta}|s^{\overline{r}}_{p,q}b||^{\omega} = c \sup_{\beta\in\mathbb{N}_0} 2^{\rho\beta\omega}\left(\sum_{\nu\in\mathbb{N}_0^2} 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})q}\left(\sum_{\overline{m}\in\mathbb{Z}^2}|\gamma^{\beta}_{\overline{\nu}\,\overline{m}}|^p\right)^{q/p}\right)^{\omega/q} \qquad (3.18) \\ &= c \sup_{\beta\in\mathbb{N}_0} 2^{\rho\beta\omega}\left(\sum_{\mu=0}^{\infty} 2^{\mu(r_1-\frac{1}{p})q}\left(\sum_{n\in\mathbb{Z}}|\lambda^{\beta}_{\mu n}|^p\right)^{q/p}\right)^{\omega/q} \qquad (3.19) \\ &= c \sup_{\beta\in\mathbb{N}_0} 2^{\rho\beta\omega}||\lambda^{\beta}|b^{r_1}_{p,q}||^{\omega} \le c||f|B^{r_1}_{p,q}(\mathbb{R})||^{\omega}. \end{aligned}$$

Furthermore, the definition of $a^{\beta}_{\overline{\nu}\,\overline{m}}$ ensures that $\operatorname{tr}\circ\operatorname{ext} f = f$

The case $r_2 < \frac{1}{p}$ follows the same scheme. We define

$$\gamma^{\beta}_{(\mu,\mu)(n,n)} = \lambda^{\beta}_{\mu,n}, \quad \mu \in \mathbb{N}, \quad n \in \mathbb{Z} \tag{3.20}$$

and $\gamma^{\beta}_{\overline{\nu}\,\overline{m}} = 0$ otherwise. We get now similarly to (3.18)

$$\begin{aligned} ||\gamma^{\beta}|s^{\overline{r}}_{p,q}b|| &= \left(\sum_{\overline{\nu}\in\mathbb{N}_0^2} 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})q}\left(\sum_{\overline{m}\in\mathbb{Z}^2}|\gamma^{\beta}_{\overline{\nu}\,\overline{m}}|^p\right)^{q/p}\right)^{1/q} \\ &= \left(\sum_{\mu=0}^{\infty} 2^{\mu(r_1+r_2-\frac{2}{p})q}\left(\sum_{n\in\mathbb{Z}}|\lambda^{\beta}_{\mu n}|^p\right)^{q/p}\right)^{1/q} = ||\lambda^{\beta}|b^{\varrho}_{p,q}||. \end{aligned}$$

Finally, in the case $r_2 = \frac{1}{p}, q > 1$ and $q > p$ we set for $0 \le \nu_2 \le \mu$

$$\gamma^{\beta}_{(\mu,\nu_2)(n,\tilde{n})} = (\mu+1)^{-1}\lambda^{\beta}_{\mu,n}, \quad \tilde{n} = [2^{\nu_2}2^{-\mu}n + \frac{1}{2}]$$

and zero otherwise. Then we get for $\beta = \frac{1}{q} - 1$

$$\begin{aligned}
||\gamma^{\beta}|s^{\overline{r}}_{p,q}b|| &= \Big(\sum_{\nu\in\mathbb{N}_0^2} 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})q}\Big(\sum_{\overline{m}\in\mathbb{Z}^2} |\gamma^{\beta}_{\overline{\nu}\,\overline{m}}|^p\Big)^{q/p}\Big)^{1/q} \\
&= \Big(\sum_{\mu=0}^{\infty} 2^{\mu(r_1-\frac{1}{p})q}(\mu+1)\Big(\sum_{n\in\mathbb{Z}} |(\mu+1)^{-1}\lambda^{\beta}_{\mu\,n}|^p\Big)^{q/p}\Big)^{1/q} = ||\lambda^{\beta}|b^{(\varrho,\beta)}_{p,q}||.
\end{aligned}$$

□

4 Traces of F spaces

Theorem 4.1. *Let*

$$0 < p < \infty, \quad 0 < q \le \infty, \quad 0 < r_1 \le r_2$$

with

$$\varrho = \min\Big(r_1, r_1 + r_2 - \frac{1}{p}\Big) > \sigma_{p,q}.$$

If $r_2 > \frac{1}{p}$ then

$$\mathrm{tr}_{\Gamma}\, S^{\overline{r}}_{p,q}F(\mathbb{R}^2) = F^{\varrho}_{p,q}(\Gamma). \tag{4.1}$$

If $r_2 < \frac{1}{p}$ then

$$\mathrm{tr}_{\Gamma}\, S^{\overline{r}}_{p,q}F(\mathbb{R}^2) = F^{\varrho}_{p,p}(\Gamma) = B^{\varrho}_{p,p}(\Gamma). \tag{4.2}$$

If $r_2 = \frac{1}{p}$ and $p \le \min(1,q)$ then

$$\mathrm{tr}_{\Gamma}\, S^{\overline{r}}_{p,q}F(\mathbb{R}^2) = F^{r_1}_{p,q}(\Gamma). \tag{4.3}$$

If $r_2 = \frac{1}{p}$ and $q < p \le 1$ then

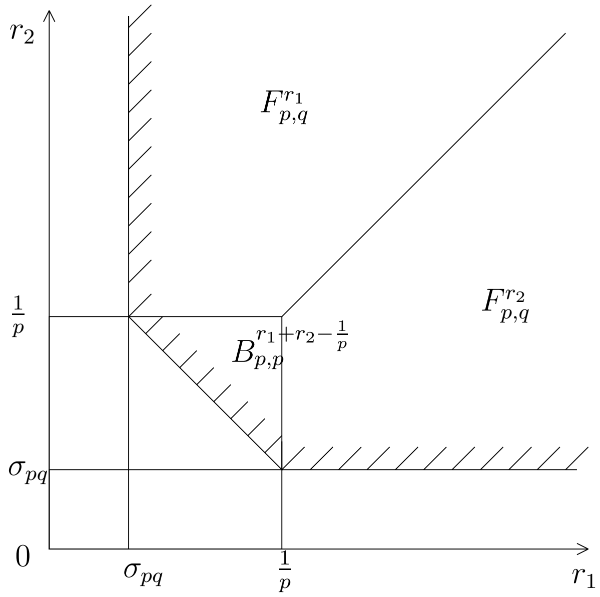
$$\mathrm{tr}_{\Gamma}\, S^{\overline{r}}_{p,q}F(\mathbb{R}^2) = F^{r_1}_{p,p}(\Gamma). \tag{4.4}$$

If $r_2 = \frac{1}{p}$ and $1 \le p \le q$ then

$$\mathrm{tr}_{\Gamma} : S^{\overline{r}}_{p,q}F(\mathbb{R}^2) \to F^{(r_1,\frac{1}{q}-1)}_{p,q}(\Gamma). \tag{4.5}$$

Finally, if $r_2 = \frac{1}{p}$ and $p \ge \max(1,q)$ then

$$\mathrm{tr}_{\Gamma} : S^{\overline{r}}_{p,q}F(\mathbb{R}^2) \to F^{(r_1,\frac{1}{p}-1)}_{p,p}(\Gamma). \tag{4.6}$$



Proof. We recall our task. We use again the notation (3.1)-(3.13).
We suppose, that the sequence

$$\lambda = \{\lambda_{\overline{\nu},\overline{m}} : \overline{\nu} \in \mathbb{N}_0^2, \overline{m} \in B_{\overline{\nu}}\}$$

is given. Then we set

$$\gamma_{\mu n} = \sum_{\substack{\overline{\nu}\in\mathbb{N}_0^2 \\ \max(\nu_1,\nu_2)=\mu}} \sum_{\overline{m}\in B_{\overline{\nu}\,n}} |\lambda_{\overline{\nu}\,\overline{m}}|. \tag{4.7}$$

and recall (3.4) for the relation of $B_{\mu n}$ and $B_{\overline{\nu}}$. We need to prove that $(r_2 > \frac{1}{p})$

$$||\{\gamma_{\mu n}\}|f^{r_1}_{p,q}|| \le c\,||\{\lambda_{\overline{\nu}\,\overline{m}}\}|s^{\overline{r}}_{p,q}f|| \tag{4.8}$$

or $(r_2 < \frac{1}{p})$

$$||\{\gamma_{\mu n}\}|f^{r_1+r_2-\frac{1}{p}}_{p,p}|| \le c\,||\{\lambda_{\overline{\nu}\,\overline{m}}\}|s^{\overline{r}}_{p,q}f|| \tag{4.9}$$

respectively.
We split (4.7) into two parts,

$$\gamma^{(1)}_{\mu n} = \sum_{\nu_2=0}^{\mu} \sum_{\overline{m}\in B_{(\mu,\nu_2),n}} |\lambda_{(\mu,\nu_2),\overline{m}}|, \qquad \gamma^{(2)}_{\mu n} = \sum_{\nu_1=0}^{\mu} \sum_{\overline{m}\in B_{(\nu_1,\mu),n}} |\lambda_{(\nu_1,\mu),\overline{m}}| \tag{4.10}$$

and prove (4.8) and (4.9) for both these parts separately.
Step 1.
We start with the case $r_2 > \frac{1}{p}$. We recall the definitions of sequence spaces involved in (4.8) and obtain

$$||\{\gamma_{\mu n}\}|f^{r_1}_{p,q}||^p = \int_{-\infty}^{\infty} \Big(\sum_{\mu=0}^{\infty} \sum_{n\in\mathbb{Z}} |2^{\mu r_1}\gamma_{\mu n}\chi_{\mu n}(x_1)|^q\Big)^{\frac{p}{q}} \mathrm{d}x_1$$

and

$$||\{\lambda_{\overline{\nu}\,\overline{m}}\}|s^{\overline{r}}_{p,q}f||^p \ge c \int_{-\infty}^{\infty} \int_{x_1-1}^{x_1+1} \Big(\sum_{\overline{\nu}\in\mathbb{N}_0^2} \sum_{\overline{m}\in B_{\overline{\nu}}} |2^{\overline{\nu}\cdot\overline{r}}\lambda_{\overline{\nu}\,\overline{m}}\chi_{\overline{\nu}\,\overline{m}}(x_1,x_2)|^q\Big)^{\frac{p}{q}} \mathrm{d}x_2\mathrm{d}x_1.$$

So, to prove (4.8) for $\gamma^{(1)}$, it is enough to prove

$$\Big(\sum_{\mu=0}^{\infty}\sum_{n\in\mathbb{Z}}|2^{\mu r_1}\gamma^{(1)}_{\mu n}\chi_{\mu n}(x_1)|^q\Big)^{\frac{p}{q}} \le c\int_{-1}^{1}\Big(\sum_{\overline{\nu}\in\mathbb{N}_0^2}\sum_{\overline{m}\in B_{\overline{\nu}}}|2^{\overline{\nu}\cdot\overline{r}}\lambda_{\overline{\nu}\,\overline{m}}\chi_{\overline{\nu}\,\overline{m}}(x_1,x_1+x_2)|^q\Big)^{\frac{p}{q}}\mathrm{d}x_2 \tag{4.11}$$

for every fixed x_1.
Finally, we try to change the notation in such a way that we could switch from integrals to sums. With x_1 being fixed, there is only one $n=n(\mu)$ such that $\chi_{\mu n}(x_1)=1$. We denote $\gamma^{(1)}_{\mu}=\gamma^{(1)}_{\mu\, n(\mu)}$. So, the left hand side of (4.11) reduces to

$$\Big(\sum_{\mu=0}^{\infty}|2^{\mu r_1}\gamma^{(1)}_{\mu}|^q\Big)^{\frac{p}{q}}.$$

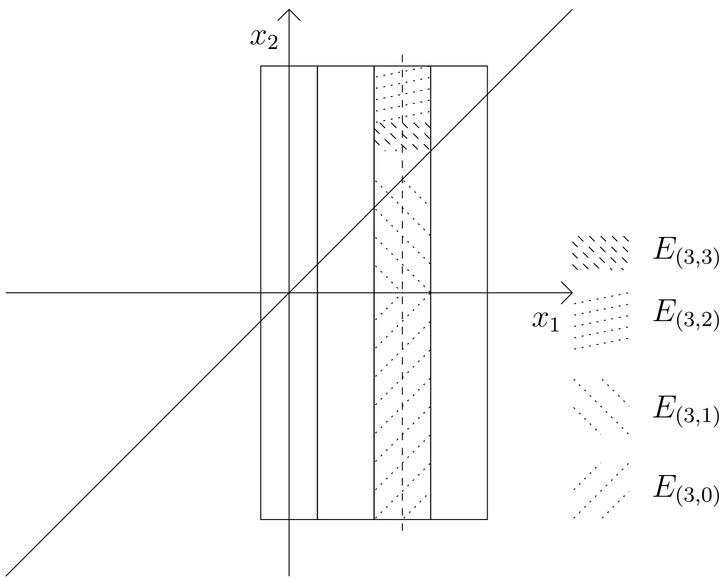
Finally, as a direct corollary of (3.5), we may suppose, that each $B_{\overline{\nu}n}$ contains only one element. So, to every $\mu\in\mathbb{N}_0$ and every $\nu_1\le\mu$ there is a unique $\overline{m}=\overline{m}(\mu,\nu_2)\in B_{(\mu,\nu_2)\,n(\mu)}$. We denote $\lambda_{(\mu,\nu_2)}=\lambda_{(\mu,\nu_2)\,\overline{m}(\mu,\nu_2)}$.
We reformulate once more our task. We start with a given sequence

$$\lambda=\{\lambda_{\overline{\nu}}:\overline{\nu}\in\mathbb{N}_0^2,\nu_1\ge\nu_2\},$$

and define

$$\gamma_\mu=\sum_{\nu_2=0}^{\mu}|\lambda_{(\mu,\nu_2)}|.$$

Finally, we use the Lemma 2.11 and choose the sets $E_{\overline{\nu}\,\overline{m}}$ such that $E_{(\mu,\nu_2),\overline{m}(\mu,\nu_2)}$ and $E_{(\mu,\nu_2'),\overline{m}(\mu,\nu_2')}$ are disjoint for $\nu_2\ne\nu_2'$.



It turns out, that it is enough to prove that

$$\Big(\sum_{\mu=0}^{\infty}|2^{\mu r_1}\gamma_\mu|^q\Big)^{\frac{p}{q}}\le c\sum_{j=0}^{\infty}2^{-j}\Big(\sum_{\mu=j}^{\infty}|2^{\mu r_1+jr_2}\lambda_{(\mu,j)}|^q\Big)^{\frac{p}{q}} \tag{4.12}$$

with c independent on the starting sequence λ. We just mention, that the $j-$sum comes from decomposition of the integral in (4.11) according to the supports of $\chi_{\overline{\nu}\,\overline{m}}$ involved.

First we discuss the case $q \le 1$. In that case,

$$\gamma_\mu^q \le \sum_{\nu_2=0}^{\mu} |\lambda_{(\mu,\nu_2)}|^q \le \sum_{\nu_2=0}^{\mu} 2^{\nu_2(r_2-\frac{1}{p})q} |\lambda_{(\mu,\nu_2)}|^q.$$

If moreover $\frac{p}{q} \le 1$,

$$\begin{aligned}\Big(\sum_{\mu=0}^{\infty} 2^{\mu r_1 q}\gamma_\mu^q\Big)^{\frac{p}{q}} &\le \Big(\sum_{\mu=0}^{\infty} 2^{\mu r_1 q} \sum_{\nu_2=0}^{\mu} 2^{\nu_2(r_2-\frac{1}{p})q}|\lambda_{(\mu,\nu_2)}|^q\Big)^{\frac{p}{q}}\\ &= \Big(\sum_{\nu_2=0}^{\infty} 2^{-\nu_2\frac{q}{p}} \sum_{\mu=\nu_2}^{\infty} |2^{\mu r_1+\nu_2 r_2}\lambda_{(\mu,\nu_2)}|^q\Big)^{\frac{p}{q}}\\ &\le c\sum_{\nu_2=0}^{\infty} 2^{-\nu_2}\Big(\sum_{\mu=\nu_2}^{\infty} |2^{\mu r_1+\nu_2 r_2}\lambda_{(\mu,\nu_2)}|^q\Big)^{\frac{p}{q}}\end{aligned}$$

This proves (4.12) for $p \le q \le 1$ and $r_2 \ge \frac{1}{p}$.

In the case $q \le 1, q < p$ we denote

$$b_{\nu_2}^q = \sum_{\mu=\nu_2}^{\infty} |2^{\mu r_1}\lambda_{(\mu,\nu_2)}|^q.$$

By this notation, the right-hand side of (4.12) may be rewritten like

$$RHS(4.12) = \sum_{\nu_2=0}^{\infty} 2^{-\nu_2}\Big(2^{\nu_2 r_2 q} b_{\nu_2}^q\Big)^{\frac{p}{q}} = \sum_{\nu_2=0}^{\infty} 2^{\nu_2(r_2-\frac{1}{p})p} b_{\nu_2}^p$$

and the left-hand side may be estimated by

$$\big(\sum_{\mu=0}^{\infty} |2^{\mu r_1}\gamma_\mu^{(1)}|^q\big)^{\frac{p}{q}} \le \big(\sum_{\nu_2=0}^{\infty} b_{\nu_2}^q\big)^{\frac{p}{q}}.$$

This (and Hölder's inequality) finishes the proof of (4.12) for $r_2 > \frac{1}{p}$ and $q \le 1, q < p$.

Next, we take $q > 1$. We denote $\beta = -\frac{1}{q'} = \frac{1}{q} - 1$ if $r_2 = \frac{1}{p}$ and $\beta = 0$ if $r_2 > \frac{1}{p}$.

By Hölder's inequality we get

$$\gamma_\mu \le c\,(\mu+1)^{-\beta}\Big(\sum_{\nu_2=0}^{\mu} 2^{\nu_2(r_2-\frac{1}{p})q}|\lambda_{\mu,\nu_2}|^q\Big)^{\frac{1}{q}}, \quad \mu \in \mathbb{N}_0.$$

Hence, for $p \le q$,

$$\begin{aligned}\big(\sum_{\mu=0}^{\infty}(\mu+1)^{\beta q} 2^{\mu r_1 q}\gamma_\mu^q\big)^{\frac{p}{q}} &\le c\Big(\sum_{\mu=0}^{\infty} 2^{\mu r_1 q} \sum_{\nu_2=0}^{\mu} 2^{\nu_2(r_2-\frac{1}{p})q}|\lambda_{\mu,\nu_2}|^q\Big)^{\frac{p}{q}}\\ &= c\Big(\sum_{\nu_2=0}^{\infty} 2^{-\nu_2\frac{q}{p}} \sum_{\mu=\nu_2}^{\infty} 2^{\mu r_1 q+\nu_2 r_2 q}|\lambda_{(\mu,\nu_2)}|^q\Big)^{\frac{p}{q}}\\ &\le c\sum_{\nu_2=0}^{\infty} 2^{-\nu_2}\Big(\sum_{\mu=\nu_2}^{\infty} 2^{\mu r_1 q+\nu_2 r_2 q}|\lambda_{(\mu,\nu_2)}|^q\Big)^{\frac{p}{q}}.\end{aligned}$$

This finishes the proof of (4.12) for $\max(p,1) \le q$ and $r_2 > \frac{1}{p}$. But for $r_2 = \frac{1}{p}$ this also proves the generalisation of (4.12), where $2^{\mu r_1}$ is replaced by $(\mu+1)^{\beta} 2^{\mu r_1}$ on the left-hand side. Hence, also the boundedness of the trace operator in (4.5) follows.
For $p > q > 1$ and $r_2 - \frac{1}{p} > \varepsilon > 0$ we get similarly

$$\gamma_\mu \le c \Big(\sum_{\nu_2=0}^{\mu} 2^{\nu_2 (r_2 - \frac{1}{p} - \varepsilon) q} |\lambda_{\mu,\nu_2}|^q \Big)^{\frac{1}{q}}$$

and

$$\Big(\sum_{\mu=0}^{\infty} 2^{\mu r_1 q} \gamma_\mu^q \Big)^{\frac{p}{q}} \le c \Big(\sum_{\nu_2=0}^{\infty} 2^{\nu_2 (r_2 - \frac{1}{p} - \varepsilon) q} b_{\nu_2}^q \Big)^{\frac{p}{q}} \le c \sum_{\nu_2=0}^{\infty} \Big(2^{\nu_2 (r_2 - \frac{1}{p}) q} b_{\nu_2}^q \Big)^{\frac{p}{q}}$$

This finishes the boundedness of the trace operator for $r_2 > \frac{1}{p}$. In the case of $r_2 = \frac{1}{p}$, we have only discussed the cases $p \le q \le 1$ and $1 \le p \le q$. To complete the proof in those cases, where the result depends on q, we consider $p \le 1 \le q$. We get by Minkowski's inequality

$$\begin{aligned}\Big(\sum_{\mu=0}^{\infty} 2^{\mu r_1 q} \Big(\sum_{\nu_2=0}^{\mu} |\lambda_{(\mu,\nu_2)}|\Big)^q \Big)^{\frac{p}{q}} &\le \Big(\sum_{\nu_2=0}^{\infty} \Big(\sum_{\mu=\nu_2}^{\infty} 2^{\mu r_1 q} |\lambda_{(\mu,\nu_2)}|^q \Big)^{\frac{1}{q}} \Big)^p \\ &\le \sum_{\nu_2=0}^{\infty} \Big(\sum_{\mu=\nu_2}^{\infty} 2^{\mu r_1 q} |\lambda_{(\mu,\nu_2)}|^q \Big)^{\frac{p}{q}} = RHS(4.12).\end{aligned}$$

Finally, to prove the boundedness of the trace operator in (4.4) and (4.6) we use the embedding

$$S^{\overline{r}}_{p,q} F(\mathbb{R}^2) \hookrightarrow S^{\overline{r}}_{p,p} B(\mathbb{R}^2),$$

which holds for $q \le p$, and Theorem 3.1.
Step 2.
Next we discuss the remaining case $0 < r_1 \le r_2 < \frac{1}{p}$, $\varrho = r_1 + r_2 - \frac{1}{p} > \sigma_{p,q}$.
We now need to prove (4.9). We introduce again the same notation as in the Step 1. and replace (4.12) by

$$\sum_{\mu=0}^{\infty} |2^{\mu \varrho} \gamma_\mu^{(1)}|^p \le c \sum_{j=0}^{\infty} 2^{-j} \Big(\sum_{\mu=j}^{\infty} |2^{\mu r_1 + j r_2} \lambda_{(\mu,j)}|^q \Big)^{\frac{p}{q}}. \tag{4.13}$$

Finally, we prove (4.13) for all $0 < q \le \infty$ if we prove it for $q = \infty$. We denote

$$a_{\nu_2} = \sup_{\mu \ge \nu_2} 2^{\mu r_1} |\lambda_{(\mu,\nu_2)}|, \quad \nu_2 \in \mathbb{N}_0.$$

Then the right-hand side of (4.13) may be (for $q = \infty$) rewritten as

$$RHS(4.13) = c \sum_{\nu_2=0}^{\infty} 2^{-\nu_2} \big(\sup_{\mu \ge \nu_2} 2^{\mu r_1 + \nu_2 r_2} |\lambda_{\mu,\nu_2}| \big)^p = c \sum_{\nu_2=0}^{\infty} 2^{-\nu_2 + \nu_2 r_2 p} a_{\nu_2}^p.$$

As for the left-hand side in (4.13), we get for $p \le 1$

$$LHS(4.13) \le \sum_{\mu=0}^{\infty} 2^{\mu \varrho p} \sum_{\nu_2=0}^{\mu} |\lambda_{\mu,\nu_2}|^p = \sum_{\nu_2=0}^{\infty} \sum_{\mu=\nu_2}^{\infty} 2^{\mu \varrho p} |\lambda_{\mu,\nu_2}|^p \le c \sum_{\nu_2=0}^{\infty} 2^{\nu_2 (r_2 - \frac{1}{p}) p} a_{\nu_2}^p.$$

For $p > 1$ we denote $\epsilon = \frac{1}{p} - r_2 > 0$ and get

$$\begin{aligned} LHS(4.13) &= \sum_{\mu=0}^{\infty} 2^{\mu r_1 p - \mu\epsilon p} \Big(\sum_{\nu_2=0}^{\mu} 2^{(\mu-\nu_2)\epsilon/2 - (\mu-\nu_2)\epsilon/2} |\lambda_{\mu,\nu_2}| \Big)^p \\ &\le \sum_{\mu=0}^{\infty} 2^{\mu r_1 p - \mu\epsilon p} \big(\sum_{\nu_2=0}^{\mu} 2^{(\mu-\nu_2)p\epsilon/2} |\lambda_{\mu,\nu_2}|^p \big) \big(\sum_{\nu_2=0}^{\mu} 2^{-(\mu-\nu_2)p'\epsilon/2} \big)^{p/p'} \\ &\le c \sum_{\nu_2=0}^{\infty} \sum_{\mu=\nu_2}^{\infty} 2^{\mu r_1 p - \mu\epsilon p + (\mu-\nu_2)p\epsilon/2} |\lambda_{\mu,\nu_2}|^p \\ &\le c \sum_{\nu_2=0}^{\infty} 2^{-\nu_2 p\epsilon/2} a_{\nu_2}^p \sum_{\mu=\nu_2}^{\infty} 2^{-\mu\epsilon p + \mu p\epsilon/2} \le c \sum_{\nu_2=0}^{\infty} 2^{-\nu_2\epsilon p} a_{\nu_2}^p. \end{aligned}$$

This finishes the proof of (4.8) and (4.9) for $\gamma^{(1)}$. One could follow the same arguments also for $\gamma^{(2)}$. Alternatively, to a given sequence

$$\lambda = \{\lambda_{\overline{\nu}} : \overline{\nu} \in \mathbb{N}_0^2, \nu_1 \le \nu_2\}$$

we consider a sequence

$$\overline{\lambda} = \{\overline{\lambda}_{\overline{\nu}} : \overline{\nu} \in \mathbb{N}_0^2, \nu_1 \ge \nu_2\}$$

defined by $\overline{\lambda}_{(\nu_1,\nu_2)} = \lambda_{(\nu_2,\nu_1)}$ and use (4.12) for $\overline{\gamma}^{(1)}$ associated with $\overline{\lambda}$. In this way, we prove (4.8) and (4.9) for $\gamma^{(2)}$ and finish the proof of boundedness of the trace operator.

Step 3.

Next, we consider the corresponding extension operators. We use the same operators as in the B-case. The first one (given by (3.16) and (3.17)) gives an extension operator in the case $r_2 > \frac{1}{p}$. To prove the corresponding inequality on the sequence space level, we again fix x_1 and prove a pointwise inequality, which now reduces to trivial

$$\Big(\sum_{\mu=0}^{\infty} 2^{\mu r_1 q} \big(\sum_{\nu_2=0}^{\mu} |\gamma_{\mu,\nu_2}| \big)^q \Big)^{p/q} = \big(\sum_{\mu=0}^{\infty} 2^{\mu r_1 q} |\lambda_\mu|^q \big)^{p/q}.$$

The same operator proves also (4.3).

The second operator, characterised by (3.17) and (3.20) gives an extension operator for $r_2 < \frac{1}{p}$ and in (4.4). We omit the trivial calculation. □

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Traces of Functions with a Dominating Mixed Derivative in $\mathbb{R}^3$

Jan Vybíral and Winfried Sickel

Abstract

We investigate traces of functions, belonging to a class of functions with dominating mixed smoothness in $\mathbb{R}^3$, with respect to planes in oblique position. In comparison with the classical theory for isotropic spaces a few new phenomenona occur. We shall present two different approaches. One is based on the use of the Fourier transform and restricted to $p = 2$. The other one is applicable in the general case of Besov-Lizorkin-Triebel spaces and based on atomic decompositions.

AMS Classification: 42B35, 46E35

Keywords and phrases: Sobolev spaces of dominating mixed smoothness, Besov and Lizorkin-Triebel classes of dominating mixed smoothness, Fourier analytic characterizations, atomic decompositions, traces on hyperplanes in oblique position.

1 Introduction

Sobolev spaces with dominating mixed smoothness $S_p^{\bar{r}}W(\mathbb{R}^d)$ have been introduced in 1962 by S.M. Nikol'skij, see [Ni1, Ni2], originally in connection with some partial differential equations. Later on there has been some interest in these type of spaces as special cases of vector-valued Sobolev spaces ($S_p^{r,\ldots,r}W(\mathbb{R}^d)$ can be interpreted as an iterated version of the Sobolev spaces $W_p^r(\mathbb{R})$), see Grisvard [Gr], Sparr [Sp] and Schmeißer [Sc]. At the end of the eighties Triebel [Tr1], motivated by problems in connection with eigenvalue distributions of integral operators, investigated the trace problem with respect to the diagonal $x_1 = x_2$ for the Besov spaces $S_{p,1}^{r,r}B(\mathbb{R}^2)$. In recent years there is an increasing interest in function spaces with a dominating mixed derivative in connection with the numerical solution of some special partial differential equations or integral equations, see e.g. Griebel, Oswald, Schiekofer [GOS], Yserentant [Ys1, Ys2], Nitzsche [Ni] or Bungartz and Griebel [BG].

We are interested in the description of the trace classes of $S_p^{r_1,r_2,r_3}W(\mathbb{R}^3)$ (and more general function spaces) with respect to an hyperplane in oblique position. Since at least twenty years the situation is well understood if the trace is taken with respect to hyperplanes parallel to the coordinate axes, cf. e.g. the monographs Amanov [Am], Gelman, Maz'ya [GM] ($p = 2$) and Schmeißer, Triebel [ST]. However, there is an essential difference in case that the hyperplane is in an oblique position. First observations in this direction have been made by Triebel [Tr1] in the two-dimensional case, later continued by Rodriguez [Ro] and complemented by the first author, see [Vy3]. To our own surprise the problem for $d = 3$ turned out to be more complicated. New phenomenona occur. Whereas for $d = 2$ almost all trace classes of Sobolev and Besov-Lizorkin-Triebel spaces are again Besov or

Lizorkin-Triebel classes (in some limiting cases of generalized smoothness, see [Vy3]) the situation changes with $d = 3$. Here it turns out that the trace classes can be described as the sum of three different function spaces of dominating mixed smoothness. In proving such a statement we offer two different approaches. The first one is restricted to $p = 2$ and uses elementary properties of the Fourier transform. In this simplified situation we are also able to establish a characterization of the trace class of $S_2^{r_1,r_2,r_3}W(\mathbb{R}^3)$ as a L_2-space with a weight in the Fourier image. For $p \neq 2$ one is confronted with difficult Fourier multiplier assertions. To circumvent this we apply the characterization of these classes by atoms which works also in the more general case of Besov and Lizorkin-Triebel spaces. However, the description of the trace classes found in this way is not very transparent. Here some further progress would be desirable.
To explain a part of the difficulties let us consider an example. We equip the hyperplane $x_1 + x_2 + x_3 = 0$ with an orthogonal basis

$$\mathcal{O} = \{\vec{\sigma}_1, \vec{\sigma}_2\}, \quad \vec{\sigma}_1 = (\sigma_{1,1}, \sigma_{1,2}, \sigma_{1,3}) \in \Gamma, \quad \vec{\sigma}_2 = (\sigma_{2,1}, \sigma_{2,2}, \sigma_{2,3}) \in \Gamma, \quad \vec{\sigma}_1 \perp \vec{\sigma}_2 \,. \tag{1.1}$$

Then we associate to this basis the corresponding "orthogonal" trace operator

$$(\mathrm{tr}_{\mathcal{O}}\, f)(z_1, z_2) = f(z_1\vec{\sigma}_1 + z_2\vec{\sigma}_2), \quad z_1, z_2 \in \mathbb{R}. \tag{1.2}$$

Now we consider the following family of functions

$$f_\lambda(x_1, x_2, x_3) := \psi(x_1)\,\psi(x_2)\,\psi(x_3)\,|x_3|^\lambda\,, \qquad (x_1, x_2, x_3) \in \mathbb{R}^3\,, \quad \lambda \in \mathbb{R}\,,$$

where $\psi : \mathbb{R} \to \mathbb{R}$ is a smooth cut-off function supported around the origin. Such a function f_λ belongs to $S_p^{r,r,r}W(\mathbb{R}^3)$ if $\lambda > r - 1/p$. But the regularity of the function

$$g_\lambda(z_1, z_2) = \psi(\sigma_{1,1}z_1 + \sigma_{2,1}z_2)\,\psi(\sigma_{1,2}z_1 + \sigma_{2,2}z_2)\,\psi(\sigma_{1,3}z_1 + \sigma_{2,3}z_2)\,|\sigma_{1,3}z_1 + \sigma_{2,3}z_2|^\lambda$$

depends on $\mathcal{O}$. The function g_λ belongs to $S_p^{r,r}W(\mathbb{R}^2)$, $\lambda > r - 1/p$, if either $\sigma_{1,3} = 0$ or $\sigma_{2,3} = 0$. If $\sigma_{1,3} \cdot \sigma_{2,3} \neq 0$, then g_λ belongs to $S_p^{t,t}W(\mathbb{R}^2)$, $\lambda > 2\,t - 1/p$. As a consequence the description of the traces of $S_p^{r_1,r_2,r_3}W(\mathbb{R}^3)$ to the hyperplane $x_1 + x_2 + x_3 = 0$ must depend on the chosen basis $\mathcal{O}$.
The paper is organized as follows. In Section 2 we start with a general discussion of the notion of the trace and continue with a detailed investigation of the trace problem for the Sobolev spaces of dominating smoothness built on $L_2(\mathbb{R}^3)$. Here we shall apply methods from Fourier analysis. In case $p \neq 2$, treated in Section 3, the situation becomes more complicated and we switch to the powerful but less transparent method of decompositions of functions into small building blocks like atoms. By means of those decompositions we are able to describe the trace classes for the general case of Besov and Lizorkin-Triebel classes. Our main results are contained in Theorems 2.11, 3.10, and 3.14.

2 The Trace of Sobolev Spaces of Dominating Mixed Smoothness $S_2^{\overline{r}}(\mathbb{R}^3)$

2.1 Sobolev Spaces of Dominating Mixed Smoothness

Let $1 < p < \infty$ and $\overline{r} = (r_1, \ldots, r_d) \in \mathbb{N}_0^d$ ($\mathbb{N}_0$ denotes the natural numbers including 0). The Sobolev space of dominating mixed smoothness $\overline{r} = (r_1, \ldots, r_d)$ is the collection of all functions $f \in L_p(\mathbb{R}^d)$ such that

$$D^\alpha f \in L_p(\mathbb{R}^d)\,, \qquad 0 \leq \alpha_i \leq r_i\,, \qquad i = 1, \ldots, d\,,$$

endowed with the norm

$$\| f \,|S_p^{\overline{r}}W(\mathbb{R}^d)\| := \sum_{\alpha \leq \overline{r}} \| D^\alpha f \,|L_p(\mathbb{R}^d)\|\,. \tag{2.1}$$

Here $\alpha \leq \overline{r}$ means $\alpha_i \leq r_i$, $i = 1, \ldots, d$.

The mixed derivative $\frac{\partial^{r_1 + \cdots + r_d} f}{\partial x_1^{r_1} \ldots \partial x_d^{r_d}}$ plays a dominant part here and this fact is responsible for the name of these classes. Based on a Fourier multiplier theorem of Lizorkin one can prove a characterization of these classes using the Fourier transfom. Let $\mathcal{S}(\mathbb{R}^d)$ denote the class of all complex-valued rapidly decreasing infinitely differentiable functions defined on $\mathbb{R}^d$. By $\mathcal{S}'(\mathbb{R}^d)$ we mean the collection of all tempered distributions and $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse, respectively, both defined on $\mathcal{S}'(\mathbb{R}^d)$. Then $f \in \mathcal{S}'(\mathbb{R}^d)$ belongs to $S_p^{\overline{r}}W(\mathbb{R}^d)$ if and only if

$$\mathcal{F}^{-1}\Big((1 + |\xi_1|^2)^{r_1/2} \ldots (1 + |\xi_d|^2)^{r_d/2}\, \mathcal{F}f(\xi)\Big)(\,\cdot\,) \in L_p(\mathbb{R}^d)\,.$$

Furthermore, the norms $\| f \,|S_p^{\overline{r}}W(\mathbb{R}^d)\|$ and

$$\| f \,|S_p^{\overline{r}}W(\mathbb{R}^d)\|^* := \Big\| \mathcal{F}^{-1}\Big(\prod_{i=1}^d (1 + |\xi_i|^2)^{r_i/2}\, \mathcal{F}f(\xi)\Big)(\,\cdot\,) \,\Big| L_p(\mathbb{R}^d)\Big\| \tag{2.2}$$

are equivalent, cf. e.g. [ST, 2.3.1]. The Fourier-analytic description can be taken to generalize these Sobolev spaces to fractional and negative order of smoothness, cf. [ST, Chapt. 2]. We will take (2.2) as the definition of $S_p^{\overline{r}}W(\mathbb{R}^d)$ if $r = (r_1, \ldots, r_d)$, $r_i \in \mathbb{R}$, $i = 1, \ldots, d$.

2.2 Some new Function Spaces

As it will become clear below the description of the trace spaces will lead to some new Sobolev-type spaces. For us it will be sufficient to introduce these classes in the two-dimensional setting. For the rest of this section we concentrate on $p = 2$.
Let $\mathcal{M}$ be a 2×2-matrix,

$$\mathcal{M} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \quad \det \mathcal{M} \neq 0\,, \quad \text{and let} \quad \vec{\eta}_1 = \begin{pmatrix} a \\ b \end{pmatrix}, \qquad \vec{\eta}_2 = \begin{pmatrix} c \\ d \end{pmatrix}. \tag{2.3}$$

Definition 2.1. *Let $\mathcal{M}, \vec{\eta_1}, \vec{\eta_2}$ be as in (2.3). Let $r_1, r_2 \in \mathbb{R}$. Then $S_2^{r_1,r_2}W(\mathcal{M}, \mathbb{R}^2)$ denotes the collection of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^2)$ such that $f \circ \mathcal{M} \in S_2^{r_1,r_2}W(\mathbb{R}^2)$. We endow this class with the norm*

$$\| f \,|S_2^{r_1,r_2}W(\mathcal{M}, \mathbb{R}^2)\| := \| f \circ \mathcal{M} \,|S_2^{r_1,r_2}W(\mathbb{R}^2)\| .$$

The following properties of these classes are immediate.

Lemma 2.2. *Let $\mathcal{M}, \vec{\eta_1}, \vec{\eta_2}$ be as in (2.3). Let $r_1, r_2 \in \mathbb{R}$.*
(i) *The classes $S_2^{r_1,r_2}W(\mathcal{M}, \mathbb{R}^2)$ are Banach spaces continuously embedded into $\mathcal{S}'(\mathbb{R}^2)$.*
(ii) *$C_0^\infty(\mathbb{R}^2)$ is a dense subset of $S_2^{r_1,r_2}W(\mathcal{M}, \mathbb{R}^2)$.*
(iii) *A function $f \in \mathcal{S}'(\mathbb{R}^2)$ belongs to $S_2^{r_1,r_2}W(\mathcal{M}, \mathbb{R}^2)$ if and only if*

$$(1 + |a\,\xi_1 + b\,\xi_2|^2)^{r_1/2}\,(1 + |c\,\xi_1 + d\,\xi_2|^2)^{r_2/2}\,|\mathcal{F}f(\xi)| \in L_2(\mathbb{R}^2) .$$

Furthermore, the expression

$$\Big\| (1 + |a\,\xi_1 + b\,\xi_2|^2)^{r_1/2}\,(1 + |c\,\xi_1 + d\,\xi_2|^2)^{r_2/2}\,|\mathcal{F}f(\xi)| \,\Big| L_2(\mathbb{R}^2)\Big\|$$

yields an equivalent norm in $S_2^{r_1,r_2}W(\mathcal{M}, \mathbb{R}^2)$.

For $r_1, r_2 \in \mathbb{N}_0$ (here $\mathbb{N}_0$ denotes the natural numbers including 0) there is an other interpretation. As usual, by $\frac{\partial f}{\partial \vec{\eta}}$ we denote the weak directional derivative of f in direction $\vec{\eta}$.

Definition 2.3. *Let $\vec{\eta_1}, \vec{\eta_2}$ be linearly independent vectors in $\mathbb{R}^2$. Let $r_1, r_2 \in \mathbb{N}_0$. Then $S_2^{r_1,r_2}W(\vec{\eta_1}, \vec{\eta_2})$ denotes the collection of all functions $f \in L_2(\mathbb{R}^2)$ such that*

$$\frac{\partial^{\alpha_1+\alpha_2} f}{\partial \vec{\eta_1}^{\,\alpha_1}\,\partial \vec{\eta_2}^{\,\alpha_2}} \in L_2(\mathbb{R}^2) \qquad \text{for all} \quad \alpha_i \le r_i,\ i = 1, 2 .$$

We endow this class with the norm

$$\| f \,|S_2^{r_1,r_2}W(\vec{\eta_1}, \vec{\eta_2})\| := \sum_{\alpha_1=0}^{r_1} \sum_{\alpha_2=0}^{r_2} \Big\| \frac{\partial^{\alpha_1+\alpha_2} f}{\partial \vec{\eta_1}^{\,\alpha_1}\,\partial \vec{\eta_2}^{\,\alpha_2}} \Big| L_2(\mathbb{R}^2)\Big\| .$$

Remark 2.4. Obviously, these classes $S_2^{r_1,r_2}W(\vec{\eta_1}, \vec{\eta_2})$ are Banach spaces. Let e_1, e_2 denote the elements of the canonical basis of $\mathbb{R}^2$. Then we have $S_2^{r_1,r_2}W(e_1, e_2) = S_2^{r_1,r_2}W(\mathbb{R}^2)$. Furthermore, $C_0^\infty(\mathbb{R}^2)$ is a dense set in $S_2^{r_1,r_2}W(\vec{\eta_1}, \vec{\eta_2})$ for arbitrary vectors $\vec{\eta_1}, \vec{\eta_2}$.

For a smooth function f it follows

$$\frac{\partial}{\partial x_1}(f \circ \mathcal{M})(x) =< \nabla f(\mathcal{M}x), \vec{\eta_1} >= \frac{\partial f}{\partial \vec{\eta_1}}(\mathcal{M}x) .$$

By an induction argument we conclude

$$\frac{\partial^{\alpha_1+\alpha_2}}{\partial x_1^{\alpha_1}\,\partial x_2^{\alpha_2}}(f \circ \mathcal{M})(x) = \frac{\partial^{\alpha_1+\alpha_2} f}{\partial \vec{\eta_1}^{\,\alpha_1}\,\partial \vec{\eta_2}^{\,\alpha_2}}(\mathcal{M}x) .$$

Using the density of smooth compactly supported functions this proves the following.

Lemma 2.5. *Let $\mathcal{M}, \vec{\eta_1}$, and $\vec{\eta_2}$ be as in (2.3). A function $f \in L_2(\mathbb{R}^2)$ belongs to $S_2^{r_1,r_2}W(\vec{\eta_1}, \vec{\eta_2})$ if and only if the function $f \circ \mathcal{M}$ belongs to $S_2^{r_1,r_2}W(\mathbb{R}^2)$. Furthermore, the norms $\| f \,|S_2^{r_1,r_2}W(\vec{\eta_1}, \vec{\eta_2})\|$ and $\| f \circ \mathcal{M} \,|S_2^{r_1,r_2}W(\mathbb{R}^2)\|$ are equivalent.*

2.3 The Trace with Respect to an Arbitrary Orthogonal Basis of the Hyperplane

Let $A_1(\mathbb{R}^3)$ be a class of functions (distributions) defined on $\mathbb{R}^3$ and let $C(\mathbb{R}^3)$ be the collection of all continuous functions on $\mathbb{R}^3$. By $\widetilde{\Gamma}$ we denote a hyperplane in $\mathbb{R}^3$. Then we consider the mapping

$$T:\, f \to f_{|\widetilde{\Gamma}}$$

which is well-defined in case of a continuous function f. The aim of this paper consists in determining a class of functions $A_2(\mathbb{R}^2) \hookrightarrow \mathcal{S}'(\mathbb{R}^2)$ such that T, originally defined on $A_1(\mathbb{R}^3) \cap C(\mathbb{R}^3)$, extends to a linear, continuous and surjective mapping belonging to $\mathcal{L}(A_1(\mathbb{R}^3), A_2(\mathbb{R}^2))$. In case, that there exists a linear and continuous operator ext $\in \mathcal{L}(A_2(\mathbb{R}^2), A_1(\mathbb{R}^3))$ such that $T \circ \mathrm{ext} = I$ (identity on $A_2(\mathbb{R}^2)$), we shall call T a retraction and ext its corresponding coretraction.
In the monographs Amanov [Am, 9.5], Gelman, Maz'ya [GM, 2.3] and Schmeißer, Triebel [ST, 2.4.2] the traces of function spaces with dominating mixed smoothness on hyperplanes parallel to the coordinate axes were studied. For simplicity let the hyperplane be given by $x_3 = 0$. Then the result is the following.

Proposition 2.6. *Let $r_3 > 1/2$. Then the mapping*

$$T:\, f(x_1, x_2, x_3) \to f(x_1, x_2, 0)$$

extends to a retraction from $S_2^{r_1,r_2}W(\mathbb{R}^3)$ onto $S_2^{r_1,r_2}W(\mathbb{R}^2)$.

Remark 2.7. A few comments are in order. First of all, $\mathcal{S}(\mathbb{R}^3)$ is dense in the class $S_p^{r_1,r_2}W(\mathbb{R}^3)$. So the trace operator is the unique linear extension of the mapping T. Secondly, there is a natural coordinate system on the hyperplane $x_3 = 0$ to measure the smoothness of the trace, namely that one induced by the unit vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Notice that the spaces $S_p^{r_1,r_2}W(\mathbb{R}^2)$ are not invariant under rotations in general.

In this paper we investigate the trace with respect to the hyperplane

$$\Gamma := \Big\{(x_1, x_2, x_3) \in \mathbb{R}^3 \;:\quad x_1 + x_2 + x_3 = 0\Big\}.$$

with Γ as a model case for a hyperplane in an oblique position. However, taking the trace with respect to the hyperplane

$$\Gamma_{\overline{\gamma}} := \{(x_1, x_2, x_3) :\quad \gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_3 = 0\}, \qquad \overline{\gamma} = (\gamma_1, \gamma_2, \gamma_3)\,,$$

where $\gamma_1 \cdot \gamma_2 \cdot \gamma_3 \neq 0$, would give us the same result (up to the norms of considered operators). This statement relies on the fact, that the mapping

$$f(x_1, x_2, x_3) \to f(\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3), \quad \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \neq 0,$$

is a bounded bijective mapping of $S_2^{\overline{r}}W(\mathbb{R}^3)$ onto itself.
The "natural" trace operators

$$(\mathrm{tr}_1\, f)(x_2, x_3) = f(-x_2 - x_3, x_2, x_3), \tag{2.4}$$

$$(\mathrm{tr}_2\, f)(x_1, x_3) = f(x_1, -x_1 - x_3, x_3), \tag{2.5}$$

$$(\mathrm{tr}_3\, f)(x_1, x_2) = f(x_1, x_2, -x_1 - x_2) \tag{2.6}$$

and the trace operator $\mathrm{tr}_{\mathcal{O}}\, f$, see (1.1) and (1.2), are connected through

$$\begin{aligned}(\mathrm{tr}_{\mathcal{O}}\, f)(z_1,z_2) &= f(z_1\vec{\sigma}_1+z_2\vec{\sigma}_2) = f\left(\sigma_{1,1}z_1+\sigma_{2,1}z_2, \sigma_{1,2}z_1+\sigma_{2,2}z_2, \sigma_{1,3}z_1+\sigma_{2,3}z_2\right)\\ &= (\mathrm{tr}_1\, f)\left(\sigma_{1,2}z_1+\sigma_{2,2}z_2, \sigma_{1,3}z_1+\sigma_{2,3}z_2\right)\\ &= (\mathrm{tr}_1\, f)(\mathcal{R}_1\vec{z}), \qquad (2.7)\end{aligned}$$

where

$$\mathcal{R}_1 = \begin{pmatrix} \sigma_{1,2} & \sigma_{2,2} \\ \sigma_{1,3} & \sigma_{2,3} \end{pmatrix} \quad \text{and} \quad \vec{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \qquad (2.8)$$

Analoguously one obtains

$$(\mathrm{tr}_{\mathcal{O}}\, f)(z_1,z_2) = (\mathrm{tr}_2\, f)(\mathcal{R}_2\vec{z}) = (\mathrm{tr}_3\, f)(\mathcal{R}_3\vec{z})\,, \qquad (2.9)$$

with

$$\mathcal{R}_2 = \begin{pmatrix} \sigma_{1,1} & \sigma_{2,1} \\ \sigma_{1,3} & \sigma_{2,3} \end{pmatrix}, \quad \mathcal{R}_3 = \begin{pmatrix} \sigma_{1,1} & \sigma_{2,1} \\ \sigma_{1,2} & \sigma_{2,2} \end{pmatrix}. \qquad (2.10)$$

The linear independence of the vectors $\vec{\sigma}_1, \vec{\sigma}_2$, combined with $\vec{\sigma}_1, \vec{\sigma}_2 \in \Gamma$, ensure that these matrices $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ are regular.
In what follows we shall determine the regularity of $\mathrm{tr}_{\mathcal{O}}\, f$ as well as of $\mathrm{tr}_i\, f$, $i = 1,2,3$.
Above we considered all orthogonal bases of Γ. Probably it would be more natural to restrict to orthonormal bases. However, all function spaces under consideration here remain invariant under the change from an orthogonal to the associated orthonormal basis (up to equivalent quasi-norms). The greater generality leads to nothing new but it simplifies the calculations. For that reason we shall work with orthogonal bases.

2.4 The Regularity of $\mathrm{tr}_{\mathcal{O}}\, f$

2.4.1 A Description of the General Case

Let $f \in C_0^\infty(\mathbb{R}^3)$. Now we introduce a very useful decomposition of f. Let $\mathcal{X}_i$ denote the characteristic function of the set

$$M_i := \Big\{(\tau_1,\tau_2,\tau_3) : \quad |\tau_i| = \min(|\tau_1|,|\tau_2|,|\tau_3|)\Big\}, \quad i = 1,2,3.$$

Hence

$$|M_i \cap M_j| = 0\,, \qquad i \neq j\,, \qquad \text{and} \qquad \bigcup_{i=1}^{3} M_i = \mathbb{R}^3\,,$$

(here $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^3$). We put

$$f_i(x) := \mathcal{F}^{-1}[\mathcal{X}_i(\xi)\, \mathcal{F}f(\xi)](x)\,,$$

and obtain $f = f_1 + f_2 + f_3$. We continue by defining

$$g_i(x_1,x_2) = (\mathrm{tr}_i\, f_i)(x_1,x_2)\,, \qquad i = 1,2,3\,. \qquad (2.11)$$

Elementary properties of the Fourier transform yield

$$\begin{aligned}
\mathcal{F}_2 g_1(\xi_1,\xi_2) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathcal{F}_3 f_1(\tau_1,\xi_1+\tau_1,\xi_2+\tau_1)\, d\tau_1 \\
\mathcal{F}_2 g_2(\xi_1,\xi_2) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathcal{F}_3 f_2(\xi_1+\tau_2,\tau_2,\xi_2+\tau_2)\, d\tau_2 \\
\mathcal{F}_2 g_3(\xi_1,\xi_2) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathcal{F}_3 f_3(\xi_1+\tau_3,\xi_2+\tau_3,\tau_3)\, d\tau_3\,,
\end{aligned} \tag{2.12}$$

where $\mathcal{F}_2 g$ denotes the Fourier transform in $\mathbb{R}^2$ and $\mathcal{F}_3 f$ the Fourier transform in $\mathbb{R}^3$, respectively. Now we are going to check the regularity of the functions g_i. To begin with we investigate $i = 1$. Let $r_1 > 1/2$. By using Hölder's inequality we obtain

$$\begin{aligned}
\int_{\mathbb{R}^2} & (1+\xi_2^2)^{r_2}(1+\xi_3^2)^{r_3}\Big|\int_{\mathbb{R}} \mathcal{F}_3 f_1(\tau_1,\xi_2+\tau_1,\xi_3+\tau_1)\ d\tau_1\Big|^2 d\xi_1\, d\xi_2 \\
&\le c_1 \int_{\mathbb{R}^3} (1+\xi_2^2)^{r_2}(1+\xi_3^2)^{r_3}(1+\tau_1^2)^{r_1}\, |\mathcal{F}_3 f_1(\tau_1,\xi_2+\tau_1,\xi_3+\tau_1)|^2\, d\tau_1\, d\xi_1\, d\xi_2 \\
&= c_1 \int_{\mathbb{R}^3} [1+(\tau_2-\tau_1)^2]^{r_2}[1+(\tau_3-\tau_1)^2]^{r_3}(1+\tau_1^2)^{r_1}|\mathcal{F}_3 f_1(\tau_1,\tau_2,\tau_3)|^2\, d\vec{\tau}
\end{aligned}$$

with $c_1 = \int_{\mathbb{R}}(1+\tau_1^2)^{-r_1}\, d\tau_1 < \infty$. Finally, we observe that if $|\tau_1| \le \min(|\tau_2|,|\tau_3|)$, then $|\tau_2-\tau_1| \le 2|\tau_2|$, $|\tau_3-\tau_1| \le 2|\tau_3|$ and

$$[1+(\tau_2-\tau_1)^2]^{r_2}\,[1+(\tau_3-\tau_1)^2]^{r_3} \le 4^{r_2+r_3}\,(1+\tau_2^2)^{r_2}\,(1+\tau_3^2)^{r_3}.$$

Because of $\operatorname{supp} \mathcal{F}_3 f_1 \subset \{(\tau_1,\tau_2,\tau_3) \in \mathbb{R}^3 : |\tau_1| \le \min(|\tau_2|,|\tau_3|)\}$, we finally conclude

$$\|\operatorname{tr}_1 f_1\,|S_2^{r_2,r_3}W(\mathbb{R}^2)\| \le c_2\, \|f_1\,|S_2^{\vec{r}}W(\mathbb{R}^3)\| \le c_2\, \|f\,|S_2^{\vec{r}}W(\mathbb{R}^3)\|\,. \tag{2.13}$$

This proves $\operatorname{tr}_1 f_1 \in S_2^{r_2,r_3}W(\mathbb{R}^2)$. Similarly one obtains $\operatorname{tr}_2 f_2 \in S_2^{r_1,r_3}W(\mathbb{R}^2)$ (if $r_2 > 1/2$) and $\operatorname{tr}_3 f_3 \in S_2^{r_1,r_2}W(\mathbb{R}^2)$ (if $r_3 > 1/2$), respectively. To summarize our findings we need to recall a further notion. For three quasi-Banach spaces $A_1, A_2, A_3 \hookrightarrow \mathcal{S}'(\mathbb{R}^2)$ of tempered distributions we put

$$A_1 + A_2 + A_3 := \Big\{g \in \mathcal{S}'(\mathbb{R}^2): \quad \exists g_i \in A_i,\ i = 1,2,3, \quad \text{s.t. } g = g_1+g_2+g_3\Big\}.$$

We equip this space with a quasi-norm by taking

$$\|g\,|A_1+A_2+A_3\| := \inf\Big\{\sum_{i=1}^{3}\|g_i|A_i\| : \quad g = g_1+g_2+g_3\,,\ g_i \in A_i,\ i = 1,2,3\Big\}.$$

Lemma 2.8. *Let* $\mathcal{O}$ *be an orthogonal basis of* Γ *and let* $\mathcal{R}_i, i = 1,2,3$ *be matrices associated with* $\mathcal{O}$ *by* (1.1), (2.8) *and* (2.10).
Suppose $\min(r_1,r_2,r_3) > 1/2$. *Then* $\operatorname{tr}_{\mathcal{O}}$ *becomes a continuous mapping of* $S_2^{r_1,r_2,r_3}W(\mathbb{R}^3)$ *into*

$$S_2^{r_1,r_2}W(\mathcal{R}_3^{-1},\mathbb{R}^2) + S_2^{r_1,r_3}W(\mathcal{R}_2^{-1},\mathbb{R}^2) + S_2^{r_2,r_3}W(\mathcal{R}_1^{-1},\mathbb{R}^2) \tag{2.14}$$

Proof. The boundedness of $\operatorname{tr}_{\mathcal{O}}$ follows from the identity

$$(\operatorname{tr}_{\mathcal{O}} f)(\vec{z}) = \sum_{i=1}^{3}(\operatorname{tr}_{\mathcal{O}} f_i)(\vec{z}) = \sum_{i=1}^{3}(\operatorname{tr}_i f_i)(\mathcal{R}_i\vec{z})\,,$$

cf. (2.7), (2.9), the definition of the spaces $S_2^{r_1,r_2}W(\mathcal{M},\mathbb{R}^2)$ and the inequality (2.13) and its counterparts for tr_2 and tr_3. $\square$

The restriction $\min(r_1, r_2, r_3) > 1/2$ has been convenient but is by no means necessary. Moreover, as we shall see by the next theorem the operator $\mathrm{tr}_{\mathcal{O}}$ is surjective in Lemma 2.8. The description of the trace class becomes more complicated than in Lemma 2.8 if $\min(r_1, r_2, r_3) < 1/2$.

Theorem 2.9. *Let $\mathcal{O}$ be an orthogonal basis of Γ and let $\mathcal{R}_i, i = 1, 2, 3$ be matrices associated with $\mathcal{O}$ by (1.1), (2.8) and (2.10).*
Let $\overline{r} = (r_1, r_2, r_3) \in \mathbb{R}^3$ with $r_i \neq 1/2, i = 1, 2, 3$ and

$$\min\left(r_1, r_2, r_3, r_1 + r_2 - \frac{1}{2}, r_1 + r_3 - \frac{1}{2}, r_2 + r_3 - \frac{1}{2}\right) > 0\,. \tag{2.15}$$

Then

$$\mathrm{tr}_{\mathcal{O}} \in \mathcal{L}\Big(S_2^{\overline{r}}W(\mathbb{R}^3), S^1(\mathbb{R}^2) + S^2(\mathbb{R}^2) + S^3(\mathbb{R}^2)\Big), \tag{2.16}$$

where

$$S^1(\mathbb{R}^2) := \begin{cases} S_2^{r_2,r_3}W(\mathcal{R}_1^{-1}, \mathbb{R}^2), & \text{if } r_1 > \frac{1}{2}\,, \\ S_2^{r_2,r_3+r_1-\frac{1}{2}}W(\mathcal{R}_1^{-1}, \mathbb{R}^2) \cap S_2^{r_2+r_1-\frac{1}{2},r_3}W(\mathcal{R}_1^{-1}, \mathbb{R}^2), & \text{if } r_1 < \frac{1}{2}\,, \end{cases}$$

and similarly for S^2 and S^3.
Conversely, to each function $g \in S^1(\mathbb{R}^2) + S^2(\mathbb{R}^2) + S^3(\mathbb{R}^2)$ there exists a function $f \in S_2^{\overline{r}}W(\mathbb{R}^3)$ such that $\mathrm{tr}_{\mathcal{O}} f = g$.

Proof. Step 1. Preparations. For $\alpha, \beta, t \in \mathbb{R}$ we define

$$I(\alpha, \beta, t) := \int_{-\infty}^{\infty} (1 + (t + \tau)^2)^{-\alpha}\, (1 + \tau^2)^{-\beta}\, d\tau\,.$$

In case $\alpha + \beta > 1/2$, $\beta < 1/2$, elementary calculations yield

$$I(\alpha, \beta, t) \le c \begin{cases} (1 + t^2)^{-\beta} & \text{if} \quad \alpha > 1/2\,, \\ (1 + t^2)^{-\beta}\,(1 + \log(1 + |t|)) & \text{if} \quad \alpha = 1/2\,, \\ (1 + t^2)^{-(\alpha+\beta)+1/2} & \text{if} \quad \alpha < 1/2\,, \end{cases} \tag{2.17}$$

for some c independent of t.
Step 2. The boundedness of $\mathrm{tr}_{\mathcal{O}}$ in case $\min(r_1, r_2, r_3) > 1/2$ has been proven before.
Now we suppose $0 < r_1 < 1/2$. We proceed as at the beginning of this subsection and obtain

$$\begin{aligned} \int_{\mathbb{R}^2} & (1 + \xi_2^2)^{r_2} (1 + \xi_3^2)^{r_3+r_1-\frac{1}{2}} \Big| \int_{\mathbb{R}} \mathcal{F}_3 f_1(\tau_1, \xi_2 + \tau_1, \xi_3 + \tau_1)\, d\tau_1 \Big|^2 d\xi_1\, d\xi_2 \\ \le & \int_{\mathbb{R}^3} (1 + \xi_2^2)^{r_2} (1 + \xi_3^2)^{r_3+r_1-\frac{1}{2}}\, I(\alpha, r_1, \xi_3)\, (1 + \tau_1^2)^{r_1}\, (1 + (\tau_1 + \xi_3)^2)^{\alpha} \\ & \qquad\qquad\qquad \times \quad |\mathcal{F}_3 f_1(\tau_1, \xi_2 + \tau_1, \xi_3 + \tau_1)|^2\, d\tau_1\, d\xi_1\, d\xi_2 \\ \le & \; c_1 \int_{\mathbb{R}^3} [1 + (\tau_2 - \tau_1)^2]^{r_2} [1 + (\tau_3 - \tau_1)^2]^{r_3-\alpha} (1 + \tau_1^2)^{r_1}\, (1 + \tau_3^2)^{\alpha}\, |\mathcal{F}_3 f_1(\tau_1, \tau_2, \tau_3)|^2\, d\vec{\tau}\,, \end{aligned}$$

where we have used (2.17) with some α satisfying $\frac{1}{2} - r_1 < \alpha < \frac{1}{2}$. Choosing α sufficiently close to $\frac{1}{2} - r_1$ our restriction $r_1 + r_3 > 1/2$, see (2.15), guarantees $r_3 - \alpha \ge 0$. Furthermore, taking into account the information on the support of $\mathcal{F} f_1$ we arrive at

$$\|\ \mathrm{tr}_1 f_1\, |S_2^{r_2,r_3+r_1-\frac{1}{2}}W(\mathbb{R}^2)\| \le c_2\, \|\, f_1\, |S_2^{\overline{r}}W(\mathbb{R}^2)\| \le c_2\, \|\, f\, |S_2^{\overline{r}}W(\mathbb{R}^2)\|$$

with some c independent of f. Interchanging the roles of ξ_1 and ξ_2 also

$$\| \operatorname{tr}_1 f_1 \,|S_2^{r_2+r_1-\frac{1}{2},r_3}W(\mathbb{R}^2)\| \le c_3 \| f\,|S_2^{\bar{r}}W(\mathbb{R}^2)\|$$

follows. Moreover, by symmetry we obtain the needed estimates of $\operatorname{tr}_i f_1$, $i = 2, 3$, as well. This completes the proof of the boundedness.

Step 3. Construction of an extension operator.

Substep 3.1. Construction of a linear extension operator for $S_2^{r_1,r_2}W(\mathbb{R}^2)$. Let $\varphi \in C_0^\infty(\mathbb{R})$ be a function such that $\int \varphi(t)\,dt = \sqrt{2\pi}$. Then, for $g \in C_0^\infty(\mathbb{R}^2)$ and $x \in \mathbb{R}^3$, we define

$$\begin{aligned}
f_1(x) &= \operatorname{ext}_1^* g(x) := \mathcal{F}_3^{-1}\big[\varphi(\xi_1)\,\mathcal{F}_2 g(\xi_2-\xi_1,\xi_3-\xi_1)\big](x)\\
f_2(x) &= \operatorname{ext}_2^* g(x) := \mathcal{F}_3^{-1}\big[\varphi(\xi_2)\,\mathcal{F}_2 g(\xi_1-\xi_2,\xi_3-\xi_2)\big](x)\\
f_3(x) &= \operatorname{ext}_3^* g(x) := \mathcal{F}_3^{-1}\big[\varphi(\xi_3)\,\mathcal{F}_2 g(\xi_1-\xi_3,\xi_2-\xi_3)\big](x)\,.
\end{aligned}$$

Hence, e.g. for f_3, we conclude

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathcal{F}_3 f_3(\xi_1+\tau_3,\xi_2+\tau_3,\tau_3)\,d\tau_3 = \mathcal{F}_2 g(\xi_1,\xi_2)$$

and from this identity we derive

$$g(x_1,x_2) = (\operatorname{tr}_3 f_3)(x_1,x_2) = f_3(x_1,x_2,-x_1-x_2)\,, \qquad (x_1,x_2)\in\mathbb{R}^2\,.$$

Similarly

$$g = \operatorname{tr}_1 f_1 \qquad \text{and} \qquad g = \operatorname{tr}_2 f_2\,.$$

The regularity of $\operatorname{ext}_3^* g$ is easily checked in view of

$$\begin{aligned}
&\int_{\mathbb{R}^3} (1+\xi_1^2)^{r_1}\,(1+\xi_2^2)^{r_2}\,(1+\xi_3^2)^{r_3}\,|\mathcal{F}_3\operatorname{ext}^* g(\xi)|^2\,d\vec{\xi}\\
&\quad= \int_{\mathbb{R}^3}(1+|\xi_1+\tau_3|^2)^{r_1}\,(1+|\xi_2+\tau_3|^2)^{r_2}\,(1+\tau_3^2)^{r_3}\,|\varphi(\tau_3)\,\mathcal{F}_2 g(\xi_1,\xi_2)|^2\,d\xi_1\,d\xi_2\,d\tau_3\\
&\quad\le c_1\int_{\mathbb{R}}(1+|\tau_3|^2)^{r_1+r_2+r_3}\,|\varphi(\tau_3)|^2\,d\tau_3\;\| g\,|S_2^{r_1,r_2}W(\mathbb{R}^2)\|^2\\
&\quad\le c_2\,\| g\,|S_2^{r_1,r_2}W(\mathbb{R}^2)\|^2\,,
\end{aligned}$$

where we also used the fact that φ has compact support. This proves $\operatorname{ext}_3^* \in \mathcal{L}(S_2^{r_1,r_2}W(\mathbb{R}^2), S_2^{r_1,r_2,r_3}W(\mathbb{R}^3))$ for any $r_3\in\mathbb{R}$. Similarly, $\operatorname{ext}_1^* \in \mathcal{L}(S_2^{r_2,r_3}W(\mathbb{R}^2), S_2^{r_1,r_2,r_3}W(\mathbb{R}^3))$ for any r_1 and $\operatorname{ext}_2^* \in \mathcal{L}(S_2^{r_1,r_3}W(\mathbb{R}^2), S_2^{r_1,r_2,r_3}W(\mathbb{R}^3))$ for any r_2, respectively.

Substep 3.2. Construction of an extension operator in case $\min(r_1,r_2,r_3) > 1/2$. We shall use the abbreviations $A_1 = S_2^{r_2,r_3}W(\mathcal{R}_1^{-1},\mathbb{R}^2)$, $A_2 = S_2^{r_1,r_3}W(\mathcal{R}_2^{-1},\mathbb{R}^2)$ and $A_3 = S_2^{r_1,r_2}W(\mathcal{R}_3^{-1},\mathbb{R}^2)$. Let $g\in A_1+A_2+A_3$. Further, let $g = g_1+g_2+g_3$, where

$$g_i\in A_i\,,\quad i=1,2,3 \qquad \text{and} \qquad \| g\,|A_1+A_2+A_3\| \le 2\sum_{i=1}^{3}\| g_i\,|A_i\|\,.$$

By definition $g_1(\mathcal{R}_1^{-1}\cdot) \in S_2^{r_2,r_3}W(\mathbb{R}^2)$ and consequently, by Step 3.1, $f_1 := \operatorname{ext}_1^* g_1(\mathcal{R}_1^{-1}\cdot) \in S_2^{r_1,r_2,r_3}W(\mathbb{R}^3)$. Similarly, $f_2 := \operatorname{ext}_2^* g_2(\mathcal{R}_2^{-1}\cdot)$, $f_3 := \operatorname{ext}_3^* g_3(\mathcal{R}_3^{-1}\cdot) \in S_2^{r_1,r_2,r_3}W(\mathbb{R}^3)$. We put

$f := f_1 + f_2 + f_3$. Because of

$$\begin{aligned}
\operatorname{tr}_{\mathcal{O}} f &= \sum_{i=1}^{3} \operatorname{tr}_{\mathcal{O}} f_i = \sum_{i=1}^{3} (\operatorname{tr}_i f_i)(\mathcal{R}_i \cdot) \\
&= \sum_{i=1}^{3} \Big(\operatorname{tr}_i \operatorname{ext}_i^* g_i(\mathcal{R}_i^{-1} \cdot) \Big)(\mathcal{R}_i \cdot) \\
&= \sum_{i=1}^{3} g_i = g \,,
\end{aligned}$$

see Substep 3.1, this proves the existence of a bounded extension of g if $\min(r_1, r_2, r_3) > 1/2$.

Substep 3.3. Let $0 < r_1 < 1/2$. We shall use the abbreviations $A_1 = S_2^{r_2, r_3 + r_1 - \frac{1}{2}} W(\mathbb{R}^2)$, $A_2 = S_2^{r_2 + r_1 - \frac{1}{2}, r_3} W(\mathbb{R}^2)$. By the arguments from the previous substep (and by symmetry) it will be sufficient to construct a function $f_1 \in S_2^{r_1, r_2, r_3} W(\mathbb{R}^3)$ such that $\operatorname{tr}_1 f_1 = g_1(\mathcal{R}_1^{-1} \cdot) \in A_1 \cap A_2$. To shorten notation we write h_1 instead of $g_1(\mathcal{R}_1^{-1} \cdot)$. To begin with we define two subsets of $\mathbb{R}^3$

$$\begin{aligned}
\Omega_1 := \Big\{ (\xi_1, \xi_2, \xi_3) \quad : \quad & |\xi_2 - \xi_1| \le |\xi_3 - \xi_1| \,, \\
& \frac{|\xi_2 - \xi_1|}{4} \le |\xi_1| \le \frac{|\xi_2 - \xi_1|}{2} \qquad \text{if} \quad |\xi_2 - \xi_1| \ge 1 \,, \\
& \qquad |\xi_1| \le 1 \qquad \text{if} \qquad |\xi_2 - \xi_1| < 1 \Big\} \,, \\
\Omega_2 := \Big\{ (\xi_1, \xi_2, \xi_3) \quad : \quad & |\xi_3 - \xi_1| < |\xi_2 - \xi_1| \,, \\
& \frac{|\xi_3 - \xi_1|}{4} \le |\xi_1| \le \frac{|\xi_3 - \xi_1|}{2} \qquad \text{if} \quad |\xi_3 - \xi_1| \ge 1 \,, \\
& \qquad |\xi_1| \le 1 \qquad \text{if} \qquad |\xi_3 - \xi_1| < 1 \Big\} \,.
\end{aligned}$$

Obviously, these sets are disjoint. Let $\mathcal{X}_i$ denote the characteristic function of Ω_i, $i = 1, 2$. Then we define

$$\begin{aligned}
f_1(x) \quad := \quad & \int e^{ix\xi} \, \mathcal{F}_2 h_1(\xi_2 - \xi_1, \xi_3 - \xi_1) \\
\times \quad & \Big(\mathcal{X}_1(\xi) \, \frac{(1 + (\xi_2 - \xi_1)^2)^{r_2 + r_1 - 1/2} \, (1 + (\xi_3 - \xi_1)^2)^{r_3}}{(1 + \xi_1^2)^{r_1} \, (1 + \xi_2^2)^{r_2} \, (1 + \xi_3^2)^{r_3}} H^1(\xi_2 - \xi_1, \xi_3 - \xi_1) \\
& + \quad \mathcal{X}_2(\xi) \, \frac{(1 + (\xi_2 - \xi_1)^2)^{r_2} \, (1 + (\xi_3 - \xi_1)^2)^{r_3 + r_1 - 1/2}}{(1 + \xi_1^2)^{r_1} \, (1 + \xi_2^2)^{r_2} \, (1 + \xi_3^2)^{r_3}} H^2(\xi_2 - \xi_1, \xi_3 - \xi_1) \Big) \, d\vec{\xi} \,,
\end{aligned}$$

where the functions H^1, H^2 will be chosen later. First we prove $\operatorname{tr}_1 f_1 = h_1$. It is sufficient to assume $h_1 \in C_0^\infty(\mathbb{R}^2)$. Setting $\tau_2 = \xi_2 - \xi_1$ and $\tau_3 = \xi_3 - \xi_1$ we find

$$\begin{aligned}
f_1(-x_2 - x_3, x_2, x_3) \quad = \quad & \int_{|\tau_2| \le |\tau_3|} e^{i(x_2 \tau_2 + x_3 \tau_3)} \, \mathcal{F}_2 h_1(\tau_2, \tau_3) \, (1 + \tau_2^2)^{r_2 + r_1 - 1/2} \, (1 + \tau_3^2)^{r_3} \, H^1(\tau_2, \tau_3) \\
& \times \int_{I(\tau_2)} \frac{1}{(1 + \xi_1^2)^{r_1} \, (1 + (\tau_2 + \xi_1)^2)^{r_2} \, (1 + (\tau_3 + \xi_1)^2)^{r_3}} \, d\xi_1 \, d\tau_2 \, d\tau_3 \\
+ \quad & \int_{|\tau_3| < |\tau_2|} e^{i(x_2 \tau_2 + x_3 \tau_3)} \, \mathcal{F}_2 h_1(\tau_2, \tau_3) \, (1 + \tau_2^2)^{r_2} \, (1 + \tau_3^2)^{r_3 + r_1 - 1/2} \, H^2(\tau_2, \tau_3) \\
& \times \int_{I(\tau_3)} \frac{1}{(1 + \xi_1^2)^{r_1} \, (1 + (\tau_2 + \xi_1)^2)^{r_2} \, (1 + (\tau_3 + \xi_1)^2)^{r_3}} \, d\xi_1 \, d\tau_2 \, d\tau_3
\end{aligned}$$

with $I(\tau_2)$ and $I(\tau_3)$ being appropriate subsets of $\mathbb{R}$. The functions H^1 and H^2 are determined through the identities

$$H^1(\tau_2,\tau_3) = \frac{1}{2\pi}\Big(\int_{I(\tau_2)} \frac{(1+\tau_2^2)^{r_2+r_1-1/2}\,(1+\tau_3^2)^{r_3}}{(1+\xi_1^2)^{r_1}\,(1+(\tau_2+\xi_1)^2)^{r_2}\,(1+(\tau_3+\xi_1)^2)^{r_3}}\,d\xi_1\Big)^{-1},$$
$$H^2(\tau_2,\tau_3) = \frac{1}{2\pi}\Big(\int_{I(\tau_3)} \frac{(1+\tau_2^2)^{r_2}\,(1+\tau_3^2)^{r_3+r_1-1/2}}{(1+\xi_1^2)^{r_1}\,(1+(\tau_2+\xi_1)^2)^{r_2}\,(1+(\tau_3+\xi_1)^2)^{r_3}}\,d\xi_1\Big)^{-1}.$$

As a consequence we obtain

$$f_1(-x_2-x_3,x_2,x_3) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i(x_2\tau_2+x_3\tau_3}\,\mathcal{F}_2 h_1(\tau_2,\tau_3)\,d\tau_2\,d\tau_3 = h_1(x_2,x_3)$$

as claimed. ¿From the definition of the sets Ω_i we derive the existence of two positive constants c_1 and c_2 such that for all τ_2,τ_3

$$c_1 \le H^1(\tau_2,\tau_3) \le c_2$$

as well as

$$c_1 \le H^2(\tau_2,\tau_3) \le c_2\,.$$

This will be used to prove that f_1 is sufficiently regular. Indeed, we have

$$\begin{aligned}
\int_{\mathbb{R}^3} & (1+\xi_1^2)^{r_1}\,(1+\xi_2^2)^{r_2}\,(1+\xi_3^2)^{r_3}|\mathcal{F}_3 f_1(\xi)|^2\,d\vec{\xi} \\
= & \int_{\mathbb{R}^3}(1+\xi_1^2)^{-r_1}\,(1+\xi_2^2)^{-r_2}\,(1+\xi_3^2)^{-r_3}|\mathcal{F}_2 h_1(\xi_2-\xi_1,\xi_3-\xi_1)|^2 \\
\times & \Big(\mathcal{X}_1(\xi)\;|H^1(\xi_2-\xi_1,\xi_3-\xi_1)|^2\,(1+(\xi_2-\xi_1)^2)^{2r_2+2r_1-1}\,(1+(\xi_3-\xi_1)^2)^{2r_3} \\
& + \mathcal{X}_2(\xi)\;|H^2(\xi_2-\xi_1,\xi_3-\xi_1)|^2\,(1+(\xi_2-\xi_1)^2)^{2r_2}\,(1+(\xi_3-\xi_1)^2)^{2r_3+2r_1-1}\Big)\,d\vec{\xi} \\
=: & \; J_1 + J_2\,.
\end{aligned}$$

A change of coordinates, the boundedness of H^1 and the definition of Ω_1 yield

$$\begin{aligned}
J_1 \le & \; c_2^2 \int_{|\tau_2|\le|\tau_3|} |\mathcal{F}_2 h_1(\tau_2,\tau_3)|^2\,(1+\tau_2^2)^{2r_2+2r_1-1}\,(1+\tau_3^2)^{2r_3} \\
& \times \int_{I(\tau_2)}(1+\xi_1^2)^{-r_1}\,(1+(\tau_2+\xi_1)^2)^{-r_2}\,(1+(\tau_3+\xi_1)^2)^{-r_3}\,d\xi_1\,d\tau_2\,d\tau_3 \\
\le & \; c_3\int_{\mathbb{R}^2}|\mathcal{F}_2 h_1(\tau_2,\tau_3)|^2\,(1+\tau_2^2)^{r_2+r_1-1/2}\,(1+\tau_3^2)^{r_3}\,d\tau_2\,d\tau_3\,.
\end{aligned}$$

The estimate of J_2 is similar. Hence

$$\|\, f_1\,|S_2^{r_1,r_2,r_3}W(\mathbb{R}^3)\| \le c_4\left(\|\, h_1\,|S_2^{r_2,r_3+r_1-\frac12}W(\mathbb{R}^2)\| + \|\, h_1\,|S_2^{r_2+r_1-\frac12,r_3}W(\mathbb{R}^2)\|\right)$$

with some constant c_4 independent of h_1. This proves the boundedness of the extension. □

Remark 2.10. Let us mention that we have not shown the existence of a linear and continuous extension operator. The step in which g is splitted into the three functions g_1, g_2 and g_3 need not be linear. This problem will be investigated in the next subsection.

2.4.2 A Description of the Trace Classes on the Fourier Side

For simplicity we concentrate on the situation $\min(r_1, r_2, r_3) > 1/2$. The sum $S_2^{r_1,r_2}W(\mathcal{R}_3^{-1}, \mathbb{R}^2) + S_2^{r_1,r_3}W(\mathcal{R}_2^{-1}, \mathbb{R}^2) + S_2^{r_2,r_3}W(\mathcal{R}_1^{-1}, \mathbb{R}^2)$ is not direct. It is obvious that

$$C_0^\infty(\mathbb{R}^3) \subset \Big(S_2^{r_1,r_2}W(\mathcal{R}_3^{-1}, \mathbb{R}^2) \cap S_2^{r_1,r_3}W(\mathcal{R}_2^{-1}, \mathbb{R}^2) \cap S_2^{r_2,r_3}W(\mathcal{R}_1^{-1}, \mathbb{R}^2)\Big) .$$

At this moment it is not clear whether the connection between g and its optimal decomposition $g_1 + g_2 + g_3$ can be realized in a linear way. But that can be seen easily by the Fourier-analytic description of the trace space.

Let $\mathcal{O}$ be an orthogonal basis of $\Gamma = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$ and let $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ be the matrices associated with $\mathcal{O}$. First, we notice that $g_3 \in S_2^{r_1,r_2}W(\mathcal{R}_3^{-1}, \mathbb{R}^2)$ if, and only if,

$$\underbrace{\Big[1 + (\sigma_{2,2}\xi_1 - \sigma_{1,2}\xi_2)^2\Big]^{r_1/2}\Big[1 + (\sigma_{2,1}\xi_1 - \sigma_{1,1}\xi_2)^2\Big]^{r_2/2}}_{m_3(\xi_1,\xi_2)} \mathcal{F}g_3(\xi_1, \xi_2) \in L_2(\mathbb{R}^2) , \tag{2.18}$$

cf. Lemma 2.2(iii). Similarly, $g_1 \in S_2^{r_2,r_3}W(\mathcal{R}_1^{-1}, \mathbb{R}^2)$ if, and only if,

$$\underbrace{\Big[1 + \Big(\sigma_{2,3}\xi_1 - \sigma_{1,3}\xi_2\Big)^2\Big]^{r_2/2}\Big[1 + \Big(\sigma_{2,2}\xi_1 - \sigma_{1,2}\xi_2\Big)^2\Big]^{r_3/2}}_{m_1(\xi_1,\xi_2)} \mathcal{F}g_1(\xi_1, \xi_2) \in L_2(\mathbb{R}^2). \tag{2.19}$$

and $g_2 \in S_2^{r_1,r_3}W(\mathcal{R}_2^{-1}, \mathbb{R}^2)$ if, and only if,

$$\underbrace{\Big[1 + \Big(\sigma_{2,3}\xi_1 - \sigma_{1,3}\xi_2\Big)^2\Big]^{r_1/2}\Big[1 + (\sigma_{2,1}\xi_1 - \sigma_{1,1}\xi_2)^2\Big]^{r_3/2}}_{m_2(\xi_1,\xi_2)} \mathcal{F}g_2(\xi_1, \xi_2) \in L_2(\mathbb{R}^2) . \tag{2.20}$$

In view of these characterizations we define

$$m(\xi_1, \xi_2) := \min\Big(m_1(\xi_1, \xi_2), m_2(\xi_1, \xi_2), m_3(\xi_1, \xi_2)\Big) . \tag{2.21}$$

and

$$L_2(\mathbb{R}^2, m) := \Big\{g \in L_2(\mathbb{R}^2) : \quad m\,\mathcal{F}g \in L_2(\mathbb{R}^2)\Big\} \tag{2.22}$$

equipped with the natural norm

$$\| g\,|L_2(\mathbb{R}^2, m)\| := \| m\,\mathcal{F}g\,|L_2(\mathbb{R}^2)\| .$$

Now we arrive at the main result of this section.

Theorem 2.11. *Let $\mathcal{O}$ be an orthogonal basis of Γ and let $\mathcal{R}_i, i = 1, 2, 3$ be matrices associated with $\mathcal{O}$ by* (1.1), (2.8) *and* (2.10)*. Suppose (2.15) and $r_i \neq 1/2$, $i = 1, 2, 3$. Then there exists a continuous function m such that* $\mathrm{tr}_{\mathcal{O}}$ *becomes a retraction of $S_2^{r_1,r_2,r_3}W(\mathbb{R}^3)$ onto $L_2(\mathbb{R}^2, m)$. There is a bounded linear extension operator* $\mathrm{ext} \in \mathcal{L}(L_2(\mathbb{R}^2, m), S_2^{r_1,r_2,r_3}W(\mathbb{R}^3))$ *such that* $\mathrm{tr}_{\mathcal{O}} \circ \mathrm{ext} = I$ *(identity on $L_2(\mathbb{R}^2, m)$).*

Proof. We concentrate on the case $\min(r_1, r_2, r_3) > 1/2$. Then the function m is given by (2.21). The modifications which have to be made for the general situation are obvious. We omit the details.

Step 1. Boundedness. Again we shall use the abbreviations $A_1 = S_2^{r_2,r_3}W(\mathcal{R}_1^{-1},\mathbb{R}^2)$, $A_2 = S_2^{r_1,r_3}W(\mathcal{R}_2^{-1},\mathbb{R}^2)$ and $A_3 = S_2^{r_1,r_2}W(\mathcal{R}_3^{-1},\mathbb{R}^2)$. Let $g \in A_1 + A_2 + A_3$ and let $g = g_1 + g_2 + g_3$ be an optimal decomposition of g with $g_i \in A_i$. Then

$$m(\xi)\,|\mathcal{F}g(\xi)| \le \sum_{i=1}^{3} m_i(\xi)\,|\mathcal{F}g_i(\xi)|\,, \qquad \xi \in \mathbb{R}^2\,.$$

But this implies

$$\|\, g\,|L_2(\mathbb{R}^2,m)\| \le \sum_{i=1}^{3} \|\, m_i\,\mathcal{F}g_i\,|L_2(\mathbb{R}^2)\| \le c\sum_{i=1}^{3}\|\, g_i\,|A_i\|\,,$$

with some c independent of g.
Vice versa, if $g \in L_2(\mathbb{R}^2,m)$, then we define

$$\Omega_i := \Big\{(\xi_1,\xi_2)\ :\quad m_i(\xi_1,\xi_2) = m(\xi_1,\xi_2)\Big\}\,, \tag{2.23}$$

$\mathcal{X}_i$ denotes its characteristic function, and

$$g_i(x) := \mathcal{F}^{-1}[\mathcal{X}_i(\xi)\,\mathcal{F}g(\xi)](x)\,, \quad i = 1,2,3. \tag{2.24}$$

Thanks to

$$|\Omega_i \cap \Omega_j| = 0\,, \quad i \neq j\,, \qquad \text{and} \qquad \bigcup_{i=1}^{3}\Omega_i = \mathbb{R}^2\,,$$

($|\cdot|$ Lebesgue measure in $\mathbb{R}^2$) this implies $g = g_1 + g_2 + g_3$ and

$$\|\, m_i\,\mathcal{F}g_i\,|L_2(\mathbb{R}^2)\| \le \|\, m\,\mathcal{F}g\,|L_2(\mathbb{R}^2)\|\,, \qquad i = 1,2,3\,.$$

Summarizing we have proved the coincidence of $S_2^{r_1,r_2}W(\mathcal{R}_3^{-1},\mathbb{R}^2) + S_2^{r_1,r_3}W(\mathcal{R}_2^{-1},\mathbb{R}^2) + S_2^{r_2,r_3}W(\mathcal{R}_1^{-1},\mathbb{R}^2)$ and $L_2(\mathbb{R}^2,m)$ in the sense of equivalent norms. Hence $\mathrm{tr}_{\mathcal{O}} \in \mathcal{L}(S_2^{r_1,r_2,r_3}W(\mathbb{R}^3), L_2(\mathbb{R}^2,m))$.
Step 2. The linear extension. Since the mappings $g \to g_i$, $i = 1,2,3$, cf. (2.24), are linear and continuous, the extension operator constructed in the proof of Theorem 2.9 is linear and bounded as well. □

2.4.3 The Trace Space for a Dominating Direction

This subsection contains an additional observation of minor importance. So we concentrate on $\min(r_1,r_2,r_3) > 1/2$.
A simplified description of the trace spaces can be given in case that one of the parameters r_1, r_2, r_3 is dominating the sum of the other.

Lemma 2.12. *Let $\mathcal{O}$ be an orthogonal basis of Γ and let $\mathcal{R}_i, i = 1,2,3$ be matrices associated with $\mathcal{O}$ by* (1.1), (2.8) *and* (2.10). *Then the embeddings*

$$S_2^{r_1,r_3}W(\mathcal{R}_2^{-1},\mathbb{R}^2) \hookrightarrow S_2^{r_1,r_2}W(\mathcal{R}_3^{-1},\mathbb{R}^2) \qquad \textit{and} \qquad S_2^{r_2,r_3}W(\mathcal{R}_1^{-1},\mathbb{R}^2) \hookrightarrow S_2^{r_1,r_2}W(\mathcal{R}_3^{-1},\mathbb{R}^2)$$

exists if, and only if, $r_3 \ge r_1 + r_2$.

Proof. Again we work in the Fourier image. Let m_1, m_2 and m_3 be the functions defined in (2.18)-(2.20). Then the first embedding is equivalent to the boundedness of m_3/m_2 and the second is equivalent to the boundedness of m_3/m_1, respectively.
Let us turn to the boundedness of the first quotient. By a change of coordinates

$$y_1 := \sigma_{2,2}\xi_1 - \sigma_{1,2}\xi_2 \qquad \text{and} \qquad y_2 := \sigma_{2,3}\xi_1 - \sigma_{1,3}\xi_2$$

and taking care of $\vec{\sigma}_1, \vec{\sigma}_2 \in \Gamma$ the boundedness of m_3/m_2 is equivalent to

$$\sup_{y_1,y_2\in\mathbb{R}} \frac{(1+y_1^2)^{r_1}\,[1+(y_1+y_2)^2]^{r_2}}{(1+y_1^2)^{r_3}(1+y_2^2)^{r_2}} < \infty\,.$$

With $y_2 = 0$ the necessity of $r_3 \ge r_1 + r_2$ follows. Sufficiency can be derived from

$$1 + (y_1+y_2)^2 \le 2(1+y_1^2)(1+y_2^2)\,.$$

□

Theorem 2.13. *Let $\mathcal{O}$ be an orthogonal basis of $\Gamma = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$ and let $\mathcal{R}_3$ be the matrix associated with $\mathcal{O}$ by* (1.1) *and* (2.10). *Let* $\min(r_1, r_2, r_3) > 1/2$ *and suppose $r_3 \ge r_1 + r_2$. Then $\mathrm{tr}_{\mathcal{O}}$ becomes a retraction of $S_2^{r_1,r_2,r_3}W(\mathbb{R}^3)$ onto $S_2^{r_1,r_2}W(\mathcal{R}_3^{-1}, \mathbb{R}^2))$ and*

$$S_2^{r_1,r_2}W(\mathcal{R}_3^{-1}, \mathbb{R}^2)) = L_2(\mathbb{R}^2, m_3) \qquad \textit{(equivalent norms).}$$

Proof. ¿From Lemma 2.12 we derive

$$S_2^{r_1,r_2}W(\mathcal{R}_3, \mathbb{R}^2) + S_2^{r_2,r_3}W(\mathcal{R}_1, \mathbb{R}^2) + S_2^{r_1,r_3}W(\mathcal{R}_2, \mathbb{R}^2) = S_2^{r_1,r_2}W(\mathcal{R}_3, \mathbb{R}^2)$$

with equivalent norms. Now the statement follows from Theorems 2.9 and 2.11. The last identity has been derived in (2.18). □

Also $\mathrm{tr}_1, \mathrm{tr}_2$ and tr_3 have additional properties if one of the smoothness parameters dominates the sum of the other.

Theorem 2.14. *Let $\mathcal{O}$ be an orthogonal basis of $\Gamma = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$ and let $\mathcal{R}_3$ be the matrix associated with $\mathcal{O}$ by* (1.1) *and* (2.10). *Let* $\min(r_1, r_2, r_3) > 1/2$ *and suppose $r_3 \ge r_1 + r_2$.*
Then tr_3 becomes a retraction of $S_2^{r_1,r_2,r_3}W(\mathbb{R}^3)$ onto $S_2^{r_1,r_2}W(\mathbb{R}^2)$, i.e. there exists a linear extension operator $\mathrm{ext}^* \in \mathcal{L}(S_2^{r_1,r_2}W(\mathbb{R}^2), S_2^{r_1,r_2,r_3}W(\mathbb{R}^3))$ s.t. $\mathrm{tr}_3 \circ \mathrm{ext}^* = I$.

Proof. Step 1. Boundedness of tr_3. To show that, we use again (2.12). Furthermore, taking $h(\xi_1, \xi_2, \tau_3) := \mathcal{F}f(\xi_1 + \tau_3, \xi_2 + \tau_3, \tau_3)$, it will be enough to show the existence of some positive constant c such that

$$\begin{aligned} &\int_{\mathbb{R}^2} (1+y_1^2)^{r_1}\,(1+y_2^2)^{r_2} \left| \int_{\mathbb{R}} h(y_1,y_2,y_3)\,dy_3 \right|^2 dy_1\,dy_2 \\ &\quad \le \; c \int_{\mathbb{R}^3} [1+(y_1+y_3)^2]^{r_1}\,[1+(y_2+y_3)^2]^{r_2}\,[1+y_3^2]^{r_3}\,|h(y_1,y_2,y_3)|^2\,dy_1\,dy_2\,dy_3\,. \end{aligned} \tag{2.25}$$

Let us denote

$$\Theta_1(y_1,y_2) := (1+y_1^2)^{r_1}\,(1+y_2^2)^{r_2}$$

and

$$\Theta_2(y_1,y_2,y_3) := [1+(y_1+y_3)^2]^{r_1}\,[1+(y_2+y_3)^2]^{r_2}\,(1+y_3^2)^{r_3}\,,$$

respectively. Then Hölder's inequality leads to

$$\begin{aligned}(1+y_1^2)^{r_1}\,(1+y_2^2)^{r_2} & \quad \Big(\int_{\mathbb{R}} |h(y_1,y_2,y_3)|\,dy_3\Big)^2 \\ & = \Big(\int_{\mathbb{R}} \frac{\sqrt{\Theta_1(y_1,y_2)}}{\sqrt{\Theta_2(y_1,y_2,y_3)}}\sqrt{\Theta_2(y_1,y_2,y_3)}|h(y_1,y_2,y_3)|\,dy_3\Big)^2 \\ & \le \underbrace{\Big(\sup_{y_1,y_2\in\mathbb{R}}\int_{\mathbb{R}} \frac{\Theta_1(y_1,y_2)}{\Theta_2(y_1,y_2,y_3)}\,dy_3\Big)}_{:=\,\Theta(r_1,r_2,r_3)} \int_{\mathbb{R}} \Theta_2(y_1,y_2,y_3)|h(y_1,y_2,y_3)|^2\,dy_3\,.\end{aligned}$$

If $\Theta(r_1,r_2,r_3)<\infty$, then it is enough to integrate this inequality with respect to $y_1,y_2\in\mathbb{R}$ to obtain (2.25). To prove finiteness of $\Theta(r_1,r_2,r_3)$ under the given restrictions is elementary.

Step 2. Surjectivity of tr_3. Here we make use of the operator ext_3^*, defined in the proof of Theorem 2.9, Substep 3.1. □

Remark 2.15. By symmetry we have similar statements with respect to tr_1 as well as to tr_2, e.g. if $\min(r_1,r_2,r_3)>1/2$ and $r_2\ge r_1+r_3$ then tr_2 becomes a retraction of $S_2^{r_1,r_2,r_3}W(\mathbb{R}^3)$ onto $S_2^{r_1,r_3}W(\mathbb{R}^2)$.

2.4.4 An Example

We consider the orthogonal basis $\vec{\sigma}_1 := (1,-1,0)$, and $\vec{\sigma}_2 := (1,1,-2)$ of Γ. Then the functions m_i, $i=1,2,3$, defined in (2.18)-(2.20), are given by

$$\begin{aligned} m_1^2(\xi_1,\xi_2) &= \Big[1+(2\xi_1)^2\Big]^{r_2}\Big[1+(\xi_1+\xi_2)^2\Big]^{r_3}, \\ m_2^2(\xi_1,\xi_2) &= \Big[1+(2\xi_1)^2\Big]^{r_1}\Big[1+(\xi_1-\xi_2)^2\Big]^{r_3}, \\ m_3^2(\xi_1,\xi_2) &= \Big[1+(\xi_1+\xi_2)^2\Big]^{r_1}\Big[1+(\xi_1-\xi_2)^2\Big]^{r_2}. \end{aligned}$$

Let $r_1=r_2=r_3=1$ and define

$$\begin{aligned} w(\xi_1,\xi_2) := \min\Big(&1+5\xi_1^2+\xi_2^2+2\xi_1\xi_2+4\xi_1^4+4\xi_1^2\xi_2^2+8\xi_1^3\xi_2, \\ &1+5\xi_1^2+\xi_2^2-2\xi_1\xi_2+4\xi_1^4+4\xi_1^2\xi_2^2-8\xi_1^3\xi_2, 1+2\xi_1^2+2\xi_2^2+\xi_1^4-2\xi_1^2\xi_2^2+\xi_2^4\Big)\end{aligned}$$

cf. (2.21). Hence, the trace space of the Sobolev space $S_2^{1,1,1}W(\mathbb{R}^3)$ with respect to this orthogonal basis is the collection of all functions $g\in L_2(\mathbb{R}^2)$ such that

$$\int_{\mathbb{R}^2} w(\xi_1,\xi_2)\,|\mathcal{F}g(\xi_1,\xi_2)|^2 d\xi < \infty\,.$$

Furthermore, the trace space of the Sobolev space $S_2^{1,1,2}W(\mathbb{R}^3)$ with respect to this orthogonal basis is the collection of all functions $g\in L_2(\mathbb{R}^2)$ such that

$$\int_{\mathbb{R}^2} \Big(1+2\xi_1^2+2\xi_2^2+\xi_1^4-2\xi_1^2\xi_2^2+\xi_2^4\Big)\,|\mathcal{F}g(\xi_1,\xi_2)|^2 d\xi < \infty\,.$$

3 Besov and Triebel-Lizorkin Spaces

Now we turn to the general case of Besov and Triebel-Lizorkin spaces. To begin with we recall the Fourier-analytic definition as well as the characterization by atoms of these classes. Since we shall need the spaces for $d = 3$ and for $d = 2$ we shall work for a while with the general d-dimensional case.

3.1 Notation

As usual, $\mathbb{R}^d$ denotes the $d-$dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The letter $\mathbb{Z}$ stands for the set of all integers and $\mathbb{C}$ denotes the complex numbers. If $x, y \in \mathbb{R}^d$, we write $x > y$ if, and only if, $x_i > y_i$ for every $i = 1, \ldots, d$. Similarly, we define the relations $x \geq y, x < y, x \leq y$. Finally, in slight abuse of notation, we write $x > \lambda$ for $x \in \mathbb{R}^d, \lambda \in \mathbb{R}$ if $x_i > \lambda, i = 1, \ldots, d$. For a real number x we denote by $x_+ := \max(x, 0)$ the nonnegative part.

Let $S(\mathbb{R}^d)$ be the Schwartz space of all complex–valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^d$.

3.2 The Fourier-analytic Approach

Let $\varphi \in S(\mathbb{R})$ with

$$\varphi(t) = 1 \quad \text{if} \quad |t| \leq 1 \quad \text{and} \quad \varphi(t) = 0 \quad \text{if} \quad |t| \geq \frac{3}{2}. \tag{3.1}$$

We put $\varphi_0 = \varphi$, $\varphi_1(t) = \varphi(t/2) - \varphi(t)$ and

$$\varphi_j(t) := \varphi_1(2^{-j+1}t), \quad t \in \mathbb{R}, \quad j \in \mathbb{N}.$$

Hence we have $\sum_{j=0}^{\infty} \varphi_j(t) = 1$ for all $t \in \mathbb{R}$. For $\overline{k} = (k_1, \ldots, k_d) \in \mathbb{N}_0^d$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we define $\varphi_{\overline{k}}(x) := \varphi_{k_1}(x_1) \cdot \ldots \cdot \varphi_{k_d}(x_d)$. Then, since

$$\sum_{\overline{k} \in \mathbb{N}_0^d} \varphi_{\overline{k}}(x) = 1 \quad \text{for every} \quad x \in \mathbb{R}^d, \tag{3.2}$$

the system $\{\varphi_{\overline{k}}\}_{\overline{k} \in \mathbb{N}_0^d}$ forms a smooth dyadic resolution of unity. This will be used to define the classes of functions we are interested in.

Definition 3.1. Let $\overline{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$, and $0 < q \leq \infty$.

(i) Let $0 < p \leq \infty$. Then the Besov space of dominating mixed smoothness $S^{\overline{r}}_{p,q}B(\mathbb{R}^d)$ is the collection of all $f \in S'(\mathbb{R}^d)$ such that

$$||f|S^{\overline{r}}_{p,q}B(\mathbb{R}^d)||_\varphi = \Big(\sum_{\overline{k} \in \mathbb{N}_0^d} 2^{q\overline{k}\cdot\overline{r}} ||\, \mathcal{F}^{-1}[\varphi_{\overline{k}} \mathcal{F}f]\, |L_p(\mathbb{R}^d)||^q\Big)^{1/q} = ||2^{\overline{k}\cdot\overline{r}} \mathcal{F}^{-1}[\varphi_{\overline{k}} \mathcal{F}f]\, |\ell_q(L_p)|| \tag{3.3}$$

is finite.

(ii) Let $0 < p < \infty$. Then the Triebel-Lizorkin space of dominating mixed smoothness $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$ is the collection of all $f \in S'(\mathbb{R}^d)$ such that

$$||f|S^{\overline{r}}_{p,q}F(\mathbb{R}^d)||_\varphi = \Big|\Big|\Big(\sum_{\overline{k} \in \mathbb{N}_0^d} |2^{\overline{k}\cdot\overline{r}} \mathcal{F}^{-1}[\varphi_{\overline{k}} \mathcal{F}f](\cdot)\,|^q\Big)^{1/q} |L_p(\mathbb{R}^d)\Big|\Big| = ||2^{\overline{k}\cdot\overline{r}} \mathcal{F}^{-1}[\varphi_{\overline{k}} \mathcal{F}f]\, |L_p(\ell_q)|| \tag{3.4}$$

is finite.

Remark 3.2. 1. Sometimes, we shall write $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ instead of $S^{\overline{r}}_{p,q}B(\mathbb{R}^d)$ or $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$.

2. Different functions φ (with properties described above) lead to equivalent quasi-norms on $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$. We shall write $||f|S^{\overline{r}}_{p,q}A(\mathbb{R}^d)||$ meaning one of these quasi-norms (which one is in general with no importance in our context). For details see [ST, Section 2.2.3].

3. For a systematic investigation of these classes we refer to the monographs [Am] and [ST]. More recent developments may be found in [Ba], [Ho] and [Vy1, Vy2, Vy3].

4. For $1 < p < \infty$ we have the coincidence of $S^{\overline{r}}_{p,2}F(\mathbb{R}^d)$ and the Sobolev space $S^{\overline{r}}_pW(\mathbb{R}^d)$ in the sense of equivalent norms, cf. [LN] and [ST, 2.3.1].

3.3 Atomic Decomposition

In the mid-eighties Frazier and Jawerth [FJ1] have been the first who studied atomic decompositions of Besov spaces. One of the applications has been a description of the solution of the trace problem with respect to hyperplanes in the isotropic situation. Here we follow the same philosophy. We shall make use of the characterization of Besov and Lizorkin-Triebel spaces by means of atoms for studiing the properties of $\mathrm{tr}_{\mathcal{O}}$.
Atomic decomposition techniques allow a certain discretization. Function spaces are replaced by sequence spaces. This method has been studied in various situations by now, cf. [FJ1, FJ2, AH, Tr2] for isotropic spaces of Besov and Lizorkin-Triebel type and [HN] for some generalizations in various directions. Besov and Lizorkin-Triebel spaces of dominating mixed smoothness have been characterized in such a way in [Vy2].

3.3.1 Sequence Spaces

For $\overline{\nu} \in \mathbb{N}_0^d, \overline{m} \in \mathbb{Z}^d$ we denote by $Q_{\overline{\nu}\,\overline{m}}$ the cube with the centre at the point $2^{-\overline{\nu}}\overline{m} = (2^{-\nu_1}m_1, \ldots, 2^{-\nu_d}m_d)$, sides parallel to the coordinate axes and of lengths $2^{-\nu_1}, \ldots, 2^{-\nu_d}$. We denote by $\chi_{\overline{\nu}\,\overline{m}} = \chi_{Q_{\overline{\nu}\,\overline{m}}}$ the characteristic function of $Q_{\overline{\nu}\,\overline{m}}$ and by $c\,Q_{\overline{\nu}\,\overline{m}}$ we mean a cube concentric with $Q_{\overline{\nu}\,\overline{m}}$ with sides c times larger.

Definition 3.3. If $0 < p, q \le \infty$, $\overline{r} \in \mathbb{R}^d$ and

$$\lambda = \{\lambda_{\overline{\nu}\,\overline{m}} \in \mathbb{C} : \overline{\nu} \in \mathbb{N}_0^d, \overline{m} \in \mathbb{Z}^d\}\,, \tag{3.5}$$

then we define

$$s^{\overline{r}}_{pq}b := \left\{\lambda : \quad ||\lambda|s^{\overline{r}}_{pq}b|| = \left(\sum_{\overline{\nu}\in\mathbb{N}_0^d} 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})q}\Big(\sum_{\overline{m}\in\mathbb{Z}^d}|\lambda_{\overline{\nu}\,\overline{m}}|^p\Big)^{q/p}\right)^{1/q} < \infty\right\} \tag{3.6}$$

and

$$s^{\overline{r}}_{pq}f := \left\{\lambda : \quad ||\lambda|s^{\overline{r}}_{pq}f|| = \left|\left|\left(\sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{m}\in\mathbb{Z}^d}|2^{\overline{\nu}\cdot\overline{r}}\lambda_{\overline{\nu}\,\overline{m}}\chi_{\overline{\nu}\,\overline{m}}(\cdot)|^q\right)^{1/q}|L_p(\mathbb{R}^d)\right|\right| < \infty\right\} \tag{3.7}$$

with the usual modification for p and/or q equal to ∞.

Remark 3.4. We shall use the same convention as in case of the distribution spaces: from time to time we shall write $\|\lambda|s^{\overline{r}}_{pq}a\|$ instead of $\|\lambda|s^{\overline{r}}_{pq}b\|$ or $\|\lambda|s^{\overline{r}}_{pq}f\|$, respectively.

3.3.2 Atomic Decompositions

We will be very brief and refer for details to [Vy1] and [Vy2]. Here we concentrate on the "regular" case, i.e.

$$\overline{r} > \begin{cases} \sigma_p = \max\Big(\frac{1}{p} - 1, 0\Big) & \text{in the B-case} \\ \sigma_{pq} = \max\Big(\frac{1}{\min(p,q)} - 1, 0\Big) & \text{in the F-case.} \end{cases} \tag{3.8}$$

The phrase "regular" indicates that only those distribution spaces are considered which consists of regular distributions. Then, compared with the general case, no moment conditions have to be satisfied by the elementary building blocks called atoms. As usual, $[x]$ denotes the integer part of the real number x. If Q is a cube and δ is a positive real number then δQ denotes the cube with the same center as Q, sides parallel to those of Q and sidelength multiplied by δ.

Definition 3.5. Let $\overline{K} = (K_1, \dots, K_d) \in \mathbb{N}_0^d$ and $\delta > 1$. A $\overline{K}$-times differentiable complex-valued function $a(x)$ is called $\overline{K}$-atom related to $Q_{\overline{\nu}\,\overline{m}}$ if

$$\operatorname{supp} a \subset \delta\, Q_{\overline{\nu}\,\overline{m}}, \tag{3.9}$$

and

$$\sup_{x \in \mathbb{R}^d} |D^\alpha a(x)| \le 2^{\alpha \cdot \overline{\nu}} \qquad \text{for} \quad 0 \le \alpha \le \overline{K} \tag{3.10}$$

Theorem 3.6. *Let* $0 < p, q \le \infty$, *(*$p < \infty$ *in the* $F-$*case) and* $\overline{r} \in \mathbb{R}^d$ *with* (3.8). *Fix* $\overline{K} \in \mathbb{N}_0^d$ *with*

$$K_i \ge (1 + [r_i])_+ \qquad i = 1, \dots, d\,, \tag{3.11}$$

and δ *sufficiently large.*
Then $f \in S'(\mathbb{R}^d)$ *belongs to* $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ *if, and only if, it can be represented as*

$$f = \sum_{\overline{\nu} \in \mathbb{N}_0^d} \sum_{\overline{m} \in \mathbb{Z}^d} \lambda_{\overline{\nu}\,\overline{m}} a_{\overline{\nu}\,\overline{m}}(x), \qquad (\textit{convergence in} \quad S'(\mathbb{R}^d)), \tag{3.12}$$

where $\{a_{\overline{\nu}\,\overline{m}}(x)\}_{\overline{\nu} \in \mathbb{N}_0^d, \overline{m} \in \mathbb{Z}^d}$ *are* $\overline{K}$*-atoms related to* $Q_{\overline{\nu}\,\overline{m}}$ *and* $\lambda \in s^{\overline{r}}_{pq}a$. *Furthermore,*

$$\inf ||\lambda | s^{\overline{r}}_{pq}a||,$$

where the infimum is taken over all admissible representations (3.12)*, yields an equivalent quasi-norm in* $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$.

Remark 3.7. To explain our philosophy, let the function a be a $\overline{K} = (K_1, K_2, K_3)$-atom related to $Q_{\overline{\nu}\,\overline{m}}$, where $\nu = (\nu_1, \nu_2, \nu_3)$ and $m = (m_1, m_2, m_3)$. Then

$$(\operatorname{tr}_3 a)(x_1, x_2) = a(x_1, x_2, -(x_1 + x_2))$$

becomes a (K_1, K_2)-atom with respect to $Q_{(\nu_1,\nu_2),(m_1,m_2)}$ if $K_3 \ge K_1 + K_2$ and $\nu_3 \le \min(\nu_1, \nu_2)$. Similarly $\operatorname{tr}_2 a$ $(\operatorname{tr}_1 a)$ becomes a (K_1, K_3)-atom $((K_2, K_3)$-atom$)$ with respect to $Q_{(\nu_1,\nu_3),(m_1,m_3)}$ $(Q_{(\nu_2,\nu_3),(m_2,m_3)})$ if $K_2 \ge K_1 + K_3$ $(K_1 \ge K_2 + K_3)$ and $\nu_2 \le \min(\nu_1, \nu_3)$ $(\ \nu_1 \le \min(\nu_2, \nu_3)\ )$. This simple observation will motivate an appropriate decomposition of the atomic decomposition of a function which turns out to be a basic step in our proof of the boundedness of $\operatorname{tr}_{\mathcal{O}}$.

3.4 Traces of Besov Spaces of Dominating Mixed Smoothness

For a better comparison we recall the properties of the mapping $f(x_1,x_2,x_3) \mapsto f(x_1,x_2,0)$ in this general context, cf. e.g. Amanov [Am, 9.5] and Schmeißer, Triebel [ST, 2.4.2] (further references are given in [ST, Remark 2.4.2]).

Proposition 3.8. *Let $0 < q \le \infty$.*
(i) *Let $0 < p \le \infty$ and $r_3 > 1/p$. Then the mapping*

$$T:\ f(x_1,x_2,x_3) \to f(x_1,x_2,0)$$

extends to a retraction from $S^{r_1,r_2,r_3}_{p,q}B(\mathbb{R}^3)$ onto $S^{r_1,r_2}_{p,q}B(\mathbb{R}^2)$.
(ii) *Let $0 < p < \infty$ and $r_3 > \frac{1}{p}$. Then the mapping T extends to a retraction from $S^{r_1,r_2,r_3}_{p,q}F(\mathbb{R}^3)$ onto $S^{r_1,r_2}_{p,q}F(\mathbb{R}^2)$.*

As mentioned in Introduction, to reflect the underlying geometry of our problem, we have to define some new spaces with dominating mixed smoothness, cf. Subsection 2.2 for $p = 2$.

Definition 3.9. Let $0 < q \le \infty$, $0 < p \le \infty$ in the B-case and $0 < p < \infty$ in the F-case. Let $\mathcal{R}$ be a $(2,2)$-matrix with $\det \mathcal{R} \neq 0$. Then we put

$$S^{\overline{r}}_{p,q}A(\mathcal{R},\mathbb{R}^2) := \Big\{ f \in S'(\mathbb{R}^2) : f \circ \mathcal{R} \in S^{\overline{r}}_{p,q}A(\mathbb{R}^2) \Big\}\,,$$
$$||f|S^{\overline{r}}_{p,q}A(\mathcal{R},\mathbb{R}^2)|| := ||\, f \circ \mathcal{R}\, |S^{\overline{r}}_{p,q}A(\mathbb{R}^2)||\,.$$

Recall that for $p = q = 2$ we have coincidence of $S^{\overline{r}}_{2,2}B(\mathcal{R},\mathbb{R}^2)$ with $S^{\overline{r}}_2W(\mathcal{R},\mathbb{R}^2)$ in the sense of equivalent norms, cf. [LN] or [ST, Thm. 2.3.1]. By means of these classes we are able to describe the trace classes for Besov as well as for Lizorkin-Triebel classes.

The counterpart of Theorem 2.9 for Besov spaces is as follows.

Theorem 3.10. *Let $\mathcal{O}$ be an orthogonal basis of Γ and let $\mathcal{R}_i, i = 1,2,3$ be matrices associated with $\mathcal{O}$ by* (1.1)*,* (2.8) *and* (2.10)*.*
Let $0 < p,q \le \infty$ and $\overline{r} = (r_1,r_2,r_3) \in \mathbb{R}^3$ with $r_i \neq \frac{1}{p}, i = 1,2,3$ and

$$\min\left(r_1, r_2, r_3, r_1 + r_2 - \frac{1}{p}, r_1 + r_3 - \frac{1}{p}, r_2 + r_3 - \frac{1}{p}\right) > \sigma_p. \tag{3.13}$$

Then

$$\mathrm{tr}_{\mathcal{O}} \in \mathcal{L}\Big(S^{\overline{r}}_{p,q}B(\mathbb{R}^3), S^1(\mathbb{R}^2) + S^2(\mathbb{R}^2) + S^3(\mathbb{R}^2)\Big), \tag{3.14}$$

where

$$S^1(\mathbb{R}^2) := \begin{cases} S^{r_2,r_3}_{p,q}B(\mathcal{R}_1^{-1},\mathbb{R}^2), & \text{if } r_1 > \frac{1}{p}\,, \\ S^{r_2,r_3+r_1-\frac{1}{p}}_{p,q}B(\mathcal{R}_1^{-1},\mathbb{R}^2) \cap S^{r_2+r_1-\frac{1}{p},r_3}_{p,q}B(\mathcal{R}_1^{-1},\mathbb{R}^2), & \text{if } r_1 < \frac{1}{p}\,, \end{cases}$$

and similarly for S^2 and S^3.

Conversely, to each function $g \in S^1(\mathbb{R}^2) + S^2(\mathbb{R}^2) + S^3(\mathbb{R}^2)$ there exists a function $f \in S^{\overline{r}}_{p,q}B(\mathbb{R}^3)$ such that $\mathrm{tr}_{\mathcal{O}} f = g$.

Proof. The restrictions in (3.13) are guaranteeing that we may apply Theorem 3.6 for $S^{\overline{r}}_{p,q}B(\mathbb{R}^3)$ as well as for all spaces appearing in the definition of the target spaces but taken with the identity matrix instead of $\mathcal{R}_i^{-1}$, $i \in \{1,2,3\}$.
Step 1. According to Theorem 3.6, each $f \in S^{\overline{r}}_{p,q}B(\mathbb{R}^3)$ may be decomposed into

$$f = \sum_{\overline{\nu}\in\mathbb{N}_0^d} \sum_{\overline{m}\in\mathbb{Z}^d} \lambda_{\overline{\nu}\,\overline{m}} a_{\overline{\nu}\,\overline{m}}(x), \tag{3.15}$$

with

$$||\,\lambda\,|s^{\overline{r}}_{p,q}b|| \le c\,||f|S^{\overline{r}}_{p,q}B(\mathbb{R}^3)|| \tag{3.16}$$

with some constant c independent of f. We require some additional regularity of the atoms, cf. Definition 3.5:

$$K_i \ge \max\Big([r_1]+[r_2]+2, [r_1]+[r_3]+2, [r_2]+[r_3]+2\Big)\,, \qquad i=1,2,3\,. \tag{3.17}$$

In view of Remark 3.7 we decompose f into three parts $f_i, i = 1,2,3$, where

$$f_1(x) \;:=\; \sum_{\nu_1=0}^{\infty}\sum_{\nu_2=\nu_1}^{\infty}\sum_{\nu_3=\nu_1}^{\infty}\sum_{\overline{m}\in\mathbb{Z}^3} \lambda_{\overline{\nu}\,\overline{m}}\, a_{\overline{\nu}\,\overline{m}}(x), \tag{3.18}$$

$$f_2(x) \;:=\; \sum_{\nu_2=0}^{\infty}\sum_{\nu_1=\nu_2+1}^{\infty}\sum_{\nu_3=\nu_2}^{\infty}\sum_{\overline{m}\in\mathbb{Z}^3} \lambda_{\overline{\nu}\,\overline{m}}\, a_{\overline{\nu}\,\overline{m}}(x), \tag{3.19}$$

$$f_3(x) \;:=\; \sum_{\nu_3=0}^{\infty}\sum_{\nu_1=\nu_3+1}^{\infty}\sum_{\nu_2=\nu_3+1}^{\infty}\sum_{\overline{m}\in\mathbb{Z}^3} \lambda_{\overline{\nu}\,\overline{m}}\, a_{\overline{\nu}\,\overline{m}}(x). \tag{3.20}$$

This allows us to decompose $\mathrm{tr}_{\mathcal{O}}\, f$ into (see (2.7))

$$(\mathrm{tr}_{\mathcal{O}}\, f)(z_1,z_2) = \sum_{i=1}^{3} (\mathrm{tr}_i\, f_i)(\mathcal{R}_i\vec{z})\,. \tag{3.21}$$

So, to establish (3.14) it is enough to prove the existence of a constant c independent of f such that

$$||\ \mathrm{tr}_1\, f_1\,|S^{r_2,r_3}_{p,q}B(\mathbb{R}^2)|| \le c\,||\,f\,|S^{\overline{r}}_{p,q}B(\mathbb{R}^3)|| \tag{3.22}$$

if $r_1 > \frac{1}{p}$ and

$$||\ \mathrm{tr}_1\, f_1\,|S^{r_2,r_3+r_1-\frac{1}{p}}_{p,q}B(\mathbb{R}^2)|| \le c\,||\,f\,|S^{\overline{r}}_{p,q}B(\mathbb{R}^3)||\,, \tag{3.23}$$

$$||\ \mathrm{tr}_1\, f_1\,|S^{r_2+r_1-\frac{1}{p},r_3}_{p,q}B(\mathbb{R}^2)|| \le c\,||f|S^{\overline{r}}_{p,q}B(\mathbb{R}^3)|| \tag{3.24}$$

if $r_1 < \frac{1}{p}$ and corresponding analoga for $\mathrm{tr}_i\, f_i$, $i = 2,3$.
Step 2. Proof of (3.22)–(3.24). We proceed similar to [Vy3]. For brevity we put

$$\begin{aligned}\Upsilon_1 \;&:=\; \{\overline{\nu}\in\mathbb{N}_0^3 : \nu_1 \le \min(\nu_2,\nu_3)\,\}\,,\\ \Upsilon_2 \;&:=\; \{\overline{\nu}\in\mathbb{N}_0^3 : \nu_2 \le \min(\nu_1,\nu_3)\,\}\,,\\ \Upsilon_3 \;&:=\; \{\overline{\nu}\in\mathbb{N}_0^3 : \nu_3 \le \min(\nu_1,\nu_2)\,\}\,.\end{aligned}$$

Then

$$\mathrm{tr}_1\, f_1(x_2,x_3) = \sum_{\overline{\nu}\in\Upsilon_1} \sum_{\overline{m}\in B_{\overline{\nu}}} \lambda_{\overline{\nu}\,\overline{m}}\, a_{\overline{\nu}\,\overline{m}}(-x_2-x_3,x_2,x_3)\,, \tag{3.25}$$

where

$$B_{\overline{\nu}} := \{\overline{m}\in\mathbb{Z}^3 : \quad \mathrm{supp}\, a_{\overline{\nu}\,\overline{m}} \cap \Gamma \neq \emptyset\}\,. \tag{3.26}$$

Due to (3.9), for given $\overline{\nu}\in\Upsilon_1$ and $m_2,m_3\in\mathbb{Z}$, there are at most N integers $m_1\in\mathbb{Z}$, such that $\overline{m}=(m_1,m_2,m_3)\in B_{\overline{\nu}}$. The number N does not depend on $\overline{\nu}$ and m_2,m_3. To simplify notation we shall work only with one number m_1, denoted by $m_1(\overline{\nu},m_2,m_3)$ or simply by m_1 if the values of $\overline{\nu},m_2$ and m_3 are clear from context. Rewriting (3.25) this gives

$$\begin{aligned} \mathrm{tr}_1\, f_1(x_2,x_3) &= \sum_{\nu_2=0}^{\infty}\sum_{\nu_3=0}^{\infty}\sum_{(m_2,m_3)\in\mathbb{Z}^2}\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)} \lambda_{\overline{\nu}\,(m_1,m_2,m_3)}\, a_{\overline{\nu}\,(m_1,m_2,m_3)}(-x_2-x_3,x_2,x_3) \\ &= \sum_{\nu_2=0}^{\infty}\sum_{\nu_3=0}^{\infty}\sum_{(m_2,m_3)\in\mathbb{Z}^2} \gamma_{(\nu_2,\nu_3)\,(m_2,m_3)}\, b_{(\nu_2,\nu_3)\,(m_2,m_3)}(x_2,x_3), \end{aligned} \tag{3.27}$$

where

$$\gamma_{(\nu_2,\nu_3)\,(m_2,m_3)} = \sum_{\nu_1=0}^{\min(\nu_2,\nu_3)} |\lambda_{\overline{\nu}\,(m_1,m_2,m_3)}|\,, \tag{3.28}$$

$b_{(\nu_2,\nu_3)\,(m_2,m_3)}(x_2,x_3)=0$ if $\gamma_{(\nu_2,\nu_3)\,(m_2,m_3)}=0$, and

$$b_{(\nu_2,\nu_3)\,(m_2,m_3)}(x_2,x_3) = \frac{1}{\gamma_{(\nu_2,\nu_3)\,(m_2,m_3)}} \sum_{\nu_1=0}^{\min(\nu_2,\nu_3)} \lambda_{\overline{\nu}\,(m_1,m_2,m_3)}\, a_{\overline{\nu}\,(m_1,m_2,m_3)}(-x_2-x_3,x_2,x_3)$$

if $\gamma_{(\nu_2,\nu_3)\,(m_2,m_3)}>0$. We recall, that in this sum m_1 is an abbreviation for $m_1(\overline{\nu},m_2,m_3)$.
Step 3. We claim

1. $b_{(\nu_2,\nu_3)\,(m_2,m_3)}$ are atoms in the sense of Definition 3.5 related to $(\nu_2,\nu_3),(m_2,m_3)$ up to a general constant.

2. $||\,\gamma\,|s^{r_2,r_3}_{p,q}b||\le c\,||\,\lambda\,|s^{\overline{r}}_{p,q}b||$ if $r_1>\frac{1}{p}$,

3. $||\,\gamma\,|s^{r_2,r_3+r_1-\frac{1}{p}}_{p,q}b||\le c\,||\,\lambda\,|s^{\overline{r}}_{p,q}b||$ and $||\,\gamma\,|s^{r_2+r_1-\frac{1}{p},r_3}_{p,q}b||\le c\,||\,\lambda\,|s^{\overline{r}}_{p,q}b||$ if $r_1<\frac{1}{p}$.

Substep 3.1. The proof of the first assertion is elementary, see Remark 3.7. Two comments are in order. The first one concerns regularity. If the components of $\overline{K}$ are large enough then b is sufficiently smooth to satisfy (3.10) for some $\widetilde{K}$ such that we can apply Theorem 3.6 with respect to the target space, cf. (3.17). The second comment concerns the estimate (3.10). As claimed this estimate is satisfied by the functions $b_{(\nu_2,\nu_3),(m_2,m_3)}$ up to a general constant c_α depending on α. Since we need to control a finite number of derivatives only we conclude that $C\,b_{(\nu_2,\nu_3),(m_2,m_3)}$ are atoms with $C^{-1}:=\max_\alpha c_\alpha$. This is enough for our purpose.
Substep 3.2. Let $r_1>\frac{1}{p}$. Let $r_1-1/p=\varepsilon_1+\varepsilon_2$, $\varepsilon_i>0$, $i=1,2$. Obviuosly, $\varepsilon_1>0$ guarantees

$$\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)} |\lambda_{\overline{\nu}\,\overline{m}}| \le c_1 \left(\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)} |\,2^{\nu_1\varepsilon_1}\,\lambda_{\overline{\nu}\,\overline{m}}|^p\right)^{1/p}.$$

Next we use $\varepsilon_2 > 0$ and obtain

$$\begin{aligned}
\|\,\gamma\,|s^{r_2,r_3}_{p,q}b\|^q &= \sum_{\nu_2=0}^{\infty}\sum_{\nu_3=0}^{\infty} 2^{[\nu_2(r_2-\frac{1}{p})+\nu_3(r_3-\frac{1}{p})]q}\Big(\sum_{(m_2,m_3)\in\mathbb{Z}^2}|\gamma_{(\nu_2,\nu_3)(m_2,m_3)}|^p\Big)^{q/p}\\
&\le c_2 \sum_{\nu_2=0}^{\infty}\sum_{\nu_3=0}^{\infty} 2^{[\nu_2(r_2-\frac{1}{p})+\nu_3(r_3-\frac{1}{p})]q}\Big(\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)} 2^{\nu_1\varepsilon_1 p}\sum_{(m_2,m_3)\in\mathbb{Z}^2}|\lambda_{\overline{\nu}\,\overline{m}}|^p\Big)^{q/p}\\
&\le c_3 \sum_{\overline{\nu}\in\Upsilon_1} 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})q}\Big(\sum_{(m_2,m_3)\in\mathbb{Z}^2}|\lambda_{\overline{\nu}\,\overline{m}}|^p\Big)^{q/p}\\
&\le c_3\,\|\,\lambda\,|s^{\overline{r}}_{p,q}b\|^q.
\end{aligned}$$

Substep 3.3. Let $r_1 < \frac{1}{p}$. To begin with let $p \ge 1$. The triangle inequality yields

$$\Big(\sum_{(m_2,m_3)\in\mathbb{Z}^2}\Big(\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)}|\lambda_{\overline{\nu}\,\overline{m}}|\Big)^p\Big)^{1/p} \le \sum_{\nu_1=0}^{\min(\nu_2,\nu_3)}\Big(\sum_{(m_2,m_3)\in\mathbb{Z}^2}|\lambda_{\overline{\nu}\,\overline{m}}|^p\Big)^{1/p}. \tag{3.29}$$

If now $q \le 1$, we get

$$\begin{aligned}
\|\,\gamma\,|s^{r_2,r_3+r_1-\frac{1}{p}}_{p,q}b\|^q &\le \sum_{\nu_2=0}^{\infty}\sum_{\nu_3=0}^{\infty} 2^{[\nu_2(r_2-\frac{1}{p})+\nu_3(r_3+r_1-\frac{1}{p}-\frac{1}{p})]q}\left[\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)}\Big(\sum_{(m_2,m_3)\in\mathbb{Z}^2}|\lambda_{\overline{\nu}\,\overline{m}}|^p\Big)^{1/p}\right]^q\\
&\le \sum_{\overline{\nu}\in\Upsilon_1} 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})q}\Big(\sum_{(m_2,m_3)\in\mathbb{Z}^2}|\lambda_{\overline{\nu}\,\overline{m}}|^p\Big)^{q/p} \le \|\,\lambda\,|s^{\overline{r}}_{p,q}b\|^q.
\end{aligned}$$

For $q > 1$, we denote

$$\varrho_{\overline{\nu}} := \Big(\sum_{(m_2,m_3)\in\mathbb{Z}^2}|\lambda_{\overline{\nu}\,\overline{m}}|^p\Big)^{1/p}$$

and apply Hölder's inequality to obtain

$$\begin{aligned}
&\|\,\gamma\,|s^{r_2,r_3+r_1-\frac{1}{p}}_{p,q}b\|^q\\
&\le \sum_{\nu_2=0}^{\infty}\sum_{\nu_3=0}^{\infty} 2^{[\nu_2(r_2-\frac{1}{p})+\nu_3(r_3+r_1-\frac{1}{p}-\frac{1}{p})]q}\left[\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)} 2^{(\nu_3-\nu_1)(r_1-\frac{1}{p})}2^{(\nu_1-\nu_3)(r_1-\frac{1}{p})}\,\varrho_{\overline{\nu}}\right]^q\\
&\le \left[\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)} 2^{(\nu_3-\nu_1)(r_1-\frac{1}{p})q'}\right]^{q/q'}\sum_{\overline{\nu}\in\Upsilon_1} 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})q}\,\varrho_{\nu}^q\\
&\le c\,\|\,\lambda\,|s^{\overline{r}}_{p,q}b\|^q.
\end{aligned}$$

This proves our claims if $p \ge 1$. Now let $p < 1$. We substitute (3.29) by

$$\Big(\sum_{(m_2,m_3)\in\mathbb{Z}^2}\Big(\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)}|\lambda_{\overline{\nu}\,\overline{m}}|\Big)^p\Big)^{1/p} \le \Big(\sum_{(m_2,m_3)\in\mathbb{Z}^2}\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)}|\lambda_{\overline{\nu}\,\overline{m}}|^p\Big)^{1/p} = \Big(\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)}\varrho_{\overline{\nu}}^p\Big)^{1/p}. \tag{3.30}$$

If $q \le p$ the monotonicity of the ℓ_r-quasinorms yields

$$\begin{aligned}
\|\,\gamma\,|s^{r_2,r_3+r_1-\frac{1}{p}}_{p,q}b\|^q &\le \sum_{\nu_2=0}^{\infty}\sum_{\nu_3=0}^{\infty} 2^{[\nu_2(r_2-\frac{1}{p})+\nu_3(r_3+r_1-\frac{1}{p}-\frac{1}{p})]q}\left[\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)}\varrho_{\overline{\nu}}^p\right]^{q/p}\\
&\le \sum_{\overline{\nu}\in\Upsilon_1} 2^{\overline{\nu}\cdot(\overline{r}-\frac{1}{p})q}\varrho_{\nu}^q \le \|\,\lambda\,|s^{\overline{r}}_{p,q}b\|^q.
\end{aligned} \tag{3.31}$$

And for $q > p$, we combine (3.30) with Hölder's inequality

$$\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)} 2^{(\nu_3-\nu_1)(r_1-\frac{1}{p})p} 2^{(\nu_1-\nu_3)(r_1-\frac{1}{p})p} \, \varrho_{\overline{\nu}}^{p} \leq c \left(\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)} 2^{(\nu_1-\nu_3)(r_1-\frac{1}{p})q} \, \varrho_{\overline{\nu}}^{q} \right)^{p/q}$$

to derive (3.31) again. Moreover, the second estimate in Claim 3 follows by interchanging the roles of r_2 and r_3. This completes the estimates claimed for γ.

Step 4. We shall prove the estimate for $\mathrm{tr}_1 f_1$. In case $r_1 > \frac{1}{p}$ we argue, by using Claim 2 and Theorem 3.6, first in $d = 2$ and later in $d = 3$, as follows

$$\| \, \mathrm{tr}_1 f_1 | S^{r_2,r_3}_{p,q} B(\mathbb{R}^2) \| \leq c_1 \| \, \gamma \, | s^{r_2,r_3}_{p,q} b \| \leq c_2 \| \, \lambda \, | s^{\overline{r}}_{p,q} b \| \leq c_3 \| \, f \, | S^{\overline{r}}_{p,q} B(\mathbb{R}^3) \| \, .$$

Mutatis mutandis the case $r_1 < \frac{1}{p}$ can be treated. The estimates of $\mathrm{tr}_i f_i$, $i = 2, 3$ follow by symmetry.

Step 5. Now we construct the (non-)linear extension operator. We start with a function $g \in S^1(\mathbb{R}^2) + S^2(\mathbb{R}^2) + S^3(\mathbb{R}^2)$. Then there are $g_i \in S^i(\mathbb{R}^2)$, $i = 1, 2, 3$, such that $g_i \in S^i(\mathbb{R}^2)$, $g = g_1 + g_2 + g_3$ and

$$\| \, g_i \, | S^i(\mathbb{R}^2) \| \leq 2 \, \| \, g \, | S^1(\mathbb{R}^2) + S^2(\mathbb{R}^2) + S^3(\mathbb{R}^2) \| \, .$$

We shall extend each g_i separately. It means, we construct three functions $f_1, f_2, f_3 \in S^{\overline{r}}_{p,q} B(\mathbb{R}^3)$ such that $\mathrm{tr}_{\mathcal{O}} f_i = g_i$, $i = 1, 2, 3$. The desirable extension will than be given by $f = f_1 + f_2 + f_3$.

Substep 5.1 We restrict ourselves to $i = 1$, the other cases follow by symmetry. To begin with we treat the case $r_1 > 1/p$. We put $h_1 := g_1 \circ \mathcal{R}_1^{-1}$. Then $h_1 \in S^{r_2,r_3}_{p,q} B(\mathbb{R}^2)$ and, according to (2.7), we get

$$g_1(z_1, z_2) = (\mathrm{tr}_{\mathcal{O}} f_1)(z_1, z_2) = (\mathrm{tr}_1 f_1)(\mathcal{R}_1 \vec{z})$$

for all $\vec{z} = (z_1, z_2) \in \mathbb{R}^2$ if, and only if,

$$g_1(\mathcal{R}_1^{-1} \vec{z}) = h_1(\vec{z}) = (\mathrm{tr}_1 f_1)(z_1, z_2) = f_1(-z_1 - z_2, z_1, z_2), \qquad \vec{z} = (z_1, z_2) \in \mathbb{R}^2.$$

Hence, our original task, namely to find f_1 such that $\mathrm{tr}_{\mathcal{O}} f_1 = g_1$, where $g_1 \in S^{r_2,r_3}_{p,q} B(\mathcal{R}_1^{-1}, \mathbb{R}^2)$ is given, can be replaced by searching for f_1 such that $\mathrm{tr}_1 f_1 = h_1$, where $h_1 \in S^{r_2,r_3}_{p,q} B(\mathbb{R}^2)$. Again we make use of atomic decompositions. According to Theorem 3.6 we can decompose

$$h_1(x_2, x_3) = \sum_{(\nu_2,\nu_3) \in \mathbb{N}_0^2} \sum_{(m_2,m_3) \in \mathbb{Z}^2} \gamma_{(\nu_2,\nu_3)\,(m_2,m_3)} b_{(\nu_2,\nu_3)\,(m_2,m_3)}(x_2, x_3),$$

where

$$c_1 \, \| \, \gamma \, | s^{r_2,r_3}_{p,q} b \| \leq \| \, h_1 \, | S^{r_2,r_3}_{p,q} B(\mathbb{R}^2) \| \leq c_2 \, \| \, \gamma \, | s^{r_2,r_3}_{p,q} b \|$$

for certain positive constants c_1 and c_2 independent of h_1. Now we choose an integer m_1 such that $|2^{-\nu_1} m_1 + 2^{-\nu_2} m_2 + 2^{-\nu_3} m_3| \leq 2^{-\nu_1}$ and define

$$a_{\overline{\nu}\,\overline{m}}(x_1, x_2, x_3) := \psi(2^{\nu_1} x_1 - m_1) \, b_{(\nu_2,\nu_3)\,(m_2,m_3)}(x_2, x_3),$$

where

$$\psi \in S(\mathbb{R}), \quad \mathrm{supp}\, \psi \subset [-2(1+\delta), 2(1+\delta)], \quad \psi(t) = 1 \text{ if } t \in [-(1+\delta), (1+\delta)]$$

and δ is the number from (3.9). For $\nu_1 \le \min(\nu_2,\nu_3)$ some easy calculations yield

$$a_{\overline{\nu}\,\overline{m}}(-x_2 - x_3, x_2, x_3) = b_{(\nu_2,\nu_3)\,(m_2,m_3)}(x_2,x_3)\,, \qquad (x_2,x_3) \in \mathbb{R}^2\,.$$

If the first component of $\overline{m}$ differs from this specific m_1 then we define $a_{\overline{\nu}\,\overline{m}} \equiv 0$. Further, we put

$$\lambda_{(\nu_1,\nu_2,\nu_3)\,(m_1,m_2,m_3)} := \begin{cases} \gamma_{(\nu_2,\nu_3)\,(m_2,m_3)} & \text{if} \quad \nu_1 = 0\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{3.32}$$

and

$$f_1 := \operatorname{ext} h_1 = \sum_{\overline{\nu}\in\Upsilon_1} \sum_{\overline{m}\in\mathbb{Z}^3} \lambda_{\overline{\nu}\,\overline{m}}\, a_{\overline{\nu}\,\overline{m}}\,.$$

Then

$$||\operatorname{ext} h_1\,|S^{\overline{r}}_{p,q}B(\mathbb{R}^3)|| \le C_1\,||\,\lambda\,|s^{\overline{r}}_{p,q}b|| = C_1\,||\,\gamma\,|s^{r_2,r_3}_{p,q}b|| \le C_2\,||\,h_1\,|S^{r_2,r_3}_{p,q}B(\mathbb{R}^2)||. \tag{3.33}$$

This showes that f_1 represents an appropriate extension of h_1 if $r_1 > \frac{1}{p}$.

Substep 5.2. Let $r_1 < 1/p$. First of all notice that this time $h_1 \in S^{r_2,r_3+r_1-\frac{1}{p}}_{p,q}B(\mathbb{R}^2) \cap S^{r_2+r_1-\frac{1}{p},r_3}_{p,q}B(\mathbb{R}^2)$. We have to modify the definition of λ, cf. (3.32). This time we use

$$\lambda_{(\nu_1,\nu_2,\nu_3)\,(m_1,m_2,m_3)} := \begin{cases} \gamma_{(\nu_2,\nu_3)\,(m_2,m_3)} & \text{if} \quad \nu_1 = \min(\nu_2,\nu_3)\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{3.34}$$

for the specific value of m_1 as chosen in Substep 5.1. In all other cases we put $\lambda_{\overline{\nu}\,\overline{m}} = 0$. Then

$$\begin{aligned}
&||\operatorname{ext} h_1\,|S^{\overline{r}}_{p,q}B(\mathbb{R}^3)||^q \le C_1\,||\,\lambda\,|s^{\overline{r}}_{p,q}b||^q \\
&= C_1 \sum_{\nu_2=0}^{\infty}\sum_{\nu_3=0}^{\infty} 2^{[\min(\nu_2,\nu_3)(r_1-1/p)+\nu_2(r_2-\frac{1}{p})+\nu_3(r_3-\frac{1}{p})]q} \Big(\sum_{(m_2,m_3)\in\mathbb{Z}^2} |\gamma_{(\nu_2,\nu_3)(m_2,m_3)}|^p\Big)^{q/p} \\
&= C_1 \Big(\sum_{\nu_2=0}^{\infty}\sum_{\nu_3=\nu_2}^{\infty} 2^{[\nu_2(r_1-1/p)+\nu_2(r_2-\frac{1}{p})+\nu_3(r_3-\frac{1}{p})]q} \Big(\sum_{(m_2,m_3)\in\mathbb{Z}^2} |\gamma_{(\nu_2,\nu_3)(m_2,m_3)}|^p\Big)^{q/p} \\
&\quad + \sum_{\nu_3=0}^{\infty}\sum_{\nu_2=\nu_3+1}^{\infty} 2^{[\nu_3(r_1-1/p)+\nu_2(r_2-\frac{1}{p})+\nu_3(r_3-\frac{1}{p})]q} \Big(\sum_{(m_2,m_3)\in\mathbb{Z}^2} |\gamma_{(\nu_2,\nu_3)(m_2,m_3)}|^p\Big)^{q/p}\Big) \\
&\le C_1 \Big(||\,\gamma\,|s^{r_2+r_1-\frac{1}{p},r_3}_{p,q}b|| + ||\,\gamma\,|s^{r_2,r_3+r_1-\frac{1}{p}}_{p,q}b||\Big)^q \\
&\le C_2 \Big(||\,h_1\,|S^{r_2+r_1-\frac{1}{p},r_3}_{p,q}B(\mathbb{R}^2)|| + ||\,h_1\,|S^{r_2,r_3+r_1-\frac{1}{p}}_{p,q}B(\mathbb{R}^2)||\Big)^q.
\end{aligned}$$

Hence, also in this situation we have an appropriate extension of g_1. The modifications for an extension of g_2 and g_3 are obvious. □

Remark 3.11. The reader may notice that the only possible failure of linearity of the extension operator comes from the (generally non-linear) decomposition of g into $g = g_1 + g_2 + g_3$.

It remains to consider the limiting cases where at least one of the r_i equals $1/p$. We concentrate on the more simple situation where $0 < p, q \le 1$.

Proposition 3.12. *Let $\mathcal{O}$ be an orthogonal basis of Γ and let $\mathcal{R}_i, i = 1,2,3$ be matrices associated with $\mathcal{O}$ by* (1.1), (2.8) *and* (2.10).
Let $0 < p,q \leq 1$. Then the statement of Theorem 3.10 remains true without the assumption $r_i \neq 1/p$, $i = 1,2,3$.

Proof. The proof of Theorem 3.10 extends to the present situation since in Substep 3.2 one can work with $\varepsilon_1 = \varepsilon_2 = 0$. □

Remark 3.13. Proposition 3.12 does not extend to values of p larger than 1. In analogy to the two-dimensional situation, cf. [Vy3] for details, more complicated spaces occur. We omit details.

3.5 Traces of Lizorkin-Triebel Spaces

Now we turn to the Lizorkin-Triebel classes. To prove an analog of Theorem 3.10 for these spaces we can proceed in the same way as in case of the Besov spaces. We shall describe the needed modifications only.

Theorem 3.14. *Let $\mathcal{O}$ be an orthogonal basis of Γ and let $\mathcal{R}_i, i = 1,2,3$ be matrices associated with $\mathcal{O}$ by* (1.1), (2.8) *and* (2.10). *Let $0 < p < \infty$ and $0 < q \leq \infty$. Let $\overline{r} = (r_1, r_2, r_3) \in \mathbb{R}^3$ with*

$$\min(r_1, r_2, r_3) > \max\left(\frac{1}{p}, \sigma_{pq}\right). \tag{3.35}$$

Then

$$\mathrm{tr}_{\mathcal{O}} \in \mathcal{L}\Big(S^{\overline{r}}_{p,q}F(\mathbb{R}^3),\, S^{r_2,r_3}_{p,q}F(\mathcal{R}_1^{-1}, \mathbb{R}^2) + S^{r_1,r_3}_{p,q}F(\mathcal{R}_2^{-1}, \mathbb{R}^2) + S^{r_1,r_2}_{p,q}F(\mathcal{R}_3^{-1}, \mathbb{R}^2)\Big). \tag{3.36}$$

Conversely, to each function $g \in S^{r_2,r_3}_{p,q}F(\mathcal{R}_1^{-1}, \mathbb{R}^2) + S^{r_1,r_3}_{p,q}F(\mathcal{R}_2^{-1}, \mathbb{R}^2) + S^{r_1,r_2}_{p,q}F(\mathcal{R}_3^{-1}, \mathbb{R}^2)$ there exists a function $f \in S^{\overline{r}}_{p,q}F(\mathbb{R}^3)$ such that $\mathrm{tr}_{\mathcal{O}} f = g$.

Proof. We shall use the same notation as in the proof of Theorem 3.10.
Step 1. Boundedness. In Step 1 of the proof of Theorem 3.10 we simply change the letter B to F. In Step 2 nothing is to change and we concentrate on Step 3 now. We have to prove that

$$\|\,\gamma\,|s^{r_2,r_3}_{p,q}f\| \leq c\,\|\,\lambda\,|s^{r_1,r_2,r_3}_{p,q}f\| \tag{3.37}$$

with some c independent of λ.
Instead we shall prove a pointwise inequality. So, first we fix a point $(x_2, x_3) \in \mathbb{R}^2$. Then there is only one element $(m_2, m_3) \in \mathbb{Z}^2$ such that $\chi_{(\nu_2,\nu_3)\,(m_2,m_3)}(x_2, x_3) = 1$. We denote $\gamma_{(\nu_2,\nu_3)} = \gamma_{(\nu_2,\nu_3)\,(m_2,m_3)}$. Similarly, for each $\overline{\nu} = (\nu_1, \nu_2, \nu_3)$, there is a unique $\overline{m}(\overline{\nu}) = (m_1, m_2, m_3)$ such that $\chi_{(\nu_1,\nu_2,\nu_3)\,(m_1,m_2,m_3)}(x_1, x_2, x_3) = 1$ and $\overline{m} \in B_{\overline{\nu}}$. We denote $\lambda_{\overline{\nu}} = \lambda_{\overline{\nu}\,\overline{m}}$.
Substep 2.1. Let $r_1 > 1/p$ and $0 < q \leq 1$. Then

$$|\gamma_{(\nu_2,\nu_3)}|^q = \Big(\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)} |\lambda_{(\nu_1,\nu_2,\nu_3)}|\Big)^q \leq \sum_{\nu_1=0}^{\min(\nu_2,\nu_3)} |\lambda_{(\nu_1,\nu_2,\nu_3)}|^q\,,$$

and

$$\left(\sum_{\nu_2,\nu_3=0}^{\infty} 2^{[\nu_2 r_2+\nu_3 r_3]q} |\gamma_{(\nu_2,\nu_3)}|^q\right)^{p/q} \le \left(\sum_{\nu_1=0}^{\infty}\sum_{\nu_2,\nu_3=\nu_1}^{\infty} 2^{[\nu_2 r_2+\nu_3 r_3]q} |\lambda_{\overline{\nu}}|^q\right)^{p/q}.$$

To continue we distinguish two cases. Let $0 < p \le q$. Then

$$\begin{aligned}\left(\sum_{\nu_1=0}^{\infty}\sum_{\nu_2,\nu_3=\nu_1}^{\infty} 2^{[\nu_2 r_2+\nu_3 r_3]q} |\lambda_{\overline{\nu}}|^q\right)^{p/q} &\le \sum_{\nu_1=0}^{\infty}\left(\sum_{\nu_2,\nu_3=\nu_1}^{\infty} 2^{[\nu_2 r_2+\nu_3 r_3]q} |\lambda_{\overline{\nu}}|^q\right)^{p/q}\\ &\le \sum_{\nu_1=0}^{\infty} 2^{-\nu_1 r_1 p}\left(\sum_{\nu_2,\nu_3=\nu_1}^{\infty} 2^{[\nu_1 r_1+\nu_2 r_2+\nu_3 r_3]q} |\lambda_{\overline{\nu}}|^q\right)^{p/q}.\end{aligned}$$

Now let $0 < q < p < \infty$. With $0 < \varepsilon < r_1 p - 1$ and applying Hölder's inequality we find

$$\begin{aligned}\left(\sum_{\nu_1=0}^{\infty}\sum_{\nu_2,\nu_3=\nu_1}^{\infty} 2^{[\nu_2 r_2+\nu_3 r_3]q} |\lambda_{\overline{\nu}}|^q\right)^{p/q} &\le c\sum_{\nu_1=0}^{\infty} 2^{-\nu_1(r_1 p-\varepsilon)}\left(\sum_{\nu_2,\nu_3=\nu_1}^{\infty} 2^{[\nu_1 r_1+\nu_2 r_2+\nu_3 r_3]q} |\lambda_{\overline{\nu}}|^q\right)^{p/q}\\ &\le c\sum_{\nu_1=0}^{\infty} 2^{-\nu_1}\left(\sum_{\nu_2,\nu_3=\nu_1}^{\infty} 2^{[\nu_1 r_1+\nu_2 r_2+\nu_3 r_3]q} |\lambda_{\overline{\nu}}|^q\right)^{p/q}.\end{aligned}$$

Substep 2.2. If $q > 1$ we use triangle inequality

$$\begin{aligned}\Big(\sum_{\nu_2,\nu_3=0}^{\infty} 2^{[\nu_2 r_2+\nu_3 r_3]q}\ \Big(\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)} |\lambda_{\overline{\nu}}|\Big)^q\Big)^{1/q} &\le \sum_{\nu_1=0}^{\infty}\left(\sum_{\nu_2,\nu_3=\nu_1}^{\infty} 2^{[\nu_2 r_2+\nu_3 r_3]q} |\lambda_{\overline{\nu}}|^q\right)^{1/q}\\ &\le \sum_{\nu_1=0}^{\infty} 2^{-\nu_1 r_1}\left(\sum_{\nu_2,\nu_3=\nu_1}^{\infty} 2^{[\nu_1 r_1+\nu_2 r_2+\nu_3 r_3]q} |\lambda_{\overline{\nu}}|^q\right)^{1/q}.\end{aligned}$$

If $0 < p \le 1$

$$\Big(\sum_{\nu_2,\nu_3=0}^{\infty} 2^{[\nu_2 r_2+\nu_3 r_3]q}\Big(\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)} |\lambda_{\overline{\nu}}|\Big)^q\Big)^{p/q} \le \sum_{\nu_1=0}^{\infty} 2^{-\nu_1 r_1 p}\left(\sum_{\nu_2,\nu_3=\nu_1}^{\infty} 2^{[\nu_1 r_1+\nu_2 r_2+\nu_3 r_3]q} |\lambda_{\overline{\nu}}|^q\right)^{p/q}$$

follows. If $p > 1$ we apply again Hölder's inequality and find

$$\begin{aligned}\Big(\sum_{\nu_2,\nu_3=0}^{\infty} 2^{[\nu_2 r_2+\nu_3 r_3]q}\ \Big(\sum_{\nu_1=0}^{\min(\nu_2,\nu_3)} |\lambda_{\overline{\nu}}|\Big)^q\Big)^{p/q} &\le c\sum_{\nu_1=0}^{\infty} 2^{-\nu_1(r_1 p-\varepsilon)}\left(\sum_{\nu_2,\nu_3=\nu_1}^{\infty} 2^{[\nu_1 r_1+\nu_2 r_2+\nu_3 r_3]q} |\lambda_{\overline{\nu}}|^q\right)^{p/q}\\ &\le c\sum_{\nu_1=0}^{\infty} 2^{-\nu_1}\left(\sum_{\nu_2,\nu_3=\nu_1}^{\infty} 2^{[\nu_1 r_1+\nu_2 r_2+\nu_3 r_3]q} |\lambda_{\overline{\nu}}|^q\right)^{p/q}.\end{aligned}$$

Substep 2.3. Summarizing in all situations we have found

$$\left(\sum_{\nu_2,\nu_3=0}^{\infty} 2^{[\nu_2 r_2+\nu_3 r_3]q} |\gamma_{(\nu_2,\nu_3)}|^q\right)^{p/q} \le c\sum_{\nu_1=0}^{\infty} 2^{-\nu_1}\left(\sum_{\nu_2,\nu_3=\nu_1}^{\infty} 2^{[\nu_1 r_1+\nu_2 r_2+\nu_3 r_3]q} |\lambda_{\overline{\nu}}|^q\right)^{p/q}, \qquad (3.38)$$

where c does not depend on λ. We have to show that this inequality implies (3.37). For fixed (x_2, x_3) we choose a sequence of intervals I_{ν_1} such that

$$I_{\nu_1} \cap I_{\nu_1'} = \emptyset\,, \quad \nu_1 \ne \nu_1'\,, \qquad |I_{\nu_1}| \ge c\,2^{-\nu_1}$$

for some $c > 0$ and

$$\Big\{(x_1, x_2, x_3): \quad x_1 \in I_{\nu_1}\Big\} \subset Q_{\overline{\nu}\,\overline{m}}\,, \qquad \overline{\nu} \in \Upsilon_1\,, \quad \overline{m} \in B_{\overline{\nu}}\,.$$

Then (3.38) implies

$$\Big(\sum_{\nu_2,\nu_3=0}^{\infty} 2^{[\nu_2 r_2+\nu_3 r_3]q} |\gamma_{(\nu_2,\nu_3)}\, \chi_{(\nu_2,\nu_3)\,(m_2,m_3)}(x_2,x_3)|^q\Big)^{p/q}$$
$$\leq \; c \sum_{\nu_1=0}^{\infty} \int_{I_{\nu_1}} \Big(\sum_{\nu_2,\nu_3=\nu_1}^{\infty} 2^{[\nu_1 r_1+\nu_2 r_2+\nu_3 r_3]q} |\lambda_{\overline{\nu}}\, \chi_{\overline{\nu}}(x_1,x_2,x_3)|^q\Big)^{p/q} dx_1.$$

Integration with respect to x_2 and x_3 completes the proof of the boundedness of $\mathrm{tr}_1\, f_1$. The rest is the same as in the B-case.

Step 2. The extension. Here the same construction as in the B-case can be applied, cf. Substep 5.1 of the proof of Theorem 3.10. □

The above proof can be used also in case that some of the r_i coincide with $1/p$, at least under additional restrictions on p and q.

Proposition 3.15. *Let $\mathcal{O}$ be an orthogonal basis of Γ and let $\mathcal{R}_i, i = 1,2,3$ be matrices associated with $\mathcal{O}$ by* (1.1)*,* (2.8) *and* (2.10)*.*
Let $0 < p \leq \min(1,q)$. Then the statement of Theorem 3.14 remains true under the weaker restriction

$$\min(r_1, r_2, r_3) \geq \frac{1}{p} \qquad \textit{and} \qquad \min(r_1, r_2, r_3) > \sigma_{p,q}.$$

Remark 3.16. A final remark. In the general situation of the Besov-Lizorkin-Triebel spaces we have proved a full counterpart of Theorem 2.9. In fact, it is not only a counterpart. Based on the identities $S^{\overline{r}}_2 W(\mathbb{R}^3) = S^{\overline{r}}_{2,2}F(\mathbb{R}^3) = S^{\overline{r}}_{2,2}B(\mathbb{R}^3)$ (in the sense of equivalent norms) we have given a new proof of Theorem 2.9. Because of $S^{\overline{r}}_p W(\mathbb{R}^3) = S^{\overline{r}}_{p,2}F(\mathbb{R}^3)$, $1 < p < \infty$, (also in the sense of equivalent norms) Theorem 3.14 contains the extension to Sobolev spaces of dominating mixed smoothness with p different from 2. However, we do not have counterparts of Theorems 2.11 and 2.13, respectively. Here a good description of the spaces $S^{r_1,r_2}_{p,q}A(\mathcal{R}, \mathbb{R}^2)$ in terms of atoms would be desirable, see Lemma 2.2(iii) for the Fourier-analytic counterpart.

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On the Interplay of Regularity and Decay in Case of Radial Functions I. Inhomogeneous spaces

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Abstract

We deal with decay and boundedness properties of radial functions belonging to Besov and Lizorkin-Triebel spaces. In detail we investigate the surprising interplay of regularity and decay. Our tools are atomic decompositions in combination with trace theorems.

1 Introduction

At the end of the seventies Strauss [38] was the first who observed that there is an interplay between the regularity and decay properties of radial functions. We recall his

Radial Lemma: Let $d \geq 2$*. Every radial function* $f \in H^1(\mathbb{R}^d)$ *is almost everywhere equal to a function* $\widetilde{f}$*, continuous for* $x \neq 0$*, such that*

$$|\widetilde{f}(x)| \leq c\,|x|^{\frac{1-d}{2}}\,\|\, f\,|H^1(\mathbb{R}^d)\|\,, \tag{1}$$

where c *depends only on* d*.*

The *Radial Lemma* contains three different assertions:

(a) the existence of a representative of f, which is continuous outside the origin;

(b) the decay of f near infinity;

(c) the limited unboundedness near the origin.

These three properties do not extend to all functions in $H^1(\mathbb{R}^d)$, of course. It will be our aim in this paper to investigate the specific regularity and decay properties of radial functions in a more general framework than Sobolev spaces. In our opinion a discussion of these properties in connection with fractional order of smoothness results in a better understanding of the announced interplay of regularity on the one side and local smoothness, decay at infinity and limited unboundedness near the origin on the other side. In the literature there are several approaches to fractional order of smoothness. Probably most popular are Bessel potential spaces $H^s_p(\mathbb{R}^d)$,

$s \in \mathbb{R}$, or Slobodeckij spaces $W^s_p(\mathbb{R}^d)$ ($s > 0$, $s \notin \mathbb{N}$). These scales would be enough to explain the main interrelations. However, for some limiting cases these scales are not sufficient. For that reason we shall discuss generalizations of the *Radial Lemma* in the framework of Besov spaces $B^s_{p,q}(\mathbb{R}^d)$ and Lizorkin-Triebel spaces $F^s_{p,q}(\mathbb{R}^d)$. These scales cover the Bessel potential and the Slobodeckij spaces since

- $W^m_p(\mathbb{R}^d) = F^m_{p,2}(\mathbb{R}^d)$, $m \in \mathbb{N}_0$, $1 < p < \infty$;
- $H^s_p(\mathbb{R}^d) = F^s_{p,2}(\mathbb{R}^d)$, $s \in \mathbb{R}$, $1 < p < \infty$;
- $W^s_p(\mathbb{R}^d) = F^s_{p,p}(\mathbb{R}^d) = B^s_{p,p}(\mathbb{R}^d)$, $s > 0$, $s \notin \mathbb{N}$, $1 \le p \le \infty$,

where all identities have to be understood in the sense of equivalent norms, see, e.g., [40].
All three phenomena (a)-(c) extend to a certain range of parameters which we shall characterize exactly. For instance, decay near infinity will take place in spaces with $s \ge 1/p$ (see Theorem 10) and limited unboundedness near the origin in the sense of (1) will happen in spaces such that $1/p \le s \le d/p$ (see Theorem 13). For $s = 1/p$ (or $s = d/p$) always the microscopic parameter q comes into play. We will study the above properties also for spaces with $p < 1$. To a certain extend this is motivated by the fact, that the decay properties of radial functions near infinity are determined by the parameter p and the decay rate increases when p decreases, see Theorem 10.
Our main tools here are the following. Based on the atomic decomposition theorem for inhomogeneous Besov and Lizorkin-Triebel spaces, which we proved in [31], we shall deduce a trace theorem for radial subspaces which is of interest on its own. Then this trace theorem will be applied to derive the extra regularity properties of radial functions. To derive the decay estimates and the assertions on controlled unboundedness near zero we shall also employ the atomic decomposition technique.
With respect to the decay it makes a difference, whether one deals with inhomogeneous or homogeneous spaces of Besov and Lizorkin-Triebel type. Homogeneous spaces (with a proper interpretation) are larger than their inhomogeneous counterparts (at least if $s > d \max(\frac{1}{p} - 1)$). Hence, the decay rate of the elements of inhomogeneous spaces can be better than that one for homogeneous spaces. This turns out to be true. However, here in this article we concentrate on inhomogeneous spaces. Radial subspaces of homogeneous spaces will be subject to the continuation of this paper, see [32]. In a further paper [33] we shall investigate a few more properties of radial subspaces like interpolation and characterization by differences.
The paper is organized as follows.

We add a few comments. In 2.1.1 we state also trace assertions for radial subspaces of Hölder-Zygmund classes. Within the borderline cases in Subsection 2.2.2 the spaces $BV(\mathbb{R}^d)$ show up. In this context we will also deal with the trace problem for the associated radial subspaces. All proofs will be given in Sections 3 and 4. There also additional material is collected, e.g., in Subsection 3.1 we deal with interpolation of radial subspaces, in Subsection 3.3.2 we recall the characterization of radial subspaces by atoms as given in [31], and finally, in Subsection 3.8 we discuss the regularity properties of some families of test functions.
Besov and Lizorkin-Triebel spaces are discussed at various places, we refer, e.g., to the monographs [26, 28, 40, 41, 43]. We will not give definitions here and refer for this to the quoted literature.
The present paper is a continuation of [31], [35] and [21].

Notation

As usual, $\mathbb{N}$ denotes the natural numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ denotes the integers and $\mathbb{R}$ the real numbers. If X and Y are two quasi-Banach spaces, then the symbol $Y \hookrightarrow X$ indicates that the embedding is continuous. By $\mathcal{L}(X, Y)$ the set of all linear and bounded operators $T : X \to Y$ is denoted equipped with the standard quasi-norm. As usual, the symbol c denotes positive constants which depend only on the fixed parameters s, p, q and probably on auxiliary functions, unless otherwise stated; its value may vary from line to line. Sometimes we will use the symbols " $\lesssim$ " and " $\gtrsim$ " instead of "$\le$" and "$\ge$", respectively. The meaning of $A \lesssim B$ is given by: there exists a constant $c > 0$ such that $A \le c\, B$. Similarly $\gtrsim$ is defined. The symbol $A \asymp B$ will be used as an abbreviation of $A \lesssim B \lesssim A$.
We shall use the following conventions throughout the paper:

- If E denotes a space of functions on $\mathbb{R}^d$ then by RE we mean the subset of radial functions in E and we endow this subset with the same quasi-norm as the original space.

- Inhomogeneous Besov and Lizorkin-Triebel spaces are denoted by $B^s_{p,q}$ and $F^s_{p,q}$, respectively. If there is no reason to distinguish between these two scales we will use the notation $A^s_{p,q}$. Similarly for the radial subspaces.
- If an equivalence class $[f]$ contains a continuous representative then we call the class continuous and speak of values of f at any point (by taking the values of the continuous representative).
- Throughout the paper $\psi \in C_0^\infty(\mathbb{R}^d)$ denotes a specific radial cut-off function, i.e., $\psi(x) = 1$ if $|x| \le 1$ and $\psi(x) = 0$ if $|x| \ge 3/2$.

2 Main results

This section consists of two parts. In Subsection 2.1 we concentrate on trace theorems which are the basis for the understanding of the higher regularity of radial functions outside the origin. Subsection 2.2 is devoted to the study of decay and boundedness properties of radial functions in dependence on their regularity. To begin with we study the decay of radial functions near infinity. Special emphasize is given to the limiting situation which arises for $s = 1/p$. Then we continue with an investigation of the behaviour of radial functions near the origin. Also here we investigate the limiting situations $s = d/p$ and $s = 1/p$ in some detail.

2.1 The characterization of the traces of radial subspaces

Let $d \ge 2$. Let $f : \mathbb{R}^d \to \mathbb{C}$ be a locally integrable radial function. By using a Lebesgue point argument its restriction

$$f_0(t) := f(t, 0, \dots, 0)\,, \qquad t \in \mathbb{R}\,.$$

is well defined a.e. on $\mathbb{R}$. However, this restriction need not be locally integrable. A simple example is given by the function

$$f(x) := \psi(x)\,|x|^{-1}\,, \qquad x \in \mathbb{R}^d\,,$$

where ψ denotes a smooth cut-off function s.t. $\psi(0) \neq 0$. Furthermore, if we start with a measurable and even function $g : \mathbb{R} \to \mathbb{C}$, s.t. g is locally integrable on all intervals (a, b), $0 < a < b < \infty$, then (again using a Lebesgue point argument) the function

$$f(x) := g(|x|)\,, \qquad x \in \mathbb{R}^d$$

is well-defined a.e. on $\mathbb{R}^d$ and is radial, of course. In what follows we shall study properties of the associated operators

$$\mathrm{tr}\ : f \mapsto f_0 \qquad \text{and} \qquad \mathrm{ext}\ : g \mapsto f\,.$$

Both operators are defined pointwise only. Later on we shall have a short look onto the existence of the trace in the distributional sense, see Subsection 2.1.4. Probably it would be more natural to deal with functions defined on $[0,\infty)$ in this context. However, that would result in more complicated descriptions of the trace spaces. So, our target spaces will be spaces of even functions defined on $\mathbb{R}$.

2.1.1 Traces of radial subspaces with $p = \infty$

The first result is maybe well-known but we did not find a reference for it.

Theorem 1 *Let $d \geq 2$. For $m \in \mathbb{N}_0$ the mapping* tr *is a linear isomorphism of $RC^m(\mathbb{R}^d)$ onto $RC^m(\mathbb{R})$ with inverse* ext *.*

Using real interpolation it is not difficult to derive the following result for the spaces of Hölder-Zygmund type.

Theorem 2 *Let $s > 0$ and let $0 < q \leq \infty$. Then the mapping* tr *is a linear isomorphism of $RB^s_{\infty,q}(\mathbb{R}^d)$ onto $RB^s_{\infty,q}(\mathbb{R})$ with inverse* ext *.*

2.1.2 Traces of radial subspaces with $p < \infty$

Now we turn to the description of the trace classes of radial Besov and Lizorkin-Triebel spaces with $p < \infty$. Again we start with an almost trivial result.

Lemma 1 *Let $d \geq 2$.*
(i) *Let $0 < p < \infty$. Then* tr $: RL_p(\mathbb{R}^d) \to RL_p(\mathbb{R}, |t|^{d-1})$ *is an linear isomorphism with inverse* ext *.*
(ii) *Let $p = \infty$. Then* tr $: RL_\infty(\mathbb{R}^d) \to RL_\infty(\mathbb{R})$ *is an linear isomorphism with inverse* ext *.*

In particular this means, that whenever the Besov-Lizorkin-Triebel space $A^s_{p,q}(\mathbb{R}^d)$ is contained in $L_1(\mathbb{R}^d) + L_\infty(\mathbb{R}^d)$, then tr is well-defined on its radial subspace. This is in sharp contrast to the general theory of traces on these spaces. To guarantee that tr is meaningful on $A^s_{p,q}(\mathbb{R}^d)$ one has to require

$$s > \frac{d-1}{p} + \max\Big(0, \frac{1}{p} - 1\Big)\,,$$

cf. e.g. [20], [14], [40, Rem. 2.7.2/4] or [12]. On the other hand we have

$$B^s_{p,q}(\mathbb{R}^d)\,,\; F^s_{p,q}(\mathbb{R}^d) \hookrightarrow L_1(\mathbb{R}^d) + L_\infty(\mathbb{R}^d)$$

if $s > d\,\max(0, \frac{1}{p} - 1)$, see, e.g., [34]. Since

$$d\,\max(0, \frac{1}{p} - 1) < \frac{d-1}{p} + \max\Big(0, \frac{1}{p} - 1\Big)$$

we have the existence of tr with repect to $RA^s_{p,q}(\mathbb{R}^d)$ for a wider range of parameters than for $A^s_{p,q}(\mathbb{R}^d)$.
Below we shall develop a description of the traces of the radial subspaces of $B^s_{p,q}(\mathbb{R}^d)$ and $F^s_{p,q}(\mathbb{R}^d)$ in terms of atoms. To explain this we need to introduce first an appropriate notion of an atom and second, adapted sequence spaces.

Definition 1 *Let $L \ge 0$ be an integer. Let I be a set either of the form $I = [-a, a]$ or of the form $I = [-b, -a] \cup [a, b]$ for some $0 < a < b < \infty$. An even function $g \in C^L(\mathbb{R})$ is called an even L-atom centered at I if*

$$\max_{t\in\mathbb{R}} |b^{(n)}(t)| \le |I|^{-n}\,, \qquad 0 \le n \le L\,.$$

and if either

$$\mathrm{supp}\ g \subset [-\frac{3\,a}{2}, \frac{3a}{2}] \qquad \textit{in case} \quad I = [-a, a]\,,$$

or

$$\mathrm{supp}\ g \subset [-\frac{3b-a}{2}, -\frac{3a-b}{2}] \cup [\frac{3a-b}{2}, \frac{3b-a}{2}] \qquad \textit{in case} \quad I = [-b, -a] \cup [a, b]\,.$$

Definition 2 *Let $0 < p < \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Let*

$$\chi^{\#}_{j,k}(t) := \begin{cases} 1 & \textit{if} \quad 2^{-j}k \le |t| \le 2^{-j}(k+1)\,, \\ 0 & \textit{otherwise}\,. \end{cases} \qquad t \in \mathbb{R}\,.$$

Then we define

$$b^s_{p,q,d} := \Big\{ s = (s_{j,k})_{j,k} \; : \quad \| \, s \, | b^s_{p,q,d}\| = \Big(\sum_{j=0}^{\infty} 2^{j(s-\frac{d}{p})q} \Big(\sum_{k=0}^{\infty} (1+k)^{d-1} \, |s_{j,k}|^p \Big)^{q/p} \Big)^{1/q} < \infty \Big\}\,.$$

and

$$\begin{aligned} f^s_{p,q,d} \quad &:= \quad \Big\{ s = (s_{j,k})_{j,k} \; : \\ \| \, s \, | f^s_{p,q,d}\| \quad &= \quad \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} \sum_{k=0}^{\infty} |s_{j,k}|^q \, \chi^{\#}_{j,k}(\cdot) \Big)^{1/q} | L_p(\mathbb{R}, |t|^{d-1}) \Big\| < \infty \Big\}\,, \end{aligned}$$

respectively.

Remark 1 *Observe $b^s_{p,q,d} = f^s_{p,q,d}$ in the sense of equivalent quasi-norms.*

Adapted to these sequence spaces we define now function spaces on $\mathbb{R}$.

Definition 3 *Let* $0 < p < \infty$, $0 < q \le \infty$, $s > 0$ *and* $L \in \mathbb{N}_0$.
(i) *Then* $TB^s_{p,q}(\mathbb{R}, L, d)$ *is the collection of all functions* $g : \mathbb{R} \to \mathbb{C}$ *such that there exists a decomposition*

$$g(t) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} s_{j,k}\, g_{j,k}(t) \tag{2}$$

(convergence in $L_{\max(1,p)}(\mathbb{R}, |t|^{d-1})$*), where the sequence* $(s_{j,k})_{j,k}$ *belongs to* $b^s_{p,q,d}$ *and the functions* $g_{j,k}$ *are even* L*-atoms centered at either* $[-2^{-j}, 2^{-j}]$ *if* $k = 0$ *or at*

$$[-2^{-j}(k+1), -2^{-j}k] \cup [2^{-j}k, 2^{-j}(k+1)]$$

if $k > 0$. *We put*

$$\| \, g \, |TB^s_{p,q}(\mathbb{R}, L, d)\| := \inf \Big\{ \| \, (s_{j,k}) \, |b^s_{p,q,d}\| \; : \quad (2) \quad \textit{holds} \Big\} .$$

(ii) *Then* $TF^s_{p,q}(\mathbb{R}, L, d)$ *is the collection of all functions* $g : \mathbb{R} \to \mathbb{C}$ *such that there exists a decomposition (2), where the sequence* $(s_{j,k})_{j,k}$ *belongs to* $f^s_{p,q,d}$ *and the functions* $g_{j,k}$ *are as in (i). We put*

$$\| \, g \, |TF^s_{p,q}(\mathbb{R}, L, d)\| := \inf \Big\{ \| \, (s_{j,k}) \, |f^s_{p,q,d}\| \; : \quad (2) \quad \textit{holds} \Big\} .$$

We need a few further notation. In connection with Besov and Lizorkin-Triebel spaces quite often the following numbers occur:

$$\sigma_p(d) := d \, \max\Big(0, \frac{1}{p} - 1\Big) \qquad \text{and} \qquad \sigma_{p,q}(d) := d \, \max\Big(0, \frac{1}{p} - 1, \frac{1}{q} - 1\Big) . \tag{3}$$

For a real number s we denote by $[s]$ the integer part, i.e. the largest integer m such that $m \le s$.

Theorem 3 *Let* $d \ge 2$, $0 < p < \infty$ *and* $0 < q \le \infty$.
(i) *Suppose* $s > \sigma_p(d)$ *and* $L \ge [s]+1$. *Then the mapping* tr *is a linear isomorphism of* $RB^s_{p,q}(\mathbb{R}^d)$ *onto* $TB^s_{p,q}(\mathbb{R}, L, d)$ *with inverse* ext .
(ii) *Suppose* $s > \sigma_{p,q}(d)$ *and* $L \ge [s]+1$. *Then the mapping* tr *is a linear isomorphism of* $RF^s_{p,q}(\mathbb{R}^d)$ *onto* $TF^s_{p,q}(\mathbb{R}, L, d)$ *with inverse* ext .

Remark 2 Let $0 < p \le 1 < q \le \infty$. Then the spaces $RB^{\sigma_p}_{p,q}(\mathbb{R}^d)$ contain singular distributions, see [34]. In particular, the Dirac delta distribution belongs to $RB^{\frac{d}{p}-d}_{p,\infty}(\mathbb{R}^d)$, see, e.g., [28, Rem. 2.2.4/3]. Hence, our pointwise defined mapping tr is not meaningful on those spaces, or, with other words, Theorem 3 does not extend to values $s < \sigma_p(d)$.

Outside the origin radial distributions are more regular. We shall discuss several examples for this claim.

Theorem 4 *Let $d \geq 2$, $0 < p < \infty$ and $0 < q \leq \infty$. Suppose $s > \max(0, \frac{1}{p} - 1)$. Let $f \in RA^s_{p,q}(\mathbb{R}^d)$ s.t. $0 \notin \operatorname{supp} f$. Then f is a regular distribution in $\mathcal{S}'(\mathbb{R}^d)$.*

Remark 3 There is a nice and simple example which explains the sharpness of the restrictions in Thm. 4. We consider the singular distribution f defined by

$$\varphi \mapsto \int_{|x|=1} \varphi(x)\, dx\,, \qquad \varphi \in \mathcal{S}(\mathbb{R}^d)\,.$$

By using the wavelet characterization of Besov spaces, it is not difficult to prove that the spherical mean distribution f belongs to the spaces $B^{\frac{1}{p}-1}_{p,\infty}(\mathbb{R}^d)$ for all p.

Theorem 5 *Let $d \geq 2$, $0 < p < \infty$ and $0 < q \leq \infty$. Suppose $s > \max(0, \frac{1}{p} - 1)$. Let $f \in RA^s_{p,q}(\mathbb{R}^d)$ s.t. $0 \notin \operatorname{supp} f$. Then $f_0 = \operatorname{tr} f$ belongs to $A^s_{p,q}(\mathbb{R})$.*

Remark 4 As mentioned above

$$A^s_{p,q}(\mathbb{R}) \hookrightarrow L_1(\mathbb{R}) + L_\infty(\mathbb{R}) \qquad \text{if} \qquad s > \sigma_p(1) = \max\left(0, \frac{1}{p} - 1\right),$$

which shows again that we deal with regular distributions. However, in Thm. 5 some additional regularity is proved.

2.1.3 Traces of radial subspaces of Sobolev spaces

Clearly, one can expect that the description of the traces of radial Sobolev spaces can be given in more elementary terms. We discuss a few examples without having the complete theory.

Theorem 6 *Let $d \geq 2$ and $1 \leq p < \infty$.*
(i) *The mapping* tr *is a linear isomorphism (with inverse* ext *) of $RW^1_p(\mathbb{R}^d)$ onto the closure of $RC^\infty_0(\mathbb{R})$ with respect to the norm*

$$\| \, g \, | L_p(\mathbb{R}, |t|^{d-1}) \| + \| \, g' \, | L_p(\mathbb{R}, |t|^{d-1}) \| \,.$$

(ii) *The mapping* tr *is a linear isomorphism (with inverse* ext *) of $RW^2_p(\mathbb{R}^d)$ onto the closure of $RC^\infty_0(\mathbb{R})$ with respect to the norm*

$$\| \, g \, | L_p(\mathbb{R}, |t|^{d-1}) \| + \| \, g' \, | L_p(\mathbb{R}, |t|^{d-1}) \| + \| \, g'/r \, | L_p(\mathbb{R}, |t|^{d-1}) \| + \| \, g'' \, | L_p(\mathbb{R}, |t|^{d-1}) \| \,.$$

Remark 5 Both statements have elementary proofs, see (11) for (i). However, the complete extension to higher order Sobolev spaces is open.

There are several ways to define Sobolev spaces on $\mathbb{R}^d$. For instance, if $1 < p < \infty$ we have

$$f \in W^{2m}_p(\mathbb{R}^d) \quad \Longleftrightarrow \quad f \in L_p(\mathbb{R}^d) \quad \text{and} \quad \Delta^m f \in L_p(\mathbb{R}^d)\,. \tag{4}$$

Such an equivalence does not extend to $p = 1$ or $p = \infty$ if $d \ge 2$, see [36, pp. 135/160]. Recall that the Laplace operator Δ applied to a radial function yields a radial function. In particular we have

$$\Delta f(x) = D_r f_0(r) := f_0''(r) + \frac{d-1}{r} f_0'(r)\,, \qquad r = |x|\,, \tag{5}$$

in case that f is radial and $\mathrm{tr}\, f = f_0$. Obviously, if $f \in RC_0^\infty(\mathbb{R}^d)$, then

$$\begin{aligned} \| f \,|L_p(\mathbb{R}^d)\| \quad &+ \quad \| \Delta^m f \,|L_p(\mathbb{R}^d)\| \\ &= \quad \Big(\frac{\pi^{d/2}}{\Gamma(d/2)}\Big)^{1/p} \Big(\| f_0 \,|L_p(\mathbb{R}, |t|^{d-1})\| + \| D_r^m f_0 \,|L_p(\mathbb{R}, |t|^{d-1})\|\Big)\,. \end{aligned} \tag{6}$$

This proves the next characterization.

Theorem 7 *Let $1 < p < \infty$ and $m \in \mathbb{N}$. Then the mapping* tr *yields a linear isomorphism (with inverse* ext *) of $RW_p^{2m}(\mathbb{R}^d)$ onto the closure of $RC_0^\infty(\mathbb{R})$ with respect to the norm*

$$\| f_0 \,|L_p(\mathbb{R}, |t|^{d-1})\| + \| D_r^m f_0 \,|L_p(\mathbb{R}, |t|^{d-1})\|\,.$$

Remark 6 By means of Hardy-type inequalities one can simplify the terms $\| D_r^m f_0 \,|L_p(\mathbb{R}, |t|^{d-1})\|$ to some extend, see Theorem 6(ii) for a comparison. We do not go into detail.

2.1.4 The trace in $\mathcal{S}'(\mathbb{R})$

Many times applications of traces are connected with boundary value problems. In such a context the continuity of tr considered as a mapping into S' is essential. Again we consider the simple situation of the L_p-spaces first.

Lemma 2 *Let $d \ge 2$ and let $0 < p < \infty$. Then $RL_p(\mathbb{R}, |t|^{d-1}) \subset S'(\mathbb{R})$ if and only if $d < p$.*

From the known embedding relations of $RA_{p,q}^s(\mathbb{R}^d)$ into L_u-spaces one obtains one half of the proof of the following general result.

Theorem 8 *Let $d \ge 2$, $0 < p < \infty$, and $0 < q \le \infty$.*
(a) *Let $s > \sigma_p(d)$ and $L \ge [s] + 1$. Then the following assertions are equivalent:*
 (i) *The mapping* tr *maps $RB_{p,q}^s(\mathbb{R}^d)$ into $S'(\mathbb{R})$.*
 (ii) *The mapping* tr : *$RB_{p,q}^s(\mathbb{R}^d) \to S'(\mathbb{R})$ is continuous.*
 (iii) *We have $TB_{p,q}^s(\mathbb{R}, L, d) \hookrightarrow S'(\mathbb{R})$.*
 (iv) *We have either $s > d(\frac{1}{p} - \frac{1}{d})$ or $s = d(\frac{1}{p} - \frac{1}{d})$ and $q \le 1$.*
(b) *Let $s > \sigma_{p,q}(d)$ and $L \ge [s] + 1$. Then following assertions are equivalent:*
 (i) *The mapping* tr *maps $RF_{p,q}^s(\mathbb{R}^d)$ into $S'(\mathbb{R})$.*
 (ii) *The mapping* tr : *$RF_{p,q}^s(\mathbb{R}^d) \to S'(\mathbb{R})$ is continuous.*
 (iii) *We have $TF_{p,q}^s(\mathbb{R}, L, d) \hookrightarrow S'(\mathbb{R})$.*
 (iv) *We have either $s > d(\frac{1}{p} - \frac{1}{d})$ or $s = d(\frac{1}{p} - \frac{1}{d})$ and $0 < p \le 1$.*

2.1.5 The trace in $\mathcal{S}'(\mathbb{R})$ and weighted function spaces of Besov and Lizorkin-Triebel type

Weighted function spaces of Besov and Lizorkin-Triebel type, denoted by $B^s_{p,q}(\mathbb{R}, w)$ and $F^s_{p,q}(\mathbb{R}, w)$, respectively, are a well-developed subject in the literature, we refer to [5, 6, 29]. Fourier analytic definitions as well as characterizations by atoms are given under various restrictions on the weights, see e.g. [4, 5, 6, 16, 18, 30]. In this subsection we are interested in these spaces with respect to the weights $w_{d-1}(t) := |t|^{d-1}$, $t \in \mathbb{R}$, $d \geq 2$. Of course, these weights belong to the Muckenhoupt class $\mathcal{A}_\infty$, more exactly $w_{d-1} \in \mathcal{A}_r$ for any $r > d$, see [37].

Theorem 9 *Let $d \geq 2$, $0 < p < \infty$, and $0 < q \leq \infty$.*
(i) *Suppose $s > \sigma_p(d)$ and let $L \geq [s] + 1$. If $TB^s_{p,q}(\mathbb{R}, L, d) \hookrightarrow S'(\mathbb{R})$ (see Theorem 8), then $TB^s_{p,q}(\mathbb{R}, L, d) = RB^s_{p,q}(\mathbb{R}, w_{d-1})$ in the sense of equivalent quasi-norms.*
(ii) *Suppose $s > \sigma_{p,q}(d)$ and let $L \geq [s]+1$. If $TF^s_{p,q}(\mathbb{R}, L, d) \hookrightarrow S'(\mathbb{R})$ (see Theorem 8), then $TF^s_{p,q}(\mathbb{R}, L, d) = RF^s_{p,q}(\mathbb{R}, w_{d-1})$ in the sense of equivalent quasi-norms.*

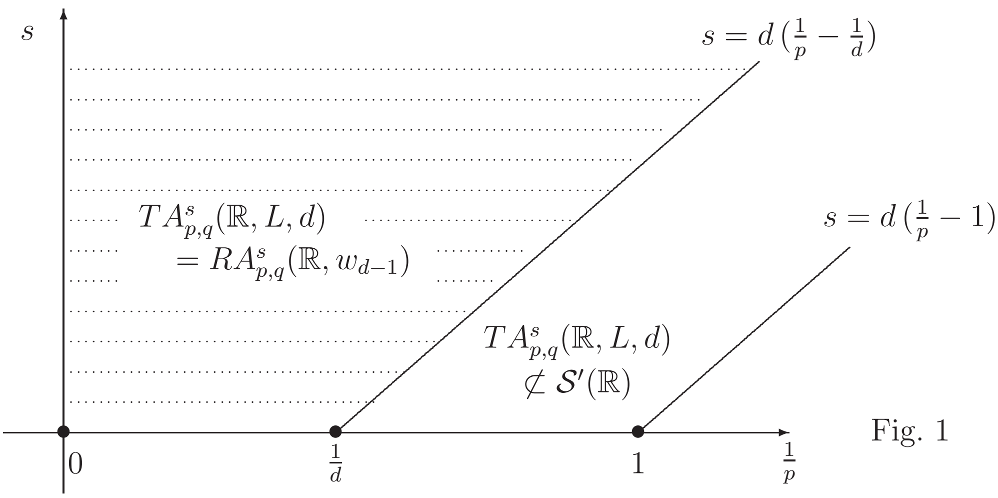

Fig. 1

Remark 7 We add some statements concerning the regularity of the most prominent singular distribution, namely $\delta : \varphi \to \varphi(0)$, $\varphi \in S(\mathbb{R}^d)$. This tempered distribution has the following regularity properties:

- First we deal with the situation on $\mathbb{R}^d$. We have $\delta \in RB^{\frac{d}{p}-d}_{p,\infty}(\mathbb{R}^d)$ (but $\delta \notin RB^{\frac{d}{p}-d}_{p,q}(\mathbb{R}^d)$ for $q < \infty$ and $\delta \notin RF^{\frac{d}{p}-d}_{p,\infty}(\mathbb{R}^d)$), see, e.g., [28, Rem. 2.2.4/3].
- Now we turn to the situation on $\mathbb{R}$. By using more or less the same arguments as on $\mathbb{R}^d$ one can show $\delta \in B^{\frac{d}{p}-1}_{p,\infty}(\mathbb{R}, w_{d-1})$ (but $\delta \notin B^{\frac{d}{p}-1}_{p,q}(\mathbb{R}, w_{d-1})$ for any $q < \infty$ and $\delta \notin F^{\frac{d}{p}-1}_{p,\infty}(\mathbb{R}, w_{d-1})$).

2.1.6 The regularity of radial functions outside the origin

Let f be a radial function such that $\operatorname{supp} f \subset \{x \in \mathbb{R}^d : \quad |x| \geq \tau\}$ for some $\tau > 0$. Then the following inequality is obvious:

$$\| f_0 \,|L_p(\mathbb{R})\| \leq \tau^{-(d-1)/p} \Big(\frac{\Gamma(d/2)}{\pi^{d/2}}\Big)^{1/p} \| f \,|L_p(\mathbb{R}^d)\| \,.$$

An extension to first or second order Sobolev spaces can be done by using Theorem 6. However, an extension to all spaces under consideration here is less obvious. Partly it could be done by interpolation, see Proposition 1, but we prefer a different way (not to exclude $p < 1$). We shall compare the atomic decompositions in Theorem 3 with the known atomic and wavelet characterizations of $B^s_{p,q}(\mathbb{R})$ and $F^s_{p,q}(\mathbb{R})$.

Corollary 1 *Let $\tau > 0$. Let $d \geq 2$, $0 < p < \infty$, and $0 < q \leq \infty$.*
(i) *We suppose $s > \sigma_p(d)$. If $f \in RB^s_{p,q}(\mathbb{R}^d)$ such that*

$$\operatorname{supp} f \subset \{x \in \mathbb{R}^d : \quad |x| \geq \tau\} \tag{7}$$

then its trace f_0 belongs to $B^s_{p,q}(\mathbb{R})$. Furthermore, there exists a constant c (not depending on f and τ) such that

$$\| f_0 \,|B^s_{p,q}(\mathbb{R})\| \leq c\, \tau^{-(d-1)/p} \, \| f \,|B^s_{p,q}(\mathbb{R}^d)\| \tag{8}$$

holds for all such functions f and all $\tau > 0$.
(ii) *We suppose $s > \sigma_{p,q}(d)$. If $f \in RF^s_{p,q}(\mathbb{R}^d)$ such that (7) holds, then its trace f_0 belongs to $F^s_{p,q}(\mathbb{R})$. Furthermore, there exists a constant c (not depending on f and τ) such that*

$$\| f_0 \,|F^s_{p,q}(\mathbb{R})\| \leq c\, \tau^{-(d-1)/p} \, \| f \,|F^s_{p,q}(\mathbb{R}^d)\| \tag{9}$$

holds for all such functions f all $\tau > 0$.

We wish to mention that Corollary 1 has a partial inverse.

Corollary 2 *Let $d \geq 2$, $0 < p < \infty$, $0 < q \leq \infty$ and $0 < a < b < \infty$.*
(i) *We suppose $s > \sigma_p(d)$. If $g \in RB^s_{p,q}(\mathbb{R})$ such that*

$$\operatorname{supp} g \subset \{x \in \mathbb{R} : \quad a \leq |x| \leq b\} \tag{10}$$

then the radial function $f :=$ ext g belongs to $RB^s_{p,q}(\mathbb{R}^d)$ and there exist positive constants A, B such that

$$A \, \| g \,|B^s_{p,q}(\mathbb{R})\| \leq \| f \,|B^s_{p,q}(\mathbb{R}^d)\| \leq B \, \| g \,|B^s_{p,q}(\mathbb{R})\| \,.$$

(ii) *We suppose $s > \sigma_{p,q}(d)$. If $g \in RF^s_{p,q}(\mathbb{R})$ such that (10) holds, then the radial function $f :=$ ext g belongs to $RF^s_{p,q}(\mathbb{R}^d)$ and there exist positive constants A, B such that*

$$A \, \| g \,|F^s_{p,q}(\mathbb{R})\| \leq \| f \,|F^s_{p,q}(\mathbb{R}^d)\| \leq B \, \| g \,|F^s_{p,q}(\mathbb{R})\| \,.$$

For our next result we need Hölder-Zygmund spaces. Recall, that $C^s(\mathbb{R}^d) = B^s_{\infty,\infty}(\mathbb{R}^d)$ in the sense of equivalent norms if $s \notin \mathbb{N}_0$. Of course, also the spaces $B^s_{\infty,\infty}(\mathbb{R}^d)$ with $s \in \mathbb{N}$ allow a characterization by differences. We refer to [40, 2.2.2, 2.5.7] and [41, 3.5.3]. We shall use the abbreviation

$$\mathcal{Z}^s(\mathbb{R}^d) = B^s_{\infty,\infty}(\mathbb{R}^d)\,, \qquad s > 0\,.$$

Taking into account the well-known embedding relations for Besov as well as for Lizorkin-Triebel spaces defined on $\mathbb{R}$ Thm. 5 implies in particular:

Corollary 3 *Let $d \geq 2$, $0 < p < \infty$, $0 < q \leq \infty$, and $s > \max(0, \frac{1}{p} - 1)$. Let φ be a smooth radial function, uniformly bounded together with all its derivatives, and such that $0 \notin \operatorname{supp} \varphi$. If $f \in RA^s_{p,q}(\mathbb{R}^d)$, then $\varphi\, f \in \mathcal{Z}^{s-1/p}(\mathbb{R}^d)$.*

Remark 8 P.L. Lions [23] has proved the counterpart of Corollary 3 for first order Sobolev spaces. We also dealt in [31] with these problems.

Finally, for later use, we would like to know when the radial functions are continuous out of the origin.

Corollary 4 *Let $\tau > 0$. Let $d \geq 2$, $0 < p < \infty$, and $0 < q \leq \infty$.*
(i) *If either $s > 1/p$ or $s = 1/p$ and $q \leq 1$ then $f \in RB^s_{p,q}(\mathbb{R}^d)$ is uniformly continuous on the set $|x| \geq \tau$.*
(ii) *If either $s > 1/p$ or $s = 1/p$ and $p \leq 1$ then $f \in RF^s_{p,q}(\mathbb{R}^d)$ is uniformly continuous on the set $|x| \geq \tau$.*

By looking at the restrictions in Cor. 4 we introduce the following set of parameters.

Definition 4 (i) *We say (s,p,q) belongs to the set $U(B)$ if (s,p,q) satisfies the restrictions in part (i) of Cor. 4.*
(ii) *The triple (s,p,q) belongs to the set $U(F)$ if (s,p,q) satisfies the restrictions in part (ii) of Cor. 4.*

Remark 9 (a) The abbreviation $(s,p,q) \in U(A)$ will be used with the obvious meaning.
(b) Let $1 \leq p = p_0 < \infty$ be fixed. Then there is always a largest space in the set

$$\{B^s_{p_0,q}(\mathbb{R}^d) \,:\, (s,p_0,q) \in U(B)\} \quad \cup \quad \{F^s_{p_0,q}(\mathbb{R}^d) \,:\, (s,p_0,q) \in U(F)\}\,.$$

This space is given either by $F^1_{1,\infty}(\mathbb{R}^d)$ if $p_0 = 1$ or by $B^{1/p_0}_{p_0,1}(\mathbb{R}^d)$ if $1 < p_0 < \infty$. If $p_0 < 1$, then obviously $B^{1/p_0}_{p_0,1}(\mathbb{R}^d)$ is the largest Besov space and $F^{1/p_0}_{p_0,\infty}(\mathbb{R}^d)$ is the largest Lizorkin-Triebel space in the above family. However, these spaces are incomparable.

2.2 Decay and boundedness properties of radial functions

We deal with improvements of Strauss' *Radial Lemma*. Decay can only be expected if we measure smoothness in function spaces built on $L_p(\mathbb{R}^d)$ with $p < \infty$.
It is instructive to have a short look onto the case of first order Sobolev spaces. Let $f = g(r(x)) \in RC_0^\infty(\mathbb{R}^d)$. Then

$$\frac{\partial f}{\partial x_i}(x) = g'(r)\,\frac{x_i}{r}, \quad r = |x| > 0\,, \qquad i = 1, \ldots, d\,.$$

Hence

$$\|\ |\nabla f(x)|\ |L_p(\mathbb{R}^d)\| = c_d \,\|\, g' \,|L_p(\mathbb{R}, |t|^{d-1})\|\,, \tag{11}$$

where $1 \le p < \infty$. Next we apply the identity

$$g(r) = -\int_r^\infty g'(t)dt$$

and obtain

$$|g(r)| \le \int_r^\infty |g'(t)|dt \le r^{-(d-1)} \int_r^\infty t^{d-1}|g'(t)|dt.$$

This extends to all functions in $RW_1^1(\mathbb{R}^d)$ by a density argument. On this elementary way we have proved the inequality

$$|x|^{d-1}\,|f(x)| = r^{d-1}\,|g(r)| \le c_d \int_{|x|>r} |\nabla f(x)|\,dx \le c_d \,\|\, \nabla f(x)\,\|_1\,. \tag{12}$$

This inequality can be interpreted in several ways:

- The possible unboundedness in the origin is limited.
- There is some decay, uniformly in f, if $|x|$ tends to $+\infty$.
- We have $\lim_{|x|\to\infty} |x|^{d-1}\,|f(x)| = 0$ for all $f \in RW_1^1(\mathbb{R}^d)$.
- It makes sense to switch to homogeneous function spaces, since in (12) only the norm of the homogeneous Sobolev space occurs.

We shall show that all these phenomena will occur also in the general context of radial subspaces of Besov and Lizorkin-Triebel spaces.

2.2.1 The behaviour of radial functions near infinity

Suppose $(s, p, q) \in U(A)$. Then $f \in RA^s_{p,q}(\mathbb{R}^d)$ is uniformly continuous near infinity and belongs to $L_p(\mathbb{R}^d)$. This implies $\lim_{|x|\to\infty} |f(x)| = 0$. However, much more is true.

Theorem 10 *Let $d \geq 2$, $0 < p < \infty$, and $0 < q \leq \infty$.*
(i) *Suppose $(s,p,q) \in U(A)$. Then there exists a constant c s.t.*

$$|x|^{(d-1)/p}\,|f(x)| \leq c \,\|\, f \,|A^s_{p,q}(\mathbb{R}^d)\| \tag{13}$$

holds for all $|x| \geq 1$ and all $f \in RA^s_{p,q}(\mathbb{R}^d)$.
(ii) *Suppose $(s,p,q) \in U(A)$. Then*

$$\lim_{|x|\to\infty} |x|^{\frac{d-1}{p}}\,|f(x)| = 0 \tag{14}$$

holds for all $f \in RA^s_{p,q}(\mathbb{R}^d)$.
(iii) *Suppose $(s,p,q) \in U(A)$. Then there exists a constant $c > 0$ such that for all x, $|x| > 1$, there exists a smooth radial function $f \in RA^s_{p,q}(\mathbb{R}^d)$, $\|\, f \,|RA^s_{p,q}(\mathbb{R}^d)\| = 1$, s.t.*

$$|x|^{\frac{d-1}{p}}\,|f(x)| \geq c\,. \tag{15}$$

(iv) *Suppose $(s,p,q) \notin U(A)$ and $\frac{1}{p} > \sigma_p(d)$. We assume also that $\frac{1}{p} > \sigma_q(d)$ in the F-case. Then, for all sequences $(x^j)_{j=1}^\infty \subset \mathbb{R}^d \setminus \{0\}$ s.t. $\lim_{j\to\infty}|x^j| = \infty$, there exists a radial function $f \in RA^s_{p,q}(\mathbb{R}^d)$, $\|\, f \,|RA^s_{p,q}(\mathbb{R}^d)\| = 1$, s.t. f is unbounded in any neighborhood of x^j, $j \in \mathbb{N}$.*

Remark 10 (i) Observe, that increasing s (for fixed p) is not improving the decay rate. In the case of Banach spaces, i.e., $p,q \geq 1$, the additional assumptions in point (iv) are always fullfiled. Hence, the largest spaces, guaranteeing the decay rate $(d-1)/p$, are spaces with $s = 1/p$, see Remark 9.
(ii) Observe that in (iii) the function depends on $|x|$. There is no function in $RA^s_{p,q}(\mathbb{R}^d)$ such that (15) holds for all x, $|x| \geq 1$, simultaneously. The naive construction $f(x) := (1-\psi(x))\,|x|^{\frac{1-d}{p}}$, $x \in \mathbb{R}^d$, does not belong to $L_p(\mathbb{R}^d)$.
(iii) If one switches from inhomogeneous spaces to the larger homogeneous spaces of Besov and Lizorkin-Triebel type, then the decay rate becomes smaller. It will depend also on s, see [7] and [32] for details.
(iv) Of course, formula (13) generalizes the estimate (1). Also Coleman, Glazer and Martin [8] have dealt with (1). P.L. Lions [23] proved a p-version of the *Radial Lemma.* Originally the *Radial Lemma* has been used to prove compactness of embeddings of radial Sobolev spaces into L_p-spaces, see [8], [23]. In the framework of radial subspaces of Besov and of Lizorkin-Triebel spaces compactness of embeddings has been investigated in [31].

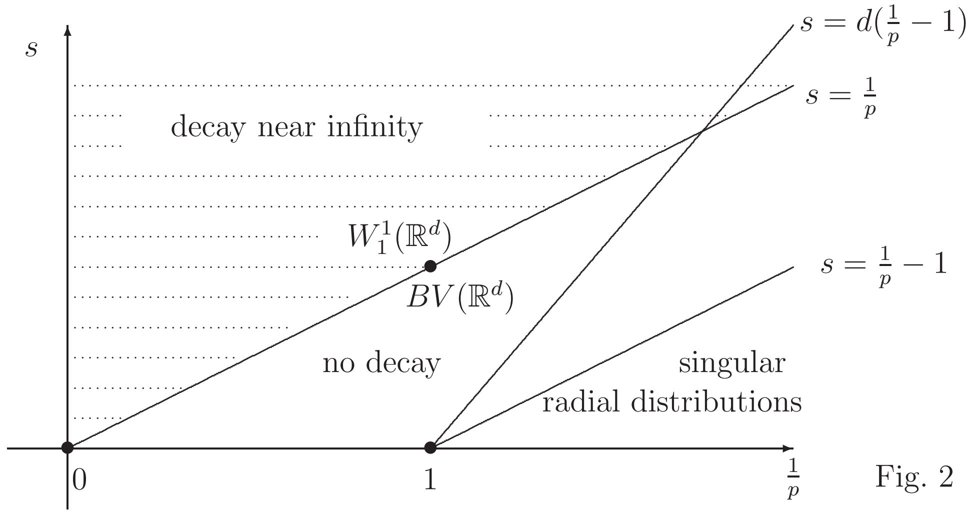

Fig. 2

2.2.2 The behaviour of radial functions near infinity – borderline cases

As indicated in Remark 9, within the scales of Besov and Lizorkin-Triebel spaces the borderline cases for the decay rate $(d-1)/p$ are either $F^1_{1,\infty}(\mathbb{R}^d)$ if $p = 1$ or $B^{1/p}_{p,1}(\mathbb{R}^d)$ if $1 < p < \infty$. Now we turn to spaces which do not belong to these scales and where the elements of the radial subspaces have such a decay rate. Hence, we are looking for spaces of radial functions with a simple norm which satisfy (13). The Sobolev space $RW^1_1(\mathbb{R}^d)$ is such a candidate for which (13) is already known, see [23]. But this is not the end of the story. Also for the radial functions of bounded variation such a decay estimate is true.

Theorem 11 *Let $d \geq 2$. Then there exists constant c s.t.*

$$|x|^{d-1}\,|f(x)| \leq c\,\|\,f\,|BV(\mathbb{R}^d)\| \tag{16}$$

holds for all $|x| > 0$ and all $f \in RBV(\mathbb{R}^d)$. Also

$$\lim_{|x|\to\infty} |x|^{d-1}\,|f(x)| = 0 \tag{17}$$

is true for all $f \in RBV(\mathbb{R}^d)$.

Remark 11 (i) Both assertions, (16) and (17), require an interpretation since, in contrast to $d = 1$, the spaces $BV(\mathbb{R}^d)$, $d \geq 2$, are spaces of equivalence classes, see Subsection 4.2. Nevertheless, in every equivalence class $[f] \in BV(\mathbb{R}^d)$, there is a representative $\tilde{f} \in [f]$, such that

$$|\tilde{f}(x)| \leq \limsup_{y\to x} |f(y)|$$

(simply take $\tilde{f}(x) := f(x)$ in every Lebesgue point x of f and $\tilde{f}(x) := 0$ otherwise). Hence, (16) and (17) have to be interpreted as follows: whenever we work with values of the equivalence class $[f]$ then we mean the function values of the above representative $\tilde{f}$.

(ii) Notice that $F^1_{1,\infty}(\mathbb{R}^d)$ and $BV(\mathbb{R}^d)$ are incomparable.

(iii) Observe, as in case of the *Radial Lemma*, that (16) holds for $x \neq 0$.

As a preparation for Theorem 11 we shall characterize the traces of radial elements in $BV(\mathbb{R}^d)$. This seems to be of independent interest. For this reason we are forced to introduce weighted spaces of functions of bounded variation on the positive half axis. We denote by $\mathbb{R}^+$ the set $(0,\infty)$.

Definition 5 (i) *A function $\varphi \in C([0,\infty))$ belongs to $C^1_c([0,\infty))$ if it is continuously differentiable on $\mathbb{R}^+$, has compact support, satisfies $\varphi(0) = 0$ and $\lim_{t\to 0^+} \varphi'(t) = \varphi'(0) = \lim_{t\to 0^+} \frac{\varphi(t)}{t}$ exists and is finite.*

(ii) *A function $g \in L_1(\mathbb{R}^+, t^{d-1})$ is said to belong to $BV(\mathbb{R}^+, t^{d-1})$ if there is a signed Radon measure ν on $\mathbb{R}^+$ such that*

$$\int_0^\infty g(t)\,[\varphi(s)s^{d-1}]'(t)\,dt = -\int_0^\infty \varphi(t)\,t^{d-1}\,d\nu(t)\,, \qquad \forall \varphi \in C^1_c([0,\infty)) \tag{18}$$

and

$$\|\, g\,|BV(\mathbb{R}^+, t^{d-1})\| := \|\, g\,|L_1(\mathbb{R}^+, t^{d-1})\| + \int_0^\infty r^{d-1}\,d|\nu|(r) \tag{19}$$

is finite.

By using these new spaces we can prove the following trace theorem.

Theorem 12 *Let g be a measurable function on $\mathbb{R}^+$. Then* ext $g \in BV(\mathbb{R}^d)$ *if, and only if $g \in BV(\mathbb{R}^+, t^{d-1})$ and*

$$\|\ \mathrm{ext}\ g\,|BV(\mathbb{R}^d)\| \asymp \|\, g\,|BV(\mathbb{R}^+, t^{d-1})\,\|\,.$$

Spaces with $1 < p < \infty$

For $1 < p < \infty$ one could use interpolation between $p = 1$ and $p = \infty$ to obtain spaces with the decay rate $(d-1)/p$. The largest spaces with this respect are obtained by the real method. Let $M_p(\mathbb{R}^d) := (RL_\infty(\mathbb{R}^d), RBV(\mathbb{R}^d))_{\Theta,\infty}$, $\Theta = 1/p$. Then (13) holds for all elements $f \in M_p(\mathbb{R}^d)$. The disadvantage of these classes $M_p(\mathbb{R}^d)$ lies in the fact that elementary descriptions of $M_p(\mathbb{R}^d)$ are not known. However, at least some embeddings are known. From

$$RB^{1/p}_{p,1}(\mathbb{R}^d) = [RB^0_{\infty,1}(\mathbb{R}^d), RB^1_{1,1,}(\mathbb{R}^d)]_\Theta \hookrightarrow (RL_\infty(\mathbb{R}^d), RBV(\mathbb{R}^d))_{\Theta,\infty}\,, \qquad \Theta = 1/p\,,$$

(combine Proposition 1 with [1, Thm. 4.7.1]), we get back Theorem 10 (i), but only in case $1 < p < \infty$.

2.2.3 The behaviour of radial functions near the origin

At first we mention that the embedding relations with respect to $L_\infty(\mathbb{R}^d)$ do not change when we switch from $A^s_{p,q}(\mathbb{R}^d)$ to its radial subspace $RA^s_{p,q}(\mathbb{R}^d)$.

Lemma 3 (i) *The embedding $RB^s_{p,q}(\mathbb{R}^d) \hookrightarrow L_\infty(\mathbb{R}^d)$ holds if and only if either $s > d/p$ or $s = d/p$ and $q \le 1$.*
(ii) *The embedding $RF^s_{p,q}(\mathbb{R}^d) \hookrightarrow L_\infty(\mathbb{R}^d)$ holds if and only if either $s > d/p$ or $s = d/p$ and $p \le 1$.*

The explicit counterexamples will be given in Lemma 8 below. Hence, unboundedness can only happen in case $s \le d/p$.

Theorem 13 *Let $d \ge 2$, $0 < p < \infty$ and $0 < q \le \infty$.*
(i) *Suppose $(s,p,q) \in U(A)$ and $s < \frac{d}{p}$. Then there exists a constant c s.t.*

$$|x|^{\frac{d}{p}-s}\, |f(x)| \le c\, \| \, f \,|RA^s_{p,q}(\mathbb{R}^d)\| \tag{20}$$

holds for all $0 < |x| \le 1$ and all $f \in RA^s_{p,q}(\mathbb{R}^d)$.
(ii) *Let $\sigma_p(d) < s < d/p$. There exists a constant $c > 0$ such that for all x, $0 < |x| < 1$, there exists a smooth radial function $f \in RA^s_{p,q}(\mathbb{R}^d)$, $\| \, f \,|RA^s_{p,q}(\mathbb{R}^d)\| = 1$, s.t.*

$$|x|^{\frac{d}{p}-s}\, |f(x)| \ge c\,. \tag{21}$$

Remark 12 (i) In case of $RB^s_{p,\infty}(\mathbb{R}^d)$ we have a function which realizes the extremal behaviour for all $|x| < 1$ simultaneously. It is well-known, see e.g. [28, Lem. 2.3.1/1], that the function

$$f(x) := \psi(x)\, |x|^{\frac{d}{p}-s}\,, \qquad x \in \mathbb{R}^d\,,$$

belongs to $RB^s_{p,\infty}(\mathbb{R}^d)$, as long as $s > \sigma_p(d)$. This function does not belong to $RB^s_{p,q}(\mathbb{R}^d)$, $q < \infty$. Since it is also not contained in $RF^s_{p,q}(\mathbb{R}^d)$, $0 < q \le \infty$ we conclude that in these cases there is no function, which realizes this upper bound for all x simultaneously. In these cases the function f in (21) has to depend on x.
(ii) These estimates do not change by switching to the larger homogeneous spaces $R\dot{A}^s_{p,q}(\mathbb{R}^d)$ of Besov and Lizorkin-Triebel type. In case of $R\dot{H}^s(\mathbb{R}^d) = R\dot{F}^s_{p,2}(\mathbb{R}^d)$ this has been observed in a recent paper by Cho and Ozawa [7], see also Ni [25], Rother [27] and Kuzin, Pohozaev [22, 8.1]. The general case is treated in [32].
(iii) In the literature one can find various types of further inequalities for radial functions. Many times preference is given to a homogeneous context, see the inequalities (1) and (16) as examples. Then one has to deal with the behaviour at infinity and around the origin simultaneously. That would be not appropriate in the context of inhomogeneous spaces. Inequalities like (1) and (16) will be investigated systematically in [32]. However, let us refer to [38], [23], [25], [27], [22, 8.1] and [7]

for results in this direction. Sometimes also decay estimates are proved by replacing on the right-hand side the norm in the space $A^s_{p,q}(\mathbb{R}^d)$ $(\dot{A}^s_{p,q}(\mathbb{R}^d))$ by products of norms, e.g., $\| f|L_p(\mathbb{R}^d)\|^{1-\Theta} \| f\,|\dot{A}^s_{p,q}(\mathbb{R}^d)\|^{\Theta}$ for some $\Theta \in (0,1)$, see [23], [25], [27] and [7]. Here we will not deal with those modifications.

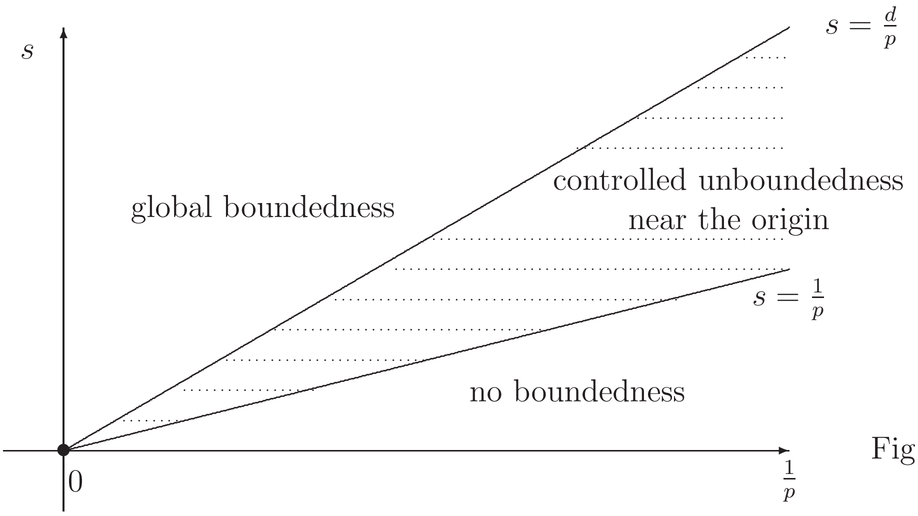

Fig. 3

Finally we have to investigate $s \le 1/p$ and $(s,p,q) \notin U(A)$.

Lemma 4 *Let $d \ge 2$, $0 < p < \infty$ and $0 < q \le \infty$. Suppose $(s,p,q) \notin U(A)$ and $\sigma_p(d) < 1/p$. Moreover let $\sigma_q(d) < 1/p$ in the F-case. Then there exists a radial function $f \in RA^s_{p,q}(\mathbb{R}^d)$, $\| f\,|RA^s_{p,q}(\mathbb{R}^d)\| = 1$, and a sequence $(x^j)_j \subset \mathbb{R}^d \setminus \{0\}$ s.t. $\lim_{j\to\infty} |x^j| = 0$ and f is unbounded in a neighborhood of all x^j.*

2.2.4 The behaviour of radial functions near the origin – borderline cases

Now we turn to the remaining limiting situation. We shall show that there is also controlled unboundedness near the origin if $s = d/p$ and $RA^{d/p}_{p,q}(\mathbb{R}^d) \not\subset L_\infty(\mathbb{R}^d)$.

Theorem 14 *Let $d \ge 2$, $0 < p < \infty$, $0 < q \le \infty$, and suppose $s = d/p$.*
(i) *Let $1 < q \le \infty$. Then there exists constant c s.t.*

$$(-\log|x|)^{-1/q'}\,|f(x)| \le c\,\| f\,|B^{d/p}_{p,q}(\mathbb{R}^d)\| \tag{22}$$

holds for all $0 < |x| \le 1/2$ and all $f \in RB^{d/p}_{p,q}(\mathbb{R}^d)$.
(ii) *Let $1 < p < \infty$. Then there exists constant c s.t.*

$$(-\log|x|)^{-1/p'}\,|f(x)| \le c\,\| f\,|F^{d/p}_{p,q}(\mathbb{R}^d)\| \tag{23}$$

holds for all $0 < |x| \le 1/2$ and all $f \in RF^{d/p}_{p,q}(\mathbb{R}^d)$.

Remark 13 Comparing Lemma 8 below and Theorem 14 we find the following. For the case $q = \infty$ in Theorem 14(i) the function $f_{1,0}$, see (61), realizes the extremal behaviour. In all other cases there remains a gap of order log log to some power.

3 Traces of radial subspaces – proofs

The main aim of this section is to prove Theorem 3. It expresses the fact that all information about a radial function is contained in its trace onto a straight line through the origin. However, a few things more will be done here. For later use one subsection is devoted to the study of interpolation of radial subspaces (Subsection 3.1) and another one is devoted to the study of certain test functions (Subsection 3.8).

3.1 Interpolation of radial subspaces

We mention two different results here, one with respect to the complex method and one with respect to the real method of interpolation.

3.1.1 Complex interpolation of radial subspaces

In [35] one of the authors has proved that in case $p, q \geq 1$ the spaces $RB^s_{p,q}(\mathbb{R}^d)$ ($RF^s_{p,q}(\mathbb{R}^d)$) are complemented subspaces of $B^s_{p,q}(\mathbb{R}^d)$ ($F^s_{p,q}(\mathbb{R}^d)$). By means of the method of retraction and coretraction, see, e.g., Theorem 1.2.4 in [39], this allows to transfer the interpolation formulas for the original spaces $B^s_{p,q}(\mathbb{R}^d)$ ($F^s_{p,q}(\mathbb{R}^d)$) to its radial subspaces. However, we prefer to quote a slightly more general result, proved in [33], concerning the complex method. It is based on the results on complex interpolation for Lizorkin-Triebel spaces from [14] and uses the method of [24] for an extension to the quasi-Banach space case.

Proposition 1 *Let* $0 < p_0, p_1 \leq \infty$, $0 < q_0$, $q_1 \leq \infty$, $s_0, s_1 \in \mathbb{R}$, *and* $0 < \Theta < 1$. *Define* $s := (1 - \Theta)\, s_0 + \Theta\, s_1$,

$$\frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1} \qquad \textit{and} \qquad \frac{1}{q} := \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}\,.$$

(i) *Let* $\max(p_0, q_0) < \infty$. *Then we have*

$$RB^s_{p,q}(\mathbb{R}^d) = \Big[RB^{s_0}_{p_0,q_0}(\mathbb{R}^d), RB^{s_1}_{p_1,q_1}(\mathbb{R}^d)\Big]_\Theta\,.$$

(ii) *Let* $p_1 < \infty$ *and* $\min(q_0, q_1) < \infty$. *Then we have*

$$RF^s_{p,q}(\mathbb{R}^d) = \Big[RF^{s_0}_{p_0,q_0}(\mathbb{R}^d), RF^{s_1}_{p_1,q_1}(\mathbb{R}^d)\Big]_\Theta\,.$$

3.1.2 Real Interpolation of radial subspaces

For later use we also formulate a result with respect to the real method of interpolation.

Proposition 2 *Let $d \geq 1$, $1 \leq q, q_0, q_1 \leq \infty$, $s_0, s_1 \in \mathbb{R}$, $s_0 \neq s_1$, and $0 < \Theta < 1$.*
(i) *Let $1 \leq p \leq \infty$. Then, with $s := (1-\Theta)\, s_0 + \Theta\, s_1$, we have*

$$RB^s_{p,q}(\mathbb{R}^d) = \Big(RB^{s_0}_{p,q_0}(\mathbb{R}^d), RB^{s_1}_{p,q_1}(\mathbb{R}^d)\Big)_{\Theta,q} .$$

(ii) *Let $1 \leq p < \infty$. Then, with $s := (1-\Theta)\, s_0 + \Theta\, s_1$, we have*

$$RB^s_{p,q}(\mathbb{R}^d) = \Big(RF^{s_0}_{p,q_0}(\mathbb{R}^d), RF^{s_1}_{p,q_1}(\mathbb{R}^d)\Big)_{\Theta,q} .$$

Proof. As mentioned above, the spaces $RB^s_{p,q}(\mathbb{R}^d)$ ($RF^s_{p,q}(\mathbb{R}^d)$) are complemented subspaces of $B^s_{p,q}(\mathbb{R}^d)$ ($F^s_{p,q}(\mathbb{R}^d)$), see [35]. Using the method of retraction and coretraction, see [40, 1.2.4], the above statements are consequences of the corresponding formulas without R, see e.g. [40, 2.4.2]. ∎

3.2 Proofs of the statements in Subsection 2.1.1

Let $m \in \mathbb{N}_0$. Then $C^m(\mathbb{R}^d)$ denotes the collection of all functions $f : \mathbb{R}^d \to \mathbb{C}$ such that all derivatives $D^\alpha f$ of order $|\alpha| \leq m$ exist, are uniformly continuous and bounded. We put

$$\| \, f \, |C^m(\mathbb{R}^d)\| := \sum_{|\alpha|\leq m} \| \, D^\alpha f \, |L_\infty(\mathbb{R}^d)\| \, .$$

By $RC^m(\mathbb{R}^d)$ we denote its subspace of radial functions.

Proof of Theorem 1

We shall use the following statement.

Let $m \in \mathbb{N}_0$, $a \in \mathbb{R}^d$ and $f \in C^m(\mathbb{R}^d) \cap C^{m+1}(\mathbb{R}^d \setminus \{a\})$. If

$$\lim_{x \to a} D^\alpha f(x)$$

exists and is finite for all $|\alpha| = m+1$, then $f \in C^{m+1}(\mathbb{R}^d)$.
Step 1. Proof in case $m \in \{0,1\}$. The case $m = 0$ is obvious. Hence, we deal with $m = 1$. Let $f \in RC^1(\mathbb{R}^d)$. Obviously,

$$\frac{\partial f}{\partial x_1}(x) = f_0'(t) \, , \qquad x = (x_1, 0, \ldots, 0) \, , \quad t = x_1 \, ,$$

which proves the estimate

$$\| \, \mathrm{tr} \, f \, |C^1(\mathbb{R})\| \leq \| \, f \, |C^1(\mathbb{R}^d)\| \tag{24}$$

and at the same time the continuity of the function $\mathrm{tr}\, f = f_0$ and its derivative.
Now we assume that $g \in RC^1(\mathbb{R})$. Let $f := \mathrm{ext}\; g$. If $x \neq 0$ we have

$$\frac{\partial f}{\partial x_1}(x) = g'(r)\, \frac{x_1}{r} \, , \qquad r = |x| > 0 \, . \tag{25}$$

This implies

$$\sup_{x\neq 0} |\frac{\partial f}{\partial x_1}(x)| \le \sup_{r>0} |g'(r)| = \sup_{t\in\mathbb{R}} |g'(t)| .$$

It remains to deal with the continuity of the derivative at the origin. Since g is even and continuously differentiable we know $g'(0) = 0$. This, together with (25) implies that $\lim_{x\to 0} \frac{\partial f}{\partial x_1}(x) = 0$. Using the *Statement*, this proves the claim for $m = 1$.

Step 2. We proceed by induction. Our induction hypothesis is as follows. If the assertion of Theorem 1 holds for m, then it holds also for $m+1$.

Substep 2.1. If $f \in RC^{m+1}(\mathbb{R}^d)$, then, of course, $f_0 = \mathrm{tr}\, f \in RC^{m+1}(\mathbb{R})$ and also the corresponding analogue of (24) follows immediately.

Substep 2.2. Now, let $g \in RC^{m+1}(\mathbb{R})$ and define $f := \mathrm{ext}\, g$. Then f is a radial function, which is $m+1$-times continuously differentiable on $\mathbb{R}^d \setminus \{0\}$. Therefore, it is enough to discuss the regularity properties of f in the origin and to prove the estimate

$$\| \mathrm{ext}\, g \,|C^{m+1}(\mathbb{R}^d)\| \lesssim \| g \,|C^{m+1}(\mathbb{R})\| . \tag{26}$$

First, let us state the following fact, which may be easily proved by induction. For every $n \in \mathbb{N}_0$ there is a constant $c > 0$ such that the function $r = r(x)$ satisfies

$$|D^\alpha r(x)| \le c r(x)^{1-|\alpha|} \tag{27}$$

for every multiindex $\alpha \in \mathbb{N}_0^d$ and all $x \in \mathbb{R}^d \setminus \{0\}$.

First we deal with a simplified situation. We assume that

$$g(0) = g'(0) = \cdots = g^{(m+1)}(0) = 0. \tag{28}$$

This clearly implies for $0 \le \ell \le m+1$

$$g^{(\ell)}(t) = o(|t|^{m+1-\ell}) \qquad \text{if} \quad t \to 0 . \tag{29}$$

Then, using chain rule and the estimates (27), (29) we find

$$\begin{aligned} |\,(D^\alpha f)(x)\,| \quad &\lesssim \quad \sum_{\ell=1}^{|\alpha|} |g^{(\ell)}(r)| \sum_{\beta^1+\ldots+\beta^\ell=\alpha} |D^{\beta^1} r(x)| \,\ldots\, |D^{\beta^\ell} r(x)| \\ &\lesssim \quad \sum_{\ell=1}^{|\alpha|} o(r^{m+1-\ell})\, r^{\ell-|\alpha|} = o(r^{m+1-|\alpha|}), \qquad r\downarrow 0\,, \end{aligned} \tag{30}$$

where $|\alpha| \le m+1$. This implies, that $\lim_{x\to 0} D^\alpha f(x) = 0$ for all α with $|\alpha| \le m+1$

Using the induction hypothesis we immediately get $D^\alpha f(0) = 0$ if $|\alpha| \le m+1$. Now let $|\alpha| = m+2$. For simplicity we concentrate on $\alpha = (m+2, 0, \ldots, 0)$. Then, as a consequence of (30), we find

$$\begin{aligned} &\frac{D^{(m+1,0,\ldots,0)} f(h,0,\ldots,0) - D^{(m+1,0,\ldots,0)} f(0,0,\ldots,0)}{h} \\ &\qquad = \quad \frac{D^{(m+1,0,\ldots,0)} f(h,0,\ldots,0)}{h} = o(1) \qquad \text{if} \quad h \to 0 . \end{aligned}$$

This yields $(D^{(m+2,0,\dots,0)}f(0) = 0$ and with the same type of argument $D^\alpha f(0) = 0$ for all derivatives of order $|\alpha| = m+2$. This proves the continuity of $D^\alpha f$, $|\alpha| \le m+2$, in the origin. Observe, that the inequality (26) follows as in (30) by using the chain rule.

Finally, we wish to remove the restriction (28). Suppose that m is even. Hence

$$g'(0) = g'''(0) = \ \dots\ = g^{(m+1)}(0) = 0,$$

but $g(0), g''(0), \dots, g^{(m+2)}(0)$ can be arbitrary. Let $\psi_0 = \mathrm{tr}\ \psi$. We introduce the function

$$h(t) := g(t) - g_1(t)\, \psi_0(t)\,, \qquad t \in \mathbb{R}\,,$$

where

$$g_1(t) := g(0) + \frac{g''(0)}{2!}t^2 + \dots + \frac{g^{(m+2)}(0)}{(m+2)!}t^{m+2}, \qquad t \in \mathbb{R}.$$

The extension of $g_1\, \psi_0$ is a radial function with compact support and continuous derivatives of arbitrary order. Furthermore, we have the obvious estimate

$$\begin{aligned} \|\ \mathrm{ext}\ g_1\, \psi_0\, |C^{m+2}(\mathbb{R}^d)\| &\le \sum_{j=0}^{\frac{m}{2}+1} \frac{|g^{(2j)}(0)|}{(2j)!}\, \|\, |x|^{2j}\, \psi(x)\, |C^{m+2}(\mathbb{R}^d)\| \\ &\lesssim \|\, g\, |C^{m+2}(\mathbb{R})\|\,. \end{aligned}$$

The function h satisfies (28). Hence, ext h belongs to $RC^{(m+2)}(\mathbb{R}^d)$ and

$$\begin{aligned} \|\ \mathrm{ext}\ h\, |C^{m+2}(\mathbb{R}^d)\| &\lesssim \|\, h\, |C^{m+2}(\mathbb{R})\| \\ &\lesssim (\|\, g\, |C^{m+2}(\mathbb{R})\| + \|\, g_1\, \psi_0\, |C^{m+2}(\mathbb{R})\|) \\ &\lesssim \|\, g\, |C^{m+2}(\mathbb{R})\|\,. \end{aligned}$$

This shows that ext $g =$ ext $h +$ ext $(g_1\, \psi_0) \in RC^{(m+2)}(\mathbb{R}^d)$ and addition we also get the estimate

$$\|\ \mathrm{ext}\ g\, |C^{m+2}(\mathbb{R}^d)\| \ \lesssim\ \|\, g\, |C^{m+2}(\mathbb{R})\|\,.$$

For odd m the proof is similar. ∎

Proof of Theorem 2

For tr $\in \mathcal{L}(B^s_{\infty,q}(\mathbb{R}^d), B^s_{\infty,q}(\mathbb{R}))$ we refer to [40, 2.7.2]. This immediately gives tr $\in \mathcal{L}(RB^s_{\infty,q}(\mathbb{R}^d), RB^s_{\infty,q}(\mathbb{R}))$. Concerning ext we argue by using real interpolation. Observe, that ext $\in \mathcal{L}(RC^m(\mathbb{R}), RC^m(\mathbb{R}^d))$ for all $m \in \mathbb{N}_0$, see Theorem 1. From the interpolation property of the real interpolation method we derive

$$\mathrm{ext}\ \in\ \mathcal{L}\Big((RC^m(\mathbb{R}), RC(\mathbb{R}))_{\Theta,q}, (RC^m(\mathbb{R}^d), RC(\mathbb{R}^d))_{\Theta,q}\Big)\,.$$

Using Proposition 2 the claim follows. ∎

3.3 Proofs of the assertions in Subsection 2.1.2

3.3.1 Proof of Lemma 1

Recall, for $f \in RL_p(\mathbb{R})$ we have

$$\int_{\mathbb{R}^d} |f(x)|^p \, dx = 2 \, \frac{\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty |f_0(r)|^p \, r^{d-1} \, dr \, .$$

Using

$$\int_0^\infty |f_0(r)|^p \, r^{d-1} \, dr = \lim_{\varepsilon \downarrow 0} \int_\varepsilon^\infty |f_0(r)|^p \, r^{d-1} \, dr \, ,$$

which implies the density of the test functions in $L_p([0,\infty), r^{d-1})$, we can read this formula also from the other side, it means

$$\int_{\mathbb{R}^d} |\operatorname{ext}\, g(x)|^p \, dx = 2 \, \frac{\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty |g(r)|^p \, r^{d-1} \, dr$$

for all $g \in L_p([0,\infty), r^{d-1})$. This proves (i). Part (ii) is obvious. ■

3.3.2 Characterizations of radial subspaces by atoms

As mentioned above our proof of the trace theorem relies on atomic decompositions of radial distributions on $\mathbb{R}^d$. We recall our characterizations of $RA^s_{p,q}(\mathbb{R}^d)$ from [31], see also [21].
In this paper we shall consider two different versions of atoms. They are not related to each other. We hope that it will be always clear fom the context with which type of atoms we are working. For the following definition of an atom we refer to [14] or [41, 3.2.2]. For an open set Q and $r > 0$ we put $r\,Q = \{x \in \mathbb{R}^d \, : \, \operatorname{dist}(x, Q) < r\}$. Observe that Q is always a subset of $r\,Q$ whatever r is.

Definition 6 *Let $s \in \mathbb{R}$ and let $1 \le p \le \infty$. Let L and M be integers such that $L \ge 0$ and $M \ge -1$. Let $Q \subset \mathbb{R}^d$ be an open connected set with* $\operatorname{diam} Q = r$.
(a) *A smooth function $a(x)$ is called an 1_L-atom centered in Q if*

$$\begin{aligned} \operatorname{supp} a \quad &\subset \quad \frac{r}{2} Q \, , \\ \sup_{y \in \mathbb{R}^d} |D^\alpha a(y)| \quad &\le \quad 1 \, , \qquad |\alpha| \le L \, . \end{aligned}$$

(b) *A smooth function $a(x)$ is called an $(s,p)_{L,M}$-atom centered in Q if*

$$\begin{aligned} \operatorname{supp} a \quad &\subset \quad \frac{r}{2} Q \, , \\ \sup_{y \in \mathbb{R}^d} |D^\alpha a(y)| \quad &\le \quad r^{s-|\alpha|-\frac{d}{p}} \, , \qquad |\alpha| \le L \\ \int_{\mathbb{R}^d} a(y)\, y^\alpha \, dy \quad &= \quad 0 \qquad |\alpha| \le M \, . \end{aligned}$$

Remark 14 If $M = -1$, then the interpretation is that no moment condition is required.

In [31] and [21] we constructed a regular sequence of coverings with certain special properties which we now recall. Consider the annuli (balls if $k = 0$)

$$P_{j,k} := \Big\{ x \in \mathbb{R}^d : \quad k\, 2^{-j} \le |x| < (k+1)\, 2^{-j} \Big\}, \qquad j = 0, 1, \ldots, \quad k = 0, 1, \ldots .$$

Then there is a sequence $(\Omega_j)_{j=0}^\infty = ((\Omega_{j,k,\ell})_{k,\ell})_{j=0}^\infty$ of coverings of $\mathbb{R}^d$ such that

(a) all $\Omega_{j,k,\ell}$ are balls with center in $x_{j,k,\ell}$ s.t. $x_{j,0,1} = 0$ and $|x_{j,k,\ell}| = 2^{-j}(k + 1/2)$ if $k \ge 1$;

(b) $\operatorname{diam} \Omega_{j,k,\ell} = 12 \cdot 2^{-j}$ for all k and all ℓ;

(c) $P_{j,k} \subset \bigcup_{\ell=1}^{C(d,k)} \Omega_{j,k,\ell}\,, \quad j = 0, 1, \ldots, \quad k = 0, 1, \ldots,$ where the numbers $C(d,k)$ satisfy the relations $C(d,k) \le (2k+1)^{d-1}$, $C(d,0) = 1$.

(d) the sums $\sum_{k=0}^\infty \sum_{\ell=1}^{C(d,k)} \mathcal{X}_{j,k,\ell}(x)$ are uniformly bounded in $x \in \mathbb{R}^d$ and $j = 0, 1, \ldots$ (here $\mathcal{X}_{j,k,\ell}$ denotes the characteristic function of $\Omega_{j,k,\ell}$);

(e) $\Omega_{j,k,\ell} = \{x \in \mathbb{R}^d : 2^j x \in \Omega_{0,k,\ell}\}$ for all j, k and ℓ;

(f) There exists a natural number K (independent of j and k) such that

$$\{(x_1, 0, \ldots, 0) : \quad x_1 \in \mathbb{R}\} \cap \frac{\operatorname{diam}(\Omega_{j,k,\ell})}{2}\, \Omega_{j,k,\ell} = \emptyset \qquad \text{if} \quad \ell > K \tag{31}$$

(with an appropriate enumeration).

We collect some properties of related atomic decompositions. To do this it is convenient to introduce some sequence spaces.

Definition 7 *Let* $0 < q \le \infty$.
(i) *If* $0 < p \le \infty$, *then we define*

$$b_{p,q,d} := \Bigg\{ s = (s_{j,k})_{j,k} : \quad \| s \,|b_{p,q,d}\| = \Bigg(\sum_{j=0}^\infty \Bigg(\sum_{k=0}^\infty (1+k)^{d-1} \, |s_{j,k}|^p \Bigg)^{q/p} \Bigg)^{1/q} < \infty \Bigg\}.$$

(ii) *By* $\widetilde{\chi}_{j,k}$ *we denote the characteristic function of the set* $P_{j,k}$. *If* $0 < p < \infty$ *we define*

$$\begin{aligned} f_{p,q,d} \ &:= \ \Bigg\{ s = (s_{j,k})_{j,k} : \\ \| s \,|f_{p,q,d}\| \ &= \ \Bigg\| \Bigg(\sum_{j=0}^\infty \sum_{k=0}^\infty |s_{j,k}|^q \, 2^{\frac{jdq}{p}} \, \widetilde{\chi}_{j,k}(\cdot) \Bigg)^{1/q} \Big| L_p(\mathbb{R}^d) \Bigg\| < \infty \Bigg\}. \end{aligned}$$

Remark 15 Observe $b_{p,p,d} = f_{p,p,d}$ in the sense of equivalent quasi-norms.

Atoms have to satisfy moment and regularity conditions. With this respect we suppose

$$L \geq \max(0, [s] + 1) \, , \qquad M \geq \max([\sigma_p(d) - s], -1) \tag{32}$$

in case of Besov spaces and

$$L \geq \max(0, [s] + 1) \, , \qquad M \geq \max([\sigma_{p,q}(d) - s], -1) \tag{33}$$

in case of Lizorkin-Triebel spaces. Under these restrictions the following assertions are known to be true:

(i) Each $f \in RB^s_{p,q}(\mathbb{R}^d)$ ($f \in RF^s_{p,q}(\mathbb{R}^d)$) can be decomposed into

$$f = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=1}^{C(d,k)} s_{j,k} \, a_{j,k,\ell} \qquad (\text{convergence in} \quad \mathcal{S}'(\mathbb{R}^d)\,), \tag{34}$$

where the functions $a_{j,k,\ell}$ are $(s,p)_{L,M}$-atoms with respect to $\Omega_{j,k,\ell}$ ($j \geq 1$), and the functions $a_{0,k,\ell}$ are 1_L-atoms with respect to $\Omega_{0,k,\ell}$.

(ii) Any formal series $\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=1}^{C(d,k)} s_{j,k} \, a_{j,k,\ell}$ converges in $\mathcal{S}'(\mathbb{R}^d)$ with limit in $B^s_{p,q}(\mathbb{R}^d)$ if the sequence $s = (s_{j,k})_{j,k}$ belongs to $b_{p,q,d}$ and if the $a_{j,k,\ell}$ are $(s,p)_{L,M}$-atoms with respect to $\Omega_{j,k,\ell}$ ($j \geq 1$), and the $a_{0,k,\ell}$ are 1_L-atoms with respect to $\Omega_{0,k,\ell}$. There exists a universal constant such that

$$\| \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=1}^{C(d,k)} s_{j,k} \, a_{j,k,\ell} \, | B^s_{p,q}(\mathbb{R}^d) \| \leq c \, \| \, s \, | b_{p,q,d} \| \tag{35}$$

holds for all sequences $s = (s_{j,k})_{j,k}$.

(iii) There exists a constant c such that for any $f \in RB^s_{p,q}(\mathbb{R}^d)$ there exists an atomic decomposition as in (34) satisfying

$$\| \, (s_{j,k})_{j,k} \, | b_{p,q,d} \| \leq c \, \| \, f \, | B^s_{p,q}(\mathbb{R}^d) \| \, . \tag{36}$$

(iv) The infimum on the left-hand side in (35) with respect to all admissible representations (34) yields an equivalent norm on $RB^s_{p,q}(\mathbb{R}^d)$.

(v) Any formal series $\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=1}^{C(d,k)} s_{j,k} \, a_{j,k,\ell}$ converges in $\mathcal{S}'(\mathbb{R}^d)$ with limit in $F^s_{p,q}(\mathbb{R}^d)$ if the sequence $s = (s_{j,k})_{j,k}$ belongs to $f_{p,q,d}$ and if the functions $a_{j,k,\ell}$ are $(s,p)_{L,M}$-atoms with respect to $\Omega_{j,k,\ell}$ ($j \geq 1$), and the functions $a_{0,k,\ell}$

are 1_L-atoms with respect to $\Omega_{0,k,\ell}$. There exists a universal constant such that

$$\| \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=1}^{C(d,k)} s_{j,k}\, a_{j,k,\ell}\, |F^s_{p,q}(\mathbb{R}^d)\| \le c \, \| \, s \, |f_{p,q,d}\| \qquad (37)$$

holds for all sequences $s = (s_{j,k})_{j,k}$.

(vi) There exists a constant c such that for any $f \in RF^s_{p,q}(\mathbb{R}^d)$ there exists an atomic decomposition as in (34) satisfying

$$\| \, (s_{j,k})_{j,k} \, |f_{p,q,d}\| \le c \, \| \, f \, |F^s_{p,q}(\mathbb{R}^d)\| \, . \qquad (38)$$

(vii) The infimum on the left-hand side in (37) with respect to all admissible representations (34) yields an equivalent norm on $RF^s_{p,q}(\mathbb{R}^d)$. Such decompositions as in (36) and (38) we shall call optimal.

Remark 16 For proofs of all these facts (even with respect to more general decompositions of $\mathbb{R}^d$) we refer to [31] and [35]. A different approach to atomic decompositions of radial subspaces has been given by Epperson and Frazier [10].

3.3.3 Proof of Theorem 3

Step 1. Let $f \in RB^s_{p,q}(\mathbb{R}^d)$. Then there exists an optimal atomic decomposition, i.e.

$$f \; = \; \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=1}^{C(d,k)} s_{j,k}\, a_{j,k,\ell} \qquad (39)$$

$$\| \, f \, |B^s_{p,q}(\mathbb{R}^d)\| \; \asymp \; \| \, (s_{j,k})_{j,k} \, |b_{p,q,d}\| \, ,$$

see (34) - (36). Since f is even we obtain

$$f(x) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=1}^{C(d,k)} s_{j,k} \, \frac{a_{j,k,\ell}(x) + a_{j,k,\ell}(-x)}{2} \, . \qquad (40)$$

We define

$$g_{j,k,\ell}(t) := 2^{j(s-d/p)} \Big(\mathrm{tr} \; \frac{a_{j,k,\ell}(\,\cdot\,) + a_{j,k,\ell}(-\,\cdot\,)}{2} \Big)(t) \, , \qquad t \in \mathbb{R} \, ,$$

and $d_{j,k} := 2^{-j(s-d/p)}\, s_{j,k}$. Of course, $a_{j,k,\ell}(\,\cdot\,) + a_{j,k,\ell}(-\,\cdot\,)$ is not a radial function. But it is an even and continuous. So, tr means simply the restriction to the x_1-axis. Clearly,

$$f_N(x) := \sum_{j=0}^{N} \sum_{k=0}^{N} \sum_{\ell=1}^{C(d,k)} s_{j,k} \, \frac{a_{j,k,\ell}(x) + a_{j,k,\ell}(-x)}{2} \, , \qquad x \in \mathbb{R}^d, \quad N \in \mathbb{N} \, ,$$

is an even (not necessarily radial) function in $C^L(\mathbb{R}^d)$. By means of property (f) of the particular coverings of $\mathbb{R}^d$, stated in the previous subsection, we obtain

$$\mathrm{tr}\ f_N = \sum_{j=0}^{N}\sum_{k=0}^{N}\sum_{\ell=1}^{\min(C(d,k),K)} d_{j,k}\, g_{j,k,\ell}$$

(here K is the natural number in (31)). Furthermore

$$\max_{0\le n\le L}\ \sup_{t\in\mathbb{R}}\ |(g_{j,k,\ell})^{(n)}(t)| \le 12^{s-d/p}\, 2^{jn}\,.$$

Obviously

$$\begin{aligned}\|\,(s_{j,k})_{j,k}\,|b_{p,q,d}\| &= \Bigg(\sum_{j=0}^{\infty}\bigg(\sum_{k=0}^{\infty}(1+k)^{d-1}\,|s_{j,k}|^p\bigg)^{q/p}\Bigg)^{1/q}\\ &= \Bigg(\sum_{j=0}^{\infty}2^{j(s-\frac{d}{p})q}\bigg(\sum_{k=0}^{\infty}(1+k)^{d-1}\,|d_{j,k}|^p\bigg)^{q/p}\Bigg)^{1/q}\,.\end{aligned}$$

This implies

$$\|\ \mathrm{tr}\ f_N\,|TB^s_{p,q}(\mathbb{R},L,d)\| \le K\,c\,\|\,f\,|B^s_{p,q}(\mathbb{R}^d)\|$$

where c and K are independent of f and N.
Next we comment on the convergence of the sequences $(f_N)_N$ and $(\mathrm{tr}\ f_N)_N$. Of course, f_N converges in $\mathcal{S}'(\mathbb{R}^d)$ against f. For the investigation of the convergence of $(\mathrm{tr}\ f_N)_N$ we choose s' such that $s > s' > \sigma_p(d)$ and conclude

$$\begin{aligned}\|\ \mathrm{tr}\ f_N - \mathrm{tr}\ f_M\,|TB^{s'}_{p,p}(\mathbb{R},L,d)\| &\lesssim \Bigg\|\sum_{j=M+1}^{N}\sum_{k=0}^{N}\sum_{\ell=1}^{\min(C(d,k),K)} d_{j,k}\,g_{j,k,\ell}\,\Bigg|TB^{s'}_{p,p}(\mathbb{R})\Bigg\|\\ &\quad + \Bigg\|\sum_{j=0}^{N}\sum_{k=M+1}^{N}\sum_{\ell=1}^{\min(C(d,k),K)} d_{j,k}\,g_{j,k,\ell}\,\Bigg|TB^{s'}_{p,p}(\mathbb{R})\Bigg\|\\ &\lesssim \Big(\sum_{j=M+1}^{\infty}\sum_{k=0}^{\infty}(1+k)^{d-1}|2^{j(s'-s)}\,s_{j,k}|^p\Big)^{1/p}\\ &\quad + \Big(\sum_{j=0}^{\infty}\sum_{k=M+1}^{\infty}(1+k)^{d-1}|2^{j(s'-s)}\,s_{j,k}|^p\Big)^{1/p}\Big)\,,\end{aligned}$$

by taking into account the different normalization of the atoms in $RB^{s'}_{p,p}(\mathbb{R}^d)$ and in $RB^{s}_{p,q}(\mathbb{R}^d)$, respectively. The right-hand side in the previous inequality tends to zero if M tends to infinity since $\|\,(s_{j,k})_{j,k}\,|b_{p,q,d}\| < \infty$. Lemma 1 in combination with $B^s_{p,q}(\mathbb{R}^d) \subset L_p(\mathbb{R}^d)$ implies the continuity of $\mathrm{tr} : RB^s_{p,1}(\mathbb{R}^d) \to L_{\max(1,p)}(\mathbb{R}, t^{d-1})$ as well as the existence of $\mathrm{tr}\ f \in L_{\max(1,p)}(\mathbb{R}, |t|^{d-1})$. Consequently

$$\lim_{N\to\infty}\mathrm{tr}\ f_N = \mathrm{tr}\,(\lim_{N\to\infty} f_N) = \mathrm{tr}\ f$$

with convergence in $L_{\max(1,p)}(\mathbb{R},|t|^{d-1})$. This proves that tr maps $RB^s_{p,q}(\mathbb{R}^d)$ into $TB^s_{p,q}(\mathbb{R},L,d)$ if L satisfies (32). Observe, that M can be chosen as -1.
Step 2. The same type of arguments proves that tr maps $RF^s_{p,q}(\mathbb{R}^d)$ into $TF^s_{p,q}(\mathbb{R})$, in particular the convergence analysis is the same. Furthermore, observe

$$\Big\| \Big(\sum_{j=0}^{\infty} \quad 2^{jsq} \quad \sum_{k=0}^{\infty} |s_{j,k}|^q \, \chi^{\#}_{j,k}(\cdot) \Big)^{1/q} |L_p(\mathbb{R}, t^{d-1}) \Big\|$$
$$= \quad c_d \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} \sum_{k=0}^{\infty} |s_{j,k}|^q \, \widetilde{\chi}_{j,k}(\cdot) \Big)^{1/q} |L_p(\mathbb{R}^d) \Big\| .$$

This proves that tr maps $RF^s_{p,q}(\mathbb{R}^d)$ into $TF^s_{p,q}(\mathbb{R},L,d)$ if L satisfies (33) (again we use $M=-1$).
Step 3. Properties of ext . Let g be an even function with a decomposition as in (2) and

$$\| \, g \, |TB^s_{p,q}(\mathbb{R},L,d)\| \asymp \| \, (s_{j,k}) \, |b^s_{p,q,d}\| \, .$$

We define

$$a_{j,k}(x) := g_{j,k}(|x|) \, , \qquad x \in \mathbb{R}^d \, .$$

The functions $a_{j,k}$ are compactly supported, continuous, and radial. Obviously

$$\mathrm{supp}\ a_{j,k} \subset \{x : \quad 2^{-j}k - 2^{-j-1} \le |x| \le 2^{-j}(k+1) + 2^{-j-1}\} \, , \qquad k \in \mathbb{N} \, ,$$

and

$$\mathrm{supp}\ a_{j,0} \subset \{x : \quad |x| \le 3 \, \cdot \, 2^{-j-1}\} \, .$$

From Theorem 1 we derive

$$|D^\alpha a_{j,k}(x)| \le \| \, a_{j,k} \, |C^{|\alpha|}(\mathbb{R}^d)\| \ \lesssim \ \| \, g_{j,k} \, |C^{|\alpha|}(\mathbb{R})\| \ \lesssim \ 2^{j|\alpha|} \, , \tag{41}$$

if $|\alpha| \le L$. Here the constants behind $\lesssim$ do not depend on j,k and $g_{j,k}$. We continue with an investigation of the sequence

$$h_N(x) := \sum_{j=0}^{N} \sum_{k=0}^{\infty} s_{j,k} \, a_{j,k}(x) \, , \qquad x \in \mathbb{R}^d \, , \quad N \in \mathbb{N} \, . \tag{42}$$

Related to our decomposition $(\Omega_{j,k,\ell})_{j,k,\ell}$ of $\mathbb{R}^d$, see Subsection 3.3.2, there is a sequence of decompositions of unity $(\psi_{j,k,\ell})_{j,k,\ell}$, i.e.

$$\sum_{k=0}^{\infty} \sum_{\ell=1}^{C(d,k)} \psi_{j,k,\ell}(x) \quad = \quad 1 \qquad \text{for all} \quad x \in \mathbb{R}^d \, , \quad j = 0,1,\ldots \, , \tag{43}$$
$$\mathrm{supp}\ \psi_{j,k,\ell} \quad \subset \quad \Omega_{j,k,\ell} \, , \tag{44}$$
$$|D^\alpha \psi_{j,k,\ell}| \quad \le \quad C_L \, 2^{j|\alpha|} \qquad |\alpha| \le L \, , \tag{45}$$

see [31]. Hence

$$
\begin{aligned}
h_N(x) &= \sum_{j=0}^{N}\sum_{k=0}^{\infty} s_{j,k}\, a_{j,k}(x) \left(\sum_{m=0}^{\infty}\sum_{\ell=1}^{C(d,m)} \psi_{j,m,\ell}(x)\right) \\
&= \sum_{j=0}^{N}\sum_{m=0}^{\infty} \left(\sum_{\ell=1}^{C(d,m)}\sum_{k=-7}^{7} s_{j,m+k}\, a_{j,m+k}(x)\psi_{j,m,\ell}(x)\right) \\
&= \sum_{j=0}^{N}\sum_{m=0}^{\infty} \lambda_{j,m} \sum_{\ell=1}^{C(d,k+m)} e_{j,m,\ell}(x)\, .
\end{aligned}
$$

where

$$
\begin{aligned}
\lambda_{j,m} &:= 2^{j(s-\frac{d}{p})} \max_{-1\le k\le 1} |s_{j,m+k}| \\
e_{j,m,\ell}(x) &:= 2^{-j(s-\frac{d}{p})}\, t_{j,m} \sum_{k=-7}^{7} s_{j,m+k}\, a_{j,m+k}(x)\, \psi_{j,m,\ell}(x) \\
t_{j,m} &:= \begin{cases} 1 & \text{if} \quad \max\limits_{k=-7,\ldots,7} |s_{j,m+k}| = 0, \\ \\ (\max\limits_{k=-7,\ldots,7} |s_{j,m+k}|)^{-1} & \text{otherwise}\, . \end{cases}
\end{aligned}
$$

We claim that the functions $e_{j,m,\ell}$ are $(s,p)_{L,-1}$-atoms (1_L-atoms if $j=0$) on $\mathbb{R}^d$ related to the covering $(\Omega_{j,k,\ell})_{j,k,\ell}$ (up to a universal constant). But this follows immediately from (44), (45), and (41). Finally we show that the sequence $\lambda = (\lambda_{j,m})_{j,m}$ belongs to $b_{p,q,d}$. The estimate

$$
\begin{aligned}
\|\,\lambda\,|b_{p,q,d}\| &= \Big(\sum_{j=0}^{N}\Big(\sum_{m=0}^{\infty}(1+m)^{d-1}\,|\lambda_{j,m}|^p\Big)^{q/p}\Big)^{1/q} \\
&\lesssim \Big(\sum_{j=0}^{N} 2^{j(s-\frac{d}{p})q}\Big(\sum_{m=0}^{\infty}(1+m)^{d-1}\,|s_{j,m}|^p\Big)^{q/p}\Big)^{1/q}
\end{aligned}
$$

is obvious. Hence ext maps $TB^s_{p,q}(\mathbb{R},L,d)$ into $RB^s_{p,q}(\mathbb{R}^d)$. Here we need that the pair $(L,-1)$ satisfies (32).
Step 4. The proof of the F-case is similar. Here we need that the pair $(L,-1)$ satisfies (33). The proof is complete. ■

3.3.4 Proof of Theorem 4

Since $RF^s_{p,q}(\mathbb{R}^d) \hookrightarrow RB^s_{p,\infty}(\mathbb{R}^d)$ it will be enough to deal with radial Besov spaces.
Step 1. Let $1 \le p < \infty$. Then $\sigma_p(d) = \sigma_p(1) = 0$. From $s > 0$ we derive $B^s_{p,q}(\mathbb{R}^d) \subset L_p(\mathbb{R}^d)$. Hence f is a regular distribution.

Step 2. Let $0 < p < 1$. Since $0 \notin \operatorname{supp} f$ there exists some $\varepsilon > 0$ s.t. the ball with radius ε and centre in the origin has an empty intersection with $\operatorname{supp} f$. Let $\lambda > 0$. Since f is a regular distribution if, and only if $f(\lambda\,\cdot\,)$ is a regular distribution we may assume $\varepsilon = 2$. Let $\varphi \in RC^\infty(\mathbb{R}^d)$ be a function s.t. $\varphi(x) = 1$ if $|x| \ge 2$ and $\varphi(x) = 0$ if $|x| \le 1$. Again we shall work with an optimal atomic decomposition of $f \in RB^s_{p,q}(\mathbb{R}^d)$, see (39). Obviously

$$f = f\varphi = \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{\ell=1}^{C(d,k)} s_{j,k}\,(\varphi\, a_{j,k,\ell})\,.$$

By checking the various support conditions we obtain

$$f = \sum_{j=0}^3 s_{j,0}\,(\varphi\, a_{j,0,1}) + \sum_{j=0}^\infty \sum_{k=\max(1,2^j-9)}^\infty \sum_{\ell=1}^{C(d,k)} s_{j,k}\,(\varphi\, a_{j,k,\ell})\,.$$

We consider the following splitting

$$\begin{aligned} f_1(x) &:= \sum_{j=0}^3 s_{j,0}\,\varphi(x)\,\frac{a_{j,0,1}(x) + a_{j,0,1}(-x)}{2} \\ f_2(x) &:= \sum_{j=0}^\infty \sum_{k=\max(1,2^j-9)}^\infty \sum_{\ell=1}^{C(d,k)} s_{j,k}\,\varphi(x)\,\frac{a_{j,k,\ell}(x) + a_{j,k,\ell}(-x)}{2}\,. \end{aligned}$$

Concerning the first part f_1 we observe that $\operatorname{tr} f_1$ is a compactly supported even C^L function. Now we concentrate on f_2. Let

$$f_{2,N}(x) := \sum_{j=0}^N \sum_{k=\max(1,2^j-9)}^{2^N} \sum_{\ell=1}^{C(d,k)} s_{j,k}\,\varphi(x)\,\frac{a_{j,k,\ell}(x) + a_{j,k,\ell}(-x)}{2}\,, \qquad N \in \mathbb{N}\,.$$

We put

$$g_{j,k,\ell}(t) := \operatorname{tr}\Big(\varphi(\,\cdot\,)\,\frac{a_{j,k,\ell}(\,\cdot\,) + a_{j,k,\ell}(-\,\cdot\,)}{2}\Big)(t)\,, \qquad t \in \mathbb{R}\,,$$

by using the same convention concerning tr as in Step 1 of the proof of Thm. 3. Since

$$\sup_{t\in\mathbb{R}} |g^{(n)}_{j,k,\ell}(t)| \le c_\varphi\,(12\,\cdot\,2^{-j})^{s-n-d/p}\,, \qquad 0 \le n \le L\,,$$

and $|\operatorname{supp} g_{j,k,\ell}| \lesssim 2^{-j}$ we obtain for a natural number m

$$\begin{aligned} \int_m^{m+1} |\operatorname{tr} f_{2,N}(t)|\,dt &\lesssim \sum_{j=0}^N \sum_{k=\max(1,2^j-9)}^{2^N} \sum_{\ell=1}^{\min(C(d,k),K)} |s_{j,k}| \int_m^{m+1} |g_{j,k,\ell}(t)|\,dt \\ &\lesssim \sum_{j=0}^N 2^{-j}\,2^{-j(s-d/p)} \sum_{k=2^jm-9}^{2^j(m+1)+6} |s_{j,k}| \\ &\lesssim m^{-(d-1)/p} \sum_{j=0}^N 2^{-j}\,2^{-j(s-d/p)}\,2^{-j(d-1)/p} \Big(\sum_{k=2^jm-9}^{2^j(m+1)+6} (1+k)^{d-1}|s_{j,k}|^p\Big)^{1/p}\,. \end{aligned}$$

Hence

$$\begin{aligned}\| \operatorname{tr} f_N \,|L_1(\mathbb{R})\| &= 2\int_1^\infty |\operatorname{tr} f_N(t)|\, dt \lesssim \| (s_{j,k})_{j,k} \,|b_{p,\infty,d}\| \sum_{m=1}^\infty m^{-(d-1)/p}\\ &\lesssim \| f \,|B^s_{p,q}(\mathbb{R}^d)\|\end{aligned}$$

since $s>0$ and $0<p<1$. Let $M \le N$. Then the same type of argument yields

$$\begin{aligned}\int_m^{m+1} |\operatorname{tr} f_{2,N}(t) &- \operatorname{tr} f_{2,M}(t)|\, dt\\ &\lesssim \sum_{j=M+1}^{N} \sum_{k=\max(1,2^j-9)}^{2^N} \sum_{\ell=1}^{\min(C(d,k),K)} |s_{j,k}| \int_m^{m+1} |g_{j,k,\ell}(t)|\, dt\\ &\quad + \sum_{j=0}^{N} \sum_{k=\max(2^M,2^jm-9)}^{2^N} \sum_{\ell=1}^{\min(C(d,k),K)} |s_{j,k}| \int_m^{m+1} |g_{j,k,\ell}(t)|\, dt\\ &\lesssim 2^{-Ms} \sup_{j=M+1,\ldots} \sum_{k=2^jm-9}^{2^j(m+1)+6} |s_{j,k}|\\ &\quad + \sum_{j=0}^{N} 2^{-j}\, 2^{-j(s-d/p)} \sum_{k=\max(2^M,2^jm-9)}^{2^j(m+1)+6} |s_{j,k}|\\ &\lesssim m^{-(d-1)/p} \Big(2^{-Ms} \| (s_{j,k})_{j,k} \,|b_{p,\infty,d}\|\\ &\qquad + \sup_{j=0,1,\ldots} \Big(\sum_{k=\max(2^M,2^jm-9)}^{2^j(m+1)+6} (1+k)^{d-1} |s_{j,k}|^p \Big)^{1/p} \Big) .\end{aligned}$$

Since

$$\lim_{M\to\infty} \sup_{j=0,1,\ldots} \Big(\sum_{k=\max(2^M,2^jm-9)}^{2^j(m+1)+6} (1+k)^{d-1} |s_{j,k}|^p \Big)^{1/p} = 0$$

for all $m \in \mathbb{N}$ we conclude

$$\| \operatorname{tr} f_{2,N} - \operatorname{tr} f_{2,M} \,|L_1(\mathbb{R})\| \longrightarrow 0 \qquad \text{if} \quad M \to \infty .$$

Hence

$$\sum_{j=0}^\infty \sum_{k=\max(1,2^j-9)}^{\infty} \sum_{\ell=1}^{\min(C(d,k),K)} s_{j,k}\, g_{j,k,\ell} \in L_1(\mathbb{R}) .$$

Let $\theta \in \mathbb{R}^d$, $|\theta| = 1$. We denote by Tr_θ the restriction of a continuous function to the line $\Theta := \{t\,\theta : \quad t \in \mathbb{R}\}$. Now we repeat, what we have done with respect to the x_1-axis, for such a line. As the outcome we obtain

$$\operatorname{Tr}_\theta \Big(\sum_{j=0}^{3} s_{j,0}\, (\varphi\, a_{j,0,1}) + \sum_{j=0}^{N} \sum_{k=\max(1,2^j-9)}^{2^N} \sum_{\ell=1}^{C(d,k)} s_{j,k}\, (\varphi\, a_{j,k,\ell}) \Big) , \qquad N \in \mathbb{N} ,$$

is a Cauchy sequence in $L_1(\Theta)$ and the limit satisfies

$$\begin{aligned}\Big\| \operatorname{Tr}_\theta \Big(\sum_{j=0}^{3} s_{j,0}\,(\varphi\, a_{j,0,1}) \quad &+ \quad \sum_{j=0}^{\infty} \sum_{k=\max(1,2^j-9)}^{\infty} \sum_{\ell=1}^{C(d,k)} s_{j,k}\,(\varphi\, a_{j,k,\ell})\Big) \Big| L_1(\Theta) \Big\| \\ &\lesssim \quad \|\, f\,|B^s_{p,q}(\mathbb{R}^d)\|\end{aligned}$$

with a constant independent of θ (of course, here, by a slight abuse of notation, Tr_θ denotes the continuous extension of the previously defined mapping). Using spherical coordinates this yields

$$\begin{aligned}\int_{\mathbb{R}^d} |f(x)|\, dx \quad &= \quad \int_{|\theta|=1} \int_0^\infty \; |f(t\,\theta)|\, dt\, d\theta \\ &\lesssim \quad \|\, f\,|B^s_{p,q}(\mathbb{R}^d)\|\,.\end{aligned}$$

But this means f is a regular distribution. ■

Remark 17 We have proved a bit more than stated. Under the given restrictions the pointwise trace tr f of a distribution $f \in RB^s_{p,q}(\mathbb{R}^d)$, $0 \not\in$ supp f, makes sense and belongs to $L_1(\mathbb{R})$.

3.3.5 Proof of Remark 3

We shall argue by using the wavelet characterization of $B^{\frac{1}{p}-1}_{p,\infty}(\mathbb{R}^d)$, see, e.g., [43, Thm. 1.20]. Let ϕ denote an appropriate univariate scaling function and Ψ an associated Daubechies wavelet of sufficiently high order. The tensor product ansatz yields $(d-1)$ generators $\Psi_1, \ldots, \Psi_{2^d-1}$ for the wavelet basis in $L_2(\mathbb{R}^d)$. Let Φ denote the d-fold tensor product of the univariate scaling function. We shall use the abbreviations

$$\Phi_k(x) := \Phi(x-k)\,, \qquad k \in \mathbb{Z}^d\,,$$

and

$$\Psi_{i,j,k}(x) := 2^{jd/2}\,\Psi_i(2^j x - k)\,, \qquad k \in \mathbb{Z}^d\,, \quad j \in \mathbb{N}_0\,, \quad i = 1, \ldots, 2^d-1\,.$$

An equivalent norm in $B^{\frac{1}{p}-1}_{p,\infty}(\mathbb{R}^d)$ is given by

$$\|\, f\,|B^{\frac{1}{p}-1}_{p,\infty}(\mathbb{R}^d)\| = \Big(\sum_{k\in\mathbb{Z}^d} |\langle f, \Phi_k\rangle|^p\Big)^{1/p} + \sup_{j=0,1,\ldots} 2^{j(\frac{1}{p}-1+d(\frac{1}{2}-\frac{1}{p}))} \Big(\sum_{i=1}^{2^d-1}\sum_{k\in\mathbb{Z}^d} |\langle f, \Psi_{i,j,k}\rangle|^p\Big)^{1/p}\,.$$

Daubechies wavelets have compact support. This implies

$$\operatorname{supp}\, \Psi_{i,j,k} \subset C\,\{x \in \mathbb{R}^d:\; 2^{-j}(k_\ell - 1) \le x_\ell \le 2^{-j}(k_\ell+1)\,,\; \ell = 1, \ldots d\}$$

and

$$\operatorname{supp}\, \Phi_k \subset C\,\{x \in \mathbb{R}^d:\; (k_\ell - 1) \le x_\ell \le (k_\ell+1)\,,\; \ell = 1, \ldots d\}$$

for an appropriate $C > 1$. By employing these relations we conclude that for fixed j the cardinality of the set of those functions $\Psi_{i,j,k}$, which do not vanish identically on $|x| = 1$ is $\lesssim 2^{j(d-1)}$. There is the general estimate

$$|\langle f, \Psi_{i,j,k}\rangle| = \Big| \int_{|x|=1} 2^{jd/2}\, \Psi_i(2^j x - k)\, dx \Big| \lesssim 2^{jd/2}\, 2^{-j(d-1)},$$

by using the information on the size of the support. Inserting this we find

$$\begin{aligned} 2^{j(\frac{1}{p}-1+d(\frac{1}{2}-\frac{1}{p}))} \Big(\sum_{i=1}^{2^d-1} \sum_{k\in\mathbb{Z}^d} |\langle f, \Psi_{i,j,k}\rangle|^p \Big)^{1/p} &\lesssim 2^{j(\frac{1}{p}-1+d(\frac{1}{2}-\frac{1}{p}))}\, 2^{j(d-1)/p}\, 2^{jd/2}\, 2^{-j(d-1)} \\ &\lesssim 1\,. \end{aligned}$$

This proves the claim. ■

3.3.6 Proof of Theorem 5

From Thm. 4 we already know that for $f \in RA^s_{p,q}(\mathbb{R}^d)$, $0 \notin \operatorname{supp} f$, the trace $\operatorname{tr} f$ makes sense and that $\operatorname{tr} f \in L_1(\mathbb{R})$.

Step 1. Let $f \in RB^s_{p,q}(\mathbb{R}^d)$. Since $0 \notin \operatorname{supp} f$ there exists some $\varepsilon > 0$ s.t. the ball with radius ε and centre in the origin has an empty intersection with $\operatorname{supp} f$. Without loss of generality we assume $\varepsilon < 1$. Let $\varphi \in RC^\infty(\mathbb{R}^d)$ be a function s.t. $\varphi(x) = 1$ if $|x| \ge \varepsilon$ and $\varphi(x) = 0$ if $|x| \le \varepsilon/2$. Again we shall work with an optimal atomic decomposition of f, see (39). It follows

$$f = \sum_{j=0}^{m} s_{j,0}\, (\varphi\, a_{j,0,1}) + \sum_{j=0}^{\infty} \sum_{k=k_j}^{\infty} \sum_{\ell=1}^{C(d,k)} s_{j,k}\, (\varphi\, a_{j,k,\ell})$$

where

$$m := 1 + [\log_2(18\, \varepsilon^{-1})] \qquad \text{and} \qquad k_j := \max(1, [2^{j-1}\, \varepsilon] - 10)\,.$$

As in the previous proof we introduce the splitting $f = f_1 + f_2$, where

$$f_1(x) := \sum_{j=0}^{m} s_{j,0}\, \varphi(x)\, \frac{a_{j,0,1}(x) + a_{j,0,1}(-x)}{2}\,.$$

Obviously, $\operatorname{tr} f_1$ is a compactly supported even C^L function. Let

$$f_{2,N}(x) := \sum_{j=0}^{N} \sum_{k=k_j}^{2^N} \sum_{\ell=1}^{C(d,k)} s_{j,k}\, \varphi(x)\, \frac{a_{j,k,\ell}(x) + a_{j,k,\ell}(-x)}{2}\,, \qquad N \in \mathbb{N}\,.$$

As above we use the notation

$$g_{j,k,\ell}(t) := \operatorname{tr} \Big(\varphi(\,\cdot\,)\, \frac{a_{j,k,\ell}(\,\cdot\,) + a_{j,k,\ell}(-\,\cdot\,)}{2} \Big)(t)\,, \qquad t \in \mathbb{R}\,.$$

Hence

$$\text{tr } f_{2,N}(t) = \sum_{j=0}^{N} \sum_{k=k_j}^{2^N} \sum_{\ell=1}^{\min(C(d,k),K)} s_{j,k}\, g_{j,k,\ell}(t)$$

Since

$$\sup_{t\in\mathbb{R}} |g^{(n)}_{j,k,\ell}(t)| \le c_\varphi\,(12\,\cdot\,2^{-j})^{s-n-d/p} = c_\varphi\,(12\,\cdot\,2^{-j})^{-(d-1)/p}\,(12\,\cdot\,2^{-j})^{s-n-1/p}$$

the functions $2^{-j(d-1)/p}12^{(d-1)/p}\, g_{j,k,\ell}/c_\varphi$ are $(s,p)_{L,-1}$-atoms in the sense of Subsection 3.3.2 (in the one-dimensional context). Applying property (ii) from this subsection we find

$$\begin{aligned}
\|\text{ tr } f_{2,N}\,|B^s_{p,q}(\mathbb{R})\| \quad &\lesssim \quad \Big(\sum_{j=0}^{N} 2^{j(d-1)q/p}\Big(\sum_{k=k_j}^{2^N} |s_{j,k}|^p\Big)^{q/p}\Big)^{1/q}\\
&\lesssim \quad \Big(\sum_{j=0}^{N} \Big(\sum_{k=k_j}^{2^N} (1+k)^{d-1}|s_{j,k}|^p\Big)^{q/p}\Big)^{1/q}\\
&\lesssim \quad \|\, f\,|RB^s_{p,q}(\mathbb{R}^d)\|\,.
\end{aligned}$$

Now we consider convergence of the sequence tr f_N. Let $\sigma_p(1) < s' < s$. Arguing as before (but taking into account the different normalization of the atoms with respect to $B^{s'}_{p,p}(\mathbb{R})$) we find

$$\begin{aligned}
\|\text{ tr } f_{2,N} - \text{tr } f_{2,M}\,|L_1(\mathbb{R})\| \quad &\le \quad \|\text{ tr } f_{2,N} - \text{tr } f_{2,M}\,|B^{s'}_{p,p}(\mathbb{R})\|\\
&\lesssim \quad \Big(\sum_{j=M+1}^{N} \sum_{k=k_j}^{2^N} (1+k)^{d-1}|2^{j(s'-s)}s_{j,k}|^p\Big)^{1/p}\\
&\qquad + \Big(\sum_{j=0}^{M} \sum_{k=\max(2^M,k_j)}^{2^N} (1+k)^{d-1}|2^{j(s'-s)}s_{j,k}|^p\Big)^{1/p}
\end{aligned}$$

Since $\|(s_{j,k})_{j,k}\,|b_{p,q,d}\| < \infty$ it follows that the right-hand side tends to 0 if $M \to \infty$. The uniform boundedness of (tr $f_{2,N})_N$ in $B^s_{p,q}(\mathbb{R})$ in combination with the weak convergence of this sequence yields $\lim_{N\to\infty}$ tr $f_{2,N} \in B^s_{p,q}(\mathbb{R})$ by means of the so-called Fatou property, see [3, 13]. Hence, tr $f_2 \in B^s_{p,q}(\mathbb{R})$. In combination with our knowledge about f_1 the claim in case of Besov spaces follows.
Step 2. Let $f \in RF^s_{p,q}(\mathbb{R}^d)$. One can argue as in Step 1. For the Fatou property of the spaces $F^s_{p,q}(\mathbb{R})$ we refer to [13]. ■

3.4 Proofs of the statements in Subsection 2.1.3

Proof of Theorem 6

Step 1. The proof of Theorem 6(i) follows from formula (11) and the density of $RC_0^\infty(\mathbb{R}^d)$ in $RW^1_p(\mathbb{R}^d)$.

Step 2. Let $f \in RC_0^\infty(\mathbb{R}^d)$. This is equivalent to $\operatorname{tr} f = f_0 \in RC_0^\infty(\mathbb{R})$, see Thm. 1. Observe, that

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \begin{cases} f_0''(r) \cdot \frac{x_i \cdot x_j}{r^2} - f_0'(r)\, \frac{x_i \cdot x_j}{r^3}, & \text{if} \quad i \neq j\,, \\ \\ f_0''(r) \cdot \frac{x_i^2}{r^2} - f_0'(r) \cdot \frac{r^2 - x_i^2}{r^3}, & \text{if} \quad i = j. \end{cases}$$

We fix $j \in \{1, 2, \ldots, d\}$ and sum up

$$\sum_{i=1}^{d} \Big(\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \Big)^2 = \frac{f_0''(r)^2}{r^2}\, x_j^2 + \frac{f_0'(r)^2}{r^4} \cdot (r^2 - x_j^2)\,.$$

Now we sum up with respect to j and find

$$\sum_{i,j=1}^{d} \Big(\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \Big)^2 = f_0''(r)^2 + \frac{d-1}{r^2} \cdot f_0'(r)^2.$$

Since the terms on the right-hand side are nonnegative this proves the claim for smooth f. As above the density argument completes the proof. ■

Proof of Theorem 7

The formulas (4)-(6) have to be combined with the density of $RC_0^\infty(\mathbb{R}^d)$ in $RW_p^{2m}(\mathbb{R}^d)$. ■

3.5 Proof of the statements in Subsection 2.1.4

3.5.1 Proof of Lemma 2

Step 1. Necessity of $p > d$. Let $\varphi \in C_0^\infty(\mathbb{R})$ be an even function s.t. $\varphi(0) \neq 0$ and $\operatorname{supp} \varphi \subset [-1/2, 1/2]$. Since $d \geq 2$ the function $g_1(t) := \varphi(t)\, |t|^{-1}$, $t \in \mathbb{R}$, belongs to $RL_p(\mathbb{R}, |t|^{d-1})$ if $p < d$. Hence $RL_p(\mathbb{R}, |t|^{d-1}) \not\subset S'(\mathbb{R})$ if $p < d$. Let $p = d$ and take $g_2(t) := \varphi(t)\, |t|^{-1}\, (-\log |t|)^{-\alpha}$, $t \in \mathbb{R} \setminus \{0\}$, for $\alpha > 0$. In case $\alpha\, d > 1$ we have $g_2 \in RL_d(\mathbb{R}, |t|^{d-1})$. However, if $\alpha < 1$ then $g_2 \notin S'(\mathbb{R})$. With $1/d < \alpha < 1$ the claim follows.
Step 2. Sufficiency of $p > d$. Using Hölder's inequality we find

$$\int_{-1}^{1} |g(t)|\, dt \leq \Big(\int_{-1}^{1} |g(t)|^p\, |t|^{d-1}\, dt \Big)^{1/p} \Big(\int_{-1}^{1} |t|^{-\frac{(d-1)p'}{p}}\, dt \Big)^{1/p'}.$$

The second factor on the right-hand side is finite if, and only if,

$$(d-1)(p'-1) < 1 \qquad \Longleftrightarrow \qquad d < p\,.$$

Complemented by the obvious inequality

$$\int_{|t|>1} |g(t)|^p \, dt \le \int_{|t|>1} |g(t)|^p \, |t|^{d-1} dt$$

we conclude $L_p(\mathbb{R}, |t|^{d-1}) \hookrightarrow L_1(\mathbb{R}) + L_p(\mathbb{R}) \subset S'(\mathbb{R})$. ■

3.5.2 Proof of Theorem 8

Step 1. We shall prove that (iv) implies (i) and (ii).

Substep 1.1. The B-case. It will be enough to deal with the limiting case. Let $s = d(\frac{1}{p} - \frac{1}{d}) > 0$ ($s > \sigma_p(d)$) and $q = 1$. In addition we assume $1 \le p < d$, where the upper bound results from the previous restriction on s, see Fig. 1 in Subsection 2.1.5. For $f \in RB^s_{p,q}(\mathbb{R}^d)$ we select an optimal atomic decomposition of the trace in the sense of Theorem 3. Let $\varphi \in S(\mathbb{R})$. Then

$$\begin{aligned}
&\Big| \int_{-\infty}^{\infty} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} s_{j,k}\, b_{j,k}(t)\, \varphi(t)\, dt \Big| \\
&\quad \le \; 4 \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} |s_{j,k}|\, 2^{-j} \, \| \, b_{j,k} \, | L_\infty(\mathbb{R}) \| \, \| \, \varphi \, | L_\infty(\mathbb{R}) \| \\
&\quad \le \; 4 \| \, \varphi \, | L_\infty(\mathbb{R}) \| \sum_{j=0}^{\infty} 2^{j(s-d/p)} \sum_{k=0}^{\infty} |s_{j,k}| \\
&\quad \le \; 4 \| \, \varphi \, | L_\infty(\mathbb{R}) \| \Big(\sum_{k=0}^{\infty} (1+k)^{-(d-1)\frac{p'}{p}} \Big)^{1/p'} \\
&\qquad\qquad \times \sum_{j=0}^{\infty} 2^{j(s-d/p)} \Big(\sum_{k=0}^{\infty} (1+k)^{d-1} |s_{j,k}|^p \Big)^{1/p} .
\end{aligned}$$

Since

$$\Big(\sum_{k=0}^{\infty} (1+k)^{-(d-1)\frac{p'}{p}} \Big)^{1/p'} < \infty$$

if $1 \le p < d$, we obtain

$$\begin{aligned}
\Big| \int_{-\infty}^{\infty} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} s_{j,k}\, b_{j,k}(t)\, \varphi(t)\, dt \Big| \; &\le \; c_1 \, \| \, \varphi \, | L_\infty(\mathbb{R}) \| \, \| \, (s_{j,k})_{j,k} \, | b^s_{p,1,d} \| \\
&\le \; c_2 \, \| \, \varphi \, | L_\infty(\mathbb{R}) \| \, \| \, f \, | B^s_{p,1}(\mathbb{R}^d) \| \, ,
\end{aligned}$$

see Theorem 3. This proves sufficiency for $1 \le p < d$ and $s = d(\frac{1}{p} - \frac{1}{d})$. Now, let $0 < p < 1$. Then it is enough to apply the continuous embedding

$$B^{\frac{d}{p}-1}_{p,1}(\mathbb{R}^d) \hookrightarrow B^{d-1}_{1,1}(\mathbb{R}^d) \, ,$$

see e.g. [40, 2.7.1] or [34].

Substep 1.2. Now we turn to the same implication in case of the F-spaces. Also here an embedding argument turns out to be sufficient. For $0 < p \le 1$ and $p < p_1 < \infty$ we have

$$F^{\frac{d}{p}-1}_{p,\infty}(\mathbb{R}^d) \hookrightarrow B^{\frac{d}{p_1}-1}_{p_1,1}(\mathbb{R}^d)\,,$$

see [19] or [34]. Now the claim follows from Substep 1.1.

Step 2. Since tr is an isomorphism of $RA^s_{p,q}(\mathbb{R}^d)$ onto $TA^s_{p,q}(\mathbb{R}, L, d)$ we deduce from Step 1 the implication (iv) $\Longrightarrow$ (iii).

Step 3. It remains to prove the implication (i) $\Longrightarrow$ (vi). We argue by contradiction.

Substep 3.1. The B-case. Let $s = \frac{d}{p} - 1$ and suppose $q > 1$. Oriented at our investigations in Lemma 5 we will use as test functions

$$f_\alpha(x) := \varphi(|x|)\,|x|^{-1}(-\log|x|)^{-\alpha}\,, \qquad x \in \mathbb{R}^d\,. \tag{46}$$

It is known, see e.g. [28, Lem. 2.3.1], that

$$f_\alpha \in B^{\frac{d}{p}-1}_{p,q}(\mathbb{R}^d) \qquad \text{if, and only if,} \qquad q\,\alpha > 1\,.$$

Since tr $f_\alpha \notin S'(\mathbb{R})$ if $\alpha < 1$, we obtain that tr does not map into $S'(\mathbb{R})$ as long as $1/q < \alpha < 1$.

Substep 3.2. The F-case. This time it holds

$$f_\alpha \in F^{\frac{d}{p}-1}_{p,\infty}(\mathbb{R}^d) \qquad \text{if, and only if,} \qquad p\,\alpha > 1\,,$$

see [28, Lem. 2.3.1]. Choosing $1/p < \alpha < 1$ we obtain that tr does not map $F^{\frac{d}{p}-1}_{p,\infty}(\mathbb{R}^d)$ into $S'(\mathbb{R})$. ■

3.6 Proof of the assertions in Subsection 2.1.5

Proof of Theorem 9

Comparing our atomic decomposition with that one for weighted spaces obtained in [16] it is essentially a question of renormalization of the atoms. This is enough to prove $TA^s_{p,q}(\mathbb{R}, L, d) \hookrightarrow RA^s_{p,q}(\mathbb{R}, w_{d-1})$. To see the converse one has to start with the fact that $f \in RA^s_{p,q}(\mathbb{R}, w_{d-1})$ is even. This allows to decompose f into sum of atoms that are even as well, see (39) and (40) for this argument. ■

Proof of Remark 7

The regularity of the δ distribution is calculated at several places, see e.g. [28, Remark 2.2.4/3]. The argument, used in this reference, comes from Fourier analysis and transfers to the weighted case. For the Fourier analytic characterization of $A^s_{p,q}(\mathbb{R}, w_{d-1})$ we refer to [5, 6] and [16]. ■

3.7 Proof of the assertions in Subsection 2.1.6

Proof of Corollary 1

We shall only prove part (i). The proof for the Triebel-Lizorkin spaces is similar. By our trace theorem we have

$$\| f_0 \,|TB^s_{p,q}(\mathbb{R}, L, d)\| \lesssim \| f \,|B^s_{p,q}(\mathbb{R}^d)\|$$

if $L > [s]+1$, cf. Theorem 3. Thus, it is sufficient to prove that

$$\| f_0 \,|B^s_{p,q}(\mathbb{R})\| \lesssim \tau^{-(d-1)/p} \| f_0 \,|TB^s_{p,q}(\mathbb{R}, L, d)\| \,. \tag{47}$$

The trace $f_0 \in TB^s_{p,q}(\mathbb{R}, L, d)$ can be represented in the form

$$f_0(t) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} s_{j,k}\, g_{j,k}(t) \tag{48}$$

(convergence in $L_{\max(1,p)}(\mathbb{R}, |t|^{d-1})$), where the sequence $(s_{j,k})_{j,k}$ belongs to $b^s_{p,q,d}$, cf. (2). Let $\varphi \in C^\infty(\mathbb{R})$ be an even function such that $\varphi(t) = 0$ if $|t| \le \frac{1}{2}$ and $\varphi(t) = 1$ if $|t| \ge 1$. For any $\tau > 0$ we define $\varphi_\tau(t) = \varphi(\tau^{-1}t)$. We will consider two cases: $\tau \ge 2$ and $0 < \tau < 2$.

Case 1. Let $\tau \ge 2$. Under this assumption any function $\varphi_\tau\, g_{j,k}$ is an even L-atom centered at the same interval as $g_{j,k}$ itself (up to a general constant depending on φ), see Definition 1. For any $j \in \mathbb{N}_0$ we define a nonnegative integer k_j by

$$k_j := \max\{k \in \mathbb{N}_0 : \quad 2^{-j}\,(k+1) \le \tau/2\}\,.$$

Hence, $\varphi_\tau\, g_{j,k} = 0$ if $k < k_j$. Furthermore, the functions $2^{-j(s-1/p)}\,\varphi_\tau\, g_{j,k}$, $k \ge 1$, restricted either to the positive or negative half axis, are $(s,p)_{L,-1}$-atoms in the sense of Definition 6 up to a universal constant c. The functions $2^{-j(s-1/p)}\,\varphi_\tau\, g_{j,0}$ are $(s,p)_{L,-1}$-atoms as well (again up to a universal constant). We obtain

$$f_0(t) = \varphi_\tau(t)\, f_0(t) = \sum_{j=0}^{\infty} \sum_{k=k_j}^{\infty} s_{j,k}\, \varphi_\tau(t)\, g_{j,k}(t)$$

and applying (35) (which is also valid for $d=1$) we arrive at the estimate

$$\begin{aligned}
\| f_0 \,|B^s_{p,q}(\mathbb{R})\| &\lesssim \Bigg(\sum_{j=0}^{\infty} 2^{j(s-\frac{1}{p})q} \Big(\sum_{k=k_j}^{\infty} |s_{j,k}|^p\Big)^{q/p}\Bigg)^{1/q} \\
&\lesssim \tau^{\frac{1-d}{p}} \Bigg(\sum_{j=0}^{\infty} 2^{j(s-\frac{d}{p})q} \Big(\sum_{k=0}^{\infty} (1+k)^{d-1}\, |s_{j,k}|^p\Big)^{q/p}\Bigg)^{1/q} \\
&= \tau^{\frac{1-d}{p}} \| s \,|b^s_{p,q,d}\|
\end{aligned} \tag{49}$$

since $k_j \sim 2^j\, \tau$. Taking the infimum with respect to all atomic representations of f_0 we have proved (47).

Case 2. Let $0 < \tau < 2$.

Step 1. We assume $s < d/p$. Then we define $j_0 \in \mathbb{N}_0$ via the relation $2^{-j_0} \le \tau < 2^{-j_0+1}$. Further, we put $K_j := \max(1, 2^{j-j_0-1} - 1)$. Now we decompose f_0 into four sums

$$\begin{aligned} f_0(t) = \varphi_\tau(t)\, f_0(t) &= \sum_{j=0}^{j_0+1} s_{j,0}\, \varphi_\tau(t)\, g_{j,0}(t) \quad + \quad \sum_{j=j_0}^{\infty} \sum_{k=K_j}^{2^{j-j_0+1}} s_{j,k}\, \varphi_\tau(t)\, g_{j,k}(t) \\ &+ \sum_{j=j_0}^{\infty} \sum_{k=2^{j-j_0+1}+1}^{\infty} s_{j,k}\, g_{j,k}(t) \quad + \quad \sum_{j=0}^{j_0-1} \sum_{k=1}^{\infty} s_{j,k}\, g_{j,k}(t) \\ &= f_1(t) + \ldots + f_4(t)\,, \end{aligned}$$

with $f_4 = 0$ if $j_0 = 0$. Observe

$$\operatorname{supp} f_i \subset \{t :\ |t| \ge \tau\}\,, \qquad i = 3, 4\,,$$

whereas the supports of the functions $\varphi_\tau\, g_{j,k}$, occuring in the defintions of f_1 and f_2, may have nontrivial intersections with the interval $(\tau/2, \tau)$. The function f_1 belongs to C^L and has compact support. The functions f_2, f_3, and f_4 are supported on $\{t :\ |t| \ge \tau/2\}$. Thus, the known convergence in $L_{\max(1,p)}(\mathbb{R}, |t|^{d-1})$ implies the convergence in $\mathcal{S}'(\mathbb{R})$. As in *Case 1* the functions $2^{-j(s-1/p)}\, g_{j,k}$, $k \ge 1$, restricted either to the positive or negative half axis, are $(s,p)_{L,-1}$-atoms in the sense of Definition 6. An easy calculation shows that also the functions $2^{-j(s-1/p)}\, \varphi_\tau\, g_{j,k}$, $j \ge j_0$, are $(s,p)_{L,-1}$-atoms (up to a universal constant). Hence we may employ (35) and obtain

$$\begin{aligned} \|\, f_2 + f_3\, | B^s_{p,q}(\mathbb{R})\| &\lesssim \Big(\sum_{j=j_0}^{\infty} 2^{j(s-\frac{1}{p})q} \Big(\sum_{k=K_j}^{\infty} |s_{j,k}|^p \Big)^{q/p} \Big)^{1/q} \\ &\lesssim \tau^{\frac{1-d}{p}} \Big(\sum_{j=j_0}^{\infty} 2^{j(s-\frac{d}{p})q} \Big(\sum_{k=K_j}^{\infty} (1+k)^{d-1}\, |s_{j,k}|^p \Big)^{q/p} \Big)^{1/q} \end{aligned}$$

as well as

$$\begin{aligned} \|\, f_4\, | B^s_{p,q}(\mathbb{R})\| &\lesssim \Big(\sum_{j=0}^{j_0-1} 2^{j(s-\frac{1}{p})q} \Big(\sum_{k=1}^{\infty} |s_{j,k}|^p \Big)^{q/p} \Big)^{1/q} \\ &\lesssim 2^{j_0 \frac{d-1}{p}} \Big(\sum_{j=0}^{\infty} 2^{j(s-\frac{d}{p})q} \Big(\sum_{k=0}^{\infty} (1+|k|)^{d-1}\, |s_{j,k}|^p \Big)^{q/p} \Big)^{1/q} \\ &\lesssim \tau^{\frac{1-d}{p}} \Big(\sum_{j=0}^{\infty} 2^{j(s-\frac{d}{p})q} \Big(\sum_{k=0}^{\infty} (1+|k|)^{d-1}\, |s_{j,k}|^p \Big)^{q/p} \Big)^{1/q} . \end{aligned}$$

Now we turn to the estimate of f_1. First we deal with the estimate of the quasi-norm of the functions $\varphi_\tau\, g_{j,0}$. Let in addition $s \ge 1/p$. Employing the Moser-type estimate of Lemma 5.3.7/1 in [28] (applied with $r = \infty$) we obtain

$$\begin{aligned}
\| \varphi_\tau \, g_{j,0} \,|B^s_{p,q}(\mathbb{R})\| &\lesssim \| \varphi_\tau \,|B^s_{p,q}(\mathbb{R})\| \, \| g_{j,0} \,|L_\infty(\mathbb{R})\| + \| \varphi_\tau \,|L_\infty(\mathbb{R})\| \, \| g_{j,0} \,|B^s_{p,q}(\mathbb{R})\| \\
&\lesssim \tau^{-(s-1/p)} \, \| \varphi \,|B^s_{p,q}(\mathbb{R})\| + \| \varphi \,|L_\infty(\mathbb{R})\| \, \| g_{j,0} \,|B^s_{p,q}(\mathbb{R})\| \\
&\lesssim \tau^{-(s-1/p)} + 2^{j(s-1/p)} \\
&\lesssim 2^{j_0(s-1/p)} \, , \qquad (50)
\end{aligned}$$

since the functions $2^{-j(s-1/p)} \, g_{j,0}$ are atoms and $j \le j_0 + 1$. If $s < 1/p$, we argue by using real interpolation. Because of

$$B^s_{p,q}(\mathbb{R}) = \Big(B^{s_0}_{p,q_0}(\mathbb{R}), L_p(\mathbb{R}) \Big)_{\Theta,q} \, , \qquad s = (1-\Theta) s_0 > \sigma_p(1) \, ,$$

see [9], an application of the interpolation inequality

$$\| \varphi_\tau \, g_{j,0} \,|B^s_{p,q}(\mathbb{R})\| \lesssim \| \varphi_\tau \, g_{j,0} \,|B^{s_0}_{p,q}(\mathbb{R})\|^{1-\Theta} \, \| \varphi_\tau \, g_{j,0} \,|L_p(\mathbb{R})\|^{\Theta}$$

yields (50) for all $s > \sigma_p(1)$. With $r := \min(1,p,q)$ and $\sigma_p(d) < s < d/p$ we conclude

$$\begin{aligned}
\| f_1 \,|B^s_{p,q}(\mathbb{R})\|^r &\le \sum_{j=0}^{j_0+1} |s_{j,0}|^r \, \| \varphi_\tau \, b_{j,0} \,|B^s_{p,q}(\mathbb{R})\|^r \\
&\lesssim 2^{j_0(s-\frac{1}{p})r} \sum_{j=0}^{j_0+1} |s_{j,0}|^r \, . \\
&\lesssim \tau^{\frac{r(1-d)}{p}} \sum_{j=0}^{j_0+1} 2^{r(j_0-j)(s-\frac{d}{p})} 2^{j(s-\frac{d}{p})r} \, |s_{j,0}|^r \\
&\lesssim \tau^{\frac{r(1-d)}{p}} \Big(\sup_{j=0,\ldots,j_0+1} 2^{j(s-\frac{d}{p})} \, |s_{j,0}| \Big)^r \\
&\lesssim \tau^{\frac{r(1-d)}{p}} \, \| f_1 \,|TB^s_{p,\infty}(\mathbb{R})\|^r \, .
\end{aligned}$$

This proves the claim for $s < d/p$.

Step 2. Let $s \ge d/p$. As in Step 1 we define $j_0 \in \mathbb{N}$ via the relation $2^{-j_0} \le \tau < 2^{-j_0+1}$. We shall use $TA^s_{p,q}(\mathbb{R}, L, d) = RA^s_{p,q}(\mathbb{R}, w_{d-1})$, cf. Theorem 9. Alternatively one could use interpolation, see Propositions 1, 2. The spaces $RA^s_{p,q}(\mathbb{R}, w_{d-1})$ allow a characterization by Daubechies wavelets, see [17] for Besov spaces and [18] for Lizorkin-Triebel spaces. The same is true with respect to the ordinary spaces $A^s_{p,q}(\mathbb{R})$, see e.g. [43, Thm. 1.20]. Let ϕ denote an appropriate scaling function and Ψ an associated Daubechies wavelet of sufficiently high order. Let

$$\phi_{0,\ell}(t) := \phi(t-\ell) \qquad \text{and} \qquad \Psi_{j,\ell}(t) := 2^{j/2} \, \Psi_{j,\ell}(2^j t - \ell) \, , \qquad \ell \in \mathbb{Z} \, , \quad j \in \mathbb{N}_0 \, .$$

Since Ψ has compact support, say supp $\Psi \subset [-2^N, 2^N]$ for some $N \in \mathbb{N}$, and supp $f_0 \subset \{t \in \mathbb{R} : |t| \ge \tau\}$ we find that

$$\langle f_0, \Psi_{j,\ell} \rangle = 0 \qquad \text{if} \qquad j - j_0 \ge N \qquad \text{and} \qquad |\ell| \le 2^{j-j_0} - 2^N \, .$$

Hence, f_0 has a wavelet expansion given by

$$\begin{aligned} f_0 &= \sum_{\ell\in\mathbb{Z}} \langle f_0, \phi_{0,\ell}\rangle\, \phi_{0,\ell} + \sum_{j=0}^{j_0+N-1} \sum_{\ell\in\mathbb{Z}} \langle f_0, \Psi_{j,\ell}\rangle\, \Psi_{j,\ell} + \sum_{j=j_0+N}^{\infty} \sum_{|\ell|>2^{j-j_0}-2^N} \langle f_0, \Psi_{j,\ell}\rangle\, \Psi_{j,\ell}\,. \\ &= f_1 + f_2 + f_3\,. \end{aligned}$$

By the references given above it follows

$$\begin{aligned} \| f_1 \,|B^s_{p,q}(\mathbb{R}, w_{d-1})\| &\asymp \Big(\sum_{\ell\in\mathbb{Z}} |\langle f_0, \phi_{0,\ell}\rangle|^p (1+|\ell|)^{d-1}\Big)^{1/p} \\ \| f_2 \,|B^s_{p,q}(\mathbb{R}, w_{d-1})\| &\asymp \Big(\sum_{j=0}^{j_0+N-1} 2^{j(s+\frac12-\frac dp)q} \Big(\sum_{\ell\in\mathbb{Z}} |\langle f_0, \Psi_{j,\ell}\rangle|^p (1+|\ell|)^{d-1}\Big)^{q/p}\Big)^{1/q} \\ \| f_3 \,|B^s_{p,q}(\mathbb{R}, w_{d-1})\| &\asymp \Big(\sum_{j=j_0+N}^{\infty} 2^{j(s+\frac12-\frac dp)q} \Big(\sum_{|\ell|\ge 2^{j-j_0}-2^N} |\langle f_0, \Psi_{j,\ell}\rangle|^p (1+|\ell|)^{d-1}\Big)^{q/p}\Big)^{1/q}. \end{aligned}$$

The quasi-norm in the unweighted spaces is obtained by deleting the factor $2^{-j(d-1)/p}\,(1+|\ell|)^{d-1}$, see [43, Thm. 1.20]. This immediately implies

$$\begin{aligned} \| f_1 \,|B^s_{p,q}(\mathbb{R})\| &\lesssim \| f_1 \,|B^s_{p,q}(\mathbb{R}, w_{d-1})\|\,, \\ \| f_2 \,|B^s_{p,q}(\mathbb{R})\| &\lesssim 2^{(j_0+N)(d-1)/p}\, \| f_2 \,|B^s_{p,q}(\mathbb{R}, w_{d-1})\|\,. \end{aligned}$$

Moreover, we also obtain

$$\| f_3 \,|B^s_{p,q}(\mathbb{R})\| \lesssim 2^{(j_0+N)(d-1)/p}\, \| f_3 \,|B^s_{p,q}(\mathbb{R}, w_{d-1})\|\,.$$

This proves (47) in case $s \ge d/p$ and $0<\tau<2$. ■

Proof of Corollary 2

We concentrate on the proof in case of Besov spaces. The proof for Lizorkin-Triebel spaces is similar.

Step 1. We claim that $g \in TB^s_{p,q}(\mathbb{R}, L, d)$. We argue as in *Case 2, Step 2* of the proof of Corollary 1. But this time we do not study the dependence of the constants on a and b.

Under the given restrictions $g \in RB^s_{p,q}(\mathbb{R})$ has a wavelet expansion of the form

$$g = \sum_{|\ell|\le c_1} \langle g, \phi_{0,\ell}\rangle\, \phi_{0,\ell} + \sum_{j=0}^{\infty} \sum_{|\ell|\le c_1 2^j} \langle g_0, \Psi_{j,\ell}\rangle\, \Psi_{j,\ell}$$

with an appropriate constant c_1. Since g is even we obtain

$$g = \sum_{|\ell|\le c_1} \langle g, \phi_{0,\ell}\rangle\, \frac{\phi_{0,\ell}(t)+\phi_{0,\ell}(-t)}{2} + \sum_{j=0}^{\infty} \sum_{|\ell|\le c_1 2^j} \langle g_0, \Psi_{j,\ell}\rangle\, \frac{\Psi_{j,\ell}(t)+\Psi_{j,\ell}(-t)}{2}\,.$$

The functions $2^{-j/2}(\Psi_{j,\ell}(t)+\Psi_{j,\ell}(-t))$ are even L-atoms (up to a universal constant) centered at $c_2\, I_{j,\ell}$, where

$$I_{j,k} := [-2^{-j}(\ell+1), -2^{-j}\ell] \cup [2^{-j}\ell, 2^{-j}(\ell+1)]$$

(modification if $\ell = 0$, see Definition 3). The constant $c_2 > 1$ depends on the size of the supports of the generators ϕ and Ψ. Without proof we mention that Theorem 3 remains true also for those more general decompositions. This implies

$$\begin{aligned}
\| g \,|TB^s_{p,q}(\mathbb{R}, L, d)\| \asymp & \Big(\sum_{|\ell|\le c_1 b} (1+|\ell|)^{d-1} |\langle g, \phi_{0,\ell}\rangle|^p \Big)^{1/p} \\
& + \quad \Big(\sum_{j=0}^{\infty} 2^{j(s-d/p)q} \Big(\sum_{|\ell|\le c_1 2^j} (1+|\ell|)^{d-1} |2^{j/2}\, \langle g, \Psi_{j,\ell}\rangle|^p \Big)^{q/p} \Big)^{1/q} \\
\lesssim & \quad \Big(\sum_{|\ell|\le c_1 b} |\langle g, \phi_{0,\ell}\rangle|^p \Big)^{1/p} + \Big(\sum_{j=0}^{\infty} 2^{j(s+\frac12-\frac1p)q} \Big(\sum_{|\ell|\le c_1 2^j} |\langle g, \Psi_{j,\ell}\rangle|^p \Big)^{q/p} \Big)^{1/q} \\
\lesssim & \quad \| g \,|B^s_{p,q}(\mathbb{R})\| \, ,
\end{aligned}$$

see e.g. [43, Thm. 1.20] for the last step. This proves the claim.
Step 2. Since g belongs to $TB^s_{p,q}(\mathbb{R}, L, d)$ we derive by means of Theorem 3 that $f := \mathrm{ext}\, g$ is an element of $RB^s_{p,q}(\mathbb{R}^d)$ and

$$\| f \,|RB^s_{p,q}(\mathbb{R}^d)\| \; \lesssim \; \| g \,|TB^s_{p,q}(\mathbb{R}, L, d)\| \; \lesssim \; \| g \,|B^s_{p,q}(\mathbb{R})\| \, .$$

Since $\mathrm{supp}\, f \subset \{x : \; |x| \ge a\}$ Corollary 1 yields

$$\| g \,|B^s_{p,q}(\mathbb{R})\| \; \lesssim \; a^{-(d-1)/p} \, \| f \,|B^s_{p,q}(\mathbb{R}^d)\| \, ,$$

because of $f_0 = g$. This completes the proof. ■

Remark 18 A closer look onto the proof shows that

$$a^{(d-1)/p} \, \| g \,|A^s_{p,q}(\mathbb{R})\| \; \lesssim \; \| f \,|RA^s_{p,q}(\mathbb{R}^d)\| \; \lesssim \; b^{(d-1)/p} \, \| g \,|A^s_{p,q}(\mathbb{R})\|$$

with constants independent of g, $a > 0$ and $b \ge 1$.

Proof of Corollaries 3, 4

Step 1. Proof of Cor. 3. The function φ is a pointwise multiplier for the spaces $A^s_{p,q}(\mathbb{R}^d)$, see e.g. [28, 4.8]. Hence, with f also the product $\varphi\, f$ belongs to $RA^s_{p,q}(\mathbb{R}^d)$ and we can apply Thm. 5 with respect to this product. Concerning the sharp embedding relations for the spaces $A^s_{p,q}(\mathbb{R})$ into Hölder-Zygmund spaces we refer to [34] and the references given there. This proves the assertion for $\varphi_0\, f_0$. A further application of Theorem 2 finishes the proof.

Observe, that we do not need the assumption $s > \sigma_{p,q}(d)$ in case of Lizorkin-Triebel spaces. We may argue with $RF^s_{p,\infty}(\mathbb{R}^d)$ first and use the elementary embedding $RF^s_{p,q}(\mathbb{R}^d) \hookrightarrow RF^s_{p,\infty}(\mathbb{R}^d)$ afterwards.
Step 2. Proof of Cor. 4. The arguments are as above. Concerning te embedding relations of the spaces $A^s_{p,q}(\mathbb{R})$ into the space of uniformly continuous and bounded functions we also refer to [34] and the references given there. ■

Remark 19 A different proof of Cor. 4, restricted to Besov spaces, has been given in [31].

3.8 Test functions

Using our previous results, in particular Corollary 2, we shall investigate the regularity of certain families of radial test functions.

Lemma 5 *Let $0 < \alpha < \min(1, 1/p)$. Let $\varphi \in C_0^\infty(\mathbb{R})$ be an even function such that* $\operatorname{supp} \varphi \subset [-2,-1/2] \cup [1/2, 2]$ *and $\varphi(1) \neq 0$.*
(i) *The function*

$$f_\alpha(x) := \varphi(|x|)\, |\, |x| - 1\, |^{-\alpha}\,, \qquad x \in \mathbb{R}^d\,, \tag{51}$$

belongs to $B^{\frac{1}{p}-\alpha}_{p,\infty}(\mathbb{R}^d)$ if

$$\alpha < \frac{1}{p} - \sigma_p(d). \tag{52}$$

(ii) *Suppose $\frac{1}{p} - \alpha > \sigma_p(1)$. Then f_α does not belong to $B^{\frac{1}{p}-\alpha}_{p,q}(\mathbb{R}^d)$ for any $q < \infty$.*
(iii) *Under the same restriction as in (ii) we have that f_α does not belong to $F^{\frac{1}{p}-\alpha}_{p,\infty}(\mathbb{R}^d)$.*

Proof. *Step 1.* Proof of (i). Let $\widetilde{\varphi} \in C_0^\infty(\mathbb{R})$ be a function such that $\operatorname{supp} \widetilde{\varphi} \subset [1/2, 2]$. Then the regularity of

$$g_\alpha(t) := \widetilde{\varphi}(t)\, |t-1|^{-\alpha}\,, \qquad t \in \mathbb{R}\,,$$

is well understood, cf. e.g. [28, Lem. 2.3.1/1]. One has $g_\alpha \in B^{\frac{1}{p}-\alpha}_{p,\infty}(\mathbb{R})$ as long as $0 < \alpha < \min(1, 1/p)$. An application of Corollary 2 yields the claim.
Step 2. Proof of (ii) and (iii). It is also known, see again [28, Lem. 2.3.1/1], that

$$g_\alpha \not\in (B^{\frac{1}{p}-\alpha}_{p,q}(\mathbb{R}) \cup F^{\frac{1}{p}-\alpha}_{p,\infty}(\mathbb{R}))\,, \qquad 0 < q < \infty\,, \quad 0 < \alpha < \min(1, 1/p)\,.$$

These properties do not change when we "add" the reflection of g_α to the left half of the real axis. With other words, if we replace $\widetilde{\varphi}$ by φ itself we do not change the regularity properties. Now we use Thm. 5. ■

Remark 20 Let $\delta > 0$. Then also the regularity of functions like

$$f_{\alpha,\delta}(x) := \varphi(|x|)\, \big| |x| - 1 \big|^{-\alpha} \, (-\log \big| |x| - 1 \big|)^{-\delta}, \qquad x \in \mathbb{R}^d\,, \tag{53}$$

can be checked in this way. With the help of the parameter δ one can see the microscopic index q. We refer to [36, 5.6.9] or [28, Lem. 2.3.1/1] for details.

Lemma 6 *Let $\alpha > 0$.*
(i) *Then the function*

$$\Phi_\alpha(x) := \max(0, (1 - |x|^2))^\alpha\,, \qquad x \in \mathbb{R}^d\,, \tag{54}$$

belongs to $B^{\frac{1}{p}+\alpha}_{p,\infty}(\mathbb{R}^d)$ if

$$\frac{1}{p} + \alpha > \sigma_p(d)\,. \tag{55}$$

(ii) *Suppose $\frac{1}{p} + \alpha > \sigma_p(1)$. Then Φ_α does not belong to $B^{\frac{1}{p}+\alpha}_{p,q}(\mathbb{R}^d)$ for any $q < \infty$.*
(iii) *Under the same restrictions as in (ii) we have that Φ_α does not belong to $F^{\frac{1}{p}+\alpha}_{p,\infty}(\mathbb{R}^d)$.*

Proof. *Step 1.* Proof of (i). First we investigate the one-dimensional case. Let $\psi_1, \psi_2 \in C_0^\infty(\mathbb{R})$ be such that

$$\operatorname{supp} \psi_1 \subset [-1/2, \infty), \quad \operatorname{supp} \psi_2 \subset (-\infty, 1/2] \quad \text{and} \quad \psi_1(t) + \psi_2(t) = 1$$

for all $t \in \mathbb{R}$. We put $\Phi_{i,\alpha} := \psi_i\, \Phi_\alpha$, $i = 1, 2$. Then $\Phi_{1,\alpha}$ behaves near 1 like

$$\phi_\alpha(t) := \begin{cases} t^\alpha & \text{if} \quad t > 0\,, \\ 0 & \text{if} \quad t < 0\,, \end{cases}$$

near the origin. The regularity of ϕ_α is well understood, we refer to [28, Lem. 2.3.1]. As above the transfer to general dimensions $d > 1$ is done by Corollary 2.
Step 2. To prove the statements in (ii) and (iii) we argue by contradiction. If Φ_α belongs to $RA^s_{p,q}(\mathbb{R}^d)$, then also $\varphi\, \Psi_\alpha$ belongs to $RA^s_{p,q}(\mathbb{R}^d)$ for any smooth radial φ. Choosing φ s.t. $0 \notin \operatorname{supp} \varphi$, we my apply Thm. 5 to conclude $\operatorname{tr} \varphi\, \Psi_\alpha \in A^s_{p,q}(\mathbb{R})$. But this implies $\Phi_{2,\alpha} \in A^s_{p,q}(\mathbb{R})$. In the one-dimensional case necessary and sufficient conditions are known, we refer again to [28, Lem. 2.3.1]. ■

Next we shall consider smooth functions supported in thin annuli.

Lemma 7 *Let $d \geq 2$, $0 < p, q \leq \infty$ and $s > \sigma_p(d)$. Let $\varphi \in C_0^\infty(\mathbb{R})$ be an even function such that $\varphi(1) = 1$ and* $\operatorname{supp} \varphi \subset [-2, -1/2] \cup [1/2, 2]$. *Then the functions*

$$f_{j,\lambda}(y) := \varphi(2^j |y| - \lambda)\,, \qquad y \in \mathbb{R}^d\,, \quad j \in \mathbb{N}\,, \quad \lambda > 0\,.$$

have the following properties:

$$\begin{aligned} \operatorname{supp} f_{j,\lambda} \quad &\subset \quad \{y: \quad (\lambda-2)\, 2^{-j} \le |y| \le (2+\lambda)\, 2^{-j}\}\,, & (56)\\ \| \, f_{j,\lambda}\, | RB^s_{p,q}(\mathbb{R}^d)\| \quad &\asymp \quad 2^{j(s-\frac{d}{p})}\, \lambda^{(d-1)/p} & (57) \end{aligned}$$

with constants in $\asymp$ independent of $\lambda > 1$ and $j \in \mathbb{N}$.

Proof. *Step 1.* Estimate from above in (57). It will be convenient to use the atomic characterizations described in Subsection 3.3.2. Therefore we shall use the decompositions of unity from (43)-(45). Thanks to the support restrictions for the functions $\psi_{j,k,\ell}$ we obtain

$$f_{j,\lambda}(y) = \sum_{\max(0,\lambda-2-n_0)\le k\le \lambda+2+n_0} \; \sum_{\ell=1}^{C(d,k)} (\varphi(2^j|y|-\lambda)\, \psi_{j,k,\ell}(y))$$

where n_0 is a fixed number ($n_0 \ge 18$ would be sufficient). The functions

$$a_{j,k,\ell}(y) := 2^{-j(s-\frac{d}{p})}\, \varphi(2^j|y|-\lambda)\, \psi_{j,k,\ell}(y)$$

are $(s,p)_{M,-1}$-atoms for any M (up to a universal constant). Hence

$$\begin{aligned} \| \, f_{j,\lambda}\, | RB^s_{p,q}(\mathbb{R}^d)\| \quad &\lesssim \quad \Big(\sum_{\max(0,\lambda-2-n_0)\le k\le \lambda+2+n_0} k^{d-1} 2^{j(s-\frac{d}{p})p}\Big)^{1/p} \\ &\lesssim \quad 2^{j(s-\frac{d}{p})} \lambda^{(d-1)/p}\,. \end{aligned}$$

Step 2. Estimate from below.
Substep 2.1. First we deal with $p = \infty$. By construction $f_{j,\lambda}(y) = 1$ if $|y| = (1+\lambda)\, 2^{-j}$. Furthermore, calculating the derivatives of $f_{j,\lambda}(y_1, 0, \ldots, 0)$, $y_1 \in \mathbb{R}$, it is immediate that

$$\| \, f_{j,\lambda}\, | C^m(\mathbb{R}^d)\| \asymp 2^{jm} \qquad (58)$$

for all $m \in \mathbb{N}_0$. Now we argue by contradiction. We fix $s > 1$, $q_1 \in (0,\infty]$ and assume that

$$\| \, f_{j,\lambda}\, | B^s_{\infty,q_1}(\mathbb{R}^d)\| \le \phi(j,\lambda)\, 2^{js}\,,$$

where $\phi:\, \mathbb{N}\times[1,\infty) \to (0,1)$ and $\lim_{\ell\to\infty} \phi(j_\ell,\lambda_\ell) = 0$ for some sequence $(j_\ell,\lambda_\ell)_\ell \subset \mathbb{N}\times[1,\infty)$. We choose $\Theta \in (0,1)$ s.t. $m = \Theta\, s$ and $q = 1$. Real interpolation between $C(\mathbb{R}^d)$ and $B^s_{\infty,q_1}(\mathbb{R}^d)$ yields

$$\| \, f_{j,\lambda}\, | B^m_{\infty,1}(\mathbb{R}^d)\| \le c\, 2^{jm}\, (\phi(j,\lambda))^{\Theta}\,,$$

where c is independent of j and λ, see the proof of Thm. 2. The continuous embedding $B^m_{\infty,1}(\mathbb{R}^d) \hookrightarrow C^m(\mathbb{R}^d)$ leads to a contradiction with (58).
Now let $0 < s < 1$. We interpolate between $B^s_{\infty,q_1}(\mathbb{R}^d)$ and $B^2_{\infty,q}(\mathbb{R}^d)$. By arguing as above we could improve the estimate from above with respect to the space $B^1_{\infty,1}(\mathbb{R}^d)$.

Since $B^1_{\infty,1}(\mathbb{R}^d) \hookrightarrow C^1(\mathbb{R}^d)$ this contradicts again (58). Hence the claim is proved with $p = \infty$, $0 < q \leq \infty$, and $s > 0$.
Substep 2.2. Also obvious is the behaviour in $L_p(\mathbb{R}^d)$. For $0 < p \leq \infty$ we have

$$\| f_{j,\lambda} \, | L_p(\mathbb{R}^d)\| \asymp 2^{-jd/p} \, \lambda^{(d-1)/p} \, . \tag{59}$$

A few more calculations yield

$$\| f_{j,\lambda} \, | W^1_p(\mathbb{R}^d)\| \asymp 2^{j(1-d/p)} \, \lambda^{(d-1)/p} \, , \tag{60}$$

as long as $1 \leq p \leq \infty$.
Substep 2.3. Let $p_1 < \infty$. We assume that for some fixed $s_1 > \sigma_{p_1}(d)$ and $q_1 \in (0, \infty]$

$$\| f_{j,\lambda} \, | B^{s_1}_{p_1,q_1}(\mathbb{R}^d)\| \leq \phi(j, \lambda) \, 2^{j(s_1 - d/p_1} \, \lambda^{(d-1)/p_1},$$

holds, where ϕ is as above. Complex interpolation between $B^{s_1}_{p_1,q_1}(\mathbb{R}^d)$ and $B^{s_2}_{\infty,q_0}(\mathbb{R}^d)$, $s_2 > 0$, yields an improvement of our estimate with respect to $B^s_{p,q}(\mathbb{R}^d)$, where $p > p_1$ is at our disposal. For s_2 large we can choose $p > 1$ s.t. $s = (1 - \Theta)s_2 + \Theta \, s_1 > 1$. Now we need a further interpolation, this time real, between $B^s_{p,q}(\mathbb{R}^d)$ and $L_p(\mathbb{R}^d)$, improving the estimate for $B^1_{p,1}(\mathbb{R}^d)$ in this way. But $B^1_{p,1}(\mathbb{R}^d) \hookrightarrow W^1_p(\mathbb{R}^d)$ and so we found a contradiction to (60). ■

Remark 21 Obviously there is no q-dependence in Lemma 7. As an immediate consequence of the elemetary embeddings

$$B^s_{p,\min(p,q)}(\mathbb{R}^d) \hookrightarrow F^s_{p,q}(\mathbb{R}^d) \hookrightarrow B^s_{p,\max(p,q)}(\mathbb{R}^d) \, ,$$

see [40,], and (57) we obtain

$$\| f_{j,\lambda} \, | F^s_{p,q}(\mathbb{R}^d)\| \asymp 2^{j(s-\frac{d}{p})} \, \lambda^{(d-1)/p} \, , \qquad \lambda > 1, \quad j \in \mathbb{N} \, .$$

Some extremal functions in $A^{d/p}_{p,q}(\mathbb{R}^d)$ have been investigated by Bourdaud [2], for $B^s_{p,p}(\mathbb{R}^d)$ see also Triebel [42]. We recall the result obtained in [2]. For $(\alpha, \sigma) \in \mathbb{R}^2$ we define

$$f_{\alpha,\sigma}(x) := \psi(x) \Big| \log |x| \Big|^{\alpha} \Big| \log | \log |x| | \Big|^{-\sigma} , \qquad x \in \mathbb{R}^d \, . \tag{61}$$

Furthermore we define a set $U_t \subset \mathbb{R}^2$ as follows:

$$U_t := \begin{cases} (\alpha = 0 \quad \text{and} \quad \sigma > 0) \quad \text{or} \quad \alpha < 0 & \text{if} \quad t = 1 \, , \\ \\ (\alpha = 1 - 1/t \quad \text{and} \quad \sigma > 1/t) \quad \text{or} \quad \alpha < 1 - 1/t & \text{if} \quad 1 < t < \infty \, , \\ \\ (\alpha = 1 \quad \text{and} \quad \sigma \geq 0) \quad \text{or} \quad \alpha < 1 & \text{if} \quad t = \infty \, , \end{cases}$$

Lemma 8 (i) *Let* $0 < p \le \infty$ *and* $1 < q \le \infty$. *Then* $f_{\alpha,\sigma}$ *belongs to* $RB^{d/p}_{p,q}(\mathbb{R}^d)$ *if, and only if* $(\alpha, \sigma) \in U_q$.
(ii) *Let* $1 < p < \infty$. *Then* $f_{\alpha,\sigma}$ *belongs to* $RF^{d/p}_{p,q}(\mathbb{R}^d)$ *if, and only if* $(\alpha, \sigma) \in U_p$.

Remark 22 Let us mention that in [2] the result is stated for $p \ge 1$ only. However, the proof extends to $p < 1$ nearly without changes (in his argument which follows formula (9) in [2] one has to choose $k > d/(2p)$).

4 Decay properties of radial functions – proofs

4.1 Proof of Theorem 10

Step 1. Proof of (i). Following Remark 9 it will be enough to prove the decay estimate (13) for $RB^{1/p}_{p,1}(\mathbb{R}^d)$, $0 < p < \infty$, and for $RF^{1/p}_{p,\infty}(\mathbb{R}^d)$, $0 < p \le 1$. A proof in case $RB^{1/p}_{p,1}(\mathbb{R}^d)$ has been given in [31]. So we are left with the proof for the Lizorkin-Triebel spaces. We will follow the ideas of the proof of Cor. 4. Let $f \in RF^{1/p}_{p,\infty}(\mathbb{R}^d)$. Let

$$f = \sum_{j=0}^{\infty} s_{j,0}\, a_{j,0} + \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \sum_{\ell=1}^{C_{d,k}} s_{j,k}\, a_{j,k,\ell},$$

be an atomic decomposition such that $\|s_{j,k}|f_{p,\infty,d}\| \asymp \| f \,|F^{1/p}_{p,\infty}(\mathbb{R}^d)\|$. We fix x, $|x| > 1$. Observe, that for all $j \ge 0$ there exists $k_j \ge 1$ such that

$$k_j\, 2^{-j} \le |x| < (k_j + 1)\, 2^{-j}\,. \tag{62}$$

Then the main part of f near $(|x|, 0, \dots, 0)$ is given by the function

$$f^M(y) = \sum_{j=0}^{\infty} s_{j,k_j}\, a_{j,k_j,0}(y)\,, \qquad y \in \mathbb{R}^d\,, \tag{63}$$

(in fact, f is a finite sum of functions of type

$$\sum_{j=0}^{\infty} s_{j,k_j+r_j}\, a_{j,k_j+r_j,t_j}(y)\,,$$

and $|r_j|$ and $|t_j|$ are uniformly bounded). For convenience we shall derive an estimate of the main part f^M only. Because of (62) and the normalization of the atoms we obtain

$$|f^M(y)| \;\lesssim\; \sum_{j=0}^{\infty} |s_{j,k_j}|\, 2^{j\frac{d-1}{p}} \;\lesssim\; |x|^{\frac{1-d}{p}} \Big(\sum_{j=0}^{\infty} |s_{j,k_j}|^p\, k_j^{d-1} \Big)^{1/p}, \tag{64}$$

since $p \le 1$. On the other hand

$$\begin{aligned} \| s\,|f_{p,\infty,d}\| &= \| \sup_{j=0,1,\dots} \sup_{k\in\mathbb{N}_0} |s_{j,k}|\, 2^{\frac{jd}{p}}\, \widetilde{\chi}_{j,k}(\cdot)\,|L_p(\mathbb{R}^d)\| \\ &\ge \| \sup_{j=0,1,\dots} |s_{j,k_j}|\, 2^{\frac{jd}{p}}\, \widetilde{\chi}_{j,k_j}(\cdot)\,|L_p(\mathbb{R}^d)\|\,. \end{aligned} \tag{65}$$

Using $P_{j+1,k_{j+1}} \subset P_{j,k_j}$ we obtain the identity

$$\sup_j |s_{j,k_j}| \, 2^{\frac{jd}{p}} \, \widetilde{\chi}_{j,k_j}(\cdot) \; = \; \sum_{j=0}^{\infty} \max_{i=0,\ldots,j} |s_{i,k_i}| \big(\widetilde{\chi}_{j,k_j}(\cdot) - \widetilde{\chi}_{j+1,k_{j+1}}(\cdot) \big) \, .$$

By the pairwise disjointness of the sets $P_{j,k_j} \setminus P_{j+1,k_{j+1}}$ this implies

$$\| \sup_{j=0,1,\ldots} |s_{j,k_j}| \, 2^{\frac{jd}{p}} \, \widetilde{\chi}_{j,k_j}(\cdot) \, | L_p(\mathbb{R}^d) \| \asymp \Big(\sum_{j=0}^{\infty} \max_{i=0,\ldots,j} \big(|s_{i,k_i}|^p 2^{id} \big) \, 2^{-jd} k_j^{d-1} \Big)^{1/p} \, . \qquad (66)$$

Obviously

$$\Big(\sum_{j=0}^{\infty} |s_{j,k_j}|^p \, k_j^{d-1} \Big)^{1/p} \; \le \; \Big(\sum_{j=0}^{\infty} \max_{i=0,\ldots,j} \big(|s_{i,k_i}|^p 2^{id} \big) \, 2^{-jd} k_j^{d-1} \Big)^{1/p} \, . \qquad (67)$$

Combining (64) - (67) we have proved (13) in case of Lizorkin-Triebel spaces.
Step 2. Proof of (ii). As in Step 1 it will be sufficient to deal with the limiting cases.
Substep 2.1. Let $f \in RF^{1/p}_{p,\infty}(\mathbb{R}^d)$. Let $B_r(0)$ be the ball in $\mathbb{R}^d$ with center in the origin and radius r. Then (64)-(67) yield

$$\begin{aligned} |f^M(x)| \quad & \lesssim \quad |x|^{\frac{1-d}{p}} \, \| \sup_{j=0,1,\ldots} |s_{j,k_j}| \, 2^{\frac{jd}{p}} \, \widetilde{\chi}_{j,k_j}(\cdot) \, | L_p(\mathbb{R}^d \setminus B_r(0)) \| \\ & \lesssim \quad |x|^{\frac{1-d}{p}} \, \| \sup_{j=0,1,\ldots} \sup_{k \in \mathbb{N}_0} |s_{j,k}| \, 2^{\frac{jd}{p}} \, \widetilde{\chi}_{j,k}(\cdot) \, | L_p(\mathbb{R}^d \setminus B_r(0)) \| \end{aligned}$$

where $r = |x| - 18 > 0$, see property (b) of the covering $(\Omega_{j,k,\ell})$ in Subsection 3.3.2. In view of this inequality an application of Lebesgue's theorem on dominated convergences proves (14).
Substep 2.2. Let $f \in RB^{1/p}_{p,1}(\mathbb{R}^d)$. We argue as in Substep 2.1 by using the notation from Step 1. Since

$$\lim_{r \to \infty} \quad \sum_{j=0}^{\infty} \Big(\sum_{k \ge r} |s_{j,k}|^p \, (1+k)^{d-1} \Big)^{1/p} = 0$$

we conclude from (64) that (14) holds in this case as well.
Step 3. Proof of (iii). We shall use the test functions constructed in Lemma 7. We choose $s_o > \max\{\sigma_p(d), s\}$. For simplicity we consider $|x| = 2^r$ with $r \in \mathbb{N}$. We choose λ s.t. $|x| = (1+\lambda)/2$. Hence $f_{1,\lambda}(x) = 1$. This implies

$$|x|^{\frac{d-1}{p}} \, |2^{-r(d-1)/p} \, f_{1,\lambda}(x)| = \; 1 \, ,$$

and

$$\| \, 2^{-r(d-1)/p} \, f_{1,\lambda} \, | A^s_{p,q}(\mathbb{R}^d) \| \;\; \lesssim \;\; \| \, 2^{-r(d-1)/p} \, f_{1,\lambda} \, | B^{s_o}_{p,q}(\mathbb{R}^d) \| \asymp 1 \, ,$$

see Rem. 21, which proves the claim.
Step 4. Proof of (iv). It will be enough to study the case $s = 1/p$.

Substep 4.1 Let $q > 1$. According to Lemma 8(i) there exists a compactly supported function g_0 which belongs to $RB^{1/p}_{p,q}(\mathbb{R})$ and is unbounded near the origin. By multiplying with a smooth cut-off function if necessary we can make the support of this functions as small as we want. For the given sequence $(x^j)_j$ we define

$$g(t) := \sum_{j=1}^{\infty} \frac{1}{\max(|x^j|, j)^{\alpha}}\, g_0(t - |x^j|)\,, \qquad t \in \mathbb{R}\,,$$

where we will choose $\alpha > 0$ in dependence on p. The function g is unbounded near $|x^j|$ and by means of the translation invariance of the Besov spaces $B^{1/p}_{p,q}(\mathbb{R})$ we obtain

$$\| \, g \,|B^{1/p}_{p,q}(\mathbb{R})\|^{\min(1,p)} \le \| \, g_0 \,|B^{1/p}_{p,q}(\mathbb{R})\|^{\min(1,p)} \sum_{j=1}^{\infty} j^{-\alpha \min(1,p)} \lesssim \| \, g_0 \,|B^{1/p}_{p,q}(\mathbb{R})\|^{\min(1,p)}\,,$$

if $\alpha \cdot \min(1,p) > 1$. We employ Rem. 18 (Cor. 2) with respect to each summand. This yields

$$\begin{aligned}
& \| \, \mathrm{ext}\; g \,|B^{1/p}_{p,q}(\mathbb{R}^d)\|^{\min(1,p)} \\
\le \quad & \sum_{j=1}^{\infty} \max(|x^j|, j)^{-\alpha \min(1,p)} \, \| \, \mathrm{ext}\,(g_0(\,\cdot\, - |x^j|)) \,|B^{1/p}_{p,q}(\mathbb{R}^d)\|^{\min(1,p)} \\
\lesssim \quad & \| \, g_0 \,|B^{1/p}_{p,q}(\mathbb{R})\|^{\min(1,p)} \sum_{j=1}^{\infty} \Big(\max(|x^j|, j)^{-\alpha} \, |x^j|^{(d-1)/p} \Big)^{\min(1,p)} \\
\lesssim \quad & \| \, g_0 \,|B^{1/p}_{p,q}(\mathbb{R})\|^{\min(1,p)}\,,
\end{aligned}$$

if $(\alpha - (d-1)/p) \min(1,p) > 1$.
Substep 4.2. We turn to the F-case. It will be enough to study the situation $s = 1/p$ and $1 < p < \infty$. We argue as above using this time Lemma 8(ii). An application of Rem. 18 (Cor. 2) yields the result but with the extra condition $1/p > \sigma_q(d)$. ■

4.2 Traces of BV-functions and consequences for the decay

We recall a definition of the space $BV(\mathbb{R}^d)$, $d \ge 2$, which will be convenient for us, see [11, 5.1] or [44, 5.1].

Definition 8 *Let $g \in L_1(\mathbb{R}^d)$. We say, that $g \in BV(\mathbb{R}^d)$ if for every $i = 1, \ldots, d$ there is a signed Radon measure μ_i of finite total variation such that*

$$\int_{\mathbb{R}^d} g(x) \frac{\partial}{\partial x_i} \phi(x)dx = - \int_{\mathbb{R}^d} \phi(x) d\mu_i(x), \qquad \phi \in C^1_c(\mathbb{R}^d),$$

where $C_c^1(\mathbb{R}^d)$ denotes the set of all continuously differentiable functions on $\mathbb{R}^d$ with compact support. The space $BV(\mathbb{R}^d)$ is equipped with the norm

$$\| \, g \, | BV(\mathbb{R}^d) \| = \| \, g \, | L_1(\mathbb{R}^d) \| + \sum_{i=1}^{d} \| \, \mu_i \, | \mathcal{M} \|,$$

where $\| \, \mu_i \, | \mathcal{M} \|$ is the total variation of μ_i.

4.2.1 Proof of Theorem 12

We need some preparations. Recall, the space $C_c^1([0,\infty))$ has been defined in Definition 5. By ω_{d-1} we denote the surface area of the unit sphere in $\mathbb{R}^d$ and by σ the $(d-1)$-dimensional Hausdorff measure in $\mathbb{R}^d$, i.e. $\omega_{d-1} := \sigma(\{x \in \mathbb{R}^d : |x| = 1\})$. As above $r = r(x) := |x|$.

Lemma 9 *(i) If $\varphi \in C_c^1([0,\infty))$, then all the functions*

$$\phi_i(x) := \begin{cases} \varphi(r(x)) \cdot \frac{x_i}{r(x)} & \text{if} \quad x = (x_1, \ldots, x_d) \neq 0 \, , \\ \\ 0 & \text{if} \quad x = 0 \, , \end{cases} \tag{68}$$

$i = 1, \ldots, d$, belong to $C_c^1(\mathbb{R}^d)$.
(ii) If $\phi \in C_c^1(\mathbb{R}^d)$, then all functions

$$\varphi_i(t) := \begin{cases} \frac{1}{\omega_{d-1} \, t^{d-1}} \int_{|x|=t} \phi(x) \cdot \frac{x_i}{r(x)} \, d\sigma(x) & \text{if} \quad t > 0 \, , \\ \\ 0 & \text{if} \quad t = 0 \, , \end{cases} \tag{69}$$

$i = 1, \ldots, d$, belong to $C_c^1([0,\infty))$.

Proof. *Step 1.* Proof of (i). Under the given assumption we immediately get $\phi_i \in C^1(\mathbb{R}^d \setminus \{0\})$ and supp ϕ_i is compact. Hence, we have to study the regularity properties in the origin. Obviously, $\phi_i(0) = 0$ and $\lim\limits_{x \to 0} \phi_i(x) = 0$. We claim that

$$\frac{\partial \phi_i}{\partial x_j}(0) = \begin{cases} \varphi'(0) & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Let $e_1, \ldots \, e_d$ denote the elements of the canonical basis of $\mathbb{R}^d$. If $i = j$, then

$$\lim_{t \to 0} \frac{\phi_i(te_i)}{t} = \lim_{t \to 0} \frac{\varphi(|t|)}{|t|} = \varphi'(0) \, .$$

Hence $\dfrac{\partial \phi_i}{\partial x_i}(0) = \varphi'(0)$. The cases $i \neq j$ are obvious.

Next, we show, that the functions $\dfrac{\partial \phi_i}{\partial x_j}$ are continuous in the origin. To begin with we investigate the case $i = j$. Then

$$\begin{aligned}\left|\frac{\partial \phi_i}{\partial x_i}(x) - \varphi'(0)\right| &= \left|\varphi'(r(x)) \cdot \frac{x_i^2}{r^2(x)} + (\varphi(r(x)) - \varphi(0)) \cdot \frac{r^2(x) - x_i^2}{r^3(x)} - \varphi'(0)\right| \\ &= \left|\varphi'(r(x)) \cdot \frac{x_i^2}{r^2(x)} + \varphi'(\theta r(x)) \cdot \frac{r^2(x) - x_i^2}{r^2(x)} - \varphi'(0)\right|,\end{aligned}$$

where we have used the Mean Value Theorem with a suitable $0 < \theta = \theta(x) < 1$. The continuity of φ' implies, that the expression on the right-hand side tends to zero if $x \to 0$. If $i \neq j$, we write

$$\begin{aligned}\frac{\partial \phi_i}{\partial x_j}(x) &= \varphi'(r(x))\frac{x_i x_j}{r^2(x)} - \varphi(r(x))\frac{x_i x_j}{r^3(x)} \\ &= \varphi'(r(x))\frac{x_i x_j}{r^2(x)} + (\varphi(0) - \varphi(r(x)))\frac{x_i x_j}{r^3(x)} \\ &= \frac{x_i x_j}{r^2(x)}\left[\varphi'(r(x)) - \varphi'(\theta r(x))\right], \qquad (70)\end{aligned}$$

with some $0 < \theta < 1$. Again the continuity of φ' implies, that the expression in (70) tends to zero as $x \to 0$. Hence, $\phi_i \in C_c^1(\mathbb{R}^d)$.

Step 2. Proof of (ii). The regularity and support properties of φ_i on $(0, \infty)$ are obvious. Hence, we are left with the study of the behaviour near 0. We shall use the identity

$$\int_{|x|=t} \frac{x_i}{r(x)}\, d\sigma(x) = 0\,, \qquad t > 0\,.$$

In case $t > 0$ this leads to the estimate

$$\begin{aligned}|\varphi_i(t)| &\leq \frac{1}{\omega_{d-1} t^{d-1}}\left(\left|\int_{|x|=t} \phi(0)\, \frac{x_i}{r(x)} d\sigma(x)\right| + \left|\int_{|x|=t} (\phi(x) - \phi(0))\, \frac{x_i}{r(x)}\, d\sigma(x)\right|\right) \\ &\leq 0 + \sup_{|x|=t} |\phi(x) - \phi(0)|.\end{aligned}$$

Hence, $\varphi_i(t)$ tends to 0 if $t \to 0^+$. Furthermore,

$$\begin{aligned}\frac{\varphi_i(t)}{t} &= \frac{1}{\omega_{d-1}\, t^d} \int_{|x|=t} (\phi(x) - \phi(0))\, \frac{x_i}{r(x)}\, d\sigma(x) \\ &= \frac{1}{\omega_{d-1}\, t^{d+1}} \int_{|x|=t} (\nabla\phi(\eta_x x) \cdot x)\, x_i\, d\sigma(x) \\ &= \frac{1}{\omega_{d-1}\, t^{d+1}} \sum_{j=1}^{d} \int_{|x|=t} \left(\frac{\partial \phi}{\partial x_j}(\eta_x x) - \frac{\partial \phi}{\partial x_j}(0)\right) x_j\, x_i\, d\sigma(x) \\ &\qquad + \frac{1}{\omega_{d-1}\, t^{d+1}} \sum_{j=1}^{d} \frac{\partial \phi}{\partial x_j}(0) \int_{|x|=t} x_j\, x_i\, d\sigma(x)\end{aligned}$$

with some $0 < \eta_x < 1$. The first term on the right-hand side tends always to zero as $t \to 0^+$ (since $\dfrac{\partial \phi}{\partial x_j}$ is continuous). From the second term those summands with

$i \neq j$ are vanishing for all t. If $i = j$, then the integrand is homogeneous. We obtain, by taking the limit with respect to t,

$$\varphi_i'(0) = \lim_{t \to 0^+} \frac{\varphi_i(t)}{t} = \frac{1}{\omega_{d-1}} \frac{\partial \phi}{\partial x_i}(0) \int_{|y|=1} y_i^2 \, d\sigma(y) \, . \tag{71}$$

It remains to check the limit of $\varphi_i'(t)$ if t tends to 0^+. Observe

$$\varphi_i'(t) = \frac{1}{\omega_{d-1}} \frac{d}{dt} \int_{|y|=1} \phi(ty) \, y_i \, d\sigma(y) = \frac{1}{\omega_{d-1}} \sum_{j=1}^{d} \int_{|y|=1} \frac{\partial \phi}{\partial y_j}(ty) \, y_j \, y_i \, d\sigma(y) \, .$$

Since $\displaystyle\int_{|y|=1} y_j \, y_i \, d\sigma(y) = 0$ if $i \neq j$ those summands (with $i \neq j$) tend to 0 if t tends to 0^+. Hence

$$\lim_{t \to 0^+} \varphi_i'(t) = \lim_{t \to 0^+} \frac{1}{\omega_{d-1}} \int_{|y|=1} \frac{\partial \phi}{\partial y_i}(ty) \, y_i^2 \, d\sigma(y) = \varphi'(0) \, ,$$

see (71). The proof is complete. ■

Proof of Theorem 12.

Step 1. Let $f(x) = g(|x|) \in BV(\mathbb{R}^d)$. We claim that $g \in BV(\mathbb{R}^+, t^{d-1})$. Let $\mu_1, \ldots, \mu_d$ denote the corresponding signed Radon measures according to Definition 8. By means of

$$d\nu := \sum_{i=1}^{d} \frac{x_i}{r(x)} \, d\mu_i$$

we define the measure ν on $\mathbb{R}^d$. Since $g(r(x))$ is radial we conclude $\mu_i(\{0\}) = 0$, $i = 1, \ldots, d$. Hence the measure $d\nu$ is well defined. In addition we introduce a measure ν^+ on $\mathbb{R}^+$ by

$$\omega_{d-1} \int_A t^{d-1} \, d\nu^+(t) := \int_{\{|x| \in A\}} d\nu(x),$$

for any Lebesgue measurable subset $A \subset \mathbb{R}^+$. We fix $\varphi \in C_c^1([0,\infty))$. Since $\varphi(r(x))\frac{x_i}{r(x)} \in C_c^1(\mathbb{R}^d)$, cf. Lemma 9(i), we calculate

$$
\begin{aligned}
\omega_{d-1} \int_0^\infty g(t)\,[\varphi(s)\,s^{d-1}]'(t)\,dt &= \omega_{d-1} \int_0^\infty t^{d-1}\,g(t)\left[\varphi'(t) + \frac{d-1}{t}\varphi(t)\right]\,dt \\
&= \int_{\mathbb{R}^d} g(r(x))\,\left[\varphi'(r(x)) + \varphi(r(x)) \cdot \frac{d-1}{r(x)}\right]\,dx \\
&= \sum_{i=1}^d \int_{\mathbb{R}^d} g(r(x))\,\left[\varphi'(r(x))\frac{x_i^2}{r^2(x)} + \varphi(r(x)) \cdot \frac{r^2(x)-x_i^2}{r^3(x)}\right]\,dx \\
&= \sum_{i=1}^d \int_{\mathbb{R}^d} g(r(x))\,\frac{\partial}{\partial x_i}\,\left[\varphi(r(x)) \cdot \frac{x_i}{r(x)}\right]\,dx \\
&= -\sum_{i=1}^d \int_{\mathbb{R}^d} \varphi(r(x))\,\frac{x_i}{r(x)}\,d\mu_i = -\int_{\mathbb{R}^d} \varphi(r(x))\,d\nu(x) \\
&= -\omega_{d-1} \int_0^\infty t^{d-1}\,\varphi(t)\,d\nu^+(t)\,.
\end{aligned}
$$

This proves (18). Moreover, (19) follows from

$$
\|\, g(r(x))\,|L_1(\mathbb{R}^d)\| = \omega_{d-1}\,\|\, g\,|L_1(\mathbb{R}, |t|^{d-1})\|
$$

and

$$
\begin{aligned}
\omega_{d-1} \int_0^\infty t^{d-1}\,d|\nu^+|(t) &= \int_{\mathbb{R}^d} d|\nu|(x) \le \sum_{i=1}^d \int_{\mathbb{R}^d} \frac{|x_i|}{r(x)}\,d|\mu_i| \\
&\le \sum_{i=1}^d \int_{\mathbb{R}^d} d|\mu_i| \le \|\, g(r(x))\,|BV(\mathbb{R}^d)\|\,.
\end{aligned}
$$

Step 2. Let g be a function in $BV(\mathbb{R}^+, t^{d-1})$. We claim, that $g(r(x)) \in BV(\mathbb{R}^d)$. Let ν^+ be the signed Radon measure associated to g according to (18). We define

$$
\nu(A) := \int_0^\infty \sigma(\{x : |x| = t\} \cap A)\,d\nu^+(t)
$$

for any Lebesgue measurable set $A \subset \mathbb{R}^d$. Further we put $\mu_i := \frac{x_i}{r(x)}\nu$, $i = 1, \ldots d$. Let χ_A denote the characteristic function of A. Then

$$
\nu(A) = \int_{\mathbb{R}^d} \chi_A(x)\,d\nu(x) = \int_0^\infty \Big[\int_{|x|=t} \chi_A(x)\,d\sigma(x)\Big]\,d\nu^+(t)
$$

and this identity can be extended to

$$
\int_{\mathbb{R}^d} \phi(x)\,d\nu(x) = \int_0^\infty \Big[\int_{|x|=t} \phi(x)\,d\sigma(x)\Big]\,d\nu^+(t)\,, \qquad \phi \in L_1(\mathbb{R}^d)\,,
$$

by using some standard arguments. Next we want to show, that $\mu_i, i = 1, \ldots, d$, are the weak derivatives of $g(r(x))$. Let $\phi \in C_c^1(\mathbb{R}^d)$ and let φ_i be the associated functions, see (69). According to Lemma 9 (ii) we know that $\varphi_i \in C_c^1([0,\infty))$. Using the normalized outer normal with respect to the surface $\{x : |x| = T\}$, which is obviously given by

$$n(x) = (n_1(x), \ldots, n_d(x)) = \frac{1}{r(x)}\,(x_1, \ldots, x_d)\ ,$$

and the Gauss Theorem, we obtain

$$\begin{aligned} -\varphi_i(T)\,\omega_{d-1}\,T^{d-1} &= -\int_{|x|=T} \phi(x)\,\frac{x_i}{r(x)}\,d\sigma(x) = -\int_{|x|=T} \phi(x)\,n_i(x)\,d\sigma(x) \\ &= -\int_{|x|\le T} \frac{\partial\phi}{\partial x_i}(x)\,dx = \int_{|x|\ge T} \frac{\partial\phi}{\partial x_i}(x)\,dx \\ &= \int_T^\infty \Big[\int_{|x|=t} \frac{\partial\phi}{\partial x_i}(x)\,d\sigma(x)\Big]\,dt. \end{aligned}$$

Hence

$$\omega_{d-1}\,\big[\varphi_i(t)\,t^{d-1}\big]'\,(T) = \int_{|x|=T} \frac{\partial\phi}{\partial x_i}(x)\,d\sigma(x)\,, \qquad T > 0.$$

This formula justifies the identity

$$\begin{aligned} \int_{\mathbb{R}^d} g(r(x))\,\frac{\partial\phi(x)}{\partial x_i}\,dx &= \int_0^\infty g(t)\int_{|x|=t} \frac{\partial\phi(x)}{\partial x_i}\,d\sigma(x)\,dt \\ &= \omega_{d-1}\int_0^\infty g(t)[\varphi_i(s)\,s^{d-1}]'(t)dt\,. \end{aligned}$$

Next we use $g \in BV(\mathbb{R}^+, t^{d-1})$. This implies

$$\begin{aligned} \int_{\mathbb{R}^d} g(r(x))\frac{\partial\phi(x)}{\partial x_i}dx &= -\omega_{d-1}\int_0^\infty \varphi_i(t)\,t^{d-1}\,d\nu^+(t) \\ &= -\int_0^\infty\int_{|x|=t} \phi(x)\cdot\frac{x_i}{r(x)}\,d\sigma(x)\,d\nu^+(t) \\ &= -\int_{\mathbb{R}^d} \phi(x)\,\frac{x_i}{r(x)}\,d\nu(x) = -\int_{\mathbb{R}^d} \phi(x)\,d\mu_i(x)\,, \end{aligned}$$

which proves that the μ_i are the weak derivatives of $g(r(x))$.
It remains to prove the estimates for the related norms. The required estimate follows easily by

$$\begin{aligned} \int_{\mathbb{R}^d} d|\mu_i| &= \int_{\mathbb{R}^d} \frac{|x_i|}{r(x)}\,d|\nu|(x) = \int_0^\infty \Big[\int_{|x|=t} \frac{|x_i|}{r(x)}\,d\sigma(x)\Big]\,d|\nu^+|(t) \\ &\le \omega_{d-1}\int_0^\infty t^{d-1}\,d|\nu^+|(t)\,. \end{aligned}$$

The proof is complete. ■

4.2.2 Proof of Theorem 11

Recall, that we will work with the particular representative $\tilde{f}$ of the equivalence class $[f]$, see Remark 11. For convenience we will drop the tilde. We shall apply standard mollifiers. Let $\varphi \in C_0^\infty(\mathbb{R})$ be a function such that $\varphi \geq 0$, supp $\varphi \subset [0,1]$, and $\int \varphi(t)\,dt = 1$. For $R > 0$ and $\varepsilon > 0$ we define

$$\varphi_\varepsilon(t) := \varepsilon^{-1} \int_R^\infty \varphi\Big(\frac{t-y}{\varepsilon}\Big)\,dy = \int_{-\infty}^{\frac{t-R}{\varepsilon}} \varphi(z)\,dz \tag{72}$$

(which is nothing but the mollification of the characteristic function of the interval (R,∞)). In addition we need a cut-off function. Let $\eta \in C_0^\infty(\mathbb{R})$ s.t. $\eta(t) = 1$ if $|t| \leq 1$ and $\eta(t) = 0$ if $|t| \geq 2$. For $M \geq 1$ we define $\eta_M(t) := \eta(t/M)$, $t \in \mathbb{R}$. It is easily checked that the functions

$$\phi_{M,\varepsilon}(t) := t^{1-d}\,\varphi_\varepsilon(t)\,\eta_M(t)\,, \qquad t \in \mathbb{R}\,,$$

belong to $C_c^1([0,\infty))$. For $g \in BV(\mathbb{R}^+, t^{d-1})$ this implies

$$\int_0^\infty g(t)\,[\phi_{M,\varepsilon}(s)\,s^{d-1}]'(t)\,dt = -\int_0^\infty \varphi_\varepsilon(t)\,\eta_M(t)\,d\nu^+(t)\,, \tag{73}$$

see (18). Since for $M > R+\varepsilon$ we have

$$\begin{aligned}\int_0^\infty g(t)\,[\phi_{M,\varepsilon}(s)\,s^{d-1}]'(t)\,dt &= \int_R^{R+\varepsilon} g(t)\,\varepsilon^{-1}\,\varphi\Big(\frac{t-R}{\varepsilon}\Big)\,dt \\ &\quad + M^{-1}\int_M^\infty g(t)\,\varphi_\varepsilon(t)\,\psi'(t/M)\,dt\end{aligned}$$

and

$$\lim_{M\to\infty} M^{-1}\int_M^\infty g(t)\,\varphi_\varepsilon(t)\,\psi'(t/M)\,dt = 0$$

($g \in L_1(\mathbb{R}^+, t^{d-1})$), we get

$$\lim_{\varepsilon\downarrow 0, M\to\infty} \int_0^\infty g(t)\,[\phi_{M,\varepsilon}(s)\,s^{d-1}]'(t)\,dt = g(R)\,, \tag{74}$$

if R is a Lebesgue point of g. But

$$\Big|\int_0^\infty \varphi_\varepsilon(t)\,\eta_M(t)\,d\nu^+(t)\Big| \leq \int_R^\infty \varphi_\varepsilon(t)\,d|\nu^+|(t) \leq R^{1-d}\int_R^\infty t^{d-1}\,d|\nu^+|(t)\,.$$

Combining (74), (73) with this estimate we have proved (16) and (17) simultaneously. ■

4.3 Proof of the assertions in Subsection 2.2.3

4.3.1 Proof of Lemma 3

Sufficiency of the conditions is obvious, see e.g. [34]. Necessity follows from the examples investigated in Lemma 8. ■

4.3.2 Proof of Theorem 13

We argue by using the atomic characterizations in Subsection 3.3.2.

Step 1. Proof of (i). For simplicity let $|x| = 2^{-r}$, $r \in \mathbb{N}$. If y satisfies $2^{-r-1} \leq |x| \leq 2^{-r+1}$, then, using the support condition for atoms, we know that f allows an optimal atomic decomposition such that

$$f(y) = \sum_{j=0}^{\infty} \sum_{k=\max(0,[2^{j-r}]-n_0)}^{[2^{j-r}]+n_0} \sum_{\ell=1}^{C(d,k)} s_{j,k}\, a_{j,k,\ell}(y)\,. \tag{75}$$

Here n_0 is a general natural number depending on the decomposition Ω, but not on r.

Substep 1.1. We assume first that $\frac{1}{p} < s < \frac{d}{p}$. From the L_∞-estimate of the atoms, property (f) of the coverings $(\Omega_{j,k,\ell})_{j,k,\ell}$, and the inequality (36) we derive

$$\begin{aligned}
|f(y_1,0,\ldots,0)| &\leq \sum_{j=0}^{\infty} \sum_{k=\max(0,[2^{j-r}]-n_0)}^{[2^{j-r}]+n_0} \sum_{\ell=1}^{K} |s_{j,k}|\, |a_{j,k,\ell}(y_1,0,\ldots,0)| \\
&\lesssim \quad \Big(\sum_{j=0}^{r+n_1} \sum_{k=0}^{n_2} |s_{j,k}|\, 2^{-j(s-d/p)} + \sum_{j=r+n_1+1}^{\infty} \sum_{k=2^{j-r}-n_0}^{2^{j-r}+n_0} |s_{j,k}|\, 2^{-j(s-d/p)} \Big) \\
&\lesssim \quad \|\, f\,|RB^s_{p,\infty}(\mathbb{R}^d)\| \Big(\sum_{j=0}^{r+n_1} 2^{-j(s-d/p)} + \sum_{j=r+n_1+1}^{\infty} 2^{-(j-r)(d-1)/p}\, 2^{-j(s-d/p)} \Big) \\
&\lesssim \quad 2^{r(\frac{d}{p}-s)}\, \|\, f\,|RB^s_{p,\infty}(\mathbb{R}^d)\|\ ,
\end{aligned}$$

for appropriate natural numbers n_1, n_2 (independent of r). For the last two steps of the estimate we used $1/p < s < d/p$. Taking into account the elementary embedding $A^s_{p,q}(\mathbb{R}^d) \hookrightarrow B^s_{p,\infty}(\mathbb{R}^d)$ we obtain the inequality (20).

Substep 1.2. Now, let $s = \frac{1}{p}$. For the Besov spaces $RB^{1/p}_{p,1}(\mathbb{R}^d)$ the inequality (20) was proved in [31]. So it remains to consider the Lizorkin -Triebel spaces $RF^{1/p}_{p,\infty}(\mathbb{R}^d)$ with $0 < p \leq 1$. For simplicity we regard the main part of $f \in RF^{1/p}_{p,\infty}(\mathbb{R}^d)$, cf. (75) and compare with (63). Now $k_j = 1$ if $0 \leq j \leq r + n_1$ and $k_j \sim |x| 2^j$ if $j > r + n_1$. Hence, we obtain

$$\begin{aligned}
|f^M(|x|,0,\ldots,0)| &\leq \sum_{j=0}^{\infty} |s_{j,k_j}|\, |a_{j,k_j,0}(y)| \ \lesssim\ \Big(\sum_{j=0}^{\infty} |s_{j,k_j}|^p\, 2^{j(d-1)} \Big)^{1/p} \\
\lesssim \quad |x|^{\frac{1-d}{p}} \Big(\sum_{j=0}^{r+n_1} |s_{j,k_j}|^p &+ \sum_{j=r+n_1+1}^{\infty} |s_{j,k}|^p\, k_j^{d-1} \Big)^{1/p} \ \lesssim\ |x|^{\frac{1-d}{p}}\, \|\, f\,|RF^{1/p}_{p,\infty}(\mathbb{R}^d)\|\ ,
\end{aligned}$$

where we used $p \leq 1$ for the second step and (65)-(66) for the last one.

Step 2. Proof of (ii). By using elementary embeddings it will be enough to prove (21) with $RA^s_{p,q}(\mathbb{R}^d) = RB^s_{p,q}(\mathbb{R}^d)$ and q small. Again we concentrate on $|x| = 2^{-r}$,

$r \in \mathbb{N}$. Our test function is taken to be $2^{-r(s-d/p)} f_{j,\lambda}$, see Lemma 7, where we choose $j := 1 + r$ and $\lambda := 1$. Then it follows

$$\| \, 2^{-r(s-d/p)} f_{j,\lambda} \, | B^s_{p,q}(\mathbb{R}^d)\| \asymp 1 \qquad \text{and} \qquad f_{j,\lambda}(x) = 2^{-r(s-d/p)} = |x|^{s-d/p} \, .$$

The proof is complete. ■

4.3.3 Proof of Lemma 4

The arguments are the same as in proof of Theorem 10(iv). ■

4.4 Proof of Theorem 14

We shall use the notation from the proof of Theorem 13, Step 1. Again we employ the formula (75) and obtain

$$|f(y_1, 0, \ldots, 0)| \lesssim \Big(\sum_{j=0}^{r+n_1} \sum_{k=0}^{n_2} |s_{j,k}| + \sum_{j=r+n_1+1}^{\infty} \sum_{k=2^{j-r}-n_0}^{2^{j-r}+n_0} |s_{j,k}| \Big)$$

since $s = d/p$.

Step 1. Proof of (i). We shall use the standard abbreviation $q' := q/(q-1)$. Since $q > 1$ we can use Hölder's inequality and conclude

$$\sum_{j=0}^{r+n_1} \sum_{k=0}^{n_2} |s_{j,k}| \lesssim r^{1/q'} \Big(\sum_{j=0}^{r+n_1} \sum_{k=0}^{n_2} |s_{j,k}|^q \Big)^{1/q} \lesssim (-\log |x|)^{1/q'} \, \| \, f \, | B^{d/p}_{p,q}(\mathbb{R}^d)\| \qquad (76)$$

as well as

$$\begin{aligned} \sum_{j=r+n_1+1}^{\infty} \sum_{k=2^{j-r}-n_0}^{2^{j-r}+n_0} |s_{j,k}| &\lesssim \sum_{j=r+n_1+1}^{\infty} 2^{-(j-r)(d-1)/p} \Big(\sum_{k \ge 2^{j-r}-n_0} (1+|k|)^{d-1} \, |s_{j,k}|^p \Big)^{1/p} \\ &\lesssim \Big(\sum_{j=r+n_1+1}^{\infty} \Big(\sum_{k \ge 2^{j-r}-n_0} (1+|k|)^{d-1} \, |s_{j,k}|^p \Big)^{q/p} \Big)^{1/q} \Big(\sum_{j=r+n_1+1}^{\infty} 2^{-(j-r)q'(d-1)/p} \Big)^{1/q'} \\ &\lesssim \| \, f \, | B^{d/p}_{p,q}(\mathbb{R}^d)\| \, . \end{aligned} \qquad (77)$$

The inequalities (76) and (77) yield (22).

Step 2. Proof of (ii). Let $1 < p < p_0 < \infty$. The Jawerth-Franke embedding $F^{d/p}_{p,1}(\mathbb{R}^d) \hookrightarrow B^{d/p_0}_{p_0,p}(\mathbb{R}^d)$, see [19] or [34], combined with (22) proves (23). ■

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